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ABSTRACT

This paper is a preliminary version of a review of the results obtained by the authors and their collaborators which mainly concern problems of quantum electrodynamics with the pair-creating external field.

In this paper the Furry picture is constructed for quantum electrodynamics with the pair-creating external field. It is shown, that various Green functions in the external field arise in the theory in a natural way. Special features of usage of the unitarity conditions for calculating the total probabilities of transitions are discussed. Perturbation theory for determining the mean electromagnetic field is constructed. Effective Lagrangians for pair--creating fields are built. One of possible ways to introduce external field in quantum electrodynamics is considered.

All the Green functions arising in the theory suggested are calculated for a constant field and a plane wave field. For the case of the electric field the total probability of creation of pairs from the vacuum accompanied by the photon irradiation and the total probability of transition from a single-electron state accompanied by the photon irradiation and creation of pairs are obtained by using the formulated rules for calculating the total probabilities of transitions.

АННОТАЦИЯ

Настоящая статья является предварительным вариантом обзора результатов авторов и их соавторов по проблемам квантовой электродинамики во внешнем поле, рождающем пары. Для таких полей построена картина Фарри. Показано, что естественным образом в теории возникают различные функции Грина, для которых нами построена полная система уравнений. Получена диаграммная техника для нахождения среднего поля. Подробно обсуждаются условия унитарности. Найден эффективный Лагранжиан для полей, рождающих пары. Все функции Грина, возникающие в теории, вычислены в постоянном поле, в поле плоской волны и в их комбинации.

Эффективность предложенной теории продемонстрирована на примерах вычислений различных эффектов во внешнем поле, рождающем пары.

KIVONAT

Ez a közlemény a kvantum elektrodinamika keretén belül a külső téren való párkeltés problémájával foglalkozik. Furry-képben előállitjuk a Green-függvényeket és perturbációszámitással meghatározzuk az átlagos elektromágneses teret. Ezután megadjuk a párkeltő terek effektiv Lagrange-függvényét. A külső terek kvantum elektrodinamikába való bevezetésének egy lehetséges módját is javasoljuk. Homogén és sikhullám külső tér esetére a javasolt elmélet összes Green-függvényeit és különböző átmeneti valószinüségeket kiszámitunk.

CONTENTS

INTRODUCTION

During the recent years the growing interest is attached to the problems of quantum electrodynamics(QED) with intense electromagnetic field. To some extent this interest is due to the achieving of strong fields in experimental conditions, further growth of the laser intensities and recognition of some situations in astrophysics where the values of the effective fields are tremendous, indeed. This interest is also provoked by the existence of analogies with problems in gravitation and in gauge theories with spontaneous symmetry breaking. In this connection solving of similar problems in QED may be thought of as, in a way, the first step in solving these problems in the mentioned theories. Finally, results for specified problems in QED with intense electromagnetic field are important for checking of its validity in the extreme domains of parameters and undoubtly are of general scientific value.

In the present paper we will consider special features of constructing of QED formalism, which are connected with the possibility of particle creation in an intense electromagnetic field, and also clarify various aspects of the external field conception in QED.

Thus, if one discusses problems of QED with an intense electromagnetic field in the frame of QED with an external field, then one of the most important is here the problem of how to keep exactly the interaction with the external field to all the orders of perturbation expansion. This problem has been investigated weel, for instance, for the spinor or scalar charged fields interacting with the external electromagnetic field (Feynman, 1949; Schwinger, 1951; 1954 a,b). During the recent years the growing interest is attached to it due to the examination of processes of particle creation from the vacuum by the external field both in electrodynamics and in gravitation (Nikishov, 1969, 1972; Popov, 1971; Bagrov, Gitman, Schwartsman, 1975, 1976; Sexl, Urbantke, 1969; Hawking, 1975; Grib, Mostepanenko, Frolov, 1972, 1976, Parker, 1976; Frolov, Gitman, 1978). For the total QED of the interacting quantized spinor and electromagnetic fields a consistent consideration of the perturbation theory when keeping exactly the external field is fulfilled(see more detail Ritus 1979) only for the fields which do not product pairs (Furry,1951). The restrictions on the field nature arise in the starting Furry approach, in particular, due to the fact that in accordance with Furry the particle and antiparticle creation and annihilation operators are built with the aid of the solutions of the Dirac equation in the external field. For the pair-creating fields there are, however, no such solutions, which may be put into correspondence to particles or antiparticles at all the time-moments. In Chapter I of the present paper we have discussed the generalisation of the Furry picture to pair-creating fields (see also Gitman, 1977; Fradkin, Gitman, 1978). In particular, a method of constructing of the vacua for the initial and final states in an intense external field is given here; perturbation theory with respect to the radiative iteraction is built with the aid of the Wick's technique generalization, wich is written out in appendix A; the quantum field representation for the electron propagator in the external field and its representation over the solutions of the Dirac equation are obtained; at the same time a consistent description is given of all the zeroth order with respect to the radiative interaction processes for an arbitrary pair-creating external field. The unitarity conditions are analysed for the case under consideration, and it is shown (see also Fradkin, Gitman, 1978; Gavrilov, Gitman, Shwartsman, 1979) that in the relations similar to the optical theorem the usage of the two types of electron propagators is essential. In accordance with these results for the pair-creating fields there arises the necessity to distinguish, for example, the two types of mass operators, the one of which discribes radiative corrections to the scattering processes and the second one is connected with the total probability of irradiation from a single-electron state.

In Chapter II an attempt is made to treat some problems of QED with the intense electromagnetic field. The matter is that the applicableness of QED with the external field is connected with a number of implicitly made assumptions. Firstly it is supposed that the real electromagnetic field in the problem may be identified with some external field which is given beforehand and does not depend on the processes proceeding in the system. Secondly the belief exists that we get, by introducing in the Lagrangian the interaction with such external field in the usual way, a theory in which the calculation of the radiative corrections makes sense to arbitrary order, moreover the accuracy of the theory itself will not be exceeded. Meanwhile the radiative corrections can contribute sufficiently in the case of large energy and intense fields and change considerably the primary given field. The proof of the second one from the above mentioned assumptions is also unknown. Therefore the approach based on the external field conception requires, undoubtly, a consistent formal substantiation, establishing of the bounds of its applicableness and ascertaining of the sense of the external field being introduced. The one of the possible ways to clarify the indicated questions is to start with QED without external field and under one or another assumption "derive" formally the QED with the external field from it. On following this way we have obtained some results. Thus, in Sec.I of Chapter II the vacuum, initial and final states are built under the condition that there is an intense mean electromagnetic field in the system. It is shown, how the problem reduces to the solving of a problem in the external field.

Here the problem of determining of the exact mean electromagnetic field in the system proved to be, in connection with the questions discussed, highly important. In Sec.2 of Chapter II we have constructed the generating functional which enables to obtain the exact mean field when arbitrary initial states are chosen. Its representation, equivalent to the perturbation theory with respect to the radiative interaction by the exact keeping the interaction with the initial mean field and external current, is given. We succeeded, by introducing matrix propagators which are composed from a whole number of propagators in the external field, in giving to the obtained theory the Feynman form. The effective Lagrangian for the exact mean field is constructed.

In Sec.3 of Chapter II the perturbation theory for matrix elements of processes is being constructed under the assumption that an intense electromagnetic field is present in the initial and final states and the system interacts in addition with an intense external current , the latter being related to the particles which are, in some approximation, external with respect to QED. When doing so the generalization of the Wick technique to unstable with respect to the particle creation vacuum is used essentially.

At last, transitions into the final states, for which the mean field is defined as the zeroth-order approximation for the exact mean field at the final time-moment with respect to the radiative interaction, are considered apart. It is shown that in this case the perturbation expansions for matrix elements of transitions coincide completely with the perturbation expansions in Furry approach to QED with the external field, equals to the mean field of the zeroth order approximation. In our opinion, this statement may be the basis for one of the possible interpretations of QED with the external field.

In appendix A we give the generalization of the Wick technique to the case when vacuum is unstable during the evolution. The definitions of the normal ordering, normal form of couplings and chronological couplings are generalized. The formulations of the Wick theorem are given, which make it possible to reduce any operator to the generalized normal form.

In appendix B the calculations of various Green functions which appear in Furry approach to QED with the external pair-creating field are presented, moreover a constant electric field and its combination with a magnetic field and a plane wave field are taken as an example. When performing these calculations we proceeded from the representations of the Green functions over the solutions of the Klein-Gordon or Dirac equation which follow from the original field theory definitions. All the results are given in the form of contour integrals in the proper time complex plane. The corresponding inverse operators for the Klein-Gordon and Dirac equations are obtained on the strength of these results.

In appendix C we write out the calculations of the total probability of the photon irradiation from the vacuum accompanied by creation of pairs and the total probability of transition from a single-electron state accompanied by the photon irradiation and creation of pairs, which are performed by cutting the diagrams. For this purpose the vacuum diagram and the diagram of the type of mass operator with the noncausal Green function are calculated according to the general theory, which is stated in Sec.2 of Chapter II. The checking of the validity of the results obtained in this way is carried out by the straightforward summing of the probabilities of transitions.

Appendix B is carried out in common with S.P. Gavrilov, Sh.M. Shwartsman and J.J. Wolfengaut, and appendix C is carried out by S.P.Gavrilov, Sh.M.Shwartsman, J.J.Wolfengaut. and D.M. Gitman.

Note in conclusion that each section in the paper has independent formulae numbering. When making reference to a formula from the same section its original number is given. When making reference to a formula from the other section of the same chapter the number of the section is placed to the left of its number. When making reference to a formula from the other chapter or from an appendix the number of the chapter or of the appendix is placed to the left of its number.

CHAPTER I. QUANTUM ELECTRODYNAMICS WITH EXTERNAL FIELD CREATING PAIRS

I. Furry approach

a) Vacuum, initial and final states.

Consider QED with an external field $A^{ext}(x)$. The corresponding Hamiltonian is

$$\mathcal{H}_A = \int :\overline{\psi}(\vec{x})[-i\vec{\gamma}\vec{\nabla} + e\hat{A}^{ext}(x) + m]\psi(\vec{x}): d\vec{x} -$$

$$-\sum_{\vec{k}\lambda} g_{\lambda\lambda} c^+_{\vec{k}\lambda} c_{\vec{k}\lambda} + \int j(\vec{x})A(\vec{x})d\vec{x} = \mathcal{H}_{eA} + \mathcal{H}_{\gamma} + \mathcal{H}_{e\gamma}, \qquad (1)$$

$$j(\vec{x}) = \frac{e}{2}[\overline{\psi}(\vec{x})\gamma, \psi(\vec{x})],$$

where $\psi(\vec{x})$, $\overline{\psi}(\vec{x})$, $A(\vec{x})$ are the spinor and electromagnetic field operators in the Schrodinger picture.

Let t_{in} and t_{out} be the initial and the final moments of time which in the final expressions will be understood as moved to infinitely remote past and future, respectively. If the vector potential of the external field is switched off in the moments t_{in}, t_{out} one may, as usual, assume that since the radiative interaction is effectively switched off when $t \to \pm\infty$ the initial and final states are free states with, say, a definite particle numbers

$$|in> = N c^+ \ldots b^+ \ldots a^+ \ldots |0>, \quad <out| = <0|a \ldots b \ldots c \ldots N, \qquad (2)$$

where $\{a^+, a, b^+, b\}$ are operators of creation and annihilation of free electrons and positrons, $\{c^+, c\}$ are the photon creation and annihilation operators, $|0>$ is the vacuum of free particles, $|0> = |0>^e \cdot |0>^\gamma$; $|0>^e$, $|0>^\gamma$ are the corresponding vacuum vectors in the Hilbert spaces of states of the spinor and electromagnetic fields, N is a normalizing factor.

Consider now a more general case when the vector potential of the external field does not disappear at t_{in}, t_{out} and propose a method of determination of vacuum, initial and final states.

Define the vacuum at the initial (final) time-moment as the state $|0\rangle_{in}$, $(|0\rangle_{out})$ which minimizes the mean value of the Hamiltonian $\mathcal{H}_A$ taken at t_{in} (t_{out}):

$$
\begin{aligned}
&{}_{in}\langle 0|\mathcal{H}_A|0\rangle_{in} - min, \quad t = t_{in}, \\
&{}_{out}\langle 0|\mathcal{H}_A|0\rangle_{out} - min, \quad t = t_{out},
\end{aligned} \tag{3}
$$

Since t_{in}, t_{out} should be understood as moved to the infinitely remote past and future time-moments one can, as before, assume that the radiative interaction is effectively switched off at t_{in}, t_{out} and therefore the problem (3) may be reduced to the following

$$
\begin{aligned}
&{}_{in}^{e}\langle 0|\mathcal{H}_{eA}|0\rangle_{in}^{e} - min, \quad t = t_{in}, \\
&{}_{out}^{e}\langle 0|\mathcal{H}_{eA}|0\rangle_{out}^{e} - min, \quad t = t_{out},
\end{aligned} \tag{4}
$$

$$
|0\rangle_{in} = |0\rangle^{\gamma}\cdot|0\rangle_{in}^{e}, \quad |0\rangle_{out} = |0\rangle^{\gamma}\cdot|0\rangle_{out}^{e}. \tag{5}
$$

To find the vectors $|0\rangle_{in}^{e}$, $|0\rangle_{out}^{e}$ suffice it to have the solutions of the eigenvalue problems for the single-particle Dirac Hamiltonian in the external fields $A^{ext}(\vec{x}, t_{in})$ and $A^{ext}(\vec{x}, t_{out})$

$$
\begin{aligned}
&\mathcal{H}_D(t_{in})\,{}_{\pm}\varphi_n(\vec{x}) = {}_{\pm}\mathcal{E}_n\cdot{}_{\pm}\varphi_n(\vec{x}), \\
&\mathcal{H}_D(t_{out})\cdot{}^{\pm}\varphi_m(\vec{x}) = {}^{\pm}\mathcal{E}_m\,{}^{\pm}\varphi_m(\vec{x}), \\
&\mathcal{H}_D(t) = \gamma^0(-i\vec{\gamma}\vec{\nabla} + e\hat{A}^{ext}(\vec{x}, t) + m),
\end{aligned} \tag{6}
$$

which obey the following requirements:

i) ${}_{\pm}\mathcal{E}_n \gtrless 0$, ${}^{\pm}\mathcal{E}_m \gtrless 0$, $\forall m, n$ and there is a gap between the negative and the posotive levels;

ii) The spinors $\{{}_{\pm}\varphi_n(\vec{x})\}$, $\{{}^{\pm}\varphi_m(\vec{x})\}$ form complete and orthonormal sets of functions in the space of $\vec{x}$ - dependent spinors. For example, for $\{{}_{\pm}\varphi_n(\vec{x})\}$

$$
\begin{aligned}
&({}_{\pm}\varphi_n, {}_{\pm}\varphi_{n'}) = \delta_{n,n'}, \quad ({}_{\pm}\varphi_n, {}_{\mp}\varphi_{n'}) = 0, \quad (\varphi\psi) = \int \varphi^+(\vec{x})\psi(\vec{x})d\vec{x}, \\
&\sum_n [{}_{+}\varphi_n(\vec{x})\,{}_{+}\varphi_n^+(\vec{x}') + {}_{-}\varphi_n(\vec{x})\,{}_{-}\varphi_n^+(\vec{x}')] = \delta(\vec{x} - \vec{x}').
\end{aligned} \tag{7}
$$

iii) The spinors $\{{}_{\pm}\varphi_n(\vec{x})\}$, $\{{}^{\pm}\varphi_m(\vec{x})\}$ obey the conditions

$$\sum_{nn'}\left\{|({}_+\varphi_n,\ {}_-\varphi^{o}_{n'})|^2+|({}_-\varphi_n,\ {}_+\varphi^{o}_{n'})|^2\right\}<\infty,$$
$$\sum_{mm'}\left\{|({}^+\varphi_m,\ {}^-\varphi^{o}_{m'})|^2+|({}^-\varphi_m,\ {}^+\varphi^{o}_{m'})|^2\right\}<\infty, \tag{8}$$

where ${}_\pm\varphi^{o}_n(\vec{x})={}_\pm\varphi_n(\vec{x})\Big|_{A^{ext}=0}$, ${}^\pm\varphi^{o}_m(\vec{x})={}^\pm\varphi_m(\vec{x})\Big|_{A^{ext}=0}$

Indeed, let us decompose, with the aid of (7), the spinor field operators $\Psi(\vec{x})$, $\bar{\Psi}(\vec{x})$ into the sums of the solutions $\{{}_\pm\varphi_n(\vec{x})\}$ and $\{{}^\pm\varphi_m(\vec{x})\}$

$$\Psi(\vec{x})=\sum_n\left\{a_n(in)\,{}_+\varphi_n(\vec{x})+b^+_n(in)\,{}_-\varphi_n(\vec{x})\right\},$$
$$\bar{\Psi}(\vec{x})=\sum_n\left\{a^+_n(in)\,{}_+\bar{\varphi}_n(\vec{x})+b_n(in)\,{}_-\bar{\varphi}_n(\vec{x})\right\}, \tag{9}$$

$$\Psi(\vec{x})=\sum_m\left\{a_m(out)\,{}^+\varphi_m(\vec{x})+b^+_m(out)\,{}^-\varphi_m(\vec{x})\right\},$$
$$\bar{\Psi}(\vec{x})=\sum_m\left\{a^+_m(out)\,{}^+\bar{\varphi}_m(\vec{x})+b_m(out)\,{}^-\bar{\varphi}_m(\vec{x})\right\}. \tag{I0}$$

Then the commutation relations for $\Psi(\vec{x})$ and $\bar{\Psi}(\vec{x})$ and equations (7) lead to the fact that the operators $\{a^+(in),\ a(in),\ b^+(in),\ b(in)\}$, $\{a^+(out),\ a(out),\ b^+(out),\ b(out)\}$ are Fermi creation and annihilation operators. The Hamiltonian $\mathcal{H}_{eA}$ diagonalizes at t_{in}, t_{out} in terms of these operators

$$\mathcal{H}_{eA}(t_{in})=\sum_n\left\{{}_+\varepsilon_n\, a^+_n(in)a_n(in)-{}_-\varepsilon_n\, b^+_n(in)b_n(in)\right\}+\chi(t_{in}),$$
$$\mathcal{H}_{eA}(t_{out})=\sum_m\left\{{}^+\varepsilon_m\, a^+_m(out)a_m(out)-{}^-\varepsilon_m\, b^+_m(out)b_m(out)\right\}+\chi(t_{out}), \tag{II}$$

where $\chi(t_{in})$, $\chi(t_{out})$ are c-numerical constants. Consequently the vectors $|0\rangle^e_{in}$ and $|0\rangle^e_{out}$ satisfy the conditions

$$a_n(in)|0\rangle^e_{in}=b_n(in)|0\rangle^e_{in}=0,\quad \forall n,$$
$$a_m(out)|0\rangle^e_{out}=b_m(out)|0\rangle^e_{out}=0,\quad \forall m. \tag{I2}$$

Equations (I2) have solutions in the original Hilbert space if the operators $\{a^+(in),\ a(in),\ b^+(in),\ b(in)\}$ and $\{a^+(out),\ a(out),\ b^+(out),\ b(out)\}$ are unitary-equivalent to the set of creation and annihilation operators for which the vacuum vector exists in this space (Berezin, 1965). The creation and annihilation operators of free particles $\{a^+,\ a,\ b^+,\ b\}$ constructed with the aid of the spinors $\{{}_{\pm}\varphi_n^{\circ}(\vec{x})\}$

$$\psi(\vec{x}) = \sum_n \{a_n\, {}_{+}\varphi_n^{\circ}(\vec{x}) + b_n^{+}\, {}_{-}\varphi_n^{\circ}(\vec{x})\},$$
$$\bar{\psi}(\vec{x}) = \sum_n \{a_n^{+}\, {}_{+}\bar{\varphi}_n^{\circ}(\vec{x}) + b_n\, {}_{-}\bar{\varphi}_n^{\circ}(\vec{x})\}. \qquad (13)$$

are that kind of operators. The comparison of (9) with (13) gives via the relations (7)

$$A_{n\lambda}(in) = \sum_{m\gamma} (\Phi_{n\lambda,m\gamma} A_{m\gamma} + \Psi_{n\lambda,m\gamma} A_{m\gamma}^{+}), \quad \lambda, \gamma = \pm, \qquad (14)$$
$$A_{n+}(in) = a_n(in), \quad A_{n-}(in) = b_n(in), \quad A_{n+} = a_n, \quad A_{n-} = b_n,$$

$$\Phi_{n+,m+} = ({}_{+}\varphi_n, {}_{+}\varphi_m^{\circ}), \quad \Phi_{n+,m-} = \Phi_{n-,m+} = 0, \quad \Phi_{n-,m-} = ({}_{-}\varphi_n, {}_{-}\varphi_m^{\circ})^*,$$

$$\Psi_{n+,m+} = \Psi_{n-,m-} = 0, \quad \Psi_{n+,m-} = ({}_{+}\varphi_n, {}_{-}\varphi_m^{\circ}), \quad \Psi_{n-,m+} = ({}_{-}\varphi_n, {}_{+}\varphi_m^{\circ})^*.$$

The transformation (14) is proper and the unitary equivalence needed holds if Ψ is a Hilbert-Schmidt operator (Berezin, 1965; Kiperman, 1970), which corresponds to the first condition (8) in our terms. The second condition (8) springs up in the same way.

It follows from (11)-(12) that $\{a^{+}(in), a(in), b^{+}(in), b(in)\}$ may be referred to as creation and annihilation operators of electrons and positrons at the initial time-moment t_{in} and $\{a^{+}(out), a(out), b^{+}(out), b(out)\}$ as creation and annihilation operators at the final time moment t_{out}. In accordance with this the states with definite numbers of electrons and positrons at t_{in}, t_{out} may be built from the vacuum vectors $|0\rangle_{in}$, $|0\rangle_{out}$ in the usual way. Consequently the general form of the initial and final states with definite numbers of electrons and positrons, in accordance with the above consideration and the relation (5), must be as follows:

$$|in\rangle = N c^{+} \ldots b^{+}(in) \ldots a^{+}(in) \ldots |0\rangle_{in},$$
$$\langle out| = {}_{out}\langle 0|\, a(out) \ldots b(out) \ldots c \ldots N \qquad (15)$$

b) Perturbation theory with respect to the radiative interaction

The probability amplitude for an arbitrary process in QED with the external field and the initial and final states (15) has the form

$$M_{in\to out} = {}_{out}\langle 0|a(out)\ldots b(out)\ldots C\ldots U_A C^+\ldots b^+(in)\ldots a^+(in)\ldots|0\rangle_{in}. \qquad (16)$$

Here $U_A = U_A(t_{out}, t_{in})$ is the evolution operator, corresponding to the Hamiltonian $\mathcal{H}_A$. (In (16) we omitted the unessential normalizing factors of the initial and final states).

Consider the construction of the perturbation theory for the matrix elements (16) with respect to the radiative interaction $\mathcal{H}_{e\gamma}$ choosing $\tilde{\mathcal{H}} = \mathcal{H}_e + \mathcal{H}_\gamma + \mathcal{H}_{eA}$ as the zeroth-order approximation Hamiltonian. Define the evolution operator corresponding to the Hamiltonian $\tilde{\mathcal{H}}$

$$\left(i\frac{\partial}{\partial t} - \tilde{\mathcal{H}}\right)\tilde{U}(t,t_{in}) = 0, \quad \tilde{U}(t,t_{in}) = T\exp\left\{-i\int_{t_{in}}^{t}\tilde{\mathcal{H}}d\tau\right\}, \qquad (17)$$

and construct, with it's aid , the field operators in the interaction picture with respect to the external field

$$\tilde{\psi}(x) = \tilde{U}^{-1}(t,t_{in})\psi(\vec{x})\tilde{U}(t,t_{in}), \quad \tilde{\bar{\psi}}(x) = \ldots, \quad \tilde{j}(x) = \ldots,$$

$$A(x) = \tilde{U}^{-1}(t,t_{in})A(\vec{x})\tilde{U}(t,t_{in}), \qquad (18)$$

$$(i\hat{\partial} - e\hat{A}^{ext}(x) - m)\tilde{\psi}(x) = 0, \quad \tilde{\bar{\psi}}(x)(i\hat{\partial} + e\hat{A}^{ext}(x) + m) = 0,$$

$$\Box A(x) = 0. \qquad (19)$$

(The operators $A(x)$ coincide in the case with the operators in the usual interaction picture). Then the total evolution operator U_A may be represented in a form for which the expansion in powers of the charge is not connected with the expansion in powers of the external field, if the operator $\tilde{U}$ is known

$$U_A = \tilde{U}\tilde{S}, \quad \tilde{S} = T\exp\left\{-i\int_{t_{in}}^{t_{out}}\tilde{j}(x)A(x)dx\right\}, \qquad (20)$$

and the matrix elements (16) assume the form

$$M_{in\to out} = {}_{out}\langle\tilde{0}|\tilde{a}(out)\ldots b(out)\ldots C\ldots\tilde{S}C^+\ldots b^+(in)\ldots a^+(in)\ldots|0\rangle_{in}, \qquad (21)$$

$$\begin{vmatrix}\tilde{a}^+(out)\\ \tilde{a}(out)\\ \tilde{b}^+(out)\\ \tilde{b}(out)\end{vmatrix} = \tilde{U}^{-1}\begin{vmatrix}a^+(out)\\ a(out)\\ b^+(out)\\ b(out)\end{vmatrix}\tilde{U}, \quad {}_{out}\langle\tilde{0}| = {}_{out}\langle 0|\tilde{U}. \qquad (22)$$

The matrix elements (21) differ from the matrix elements of the processes in QED without the external field in that the vacuum vectors as well as the creation and annihilation operators which stand to the right and left of the $\tilde{S}$- matrix are different. (These distinctions are essential, as it will be clear from the below consideration, only for the fields creating pairs.) Therefore the conventional Wick's technique based on the reduction of the $\tilde{S}$- matrix to a normal form with respect to one vacuum is not efficient here when evaluating the perturbation expansion. The main idea which allows us to obtain an analogue of conventional perturbation theory is to express any operators of the spinor field, and specifically the $\tilde{S}$- matrix, only in terms of the creation $\tilde{a}^+(out)$, $\tilde{b}^+(out)$ and annihilation $a(in)$, $b(in)$ operators, all the $\tilde{a}^+(out)$, $\tilde{b}^+(out)$ being placed on the left of all the $a(in)$, $b(in)$. The corresponding formalized computational technique is discussed in detail in the appendix A. To use it when evaluating the matrix element (21) one should obtain the explicit form of the canonical transformation from in- to $\tilde{out}$- operators; the decomposition of (A.5) - form for the operators $\tilde{\psi}(x)$ and $\tilde{\bar{\psi}}(x)$

$$\tilde{\psi}(x) = \tilde{\psi}^{(-)}(x) + \tilde{\psi}^{(+)}(x), \quad \tilde{\bar{\psi}}(x) = \tilde{\bar{\psi}}^{(-)}(x) + \tilde{\bar{\psi}}^{(+)}(x),$$

$$\tilde{\psi}^{(-)}(x)|0\rangle_{in} = \tilde{\bar{\psi}}^{(-)}(x)|0\rangle_{in} = {}_{out}\langle\tilde{0}|\tilde{\psi}^{(+)}(x) = {}_{out}\langle\tilde{0}|\tilde{\bar{\psi}}^{(+)}(x) = 0;$$

the generalized chronological coupling of the operators $\psi(x)$ and $\bar{\psi}(y)$ which is the perturbation theory propagator; the anticommutators of the operators $\tilde{\psi}^{(+)}(x)$, $\tilde{\bar{\psi}}^{(+)}(x)$ and $\tilde{a}(out)$, $\tilde{b}(out)$ as well as those of the operators $\tilde{\psi}^{(-)}(x)$, $\tilde{\bar{\psi}}^{(-)}(x)$ and $a^+(in)$, $b^+(in)$; the probability amplitude for the vacuum to remain the vacuum to the zeroth order with respect to the radiative interaction C_v

$$C_v = {}_{out}\langle 0|\tilde{U}|0\rangle_{in} = {}_{out}\langle\tilde{0}|0\rangle_{in} \; ; \tag{23}$$

the relative probability amplitudes of the processes in the external field which are of the zeroth order with respect to the radiative interaction

$$w(\overset{+}{m}\ldots\bar{s}\ldots|\bar{n}\ldots\overset{+}{\ell}\ldots) =$$

$$= {}_{out}\langle\tilde{0}|\tilde{a}_m(out)\ldots\tilde{b}_s(out)\ldots b_n^+(in)\ldots a_\ell^+(in)\ldots|0\rangle_{in} \cdot C_v^{-1} \, . \tag{24}$$

Let us now find the coefficients of the above-mentioned transformation and the representation of the (A.3)-type. Consider the function $\tilde{G}(x,x')$ which is the $\vec{x}$ representation for the matrix element of the evolution operator of the Dirac equation with an external field $A^{ext}(x)$. The function $\tilde{G}(x,x')$ satisfies the Dirac equation and the condition

$$\tilde{G}(x,x')\Big|_{t=t'} = \delta(\vec{x}-\vec{x}'). \tag{25}$$

For it the relations

$$\begin{gathered}\int \tilde{G}(x,y)\tilde{G}(y,x')dy = \tilde{G}(x,x') \\ \tilde{G}^{+}(x,x') = \tilde{G}(x',x)\end{gathered} \tag{26}$$

hold. Note also, that $\tilde{G}(x,x')$ is the anticommutator of spinor field operators in the interaction picture with respect to the field

$$[\tilde{\psi}(x),\ \tilde{\psi}^{+}(x')]_{+} = \tilde{G}(x,x').$$

The function $\tilde{G}(x,x')$ may be constructed using any complete and orthonormal set of solutions $\{\varphi_K(x)\}$ of the Dirac equation in the usual way

$$\tilde{G}(x,x') = \sum_{K} \varphi_K(x)\varphi_K^{+}(x'). \tag{27}$$

The properties of the function $\tilde{G}(x,x')$ imply that the operators $\tilde{\psi}(x)$, $\tilde{\bar{\psi}}(x)$ satisfying the Dirac equation in the field $A^{ext}(x)$ are connected for different time-moments by means of the function $\tilde{G}(x,x')$ in the following way

$$\begin{gathered}\tilde{\psi}(x) = \int \tilde{G}(x,x')\tilde{\psi}(x')d\vec{x}', \\ \tilde{\bar{\psi}}(x) = \int \tilde{\bar{\psi}}(x')\gamma^0\tilde{G}(x',x)\gamma^0 d\vec{x}'.\end{gathered} \tag{28}$$

The relations (28) allow us to find the connection between the operators $\{\tilde{a}^{+}(out), \tilde{a}(out), \tilde{b}^{+}(out), \tilde{b}(out)\}$ and $\{a^{+}(in), a(in), b^{+}(in), b(in)\}$. Put $t=t_{out}$, $t'=t_{in}$ in (28), write the left-hand sides with the aid of the representation (I8) and substitute the decompositions (IO) into them, while the decompositions (9) must be substituted into the right-hand sides. This yields

$$\tilde{a}(out) = \tilde{G}({}^{+}|{}_{+})a(in) + \tilde{G}({}^{+}|{}_{-})b^{+}(in), \tag{29}$$

$$\tilde{a}^+(out) = a^+(in)\,\tilde{G}(+|^+) + b(in)\,\tilde{G}(-|^+),$$

$$\tilde{b}^+(out) = \tilde{G}(^-|_+)a(in) + \tilde{G}(^-|_-)\,b^+(in),$$

$$\tilde{b}(out) = a^+(in)\,\tilde{G}(+|^-) + b(in)\,\tilde{G}(-|^-),$$

$$\tilde{G}(^\pm|_\pm)_{mn} = \int {}^\pm\varphi_m^+(\vec{x})\tilde{G}(\vec{x}t_{out}, \vec{x}'t_{in})_\pm\varphi_n(\vec{x}')d\vec{x}\,d\vec{x}';$$

$$\tilde{G}(\pm|^\pm)_{mn} = \int {}_\pm\varphi_m^+(\vec{x}')\tilde{G}(\vec{x}'t_{in}, \vec{x}t_{out})^\pm\varphi_n(\vec{x})d\vec{x}\,d\vec{x}'. \qquad (30)$$

Put $t = t_{in},\ t' = t_{out}$ in (28), write the right-hand sides with the aid of the representation (I8) and substitute the decompositions (I0) into the right-hand sides and (9) into the left-hand sides. This yields

$$a(in) = \tilde{G}(+|^+)\tilde{a}(out) + \tilde{G}(+|^-)\tilde{b}^+(out),$$

$$a^+(in) = \tilde{a}^+(out)\tilde{G}(^+|_+) + \tilde{b}(out)\tilde{G}(^-|_+),$$

$$b^+(in) = \tilde{G}(-|^+)\tilde{a}(out) + \tilde{G}(-|^-)\,\tilde{b}^+(out), \qquad (3I)$$

$$b(in) = \tilde{a}^+(out)\,\tilde{G}(^+|_-) + \tilde{b}(out)\,\tilde{G}(^-|_-).$$

The matrices $\tilde{G}(^\pm|_\pm)$ and $\tilde{G}(\pm|^\pm)$ satisfy the completeness and orthonormality relations which follow from the relations of the (7)-type for the functions ${}_\pm\varphi_n(\vec{x})$ and ${}^\pm\varphi_m(\vec{x})$ and the properties (26) of the function $\tilde{G}(x,x')$:

$$\tilde{G}(^\pm|_+)\tilde{G}(+|^\pm) + \tilde{G}(^\pm|_-)\tilde{G}(-|^\pm) = I,\quad \tilde{G}(^\pm|_+)\tilde{G}(+|^\mp) + \tilde{G}(^\pm|_-)\tilde{G}(-|^\mp) = 0,$$

$$\tilde{G}(\pm|^+)\tilde{G}(^+|_\pm) + \tilde{G}(\pm|^-)\tilde{G}(^-|_\pm) = I,\quad \tilde{G}(\pm|^+)\tilde{G}(^+|_\mp) + \tilde{G}(\pm|^-)\tilde{G}(^-|_\mp) = 0.$$

Moreover

$$\tilde{G}(^\pm|_\pm)^+ = \tilde{G}(\pm|^\pm).$$

By applying the relations (29), (3I) one may find the simplest amplitudes (24) for the processes of scattering, annihilation and pair creation, which at the same time are the generalized couplings of the corresponding creation and annihilation operators with respect to the vacua ${}_{out}\langle\tilde{0}|$ and $|0\rangle_{in}$

$$\tilde{a}_m(out)\,\tilde{b}_n(out) = w(\overset{+}{m}\,\bar{n}\,|0) =$$

$$= \{\tilde{G}^{-1}(+|^+)\,\tilde{G}(+|^-)\}_{mn} = -\{\tilde{G}(+|_-)\,\tilde{G}^{-1}(-|_-)\}_{mn}, \qquad (32)$$

$$b^+_m(in)\,a^+_n(in) = w(0|\bar{m}\,\overset{+}{n}) =$$

$$= \{\tilde{G}(-|^+)\,\tilde{G}^{-1}(+|^+)\}_{mn} = -\{\tilde{G}^{-1}(-|_-)\,\tilde{G}(-|_+)\}_{mn},$$

$$\tilde{a}_m(out)\,a^+_n(in) = w(\overset{+}{m}|\overset{+}{n}) = \tilde{G}^{-1}(+|^+)_{mn},$$

$$\tilde{b}_m(out)\,b^+_n(in) = w(\bar{m}|\bar{n}) = \tilde{G}^{-1}(-|_-)_{nm}.$$

From (27), (29) and (32) it follows that

$$\tilde{a}_m(out) = \sum_n w(\overset{+}{m}|\overset{+}{n})\,a_n(in) - \sum_s w(\overset{+}{m}\,\bar{s}\,|0)\,\tilde{b}^+_s(out),$$

$$\tilde{b}_m(out) = \sum_n w(\bar{m}|\bar{n})\,b_n(in) + \sum_s w(\overset{+}{s}\,\bar{m}\,|0)\,\tilde{a}^+_s(out), \qquad (33)$$

$$a^+_n(in) = \sum_m w(\overset{+}{m}|\overset{+}{n})\,\tilde{a}^+_m(out) - \sum_\ell w(0|\bar{\ell}\,\overset{+}{n})\,b_\ell(in),$$

$$b^+_n(in) = \sum_m w(\overset{+}{m}|\bar{n})\,\tilde{b}^+_m(out) + \sum_\ell w(0|\bar{n}\,\overset{+}{\ell})\,a_\ell(in).$$

The relations (33) are the specification of the general representation (A.3) for the case under consideration. It is seen that the transformation (29) admits transition to the generalized normal form with respect to the vacua ${}_{out}\langle\tilde{0}|$ and $|0\rangle_{in}$ if the inverse matrices $\tilde{G}^{-1}(+|^+)$ and $\tilde{G}^{-1}(-|_-)$ exist, in full accordance with the general requirement (A.4).

From (33) it follows, that (32) are the only nonzero generalized couplings of the in - and $\widetilde{out}$ - operators. Therefore any matrix element (24) may be expressed, in accordance with (A.17), in terms of the amplitudes (32) only. For example the probability amplitude for electron scattering accompanied by the creation of a pair is expressed in the following way

$$w(\overset{+}{m}\overset{+}{s}\bar{\kappa}|\overset{+}{n}) = w(\overset{+}{s}\bar{\kappa}|0)\, w(\overset{+}{m}|\overset{+}{n}) - w(\overset{+}{m}\bar{\kappa}|0)\, w(\overset{+}{s}|\overset{+}{n}).$$

Let us evaluate C_V. To do so consider the operator V which performs the proper canonical transformation of the in-to $\widetilde{out}$-operators. (The condition under which such an operator exists nonformally will be obtained below when discussing the unitarity of the operator $\tilde{\mathcal{U}}$)

$$\begin{vmatrix} \tilde{a}^+(out) \\ \tilde{a}(out) \\ \tilde{b}^+(out) \\ \tilde{b}(out) \end{vmatrix} = V^{-1} \begin{vmatrix} a^+(in) \\ a(in) \\ b^+(in) \\ b(in) \end{vmatrix} V, \quad {}_{out}\langle \tilde{0}| = {}_{in}\langle 0|V. \tag{34}$$

The explicit form of V may be obtained from the relations (29), (31) by operator methods (Bagrov, Gitman, Schwartsman, 1975; Gitman, 1977) or by using the general expression for the generating functional of the proper canonical transformation operator (Berezin, 1965).

For the case under consideration we have, with an accuracy to a phase factor, the following

$$\begin{aligned} V = & \exp\{-a^+(in)\, w(+-|0)\, b^+(in)\} \exp\{a^+(in) \ln w(+|+)\, a(in)\} \cdot \\ & \cdot \exp\{-b(in) \ln w(-|-)^T b^+(in)\} \exp\{-b(in)\, w(0|-+)\, a(in)\}. \end{aligned} \tag{35}$$

From (23), (34), (35) and (32) we get

$$C_V = {}_{in}\langle 0|V|0\rangle_{in} = \exp\{-\operatorname{tr} \ln w(-|-)^T\} = \det \tilde{G}(-|-). \tag{36}$$

Let us find the explicit form of the representation (A.5) for the operators $\tilde{\psi}(x)$ and $\tilde{\bar{\psi}}(x)$. By setting $t' = t_{in}$ in (28) and using the decompositions (9) in the right-hand sides we obtain

$$\begin{aligned} \tilde{\psi}(x) &= \sum_n \{a_n(in)\, {}_+\tilde{\psi}_n(x) + b_n^+(in)\, {}_-\tilde{\psi}_n(x)\}, \\ \tilde{\bar{\psi}}(x) &= \sum_n \{a_n^+(in)\, {}_+\tilde{\bar{\psi}}_n(x) + b_n(in)\, {}_-\tilde{\bar{\psi}}_n(x)\}, \end{aligned} \tag{37}$$

where

$$
{}_{\pm}\tilde{\varphi}_n(x) = \int \tilde{G}(x, \vec{x}'t_{in})\, {}_{\pm}\varphi_n(\vec{x}')d\vec{x}',
\qquad (38)
$$

$$
{}_{\pm}\tilde{\tilde{\varphi}}_n(x) = \int {}_{\pm}\varphi_n^+(\vec{x}')G(\vec{x}'t_{in}, x)\gamma^0 d\vec{x}'.
$$

By setting $t' = t_{out}$ in (28) and using the representation (I8) in the right-hand sides and the decompositions (9) we obtain

$$
\tilde{\psi}(x) = \sum_m \{\tilde{a}_m(out)\,{}^+\tilde{\varphi}_m(x) + \tilde{b}_m^+(out)\,{}^-\tilde{\varphi}_m(x)\},
\qquad (39)
$$

$$
\tilde{\tilde{\psi}}(x) = \sum_m \{\tilde{a}_m^+(out)\,{}^+\tilde{\tilde{\varphi}}_m(x) + \tilde{b}_m(out)\,{}^-\tilde{\tilde{\varphi}}_m(x)\},
$$

where

$$
{}^{\pm}\tilde{\varphi}_m(x) = \int \tilde{G}(x, \vec{x}'t_{out})\,{}^{\pm}\varphi_m(\vec{x}')d\vec{x}',
\qquad (40)
$$

$$
{}^{\pm}\tilde{\tilde{\varphi}}_m(x) = \int {}^{\pm}\varphi_m^+(\vec{x}')\tilde{G}(\vec{x}'t_{out}, x)\gamma^0 d\vec{x}'.
$$

By combining the relations (33), (37) and (39) we find

$$
\tilde{\psi}^{(-)}(x) = \sum_n {}_+\tilde{\varphi}_n(x)a_n(in), \qquad \tilde{\psi}^{(+)}(x) = \sum_m {}^-\tilde{\varphi}_m(x)\tilde{b}_m^+(out),
$$

$$
\tilde{\tilde{\psi}}^{(-)}(x) = \sum_n {}_-\tilde{\tilde{\varphi}}_n(x)b_n(in), \qquad \tilde{\tilde{\psi}}^{(+)}(x) = \sum_m {}^+\tilde{\tilde{\varphi}}_m(x)\tilde{a}_m^+(out),
\qquad (4I)
$$

$$
{}_+\tilde{\varphi}_n(x) = {}_+\tilde{\varphi}_n(x) + \sum_m w(0|\bar{m}\overset{+}{n})\,{}_-\tilde{\varphi}_m(x) = \sum_m w(\overset{+}{m}|\overset{+}{n})\,{}^+\tilde{\varphi}_m(x),
$$

$$
{}^-\tilde{\varphi}_n(x) = {}^-\tilde{\varphi}_n(x) - \sum_m w(\overset{+}{m}\bar{n}|0)\,{}^+\tilde{\varphi}_m(x) = \sum_m w(\bar{n}|\bar{m})\,{}_-\tilde{\varphi}_m(x),
$$

$$
{}_-\tilde{\tilde{\varphi}}_n(x) = {}_-\tilde{\tilde{\varphi}}_n(x) - \sum_m w(0|\bar{n}\overset{+}{m})\,{}_+\tilde{\tilde{\varphi}}_m(x) = \sum_m w(\bar{m}|\bar{n})\,{}^-\tilde{\tilde{\varphi}}_m(x),
$$

$$
{}^+\tilde{\tilde{\varphi}}_n(x) = {}^+\tilde{\tilde{\varphi}}_n(x) + \sum_m w(\overset{+}{n}\bar{m}|0)\,{}^-\tilde{\tilde{\varphi}}_m(x) = \sum_m w(\overset{+}{n}|\overset{+}{m})\,{}_+\tilde{\tilde{\varphi}}_m(x).
$$

Consequently the following anticommutators are different from zero:

$$
[\tilde{a}_m(out), \tilde{\tilde{\psi}}^{(+)}(x)]_+ = {}^+\tilde{\tilde{\varphi}}_m(x), \quad [\tilde{b}_m(out), \tilde{\psi}^{(+)}(x)]_+ = {}^-\tilde{\varphi}_m(x),
\qquad (42)
$$

$$[\tilde{\psi}^{(-)}(x), a_n^+(in)]_+ = {}_+\tilde{\varphi}_n(x), \quad [\tilde{\bar{\psi}}^{(-)}(x), b_n^+(in)]_+ = {}_-\tilde{\bar{\varphi}}_n(x).$$

The generalized chronological coupling of the spinor field operators has, in accordance with (A.I2), (37) and (39), the form

$$\overbrace{\tilde{\psi}(x)\tilde{\bar{\psi}}(y)} = {}_{out}\langle\tilde{0}|T\tilde{\psi}(x)\tilde{\bar{\psi}}(y)|0\rangle_{in}\cdot C_v^{-1} = -i\tilde{S}^c(x,y), \tag{43}$$

$$\tilde{S}^c(x,y) = \begin{cases} \tilde{S}^{(-)}(x,y), & x^0 > y^0, \\ -\tilde{S}^{(+)}(x,y), & x^0 < y^0, \end{cases} \tag{44}$$

$$\tilde{S}^{(-)}(x,y) = i[\tilde{\psi}^{(-)}(x), \tilde{\bar{\psi}}^{(+)}(y)]_+ = i\sum_{nm} {}^+\tilde{\varphi}_m(x)\, w(\overset{+}{m}|\overset{+}{n})_+\, {}_+\tilde{\bar{\varphi}}_n(y),$$

$$\tilde{S}^{(+)}(x,y) = i[\tilde{\psi}^{(+)}(x), \tilde{\bar{\psi}}^{(-)}(y)]_+ = i\sum_{nm} {}_-\tilde{\varphi}_n(x)\, w(\bar{m}|\bar{n})^-\, {}^-\tilde{\bar{\varphi}}_m(y).$$

moreover $\tilde{S}^c(x,y)$ satisfies the Green function Dirac equation in the external field $A^{ext}(x)$

$$(i\hat{\partial} - e\hat{A}^{ext}(x) - m)\tilde{S}^c(x,y) = -\delta(x-y) \tag{45}$$

and is the generalization of the Feynman causal Green function for the present case.

One can express in terms of $\tilde{S}^c(x,y)$ the anticommutators (42)

$${}_+\tilde{\varphi}_n(x) = -i\int \tilde{S}^c(x, \vec{x}'t_{in})\gamma^0\, {}_+\varphi_n(\vec{x}')\,d\vec{x}',$$

$${}^-\tilde{\varphi}_n(x) = i\int \tilde{S}^c(x, \vec{x}'t_{out})\gamma^0\, {}^-\varphi_n(\vec{x}')\,d\vec{x}',$$

$${}_-\tilde{\bar{\varphi}}_n(x) = i\int {}_-\varphi_n^+(\vec{x}')\,\tilde{S}^c(\vec{x}'t_{in}, x)\,d\vec{x}',$$

$${}^+\tilde{\bar{\varphi}}_n(x) = -i\int {}^+\varphi_n^+(\vec{x}')\,\tilde{S}^c(\vec{x}'t_{out}, x)\,d\vec{x}',$$

the current operator $\tilde{j}(x)$ reduced to the generalized normal form with respect to the vacua ${}_{out}\langle 0|$ and $|0\rangle_{in}$

$$\tilde{j}(x) = eN\tilde{\bar{\psi}}(x)\gamma\tilde{\psi}(x) + \tilde{J}(x),$$

$$\tilde{\mathcal{J}}(x) = {}_{out}\langle \tilde{0} | \tilde{j}(x) | 0 \rangle_{in} \cdot C_v^{-1} = ie\, tr\, \gamma\, \tilde{S}^c(x,x),$$

$$\tilde{S}^c(x,x) = \frac{1}{2}\left[\tilde{S}^c(x+0,x) + \tilde{S}^c(x,x+0)\right],$$

and also the amplitude C_v. Indeed

$$\frac{\delta\, i \ln C_v}{\delta A^{ext}(x)} = \tilde{\mathcal{J}}(x) = ie\, tr\, \gamma\, \tilde{S}^c(x,x') =$$

$$= -i \frac{\delta}{\delta A^{ext}(x)} Tr \ln \tilde{S}^c,$$

where the operation Tr includes also the coordinate integration. Then by using (36) we find

$$C_v = \det \tilde{G}(-|-)\Big|_{A^{ext}=0} \cdot \exp\left\{-Tr \ln \frac{\tilde{S}^c}{\tilde{S}^c(A^{ext}=0)}\right\}. \tag{46}$$

The perturbation expansion for the matrix elements of (21)-type may be obtained by representing the $\tilde{S}^c$- matrix in the generalized normal form with respect to the vacua ${}_{out}\langle 0|$ and $|0\rangle_{in}$ This can be done, as it is shown in appendix A, with the help of the usual Wick's theorem for the T- products if instead of the normal products and chronological couplings their generalized counterparts are taken. Thus the problem reduces to calculating the matrix elements of the generalized normal products of the following form:

$${}_{out}\langle \tilde{0} | \tilde{a}(out) \ldots \tilde{b}(out) \ldots c \ldots \tilde{N}(\ldots) c^{+} \ldots b^{+}(in) \cdots a^{+}(in) \cdots | 0 \rangle_{in}.$$

It is evident that this matrix element is different from zero if the sum of numbers of particles of each field in the initial and final states is greater than or equal to the number of operator functions of the given field in the generalized normal product.

Consider the case when for each field operator $\tilde{\Psi}(x)$, $\tilde{\bar{\Psi}}(x)$, $A(x)$ taken from the generalized normal product there may be found a corresponding operator $a^{+}(in)$, $b^{+}(in)$, c^{+} from the initial state or $\tilde{a}(out)$, $\tilde{b}(out)$, c from the final state which will cancel it after the commutation. Such matrix element can be represented by the usual Feynman diagrams with the following rules of correspondence:

I. Electron in initial (final) state with the quantum number $n(m)$ is represented by the factor ${}_{+}\tilde{\varphi}_n(x)$ $(+\tilde{\tilde{\varphi}}_m(x))$.

2. Positron in initial (final) state with the quantum number $n(m)$ is represented by the factor ${}_{-}\tilde{\tilde{\varphi}}_n(x)$ $(-\tilde{\varphi}_m(x))$.

3. Internal electron line directed from the point x' into the point x is represented by the generalized coupling $-i\tilde{S}^c(x,x')$.

4. To the closed electron line the generalized vacuum current $\tilde{J}(x)$ is put into correspondence.

5. Contribution of every diagram contains the amplitude C_v of probability for the vacuum to remain vacuum as a factor.

The rest of the rules of correspondence are the same as those in the standard QED (Bogoliubov, Shirkov, 1959).

In the case when the number of the spinor field operators in the initial and final states is greater than that which is necessary for the compensation of the generalized normal product, the matrix element is equal to the sum of products of contributions, coming from the Feynman graphs arising due to the "interaction" of the generalized normal product with the operators of the initial and final states, by the amplitudes $w(\overset{+}{m}\ldots\bar{s}\ldots|\bar{n}\ldots\overset{+}{\ell})$ coming from the noncompensated creation and annihilation operators of these states.

2. Unitarity conditions

a) Consider first the problem of unitarity of the spinor field evolution operator $\tilde{\mathcal{U}}$ in an external electromagnetic field.

The conditions (I.8), assumed for the spinors of the initial and final states, ensure the unitary equivalence of the in- and out- states operators. Therefore from (I.22) it follows that the existence and unitarity of the operator $\tilde{\mathcal{U}}$ are connected single-valuedly with the existence and unitarity of the operator V, fixed by the conditions (I.34). The latter exists and is unitary if the canonical transformation (I.29) is proper. The question whether the linear canonical transformation of the Fermi-operators is proper may be solved according to the theorems suggested in (Berezin, 1965; Kiperman, 1970) in the same way as it is done in item a) of Sec.I when investigating the transformation (I.14). Taking into account properties of the matrices $\tilde{G}(\pm|\pm)$ we obtain the corresponding criterion

$$tr\{\tilde{G}(^+|_-)\tilde{G}(_-|^+)+\tilde{G}(^-|_+)\tilde{G}(_+|^-)\}<\infty . \tag{I}$$

We will show that the left-hand side of the inequality represents the total number of particles created by the field. To do this we calculate the absolute probabilities of electron creation at the given quantum state n_m^+ and positron creation at the given quantum state n_m^- By using the formulae (I.29) and assuming that $\tilde{U}$ is the unitary operator, we get

$$n_m^+ = \sum_{\ell,\kappa=0}^{\infty} \sum_{\{m_i\}\{s_i\}} |_{out}\langle 0|a_m(out)a_{m_1}(out)\ldots a_{m_\ell}(out)b_{s_1}(out)\ldots$$

$$b_{s_\kappa}(out)\tilde{U}|0\rangle_{in}|^2 = {}_{in}\langle 0|\tilde{U}^{-1}a_m^+(out)a_m(out)\tilde{U}|0\rangle_{in} = \tag{2}$$

$$= \{\tilde{G}(^+|_-)\tilde{G}(_-|^+)\}_{mm},$$

$$n_s^- = \sum_{\ell,\kappa=0} \sum_{\{m_i\}\{s_i\}} |_{out}\langle 0|a_{m_1}(out)\ldots a_{m_\ell}(out)b_s(out)b_{s_1}(out)\ldots$$

$$b_{s_\kappa}(out)\tilde{U}|0\rangle_{in}|^2 = {}_{in}\langle 0|\tilde{U}^{-1}b_s^+(out)b_s(out)\tilde{U}|0\rangle_{in} = \tag{3}$$

$$= \{\tilde{G}(^-|_+)\tilde{G}(_+|^-)\}_{ss}.$$

According to the Pauli principle, expressions (2) and (3) are also the mean numbers of electrons and positrons created at the given quantum state. Thus the total numbers of electrons n^+ and positrons n^- created are equal to

$$n^+ = tr\,\tilde{G}(^+|_-)\tilde{G}(_-|^+), \quad n^- = tr\,\tilde{G}(^-|_+)\tilde{G}(_+|^-), \tag{4}$$

respectively, and the left-hand side (I) really represents the total number of particles created. (It is possible to verify that $n^+ = n^-$, so the charge conservation law is valid for this case.)

Thus the operator $\tilde{U}$ is unitary if the total number of created particles is not equal to infinity. It is evident from the physical consideration that the latter is always valid for a system placed in a finite volume V and for a pair-creating field acting during the finite time interval. If the external electromagnetic field is such that at $V\to\infty$ and during the infinite time interval it creats the infinite number of pairs, then according to the previous discussion the evolution operator $\tilde{\mathcal{U}}$ can not be unitary.

Note, that the unitarity of the operator $\tilde{\mathcal{U}}$ for the case of a constant electric field has been proved in (Nikishov, 1974). However, the problem of the conservation of unitarity under $V \to \infty$ and for the field acting during the infinite time interval has not been investigated.

b) Let us assume that the conditions under which the operator $\tilde{\mathcal{U}}$ is unitary hold. Then the scattering $\tilde{S}$- matrix in the external field is unitary (see (I.20)) due to the unitarity of the total evolution operator of QED (Bogoliubov, Shirkov, 1959; Akhiezer, Bereztetski, 1963)

$$\tilde{S}^+\tilde{S} = \tilde{S}\tilde{S}^+ = I.$$

Write in the usual manner $\tilde{S} = I + i\tilde{T}$, then $\tilde{T}^+T = -i(\tilde{T} - \tilde{T}^+)$. If $|in\rangle$ is some initial state and $\{\underset{f}{<}out|\}$ is a complete set of final out- states, then one can get

$$\sum_f |\underset{f}{<}\widetilde{out}|\tilde{T}|in\rangle|^2 = 2\,\mathrm{Im}\,\langle in|\tilde{T}|in\rangle, \qquad (5)$$
$$\underset{f}{<}\widetilde{out}| = \underset{f}{<}out|\tilde{\mathcal{U}}.$$

The perturbational analysis of (5) creates a number of differencies from the relations which are usually obtained in this way. The matter is that the propagators for perturbation expansions of the matrix elements $\underset{f}{<}\widetilde{out}|\tilde{T}|in\rangle$ and $\langle in|\tilde{T}|in\rangle$ are different: in the first case it is the generalized chronological coupling (I.43)

$${}_{out}\langle\tilde{0}|T\tilde{\psi}(x)\tilde{\bar{\psi}}(x')|0\rangle_{in}\cdot C_v^{-1} = -i\tilde{S}^c(x,x'),$$

and in the second case it is the chronological coupling of the following form

$${}_{in}\langle 0|T\tilde{\psi}(x)\tilde{\bar{\psi}}(x')|0\rangle_{in} = -i\tilde{\tilde{S}}^c(x,x'),$$

$$\tilde{\tilde{S}}^c(x,x') = \begin{cases} \tilde{\tilde{S}}^{(-)}(x,x'), & x_0 > x_0', \\ -\tilde{\tilde{S}}^{(+)}(x,x'), & x_0 < x_0', \end{cases} \qquad (6)$$

$$\tilde{\tilde{S}}^{(-)}(x,x') = i\sum_n {}_+\tilde{\psi}_n(x)\,{}_+\tilde{\bar{\psi}}_n(x'),$$

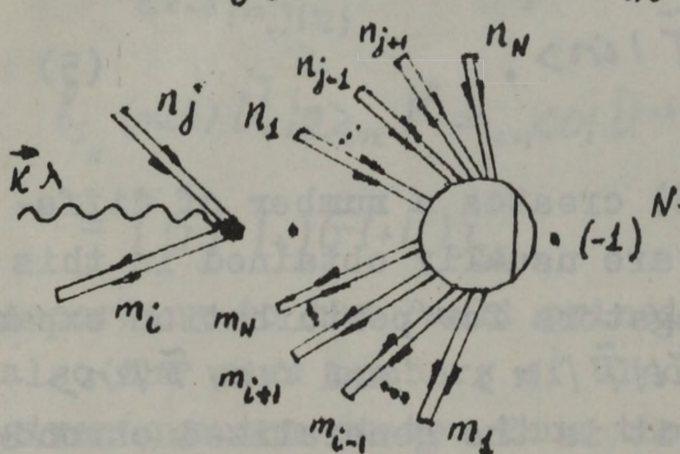
$$\tilde{\tilde{S}}^{(+)}(x,x') = i\sum_{n} {}_{-}\tilde{\varphi}_n(x)\,{}_{-}\tilde{\tilde{\varphi}}_n(x')$$

Besides, there arise singularities which are due to the possible particle creation already in the zeroth order of the perturbation theory with respect to the radiative interaction. To illustrate the abovesaid we will write the relations which follow from (5) in a number of cases, corresponding to the different choice of *in*-states. While doing so we will restrict ourselves to the comparison of the left- and right-hand sides of (5) in the second order of the perturbation theory with respect to the radiative interaction.

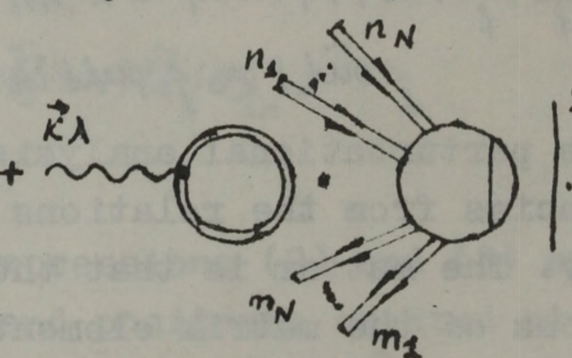
a) Let $|in\rangle = |0\rangle_{in}$, then

$$\rho = \sum_{N\{m\}\{n\}\vec{k}\lambda} (N!)^{-2}\Big|\,{}_{out}\langle\tilde{0}|\tilde{a}_{m_1}(out)\dots\tilde{a}_{m_N}(out)\,\tilde{b}_{n_1}(out)\dots\tilde{b}_{n_N}(out)\cdot$$

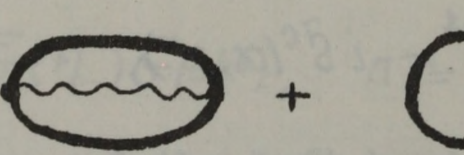
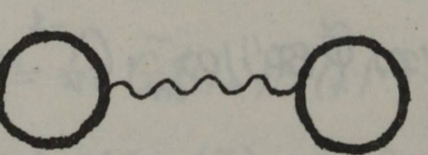
$$\cdot\, C^{+}_{\vec{k}\lambda}\Big\{-i\int\tilde{j}(x)A(x)\,dx\Big\}|0\rangle_{in}\Big|^2 = \sum_{N\{m\}\{n\}\vec{k}\lambda}(N!)^{-2}\Big|\sum_{i,j}^{N}\Big\{$$



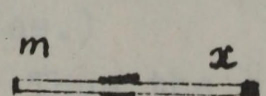
$\cdot(-1)^{N-i-j}\Big\} +$

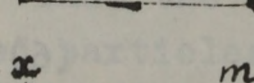
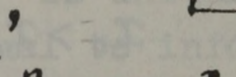
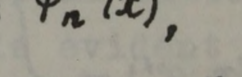


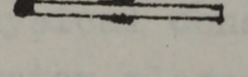
$\Big|^2\cdot P_v =$

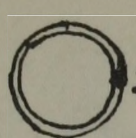
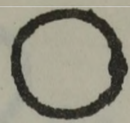
$= 2\,\mathrm{Im}\Big\{$  $+$  $\Big\}$. (7)

Here and elsewhere the following abbreviations are used:

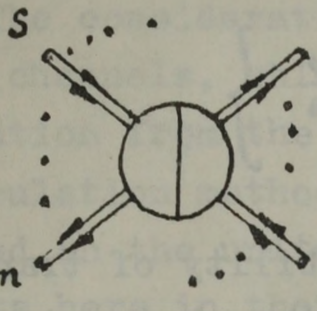
 $= {}^{+}\tilde{\tilde{\varphi}}_m(x)$,  $= {}^{-}\tilde{\varphi}_n(x)$,

 $= {}_{+}\tilde{\varphi}_m(x)$,  $= {}_{+}\tilde{\tilde{\varphi}}_n(x)$,  $= {}_{+}\tilde{\varphi}_n(x)$,

 $= \frac{1}{i}\tilde{\tilde{S}}^{c}(x,x')$, $P_v = |C_v|^2$,

 $x = \tilde{\mathcal{J}}(x),$  $x = \tilde{\tilde{\mathcal{J}}}(x),$

$$\tilde{\tilde{\mathcal{J}}}(x) = \frac{ie}{2}\, tr\gamma \left[\tilde{\tilde{S}}^c(x+0,0) + \tilde{\tilde{S}}^c(x,x+0)\right],$$



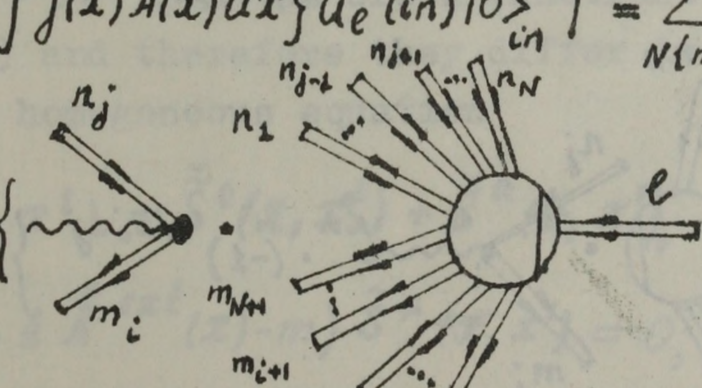
$$= w(\overset{+}{m}\ldots\bar{s}\ldots|\bar{n}\ldots\overset{+}{l}\ldots).$$

The left-hand side of (7) is the total probability of the photon irradiation when electron-positron pairs are created from vacuum by the external field to the lowest order of the perturbation expansion.

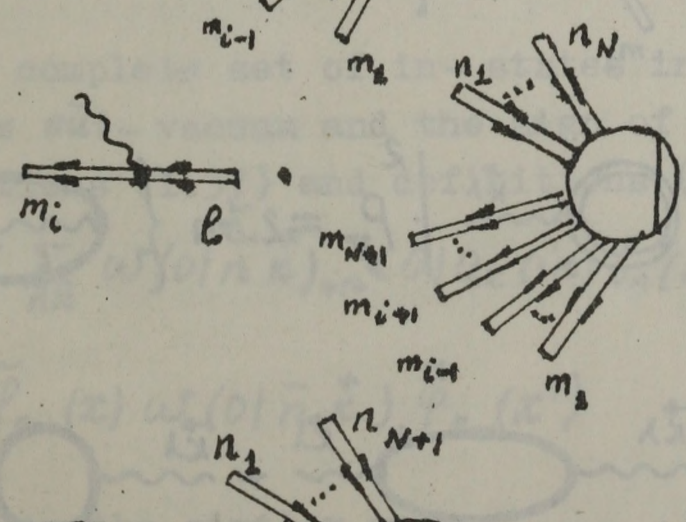
b) Let $|in\rangle = a_e^+(in)|0\rangle_{in}$ then

$$P_e = \sum_{N\{m\}\{n\}\vec{k}\lambda} [N!(N+1)!]^{-1} \Big|_{out}\langle\tilde{0}|\tilde{a}_{m_1}(out)\ldots\tilde{a}_{m_{N+1}}(out)\tilde{b}_{n_1}(out)\ldots\tilde{b}_{n_N}(out).$$

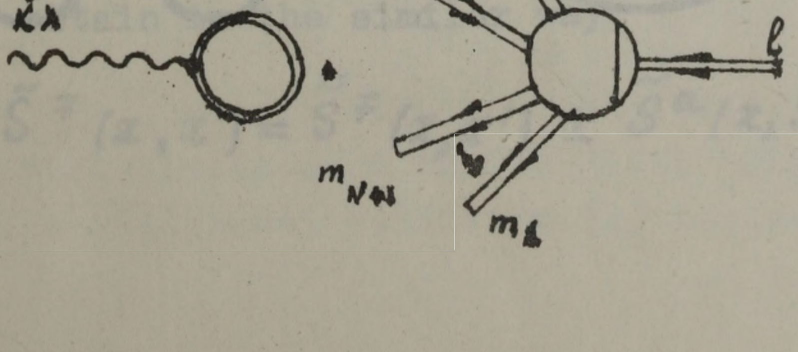
$$\cdot \vec{C}_{k\lambda}\left\{-i\int \tilde{j}(x)A(x)dx\right\} a_e^+(in)|0\rangle_{in}\Big|^2 = \sum_{N\{m\}\{n\}\vec{k}\lambda} [N!(N+1)!]^{-1}\cdot \tag{8}$$

$$\cdot\Big|\sum_{i,j}^{N+1,N}\Big\{$$ 

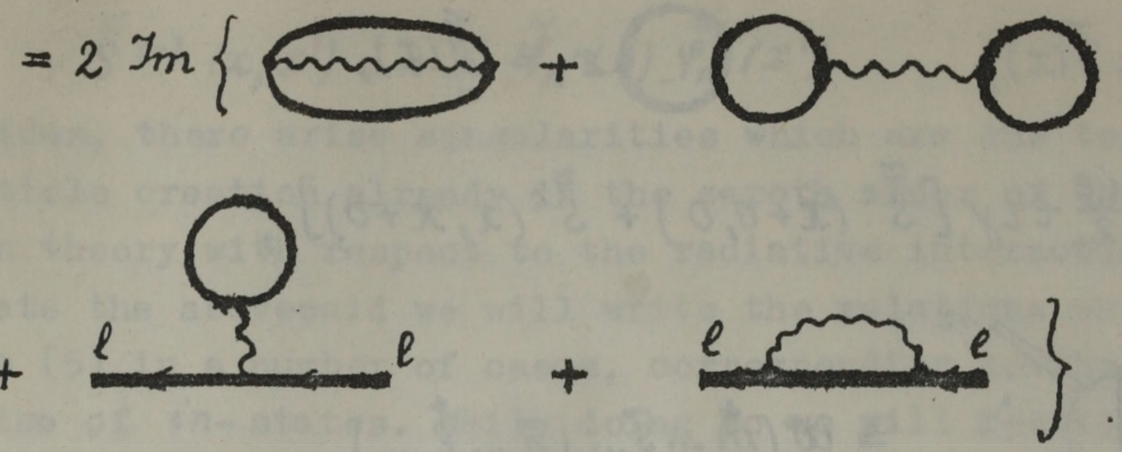
$$(-1)^{N-i-j+1}\Big\} +$$

$$+\sum_i \Big\{$$ 

$$\cdot(-1)^{i-1}\Big\} +$$

$$+$$ 

$$\Big|^2 P_v =$$



(8)

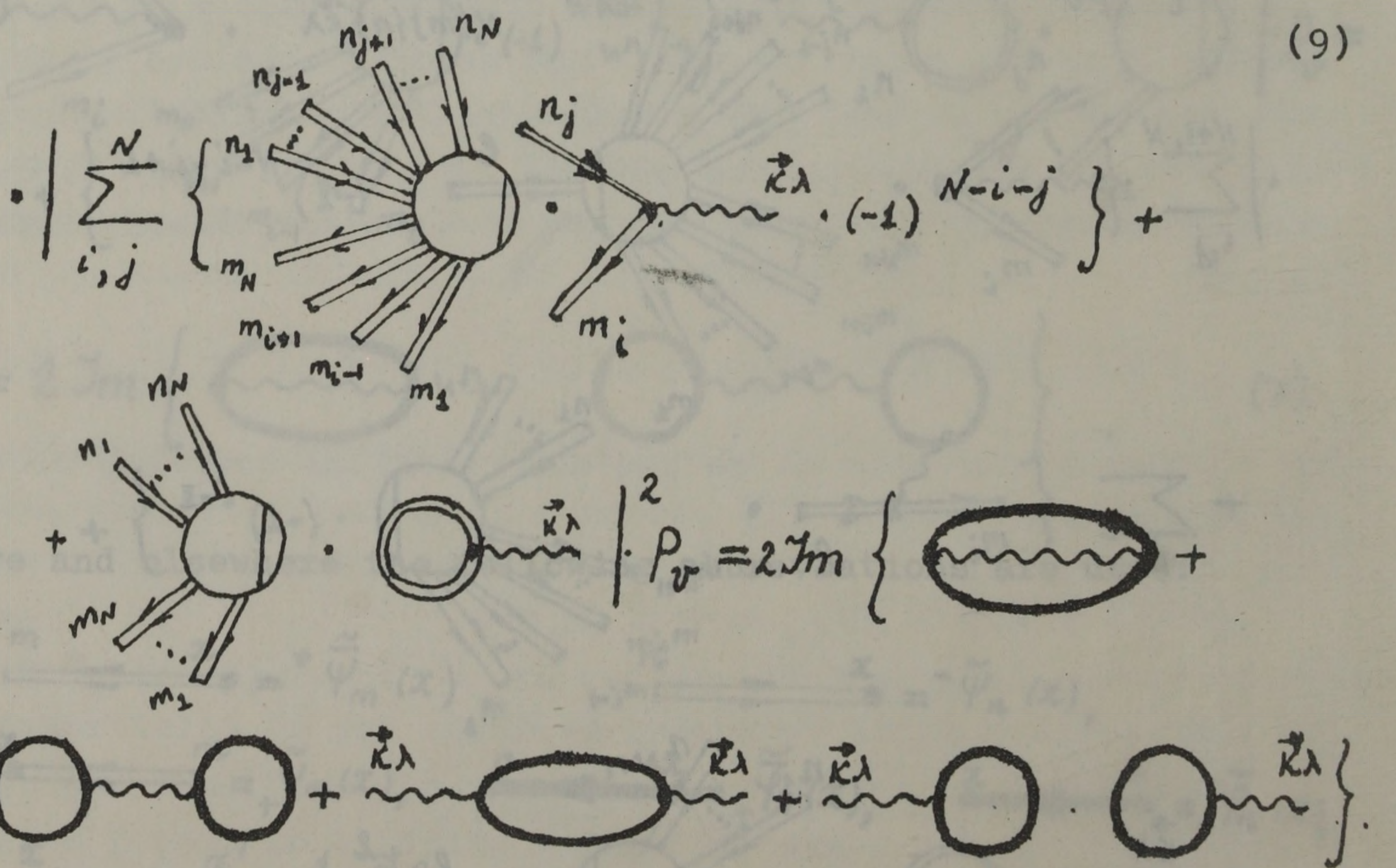
The left-hand side of (8) is the total probability of transition from a single-electron state accompanied by the photon irradiation and creation of pairs in the external field to the lowest order of the perturbation expansion.

c) Let $|in\rangle = C^{+}_{\vec{k}\lambda}\,|0\rangle_{in}$ then

$$P_{\vec{k}\lambda} = \sum_{N\{m\}\{n\}} (N!)^{-2} \Big|{}_{out}\langle\tilde{0}|\,\tilde{a}_{m_1}(out)\dots\tilde{a}_{m_N}(out)\,\tilde{b}_{n_1}(out)\dots\tilde{b}_{n_N}(out)\,\cdot$$

$$\cdot\Big\{-i\int\tilde{j}(x)A(x)\,dx\Big\}C^{+}_{\vec{k}\lambda}\,|0\rangle_{in}\Big|^2 = \sum_{N\{m\}\{n\}}(N!)^{-2}.$$

(9)



The left-hand side of (9) is the total probability of the pair creation by a photon in the external field to the lowest order of the perturbation expansion.

The consideration presented here shows that the existence of new channels, which is connected with the possibility of pair creation from the vacuum by the external field, modifies the calculation method of the total probabilities of transitions, based on the unitarity conditions. The main special feature consists here in that one should calculate the diagrams, which are subject to cutting, with the aid of the electron Green function $\tilde{\tilde{S}}^c$, the latter differs from the causal Green function $\tilde{S}^c$, which appears in the perturbation theory for the matrix elements of transitions. Moreover, the external electron lines of the diagrams, which are subject to cutting, differ from the external electron lines of the corresponding diagrams in the perturbation theory for matrix elements of transitions.

In conclusion we will find, by using the definitions (I.43) and (6), the connection between the Green functions $\tilde{S}^c(x,x')$ and $\tilde{\tilde{S}}^c(x,x')$. Both the Green functions satisfy the same equation (I.45) and therefore they differ in a solution of the corresponding homogeneous equation

$$\tilde{S}^c(x,x') = \tilde{\tilde{S}}^c(x,x') + \tilde{S}^a(x,x'), \qquad (I0)$$
$$(i\hat{\partial} - e\hat{A}^{ext}(x) - m)\,\tilde{S}^a(x,x') = 0,$$

By placing the complete set of in- states in the expression (I.43) between the $\widetilde{out}$- vacuum and the sign of T- product and using the relations (I.37) and definitions (I.24), we find

$$\tilde{S}^a(x,x') = i\sum_{nk} \omega(0|\bar{n}\overset{+}{k})_{in} \langle 0| a_k(in)\, b_n(in)\, T\, \tilde{\psi}(x)\tilde{\bar{\psi}}(x')|0\rangle_{in} =$$
$$= i\sum_{nk} {}_-\tilde{\varphi}_n(x)\,\omega(0|\bar{n}\overset{+}{k})\, {}_+\tilde{\bar{\varphi}}_k(x') \qquad (II)$$

One can obtain in the similar way

$$\tilde{S}^{\mp}(x,x') = \tilde{\tilde{S}}^{\mp}(x,x') \pm \tilde{S}^a(x,x') \qquad (I2)$$

For the external field in which the *in* - vacuum is stable, that is invariant with respect to the operator $\tilde{\mathcal{U}}$ the difference between $\tilde{S}^c$ and $\tilde{\tilde{S}}^c$ disappears. In appendix B the functions $\tilde{S}^c$ and $\tilde{\tilde{S}}^c$ will be calculated explicitly in a number of configurations of the external electromagnetic field.

CHAPTER II. QUANTUM ELECTRODYNAMICS WITH INTENSE MEAN ELECTROMAGNETIC FIELD

§1. Vacuum, initial and final states

Here we will consider the problem of building of initial and final states for the processes in QED with intense mean electromagnetic field. When doing so we will start formally with QED with the external current $\mathcal{J}(x)$ but without the external field.

Let t_{in} and t_{out} be the initial and final moments of time which in the final expressions will be understood as moved to infinitely remote future and past, resp. If the external current is switched off in the moments t_{in}, t_{out} and the mean values of the electromagnetic field disappear one may as usual assume that since the interaction is switched off when $t \to \pm\infty$ the initial and final states are free states with, say, a definite particle number.

$$|in\rangle = N c^+ \dots b^+ \dots a^+ \dots |0\rangle_{in}\,, \quad \langle out| = \langle 0|a \dots b \dots c \dots N. \tag{1}$$

Here $\{a^+, a, b^+, b\}$ are operators of creation and annihilation of free electrons and positrons, $\{c^+, c\}$ are the photon creation and annihilation operators, $|0\rangle$ is the vacuum of free particles $|0\rangle = |0\rangle^e \cdot |0\rangle^\gamma$, $|0\rangle^e$, $|0\rangle^\gamma$ are the corresponding vacuum vectors in the Hilbert spaces of states of the spinor and electromagnetic fields, N is a normalizing factor.

Consider the case when the external current and the mean values of the electromagnetic field do not disappear at t_{in}, t_{out}. Note, that whereas the external current and its values at t_{in}, t_{out} may be given arbitrarily and the mean value of the electromagnetic field at t_{in} may be also chosen rather arbitrarily, after that the mean value of the electromagnetic field at t_{out} is to be determined by the QED-equations and initial conditions.

Assume that at the initial moment t_{in}

$$\langle A\rangle^M\big|_{t_{in}} = A^{in}(\vec{x}), \quad \langle \dot{A}\rangle^M\big|_{t_{in}} = \dot{A}^{in}(\vec{x}), \tag{2}$$

where $\langle\dots\rangle^{Mean}$ stands for the mean value. Define the vacuum at the initial time-moment as the state $|0\rangle_{in}$ which minimizes under the additional condition (2) the mean value of the total

Hamiltonian of QED with the external current which is taken at

We write the total Hamiltonian with the external current as

$$\mathcal{H}_J = \int : \bar{\Psi}(\vec{x})(-i\vec{\gamma}\vec{\nabla}+m)\Psi(\vec{x}) : d\vec{x} - \sum_{\vec{\kappa}\lambda} g_{\lambda\lambda}\kappa C^+_{\vec{\kappa}\lambda} C_{\vec{\kappa}\lambda} + \tag{3}$$

$$+\int j(\vec{x})A(\vec{x})d\vec{x} + \int J(x)A(\vec{x})d\vec{x} = \mathcal{H}_e + \mathcal{H}_\gamma + \mathcal{H}_{e\gamma} + \mathcal{H}_{J\gamma},$$

$$A(\vec{x}) = \sum_{\vec{\kappa}\lambda} (2V\kappa)^{-\frac{1}{2}} (C_{\vec{\kappa}\lambda} e^{i\vec{\kappa}\vec{x}} + C^+_{\vec{\kappa}\lambda} e^{-i\vec{\kappa}\vec{x}}) e_{\vec{\kappa}\lambda},$$

$$j(\vec{x}) = \frac{e}{2}[\bar{\Psi}(\vec{x})\gamma, \Psi(\vec{x})].$$

Then the vacuum $|0\rangle_{in}$ is fixed by the conditions

$$_{in}\langle 0|\mathcal{H}_J|0\rangle_{in} - min, \quad t = t_{in}, \tag{4}$$

$$_{in}\langle 0|A(\vec{x})|0\rangle_{in} = A^{in}(\vec{x}), \quad _{in}\langle 0|\dot{A}(\vec{x})|0\rangle_{in} = \dot{A}^{in}(\vec{x}), \tag{5}$$

where the operator

$$\dot{A}(\vec{x}) = -i\sum_{\vec{\kappa}\lambda}\left(\frac{\kappa}{2V}\right)^{\frac{1}{2}}(C_{\vec{\kappa}\lambda}e^{i\vec{\kappa}\vec{x}} - C^+_{\vec{\kappa}\lambda}e^{-i\vec{\kappa}\vec{x}})e_{\vec{\kappa}\lambda}$$

whould be normalized and belong to the Lorentz set. The latter requirement is fulfilled automatically if $\partial^\mu A^{in}_\mu(\vec{x}) = 0$ and we satisfy conditions (5).

Since t_{in} is recognized as inifinitely remote past time-moment one may as usual assume that the radiative interaction between particles is effectively switched off at t_{in}. This gives the reason to look for the vacuum $|0\rangle_{in}$ among the vectors which are the direct product of the vectors $|0\rangle^e_{in}$ and $|0\rangle^\gamma_{in}$ taken from the space of states of the spinor and electromagnetic fields, resp.

$$|0\rangle_{in} = |0\rangle^e_{in} \cdot |0\rangle^\gamma_{in}. \tag{6}$$

In this case the problem (4)-(5) reduced to finding independently the vector $|0\rangle^\gamma_{in}$ subjected to the conditions

$$^\gamma_{in}\langle 0|\mathcal{H}_\gamma|0\rangle^\gamma_{in} - min \tag{7}$$

$$ {}^{\gamma}_{in}\langle 0|A(\vec{x})|0\rangle^{\gamma}_{in} = A^{in}(\vec{x}), \quad {}^{\gamma}_{in}\langle 0|\dot{A}(\vec{x})|0\rangle^{\gamma}_{in} = \dot{A}^{in}(\vec{x}) \tag{8} $$

and the vector $|0\rangle^{e}_{in}$ subjected to the condition

$$ {}^{e}_{in}\langle 0|\mathcal{H}_{eA}|0\rangle^{e}_{in} - min, \tag{9} $$

$$ \mathcal{H}_{eA} = \int :\bar{\Psi}(\vec{x})(-i\vec{\gamma}\vec{\nabla} + e\hat{A}^{in}(\vec{x}) + m)\Psi(\vec{x}): d\vec{x} \tag{10} $$

We look for the $|0\rangle^{\gamma}_{in}$ among the vectors that minimize the mean value of the operator

$$ \mathcal{H}'_{\gamma} = \mathcal{H}_{\gamma} + \sum_{\vec{k}\lambda} g_{\lambda\lambda} \kappa \{ z^{*}_{\vec{k}\lambda}(in) c_{\vec{k}\lambda} + z_{\vec{k}\lambda}(in) c^{+}_{\vec{k}\lambda} \}, \tag{11} $$

where $z_{\kappa\lambda}(in)$ are the undetermined Lagrange multipliers which are to be found from the conditions (8)

$$ {}^{\gamma}_{in}\langle 0|\mathcal{H}'_{\gamma}|0\rangle^{\gamma}_{in} - min. \tag{12} $$

It is sufficient, however, to demand the minimization only of the transversal part of the mean value (12) since the states of longitudinal and time photons do not contribute into expression (7) which as a matter of fact is to be minimized.

Let us diagonalize the operator (11) by the shift

$$ c_{\vec{k}\lambda} = c_{\vec{k}\lambda}(in) + z_{\vec{k}\lambda}(in), \quad c^{+}_{\vec{k}\lambda} = c^{+}_{\vec{k}\lambda}(in) + z^{*}_{\vec{k}\lambda}(in) $$
$$ \mathcal{H}'_{\gamma} = -\sum_{\vec{k}\lambda} g_{\lambda\lambda} \kappa c^{+}_{\vec{k}\lambda}(in) c_{\vec{k}\lambda}(in) + \sum_{\vec{k}\lambda} g_{\lambda\lambda} \kappa |z_{\vec{k}\lambda}(in)|^{2}, \tag{13} $$

which is a canonical transformation. It is seen that due to the above mentioned remark one may choose for $|0\rangle^{\gamma}_{in}$ the vector subjected to the condition

$$ c_{\vec{k}\lambda}(in)|0\rangle^{\gamma}_{in} = 0, \quad \forall \vec{k}, \lambda. \tag{14} $$

If $\sum_{\vec{k},\lambda=1,2} |z_{\vec{k}\lambda}(in)|^{2} < \infty$ the transformation (13) is proper and equation (14) has the solution $|0\rangle^{\gamma}_{in}$ with the transversal part lying in the original Hilbert space. The operator of the proper canonical transformation (13) can be found:

$$ c_{\vec{k}\lambda}(in) = D(z(in)) c_{\vec{k}\lambda} D^{-1}(z(in)), $$
$$ c^{+}_{\vec{k}\lambda}(in) = D(z(in)) c^{+}_{\vec{k}\lambda} D^{-1}(z(in)), \tag{15} $$

$$D(z) = \exp \sum_{\vec{k}\lambda} g_{\lambda\lambda} \{ z^*_{\vec{k}\lambda} c_{\vec{k}\lambda} - z_{\vec{k}\lambda} c^+_{\vec{k}\lambda} \} \tag{16}$$

Consequently

$$|0\rangle^{\gamma}_{in} = D(z(in))|0\rangle^{\gamma}. \tag{17}$$

The vector $|0\rangle^{\gamma}_{in}$ is the coherent state of the free electromagnetic field (Glauber, 1970). By substituting (17) into (8) one gets:

$$z_{\vec{k}\lambda}(in) = \frac{g_{\lambda\lambda}}{\sqrt{2Vk}} \int (e_{\vec{k}\lambda}, [k A^{in}(\vec{x}) + i \dot{A}^{in}(\vec{x})]) e^{-i\vec{k}\vec{x}} d\vec{x} \tag{I8}$$

Note, that when defining the vacuum vector $|0\rangle^{\gamma}_{in}$ we do not exploit explicitly any information about the value of the external current at t_{in}. It is however, implicitly present in the initial mean values of the electromagnetic field $\{A^{in}(\vec{x}), \dot{A}^{in}(\vec{x})\}$ which are formed by the field for which the current at t_{in} is responsible and by the free initial field.

We define the excited states of the electromagnetic field above the vacuum (17) by requiring that the relations (8) are fulfilled in these states while their energy differs from that of the vacuum $|0\rangle^{\gamma}_{in}$ by the energy of the corresponding number of photons. These states are

$$|z(in), n\rangle^{\gamma} = D(z(in))|n\rangle^{\gamma} = D(z(in)) \prod_{\vec{k}, \lambda=1,2} \frac{(c^+_{\vec{k}\lambda})^{n_{\vec{k}\lambda}}}{\sqrt{n_{k\lambda}!}} |0\rangle^{\gamma}. \tag{19}$$

We call them semicoherent (Bagrov, Gitman, Kutchin, 1976). It is evident that $|z, 0\rangle^{\gamma}$ is a coherent state, $|0, n\rangle^{\gamma}$ is n-photon state. For the fixed z the set $|z, n\rangle^{\gamma}$ is complete and orthonormal. For the fixed n the completeness relation

$$\int \prod \frac{d^2 z}{\pi} |z, n\rangle^{\gamma} \cdot {}^{\gamma}\langle z, n| = I$$

is fulfilled. The mean energy and the mean values of electromagnetic potentials in the state $|z(in), n\rangle^{\gamma}$ are:

$${}^{\gamma}\langle z(in), n | \mathcal{H}_{\gamma} | z(in), n\rangle^{\gamma} = \sum_{\vec{k}, \lambda=1,2} k \{ |z_{\vec{k}\lambda}(in)|^2 + n_{\vec{k}\lambda} \},$$

$${}^{\gamma}\langle z(in), n | A(\vec{x}) | z(in), n\rangle^{\gamma} = A^{in}(\vec{x}),$$

$${}^{\gamma}\langle z(in), n | \dot{A}(\vec{x}) | z(in), n\rangle^{\gamma} = \dot{A}^{in}(\vec{x})$$

The states $|z(in), n\rangle^{\gamma}$ may be written as

$$|z(in), n\rangle^{\gamma} = \prod_{\vec{k},\lambda=1,2} \frac{\{C^{+}_{\vec{k}\lambda}(in)\}^{n_{\vec{k}\lambda}}}{\sqrt{n_{\vec{k}\lambda}!}} \cdot |0\rangle^{\gamma}_{in}.$$

To find the vector $|0\rangle^{e}_{in}$ subjected to the condition (9) suffice it to have the solution of the eigenvalue problem of the Dirac Hamiltonian in the external field $A^{in}(\vec{x})$:

$$\mathcal{H}^{in}_{D} \cdot {}_{\pm}\varphi_n(\vec{x}) = {}_{\pm}\mathcal{E}_n \cdot {}_{\pm}\varphi_n(\vec{x}),$$

$$\mathcal{H}^{in}_{D} = \gamma^0(-i\vec{\gamma}\vec{\nabla} + e\hat{A}^{in}(\vec{x}) + m)$$

which obeys the following requirements:

I) ${}_{\pm}\mathcal{E}_n \gtrless 0 \quad \forall n$ and there is a gap between the positive and negative levels

II) The spinors ${}_{\pm}\varphi_n(\vec{x})$ form a complete orthonormal set of the functions in the space of $\vec{x}$ -dependent spinors

$$({}_{\pm}\varphi_n, {}_{\pm}\varphi_{n'}) = \delta_{nn'}, \quad ({}_{\pm}\varphi_n, {}_{\mp}\varphi_{n'}) = 0, \quad (\varphi, \psi) = \int \varphi^{+}(\vec{x})\psi(\vec{x})d\vec{x},$$
$$\sum_n [{}_{+}\varphi_n(\vec{x})\,{}_{+}\varphi^{+}_n(\vec{x}') + {}_{-}\varphi_n(\vec{x})\,{}_{-}\varphi^{+}_n(\vec{x}')] = \delta(\vec{x}-\vec{x}') \tag{20}$$

III) The spinors ${}_{\pm}\varphi_n(\vec{x})$ obey the condition

$$\sum_{nm} \{|({}_{+}\varphi_n, {}_{-}\varphi^{o}_m)|^2 + |({}_{-}\varphi_n, {}_{+}\varphi^{o}_m)|^2\} < \infty, \tag{21}$$

where

$${}_{\pm}\varphi^{o}_m(\vec{x}) = {}_{\pm}\varphi_m(\vec{x})\Big|_{A^{in}=0}$$

Indeed with the use of (20) let us decompose the spinor field operators $\psi(\vec{x})$ and $\bar{\psi}(\vec{x})$ into sums of the solutions ${}_{\pm}\varphi_n(\vec{x})$

$$\psi(\vec{x}) = \sum_n \{a_n(in)\,{}_{+}\varphi_n(\vec{x}) + b^{+}_n(in)\,{}_{-}\varphi_n(\vec{x})\},$$
$$\bar{\psi}(\vec{x}) = \sum_n \{a^{+}_n(in)\,{}_{+}\bar{\varphi}_n(\vec{x}) + b_n(in)\,{}_{-}\bar{\varphi}_n(\vec{x})\}. \tag{22}$$

Then the commutation relations for $\psi(\vec{x}), \bar{\psi}(\vec{x})$ and eqs. (20) lead to the fact that the operators $\{a^{+}(in), a(in), b^{+}(in), b(in)\}$ **are Fermi creation and annihilation operators.** The Hamiltonian (10) diagonalizes in terms of them

$$\mathcal{H}_{eA} = \sum_n \{{}_{+}\mathcal{E}_n\, a^{+}_n(in)a_n(in) - {}_{-}\mathcal{E}_n\, b^{+}_n(in)b_n(in)\} + \chi, \tag{23}$$

Here χ is an undetermined constant. Consequently the vector $|0\rangle^e_{in}$ satisfies the condition

$$a_n(in)|0\rangle^e_{in} = b_n(in)|0\rangle^e_{in} = 0 \quad \forall n . \tag{24}$$

Equation (24) has solutions in the original Hilbert space if the operators $\{a^+(in), a(in), b^+(in), b(in)\}$ are unitary-equivalent to a complete set of creation and annihilation operators for which the vacuum vector exists (Berezin, 1965). The set of creation and annihilation operators of free particles $\{a^+, a, b^+, b\}$ is an example of such a set:

$$\begin{aligned} \psi(\vec{x}) &= \sum_n \{a_n\, {}_+\varphi^0_n(\vec{x}) + b^+_n\, {}_-\varphi^0_n(\vec{x})\}, \\ \bar{\psi}(\vec{x}) &= \sum_n \{a^+_n\, {}_+\bar{\varphi}^0_n(\vec{x}) + b_n\, {}_-\bar{\varphi}^0_n(\vec{x})\} . \end{aligned} \tag{25}$$

The comparison of (22) with (25) gives via the relations (20)

$$\begin{gathered} A_{n\lambda}(in) = \sum_{m\gamma} (\Phi_{n\lambda,m\gamma} A_{m\gamma} + \Psi_{n\lambda,m\gamma} A^+_{m\gamma}), \\ A_{n+}(in) = a_n(in), \quad A_{n-}(in) = b_n(in), \quad A_{n+} = a_n, \quad A_{n-} = b_n, \\ \Phi_{n+,m+} = ({}_+\varphi_n, {}_+\varphi^0_m), \quad \Phi_{n+,m-} = \Phi_{n-,m+} = 0, \quad \Phi_{n-,m-} = ({}_-\varphi_n, {}_-\varphi^0_m), \\ \Psi_{n+,m+} = \Psi_{n-,m-} = 0, \quad \Psi_{n+,m-} = ({}_+\varphi_n, {}_-\varphi^0_n), \quad \Psi_{n-,m+} = ({}_-\varphi_n, {}_+\varphi^0_m)^* . \end{gathered} \tag{26}$$

The transformation (26) is proper and the unitary equivalence needed holds if Ψ is a Hilbert-Schmidt operator (Berezin, 1965; Kiperman, 1970) which corresponds to condition (21) in our terms. It follows from (23)-(24) that $\{a^+(in), a(in), b^+(in), b(in)\}$ may be referred to as creation and annihilation operators of electrons and positrons in the initial time-moment t_{in}. In accordance with this the states with definite numbers of electrons and positrons at t_{in} are built in the usual way.

The general form of the initial states with definite numbers of particles at t_{in} in accordance with the above mentioned considerations must be as follows

$$|in\rangle = N\, C^+(in) \ldots b^+(in) \ldots a^+(in) \ldots |0\rangle_{in} . \tag{27}$$

Analogously one may build the final states at t_{out} under the condition that the mean values are found:

$$\langle A \rangle^{M}\Big|_{t_{out}} = A^{out}(\vec{x}), \quad \langle \dot{A} \rangle^{M}\Big|_{t_{out}} = \dot{A}^{out}(\vec{x}). \tag{28}$$

(The mean values (28) are, generally, different for different $|in\rangle$ states. The problem of their determination is discussed in Sec.2.) Then

$$C_{\vec{k}\lambda}(out) = D(z(out))C_{\vec{k}\lambda}D^{-1}(z(out)) = C_{\vec{k}\lambda} - z_{\vec{k}\lambda}(out),$$
$$C^{+}_{\vec{k}\lambda}(out) = D(z(out))C^{+}_{\vec{k}\lambda}D^{-1}(z(out)) = C^{+}_{\vec{k}\lambda} - z^{*}_{\vec{k}\lambda}(out), \tag{29}$$
$$z_{\vec{k}\lambda}(out) = \frac{g_{\lambda\lambda}}{\sqrt{2V\kappa}} \int (e_{\vec{k}\lambda}, [\kappa A^{out}(\vec{x}) + i\dot{A}^{out}(\vec{x})])\, e^{-i\vec{k}\vec{x}} d\vec{x},$$

$${}_{out}^{\gamma}\langle 0| = {}^{\gamma}\langle 0| D^{-1}(z(out)). \tag{30}$$

Note here that from the fact that the vacua $|0\rangle^{\gamma}_{in}$ and ${}_{out}^{\gamma}\langle 0|$ belong to the Lorentz set it follows that $z_{\vec{k}3}\binom{out}{in} = z_{\vec{k}0}\binom{out}{in}$. The electron and positron creation and annihilation operators at t_{out} are fixed by the decomposition

$$\psi(\vec{x}) = \sum_m \{a_m(out)\,{}^{+}\varphi_m(\vec{x}) + b^{+}_m(out)\,{}^{-}\varphi_m(\vec{x})\},$$
$$\bar{\psi}(\vec{x}) = \sum_m \{a^{+}_m(out)\,{}^{+}\bar{\varphi}_m(\vec{x}) + b_m(out)\,{}^{-}\bar{\varphi}_m(\vec{x})\}, \tag{31}$$

where the spinors ${}^{\pm}\varphi(\vec{x})$ are chosen from the solution of the eigenvalue problem

$$\mathcal{H}_D^{out}\,{}^{\pm}\varphi_m(\vec{x}) = {}^{\pm}\varepsilon_m\,{}^{\pm}\varphi_m(\vec{x}),$$
$$\mathcal{H}_D^{out} = \gamma^0(-i\vec{\gamma}\vec{\nabla} + e\hat{A}^{out}(\vec{x}) + m)$$

satisfying conditions I) - III).

$${}_{out}^{e}\langle 0| a^{+}_m(out) = {}_{out}^{e}\langle 0| b^{+}_m(out) = 0, \quad \forall m.$$

Consequently the general form of the final states with a definite number of particles at t_{out} is

$$\langle out| = {}_{out}\langle 0| a(out) \ldots b(out) \ldots C(out) \ldots N,$$
$${}_{out}\langle 0| = {}_{out}^{e}\langle 0| \cdot {}_{out}^{\gamma}\langle 0|. \tag{32}$$

For future it is important to note that the above introduced *in*- and *out*- creation and annihilation operators are interconnected by linear canonical transformations, these transformations being mere shifts for the case of electromagnetic field operators.

2. Mean field in quantum electrodynamics

Consider here the problem of determining of the mean electromagnetic field in QED with the external current. We assume, for definiteness, that the initial states maybe given in the form (I.27). Then the problem reduces to the calculation of the following average

$$\langle A(x)\rangle^{M} = \langle in|\check{A}(x)|in\rangle, \tag{I}$$

where $\check{A}(x) = \mathcal{U}_J^{-1}(tt_{in})A(\vec{x})\mathcal{U}_J(tt_{in})$, and $\mathcal{U}_J(tt_{in})$ is the evolution operator of QED corresponding to the Hamiltonian $\mathcal{H}_J$ (I.3).

It is convenient to treat the proposed problem with functional methods. To do so let us add the terms, corresponding to the interaction with the external sources $I(x), \bar{\eta}(x), \eta(x)$ of the electromagnetic and spinor fields to the Hamiltonian of QED with the external current $\mathcal{J}(x)$

$$\mathcal{H}_J' = \mathcal{H}_J + \int[I(x)A(\vec{x}) + \bar{\Psi}(\vec{x})\eta(x) + \bar{\eta}(x)\Psi(\vec{x})]d\vec{x}.$$

Here and elsewhere we shall prime the quantities taken when all the external sources are present. Denote as $\mathcal{U}_J'(t,t_{in})$ the evolution operator corresponding to the Hamiltonian $\mathcal{H}_J'$ $(\mathcal{U}_J' = \mathcal{U}_J'(t_{out},t_{in}))$ and introduce the generating functional Z^M which depends on the doubled number of sources

$$Z^M = {}_{in}\langle 0|\mathcal{U}_J'^{-1}(I_2\bar{\eta}_2\eta_2)\mathcal{U}_J'(I_1\bar{\eta}_1\eta_1)|0\rangle_{in}, \tag{2}$$

$$Z^M\Big|_{I_2=I_1,\ \bar{\eta}_2=\bar{\eta}_1,\ \eta_2=\eta_1} = 1,$$

and may be also written in terms of the matrix of scattering by the external sources in the Heisenberg picture (Fradkin, I954; I965a)

$$Z^M = {}_{in}\langle 0|\check{S}'^{-1}(I_2\bar{\eta}_2\eta_2)\check{S}'(I_1\bar{\eta}_1\eta_1)|0\rangle_{in},$$

$$\mathcal{U}_J'(tt_{in}) = \mathcal{U}_J(tt_{in})\check{S}'(tt_{in}),\quad \check{S}' = \check{S}'(t_{out}t_{in}), \tag{3}$$

$$\check{S}'(I\bar{\eta}\eta) = \check{T}\exp\{-i\int[I(x)\check{A}(x) + \check{\bar{\psi}}(x)\eta(x) + \bar{\eta}(x)\check{\psi}(x)]dx\},$$

$$\check{S}'^{-1}(I\bar{\eta}\eta) = \exp\{i\int[I(x)\check{A}(x) + \check{\bar{\psi}}(x)\eta(x) + \bar{\eta}(x)\check{\psi}(x)]dx\}T,$$

$$\check{\psi}(x) = \mathcal{U}_J^{-1}(tt_{in})\psi(\vec{x})\mathcal{U}_J(tt_{in}),\quad \check{\bar{\psi}}(x) = \ldots,\ \check{A}(x) = \ldots$$

It is adopted here that the symbol T when placed to the right of an operator functional arranges the operators involved into it in the antichronological order. (The functional Z^M in the form (3) was considered in (Fradkin, 1964) when the generalized unitarity relations for the exact Green functions were being obtained.)

Define the Green functions as the functional derivatives of the functional Z^M with respect to all the sources by using the relations

$$\frac{\vec{\delta}^{\,n+m+\nu}\check{S}'}{\delta\bar{\eta}_1(x_1)\ldots\delta\bar{\eta}_1(x_n)\delta\eta_1(y_1)\ldots\delta\eta_1(y_m)\delta I_1(z_1)\ldots\delta I_1(z_\nu)} =$$

$$= (-i)^{n+\nu}(i)^m\check{S}'\check{T}[\check{\psi}'(x_1)\ldots\check{\psi}'(x_n)\check{\bar{\psi}}'(y_1)\ldots\check{\bar{\psi}}'(y_m)\check{A}'(z_1)\ldots\check{A}'(z_\nu)],$$

$$\frac{\vec{\delta}^{\,n+m+\nu}\check{S}'^{-1}}{\delta\bar{\eta}_2(x_1)\ldots\delta\bar{\eta}_2(x_n)\delta\eta_2(y_1)\ldots\delta\eta_2(y_m)\delta I_2(z_1)\ldots\delta I_2(z_\nu)} = \tag{4}$$

$$= (i)^{n+\nu}(-1)^m[\check{\psi}'(x_1)\ldots\check{\psi}'(x_n)\check{\bar{\psi}}'(y_1)\ldots\check{\bar{\psi}}'(y_m)\check{A}'(z_1)\ldots\check{A}'(z_\nu)T]\check{S}'^{-1}.$$

Then $\mathcal{G}^M_{nm\nu,n'm'\nu'}(xyz,\, x'y'z') =$

$$= \left.\frac{(-i)^{n+\nu+m'}(i)^{m+\nu'+n'}\vec{\delta}^{\,n+m+\nu+n'+m'+\nu'}Z^M}{\delta\bar{\eta}_2(x_1)\ldots\delta\bar{\eta}_2(x_n)\delta\eta_2(y_1)\ldots\delta\eta_2(y_m)\delta I_2(z_1)\ldots\delta I_2(z_\nu)\delta\bar{\eta}_1(x_1')\ldots\delta I_1(z_{\nu'}')}\right|_{\substack{I_\lambda=0\\ \eta_\lambda=\\ =\bar{\eta}_\lambda=0\\ \lambda=1,2}} =$$

$$= {}_{in}\langle 0|\check{\psi}(x_1)\ldots\check{\psi}(x_n)\check{\bar{\psi}}(y_1)\ldots\check{\bar{\psi}}(y_m)\check{A}(z_1)\ldots\check{A}(z_\nu)T\check{\psi}(x'_1)\ldots\check{\psi}(x'_{n'})\cdot$$

$$\cdot\check{\bar{\psi}}(y'_1)\ldots\check{\bar{\psi}}(y'_{m'})\check{A}(z'_1)\ldots\check{A}(z'_{\nu'})|0\rangle_{in} \,. \qquad (5)$$

The sign of the T- product in (5) acts both to the right and to the left.

The Green functions (5) give the possibility to find the expectation values of the Heisenberg operators with respect to any in- states, for which the creation and annihilation operators may be expressed explicitly through the field operators in the Schrodinger picture or, what is the same, in the Heisenberg picture at $t=t_{in}$. For example we will get by using the relations (I.22) that the mean field in the system with the initial vacuum-state and in the system with an electron in the initial state is

$$\langle A(x)\rangle^M = {}_{in}\langle 0|\check{A}(x)|0\rangle_{in} = \mathcal{G}^M_{000,\,001}(x),$$

$$\langle A(x)\rangle^M = {}_{in}\langle 0|a_n(in)\,\check{A}(x)\,a^+_n(in)|0\rangle_{in} = \qquad (6)$$

$$= \int {}_+\varphi^+_n(\vec{y})\,\mathcal{G}^M_{100,\,001}(y,z\,x)\,\gamma^0\,{}_+\varphi_n(\vec{z})\,d\vec{y}\,d\vec{z}\,\Big|_{y^0=z^0=t_{in}}$$

respectively.

To determine the mean field (I) let us build the perturbation theory in which the interaction with the external current $\mathcal{J}(x)$ and the mean initial field is kept exactly. For this purpose represent the evolution operator $\mathcal{U}'_J$ in the following way

$$\mathcal{U}'_J(tt_{in}) = \tilde{\tilde{\mathcal{U}}}(tt_{in})\,\tilde{\tilde{S}}'(tt_{in}),\quad \tilde{\tilde{\mathcal{U}}}(tt_{in}) = T\exp\Big\{-i\int_{t_{in}}^{t}\tilde{\tilde{\mathcal{H}}}\,d\tau\Big\},$$

$$\tilde{\tilde{\mathcal{H}}} = \mathcal{H}_e + \mathcal{H}_\gamma + \mathcal{H}_{J\gamma} + \int j(\vec{x})\,A^M(x)\,d\vec{x},$$

$$\tilde{\tilde{S}}'(tt_{in}) = \exp\Big\{-i\int_{t_{in}}^{t} I(x)\,A^M(x)\,dx\Big\}\,T\exp\Big\{-i\int_{t_{in}}^{t}[\tilde{\tilde{j}}(x)\,\tilde{\tilde{\underline{A}}}(x) +$$

$$+ I(x)\,\tilde{\tilde{\underline{A}}}(x) + \tilde{\tilde{\bar{\psi}}}(x)\,\eta(x) + \bar{\eta}(x)\,\tilde{\tilde{\psi}}(x)]\,dx, \qquad (7)$$

$$A^M(x) = {}_{in}\langle 0|\tilde{\tilde{A}}(x)|0\rangle_{in}\,,\quad \tilde{\tilde{\underline{A}}}(x) = \tilde{\tilde{A}}(x) - A^M(x),$$

$$\tilde{\tilde{\Psi}}(x)=\tilde{\tilde{U}}^{-1}(tt_{in})\psi(\vec{x})\tilde{\tilde{U}}(tt_{in}),\quad \tilde{\tilde{\bar{\Psi}}}(x)=\dots,\quad \tilde{\tilde{A}}(x)=\dots,$$

$$\tilde{\tilde{j}}(x)=\tilde{\tilde{U}}^{-1}(tt_{in})j(\vec{x})\tilde{\tilde{U}}(tt_{in}),$$

$$(i\hat{\partial}-e\hat{A}^{M}(x)-m)\tilde{\tilde{\Psi}}(x)=0,\quad \Box\tilde{\tilde{A}}(x)=J(x),\quad \Box\underline{\tilde{\tilde{A}}}(x)=0.$$

Then the generating functional (2) may be written in the form

$$Z^{M}=\exp(iA^{M}[I_2-I_1])\exp e\left(\frac{\vec{\delta}}{\delta\eta_1}\gamma\frac{\vec{\delta}}{\delta\bar{\eta}_1}\frac{\delta}{\delta I_1}+\frac{\vec{\delta}}{\delta\eta_2}\gamma\frac{\vec{\delta}}{\delta\bar{\eta}_2}\frac{\delta}{\delta I_2}\right)Z_0^{M},$$

$$Z_0^{M}={}_{in}\langle 0|\,\tilde{\tilde{S}}_0'(I_2\bar{\eta}_2\eta_2)\,\tilde{\tilde{S}}_0'(I_1\bar{\eta}_1\eta_1)\,|0\rangle_{in},$$

$$\tilde{\tilde{S}}_0'(I\bar{\eta}\eta)=T\exp\{-i(I\underline{\tilde{\tilde{A}}}+\tilde{\tilde{\bar{\Psi}}}\eta+\bar{\eta}\tilde{\tilde{\Psi}})\},$$

$$\tilde{\tilde{S}}_0'^{-1}(I\bar{\eta}\eta)=\exp\{i(I\underline{\tilde{\tilde{A}}}+\tilde{\tilde{\bar{\Psi}}}\eta+\bar{\eta}\tilde{\tilde{\Psi}})\}T.$$

One can derive explicitly the functional Z_0^{M} by using the following formulae which are the generalization of the formulae (A.I6) and may be easily proved

$$\langle 0|\Phi(\varphi)TF(\varphi)|0\rangle=\exp\frac{\varepsilon}{2}\left(\frac{\vec{\delta}}{\delta\varphi_1}\overbrace{\varphi\varphi}\frac{\vec{\delta}}{\delta\varphi_1}+2\frac{\vec{\delta}}{\delta\varphi_2}\underbrace{\varphi\varphi}\frac{\vec{\delta}}{\delta\varphi_1}+\right.$$

$$\left.+\frac{\vec{\delta}}{\delta\varphi_2}\overbrace{\varphi\varphi}^{\leftarrow}\frac{\vec{\delta}}{\delta\varphi_2}\right)\Phi(\varphi_2)F(\varphi_1)\Big|_{\varphi_{1,2}=\varphi^{(0)}},\quad \varphi^{(0)}=\langle 0|\varphi|0\rangle,$$

$$\overbrace{\varphi\varphi}=\langle 0|T\varphi\varphi|0\rangle,\quad \underbrace{\varphi\varphi}=\langle 0|\varphi\varphi|0\rangle,\quad \overbrace{\varphi\varphi}^{\leftarrow}=\langle 0|\varphi\varphi T|0\rangle.$$

By deciphering the abbreviated notation for the case under consideration we get

$$Z_0^{M}=\exp i\,(\bar{\eta}_1\tilde{\tilde{S}}^{c}\eta_1+\bar{\eta}_2\tilde{\tilde{S}}^{\bar{c}}\eta_2-\bar{\eta}_1\tilde{\tilde{S}}^{(+)}\eta_2-\eta_2\tilde{\tilde{S}}^{(-)}\eta_1-$$

$$-\frac{1}{2}[I_1D_0^{c}I_1+I_2D_0^{\bar{c}}I_2+I_1D_0^{(+)}I_2-I_2D_0^{(-)}I_1]),$$

where

$$\tilde{\tilde{S}}^{c}(xy)=i\,{}_{in}\langle 0|T\tilde{\tilde{\Psi}}(x)\tilde{\tilde{\bar{\Psi}}}(y)|0\rangle_{in},$$

$$\tilde{\tilde{S}}^{\bar{c}}(xy)=i\,{}_{in}\langle 0|\tilde{\tilde{\Psi}}(x)\tilde{\tilde{\bar{\Psi}}}(y)T|0\rangle_{in},\qquad (8)$$

$$\tilde{\tilde{S}}^{(-)}(xy)=i\,{}_{in}\langle 0|\tilde{\tilde{\Psi}}(x)\tilde{\tilde{\bar{\Psi}}}(y)|0\rangle_{in},$$

$$\tilde{\tilde{S}}^{(+)}(xy)=i_{in}\langle 0|\bar{\tilde{\tilde{\Psi}}}(y)\tilde{\tilde{\Psi}}(x)|0\rangle_{in};$$

$$D_0^c(xy)=-i_{in}\langle 0|T\underline{\tilde{\tilde{A}}}(x)\underline{\tilde{\tilde{A}}}(y)|0\rangle_{in},$$

$$D_0^{\bar{c}}(xy)=-i_{in}\langle 0|\underline{\tilde{\tilde{A}}}(x)\underline{\tilde{\tilde{A}}}(y)T|0\rangle_{in}=-D_0^{c}\overset{*}{(xy)},$$

$$D_0^{(-)}(xy)=-i_{in}\langle 0|\underline{\tilde{\tilde{A}}}(x)\underline{\tilde{\tilde{A}}}(y)|0\rangle_{in}, \tag{9}$$

$$D_0^{(+)}(xy)=i_{in}\langle 0|\underline{\tilde{\tilde{A}}}(y)\underline{\tilde{\tilde{A}}}(x)|0\rangle_{in}=-D_0^{(-)}(yx)$$

The final expression for Z^M may be written in a compact form by introducing the matrix propagators and the vertex

$$Z^M=\exp i(I_2-I_1)A^M\colon\exp e\frac{\overrightarrow{\delta}}{\delta\eta_\lambda}\Gamma_{\lambda\alpha\beta}\frac{\overrightarrow{\delta}}{\delta\bar{\eta}_\alpha}\cdot\frac{\delta}{\delta I_\beta}\cdot \tag{I0}$$

$$\cdot\exp i(\bar{\eta}_\lambda\tilde{\tilde{S}}_{\lambda\alpha}\eta_\alpha-\tfrac{1}{2}I_\lambda D_{\lambda\alpha}I_\alpha),\quad \Gamma_{\lambda\alpha\beta}=\delta_{\lambda\alpha}\delta_{\lambda\beta}\gamma,\quad \alpha,\lambda,\beta=1,2;$$

$$\tilde{\tilde{S}}=\begin{vmatrix}\tilde{\tilde{S}}^c, & -\tilde{\tilde{S}}^{(+)}\\ -\tilde{\tilde{S}}^{(-)}, & \tilde{\tilde{S}}^{\bar{c}}\end{vmatrix},\qquad D=\begin{vmatrix}D_0^c, & D_0^{(+)}\\ -D_0^{(-)}, & D_0^{\bar{c}}\end{vmatrix}. \tag{II}$$

Note, that the canonical transformations connecting $\underline{\tilde{\tilde{A}}}(x)$, $|0\rangle_{in}$ with the corresponding free quantities are shift transformations of the free creation and annihilation operators of the electromagnetic field. Therefore the propagators (9) coincide with the corresponding free propagators and are denoted in the standard way.

It may be shown, for example (Bagrov, Gitman, Kutchin, I976) that

$$\tilde{\tilde{A}}(x)=\int D_0^{ret}(x-y)\mathcal{J}(y)dy+A(x); \tag{I2}$$

where $A(x)$ is the vector potential operator in the usual interaction picture. Then from (7), (I2) and (I.I7) it follows that

$$A^M(x)=\int D_0^{ret}(x-y)\mathcal{J}(y)dy+A^{in}(x),\quad A^{in}(x)={}_{in}\langle 0|A(x)|0\rangle_{in},$$

$$\Box A^M(x)=\mathcal{J}(x),\quad \Box A^{in}(x)=0, \tag{I3}$$

where $A^{in}(x)$ is a free electromagnetic field, which at the initial time-moment coincides with the mean field in the system (see (I.2), (I.5))

$$A^{in}(x)\big|_{t_{in}} = A^{in}(\vec{x}), \qquad \dot{A}^{in}(x)\big|_{t_{in}} = \dot{A}^{in}(x).$$

One can obtain an analogue for the Feynman representation of the propagators (8), the first two of which obey the equation (I.I.45) with the external field $A^M(x)$ while the last two obey the Dirac equation (I.2.IO) with the same field. Note, that the propagator $\tilde{\tilde{S}}^c$ has been already mentioned in Chapter I, Sec.2 in connection with the unitarity relations, and at the same place their representation over the solutions of the Dirac equation (I.2.6) was given. In the same way we get

$$\begin{aligned}
&\tilde{\tilde{S}}^{(-)}(xy) = i\sum_n {}_+\tilde{\varphi}_n(x)\,{}_+\bar{\tilde{\varphi}}_n(y),\\
&\tilde{\tilde{S}}^{(+)}(xy) = i\sum_n {}_-\tilde{\varphi}_n(x)\,{}_-\bar{\tilde{\varphi}}_n(y),\\
&\tilde{\tilde{S}}^{c}(xy) = \begin{cases} \tilde{\tilde{S}}^{(-)}(xy), & x^0 > y^0,\\ -\tilde{\tilde{S}}^{(+)}(xy), & x^0 < y^0,\end{cases}\\
&\tilde{\tilde{S}}^{\bar{c}}(xy) = \begin{cases} -\tilde{\tilde{S}}^{(+)}(xy), & x^0 > y^0,\\ \tilde{\tilde{S}}^{(-)}(xy), & x^0 < y^0.\end{cases}
\end{aligned} \tag{I4}$$

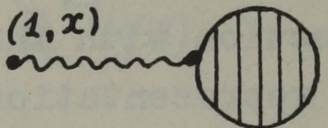
Here ${}_\pm\tilde{\varphi}_n(x)$ are the solutions of the Dirac equation in the external field $A^M(x)$ satisfying at the initial time-moment the conditions ${}_\pm\tilde{\varphi}_n(x)\big|_{t=t_{in}} = {}_\pm\varphi_n(\vec{x})$, where the spinors ${}_\pm\varphi_n(\vec{x})$ are defined in Sec.I.

The representation (IO) for the generating functional Z^M is equivalent to the perturbation expansion and diagrammatic technique for the Green functions (5), wherein the interaction with the current and the initial field is kept exactly. The same perturbation theory one can obtain by the straightforward usage of the Wick technique and their generalization in the sense of the appendix A, if one writes with the aid of (7) the Green functions (5) in the form[I]:

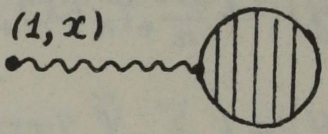
I) The construction of the perturbation expansion and diagrammatic technique for the Green functions of the type of $\mathcal{G}^M_{\dots,nm\lambda}$ in statistical physics was considered in (Keldysh, 1964) by ordering along a contour.

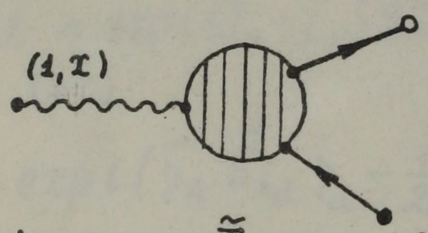
$$\mathcal{G}^{M}_{nm\nu,\,n'm'\nu'}(xyz,\,x'y'z') = {}_{in}\langle 0|\tilde{\tilde{S}}^{-1}\tilde{\tilde{\psi}}(x_1)\dots\tilde{\tilde{\psi}}(x_n)\bar{\tilde{\tilde{\psi}}}(y_1)\dots\bar{\tilde{\tilde{\psi}}}(y_m)\cdot$$
$$\tilde{\tilde{A}}(z_1)\dots\tilde{\tilde{A}}(z_\nu)\,T\,\tilde{\tilde{\psi}}(x'_1)\dots\tilde{\tilde{\psi}}(x'_n)\bar{\tilde{\tilde{\psi}}}(y'_1)\dots\bar{\tilde{\tilde{\psi}}}(y'_m)\tilde{\tilde{A}}(z'_1)\dots\tilde{\tilde{A}}(z'_\nu)\tilde{\tilde{S}}\,|0\rangle_{in}\,.$$

The diagrammatic technique in terms of the matrix quantities (II) has the Feynman form. Thus, for example, the expansions for the mean field $\langle A(x)\rangle^M$ in the cases when the initial state is the vacuum state or a single-electron state have, respectively, the form

a) $|in\rangle = |0\rangle_{in}\,,\quad \langle A(x)\rangle^M = A^M(x) +$ $(1,x)$  $=$

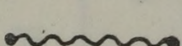
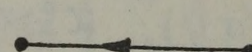
$$= A^M(x) + \int D_0^{ret}(x-y)\,\tilde{\tilde{J}}(y)\,dy + \dots,\quad \tilde{\tilde{J}}(y) = ie\,tr\,\gamma\,\tilde{\tilde{S}}^c(y,y). \qquad (I5)$$

b) $|in\rangle = a_n^+(in)|0\rangle_{in}\,,\quad \langle A(x)\rangle^M = A^M(x) +$ $(1,x)$  $+$

$+$ $(1,x)$  $= A^M(x) + \int D_0^{ret}(x-y)\,[\tilde{\tilde{J}}(y) + j_n(y)]\,dy + \dots,$

$$j_n(y) = e\,{}_{+}\bar{\tilde{\varphi}}_n(y)\gamma\cdot{}_{+}\tilde{\varphi}_n(y). \qquad (I6)$$

Here

 $(\alpha,x)\quad(\beta,y)$ $= D_{\alpha\beta}(x,y),$  $(\alpha,x)\quad(\beta,y)$ $= \tilde{\tilde{S}}_{\alpha\beta}(xy),$

$\circ = \delta_{2\alpha}\,{}_{+}\tilde{\varphi}_n^+(x)\,\delta(x^0 - t_{in}),$

$\bullet = \delta_{1\alpha}\,\gamma^0\,{}_{+}\tilde{\varphi}_n(x)\,\delta(x^0 - t_{in}),$ $\qquad \alpha,\beta = 1,2,$

The shaded circles denote the sums of all the connected Feynman diagrams with the corresponding number of the external lines and with the matrix vertices and propagators (II).

For the generating functional one can get the following set of the functional equations

$$\Box\,\frac{\delta Z^M}{\delta I_\lambda(x)} = (-1)^\lambda i\left[(J(x) + I_\lambda(x))Z^M - e\,tr\,\gamma\,\frac{\delta^2 Z^M}{\delta\bar{\eta}_\lambda(x)\delta\eta_\lambda(x)}\right],$$

$$\left(i\hat{\partial} + (-1)^\lambda ie\frac{\hat{\delta}}{\delta I_\lambda(x)} - m\right)\frac{\delta Z^M}{\delta\bar{\eta}_\lambda(x)} = (-1)^\lambda i\,\eta_\lambda(x)Z^M, \qquad (I7)$$

$$\frac{\delta Z^M}{\delta\eta_\lambda(x)}\left(i\hat{\partial} - (-1)^\lambda ie\frac{\hat{\delta}}{\delta I_\lambda(x)} + m\right) = (-1)^\lambda i\,\bar{\eta}_\lambda(x)Z^M.$$

The equations (I7) under $\lambda=1$ coincide formally with the equations for the generating functional obtained in (Fradkin, 1965a). The equations (I7) generate a set of equations for the Green functions $\mathcal{G}^M$. Let us introduce, as usual, the functional

$$W^M = i \ln Z^M, \tag{I8}$$

which is the generating functional for the connected Green functions, and the following definitions

$$\frac{\delta W^M}{\delta I_\lambda(x)}\bigg|_{\bar\eta=\eta=0} = \alpha_\lambda(x), \quad \frac{\delta^2 W^M}{\delta I_\lambda(x)\delta I_\beta(y)}\bigg|_{\bar\eta=\eta=0} = D_{\lambda\beta}(xy),$$

$$\frac{\delta^2 W^M}{\delta\bar\eta_\lambda(x)\delta\eta_\beta(y)}\bigg|_{\bar\eta=\eta=0} = S_{\lambda\beta}(xy), \quad \lambda,\beta=1,2. \tag{I9}$$

After the differentiation of the set (I7) with respect to the sources we get, by taking into account (I9), the following

$$\Box D_{\lambda\beta}(xy) = (-1)^{\lambda-1}\left[\delta_{\lambda\beta}\delta(x-y) + i\,tr\,\gamma\cdot\frac{\delta S_{\lambda\lambda}(xx)}{\delta I_\beta(y)}\right],$$

$$\left(i\hat\partial + (-1)^\lambda e\hat\alpha_\lambda(x) - m + i(-1)^\lambda e\frac{\hat\delta}{\delta I_\lambda(x)}\right)S_{\lambda\beta}(xy) = (-1)^\lambda\delta_{\lambda\beta}\delta(x-y),$$

$$S_{\beta\lambda}(yx)\left(i\hat\partial + m - (-1)^\lambda e\hat\alpha_\lambda(x) - i(-1)^\lambda e\frac{\hat\delta}{\delta I_\lambda(x)}\right) = (-1)^{\lambda-1}\delta_{\lambda\beta}\delta(x-y),$$

$$\Box\alpha_\lambda(x) = (-1)^{\lambda-1}\left[J(x) + I_\lambda(x) + ie\,tr\,\gamma S_{\lambda\lambda}(xx)\right]. \tag{20}$$

The set (20) is an analogue of the Schwinger set for the case under consideration. It may be transformed, like in the usual case, to the integral form

$$(-1)^{\lambda-1} ie\,tr\,\gamma\,\frac{\delta S_{\lambda\lambda}(xx)}{\delta I_\beta(y)} = \sum_{\varkappa}\int \Pi_{\lambda\varkappa}(xz)D_{\varkappa\beta}(zy)dz,$$

$$(-1)^{\lambda-1} ie\gamma\,\frac{\delta S_{\lambda\beta}(xy)}{\delta I_\lambda(x)} = \sum_{\varkappa}\int \Sigma_{\lambda\varkappa}(xz)S_{\varkappa\beta}(zy)dz,$$

$$\Gamma_{\lambda\beta\varkappa}(xyz) = \frac{\delta S^{-1}_{\lambda\beta}(xy)}{e\delta\alpha_\varkappa(z)} = (-1)^{\lambda-1}\frac{\delta\Sigma_{\lambda\beta}(xy)}{e\delta\alpha_\varkappa(z)} + \gamma\delta_{\lambda\varkappa}\delta(x-z)\cdot$$
$$\cdot\delta_{\lambda\beta}\delta(x-y).$$

$$\Pi_{\lambda\beta}(xy)=(-1)^{\lambda}ie^{2}tr\gamma\int S_{\lambda\varkappa}(xz)\Gamma_{\varkappa\mu\beta}(zz'y)S_{\mu\lambda}(z'x)dz\,dz',$$

$$\Sigma_{\lambda\beta}(xy)=(-1)^{\lambda}ie^{2}\gamma\int S_{\lambda\varkappa}(xx')\Gamma_{\varkappa\beta\nu}(x'yz)D_{\nu\lambda}(zx)dx'dz,$$

$$\Box D_{\lambda\beta}(xy)-\int \Pi_{\lambda\varkappa}(xz)D_{\varkappa\beta}(zy)dz=(-1)^{\lambda-1}\delta_{\lambda\beta}\,\delta(x-y),$$

$$[i\hat{\partial}+(-1)^{\lambda}e\hat{\alpha}_{\lambda}(x)-m]S_{\lambda\beta}(xy)-\int\Sigma_{\lambda\varkappa}(xz)S_{\varkappa\beta}(zy)dz=$$

$$=(-1)^{\lambda}\delta_{\lambda\beta}\,\delta(x-y),$$

$$\Box\alpha_{\lambda}(x)=(-1)^{\lambda-1}[J(x)+I_{\lambda}(x)+ietr\gamma S_{\lambda\lambda}(xx)] \qquad (21)$$

Under $I_{\lambda}=0$ the iteration of the set (2I), starting with the bare Green functions, the vertex (II) and the field $\alpha^{0}_{\lambda}(x)=(-1)^{\lambda-1}A^{M}(x)$ leads to the correct perturbation expansions for the corresponding exact quantities.

Let us now construct the effective action $\Gamma^{M}(\alpha)$ which is connected with the functional W^{M} by the Legendre transformation. (Later on we will put everywhere the sources $\bar{\eta}$ and η equal to zero).

$$\Gamma^{M}(\alpha)=I\alpha-W^{M}, \qquad (22)$$

where the sources I in the right-hand side (22) should be expressed through α with the aid of (I9). From $\delta\Gamma^{M}/\delta\alpha=I$ and the relation

$$\langle A(x)\rangle^{M}={}_{in}\langle 0|\check{A}(x)|0\rangle_{in}=\alpha_{1}(x)\Big|_{I_{\lambda}=0}=-\alpha_{2}(x)\Big|_{I_{\lambda}=0}, \qquad (23)$$

it follows that the column $\alpha_{\lambda}(x)=(-1)^{\lambda-1}\langle A(x)\rangle^{M}$ gives the extremum to the functional $\Gamma^{M}(\alpha)$. Thus the finding of the functional $\Gamma^{M}(\alpha)$ is useful, in particular, since it enables to get a closed equation for the exact mean field (23).

When obtaining the effective action for the mean field which is related to the more complicated initial states with the nonzero number of charged particles, one should construct the Legendre transformations of the higher order.

We will obtain the explicit form of the functional $\Gamma^M(\alpha)$ in terms of powers of radiative interaction, the interaction with the field $A^M(x)$ will be kept exactly. To do so introduce the quantities

$$\bar{Z}^M = exp(i I \alpha^0) Z^M,$$
$$\bar{W}^M = i ln \bar{Z}^M = W^M - I\alpha^0, \quad \frac{\delta \bar{W}^M}{\delta I} = \bar{\alpha} = \alpha - \alpha^0; \qquad (24)$$
$$\bar{\Gamma}^M(\bar{\alpha}) = \Gamma^M(\alpha) = I\bar{\alpha} - \bar{W}^M, \quad \frac{\delta \bar{\Gamma}^M}{\delta \bar{\alpha}} = I.$$

We will get to the zeroth order with respect to the radiative interaction, by using the explicit form of the functional Z^M (IO), the following

$$\bar{Z}_{(0)}^M = exp(-\tfrac{i}{2} I D I), \quad \bar{W}_{(0)}^M = \tfrac{1}{2} I D I,$$
$$I_{(0)} = D^{-1}\bar{\alpha}, \quad \bar{\Gamma}_{(0)}^M(\bar{\alpha}) = \tfrac{1}{2}\bar{\alpha} D^{-1}\bar{\alpha}, \qquad (25)$$
$$D^{-1}_{\alpha\beta} = (-1)^{\alpha-1}\delta_{\alpha\beta}\cdot\Box \quad , \quad \alpha, \beta = 1, 2.$$

For the quantities

$$\Delta\bar{W}^M = \bar{W}^M - \bar{W}^M_{(0)}, \quad \Delta\bar{\Gamma}^M(\bar{\alpha}) = \bar{\Gamma}^M(\bar{\alpha}) - \bar{\Gamma}^M_{(0)}(\bar{\alpha})$$

we will get from (22), (24), (25)

$$\Delta\bar{\Gamma}^M(\bar{\alpha}) = -\frac{1}{2}\frac{\delta\Delta\bar{\Gamma}^M}{\delta\bar{\alpha}}\cdot D\frac{\delta\Delta\bar{\Gamma}^M}{\delta\bar{\alpha}} - \Delta\bar{W}^M\Big(I = D^{-1}\bar{\alpha} + \frac{\delta\Delta\bar{\Gamma}^M}{\delta\bar{\alpha}}\Big). \qquad (26)$$

One can show (Vasil'ev, 1976) that the Legendre transformation of the functional of the type of $\bar{W}^M$ leads to the single-indecomposable diagrams for the $\Delta\bar{\Gamma}^M$ only. Therefore we get, finally, by taking the single-indecomposable part of (25)

$$\Gamma^M(\alpha) = \bar{\Gamma}^M(\bar{\alpha}) = \tfrac{1}{2}\bar{\alpha} D^{-1}\bar{\alpha} - \text{single-ind.part } \Delta\bar{W}^M(I = D^{-1}\bar{\alpha}) \qquad (27)$$

The functional $\Gamma^M(\alpha)$ is equal to zero in the stationary point since it coincides with $W^M(I=0)=0$ in this point.

By doing the partial summing in (27) which is equivalent to the exact keeping of the interaction with the whole field α, we get

$$\Gamma^M(\alpha) = \bar{\Gamma}^M(\bar{\alpha}) = \frac{1}{2}(\alpha-\alpha^\circ)D^{-1}(\alpha-\alpha^\circ) + \tag{28}$$

$$+ i\,Tr\,\ln\frac{\tilde{\tilde{S}}(\alpha)}{\tilde{\tilde{S}}(\alpha_\circ)} - \quad \text{single-ind. vacuum diag. } \Delta\bar{W}(\tilde{\tilde{S}}(\alpha)),$$

where

$$\tilde{\tilde{S}}^{-1}_{\lambda\beta}(\alpha) = (-1)^\lambda \delta_{\lambda\beta}(i\hat{\partial} - e(-1)^{\lambda-1}\hat{\alpha}_\lambda - m),$$

$$\tilde{\tilde{S}}(\alpha) = \begin{vmatrix} \tilde{\tilde{S}}^c(\alpha_1), & -\tilde{\tilde{S}}^{(+)}(\alpha_1) \\ -\tilde{\tilde{S}}^{(-)}(-\alpha_2), & \tilde{\tilde{S}}^{\bar{c}}(-\alpha_2) \end{vmatrix}.$$

Those external fields, for which the Green functions are determined, are shown as arguments in the matrix $\tilde{\tilde{S}}(\alpha)$.

The two first terms in the expression (28) correspond to the one-loop approximation. The equations for the fields $\alpha_1(x)$ and $\alpha_2(x)$ are independent in this approximation

$$\Box\alpha_1(x) = \mathcal{J}(x) + ie\,tr\,\gamma\,\tilde{\tilde{S}}^c(x,x|\alpha_1),$$

$$-\Box\alpha_2(x) = \mathcal{J}(x) + ie\,tr\,\gamma\,\tilde{\tilde{S}}^{\bar{c}}(x,x|-\alpha_2). \tag{29}$$

In view of the fact the functions $\tilde{\tilde{S}}^c$ and $\tilde{\tilde{S}}^{\bar{c}}$ are defined in the same way in the case when the time variables are equal

$$\tilde{\tilde{S}}^c(x,x) = \frac{1}{2}\left[\tilde{\tilde{S}}^c(x+0,x) + \tilde{\tilde{S}}^c(x,x+0)\right] =$$

$$= \tilde{\tilde{S}}^{\bar{c}}(x,x) = \frac{1}{2}\left[\tilde{\tilde{S}}^{\bar{c}}(x+0,x) + \tilde{\tilde{S}}^{\bar{c}}(x,x+0)\right]$$

It is clear that the set 29 has a solution

$$\alpha(x) = \begin{cases} \alpha_1(x) \\ \alpha_2(x) = -\alpha_1(x) \end{cases}$$

what is in conformity the expression (23).

The equation (29) for the mean field can be rewritten in a form

$$\int \{ \Box \delta(x-z) - \Pi(x-z) \} \alpha_1(z)\, dz = J(x) + \\ + ie\, tr\, \gamma \left\{ \tilde{\tilde{S}}^c(x,x|\alpha_1) - \int \frac{\delta \tilde{\tilde{S}}^c(x,x|\alpha_1)}{\delta \alpha_1(z)} \bigg|_{\alpha_1=0} \alpha_1(z)\, dz \right\}, \tag{30}$$

where

$$\Pi(x-z) = ie^2\, tr\, \gamma \int dz \{ \tilde{\tilde{S}}^c(x-z) \gamma \tilde{\tilde{S}}^c(z-x) + \\ + \tilde{\tilde{S}}^{(+)}(x-z) \gamma \tilde{\tilde{S}}^{(-)}(z-x) \}, \tag{31}$$

$$\Pi(K) = \Pi^c(K) - i\, \mathrm{Im}\, \Pi^c(K) = \mathrm{Re}\, \Pi^c(K)$$

$$\Pi^c(K) = \frac{ie^2}{(2\pi)^4}\, tr \left\{ \gamma \int d^4p\, \tilde{\tilde{S}}^c(p+K) \gamma \tilde{\tilde{S}}^c(p) \right\}. \tag{32}$$

In Eq. (31-33)

The program of renormalization and the subtraction of infinities in the set of Eq.(21) and (30) can be performed by the same method as in the usual set of Green-function equations in an external field /see E.S. Fradkin (1955) and (1965a)/. This is a consequence of the reality of the renormalization constants and the infinite mass corrections. Thus, for example, the renormalized equation for the mean field $\alpha_1(x) = \langle A(x) \rangle$ has the form

$$\int (\Box \delta(x-z) - \Pi^R(x-z)) \langle A(z) \rangle dz = J(x) + \\ + ie\, tr\, \gamma \left\{ \tilde{\tilde{S}}^c(x,x|\langle A \rangle) - \int \frac{\delta \tilde{\tilde{S}}^c(x,x|\langle A \rangle)}{\delta \langle A \rangle} \bigg|_{\langle A \rangle = 0} \langle A(z) \rangle dz \right\} \tag{30a}$$

$$\Pi^R(K) = \mathrm{Re} \left\{ \Pi^c(K^2) - \Pi^c(0) - \frac{\partial \Pi^c(K^2)}{\partial K^2} \bigg|_{K^2=0} K^2 \right\} \tag{34}$$

In two-loop approximation we have

$$\left.\frac{\delta\Gamma^{M}}{\delta\alpha_1(x)}\right|_{\alpha_1(x)=\alpha_2=\langle A(x)\rangle} = 0 = \Box A(x) - \mathcal{J}(x) - ie\,\mathrm{tr}\,\gamma\,\tilde{\tilde{S}}^{c}(x,x|\langle A\rangle) +$$
$$+ e^{3}\,\mathrm{tr}\,\gamma\int dy\,dz\Big\{\tilde{\tilde{S}}^{c}(y,x|\langle A\rangle)\gamma\,\tilde{\tilde{S}}^{c}(x,z|\langle A\rangle)\gamma\,\tilde{\tilde{S}}^{c}(y,x|\langle A\rangle)D^{c}(y,z) -$$
$$- \tilde{\tilde{S}}^{(-)}(y,x|\langle A\rangle)\gamma\,\tilde{\tilde{S}}^{c}(x,z|\langle A\rangle)\gamma\,\tilde{\tilde{S}}^{(+)}(z,y|\langle A\rangle)D^{(-)}(y,z) + \tag{35}$$
$$+ \tilde{\tilde{S}}^{c}(y,x|\langle A\rangle)\gamma\,\tilde{\tilde{S}}^{(+)}(x,z|\langle A\rangle)\gamma\,\tilde{\tilde{S}}^{(-)}(z,y|\langle A\rangle)D^{(+)}(y,z) +$$
$$+ \tilde{\tilde{S}}^{(-)}(y,x|\langle A\rangle)\gamma\,\tilde{\tilde{S}}^{(+)}(x,z|\langle A\rangle)\gamma\,\tilde{\tilde{S}}^{\bar{c}}(z,y|\langle A\rangle)D^{\bar{c}}(y,z)\Big\}$$

3. Perturbation theory for matrix elements of processes. Contact with the Furry approach in and external field

Let us assume here that the initial and final states in QED with the intense mean field are constructed in the way suggested in Sec. I of this Chapter. Then the matrix element of arbitrary process between the states (I.27), (I.32) has the form

$$M_{in\to out} = {}_{out}\langle 0|a(out)\ldots b(out)\ldots c(out)\ldots U_{\mathcal{J}}c^{+}(in)\ldots b^{+}(in)\ldots a^{+}(in)\ldots|0\rangle_{in}$$

Here $U_{\mathcal{J}} = U_{\mathcal{J}}(t_{out}, t_{in})$ is the evolution operator[I] corresponding to the Hamiltonian (I.3). (The unessential normalizing factor in (I) are omitted.)

The problem is to construct the perturbation theory and diagrammatic thechnique for the matrix elements (I) under the condition that the initial mean field and external current are not small. At this stage we will consider transitions into the final states with an arbitrary mean field. One may assume, that experimentally the transitions into the state with the mean field which equals to the exact mean field in the system at the final time-moment, are measured. The one should determine this mean field for example, with the aid of perturbation theory considered in Sec.2 of this Chapter. Under this assumption the perturbation theor which will be constructed below is, in a sense, inconsistent since it contains exact

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I Further the similar abbreviations for the evolution operators and scattering matrices will be used.

quantities which must be determined separately. However, this inconsistence is compensated for convenience because it is possible to give the Feynman form to the obtained expressions. In conclusion we will discuss the other possible approaches too.

Represent the evolution operator in the following way

$$\mathcal{U}_J(tt_{in}) = \tilde{\mathcal{U}}(tt_{in})\tilde{S}(tt_{in}), \quad \tilde{\mathcal{U}}(tt_{in}) = T\exp\{-i\int_{t_{in}}^{t}\tilde{\mathcal{H}}dx^0\},$$
$$\tilde{\mathcal{H}} = \mathcal{H}_e + \mathcal{H}_\gamma + \mathcal{H}_{J\gamma} + \int j(\vec{x})A^{\mathcal{P}}(x)d\vec{x}, \tag{2}$$
$$\tilde{S}(tt_{in}) = T\exp\{-i\int\tilde{j}(x)\underline{\tilde{A}}(x)dx\},$$

$$\underline{\tilde{A}}(x) = \tilde{A}(x) - A^{\mathcal{P}}(x), \tag{3}$$

$$\tilde{\psi}(x) = \tilde{\mathcal{U}}^{-1}(tt_{in})\psi(\vec{x})\tilde{\mathcal{U}}(tt_{in}), \quad \tilde{\bar{\psi}}(x) = \dots,$$
$$\tilde{A}(x) = \dots, \quad \tilde{j}(x) = \dots \tag{4}$$

Here $A^{\mathcal{P}}(x)$ are C- numerical, generally complex vector potentials whose form will be established below. The tilded $(\sim)$ field operators satisfy the equations

$$(i\hat{\partial} - e\hat{A}^{\mathcal{P}}(x) - m)\tilde{\psi}(x) = 0,$$
$$\tilde{\bar{\psi}}(x)(i\hat{\partial} + e\hat{A}^{\mathcal{P}}(x) + m) = 0, \tag{5}$$
$$\Box\tilde{A}(x) = J(x), \quad \Box = \partial_\mu\partial^\mu.$$

Note, that for the complex $A^{\mathcal{P}}(x)$ the operators $\tilde{\mathcal{U}}$ and $\tilde{S}$ are not, generally, unitary, although the total operator $\mathcal{U}_J$ is unitary. In this case the tilding $(\sim)$ does not commute with the Hermitian conjugation in the relations (4).

The transformation (2) leads to

$$M_{in\to out} = {}_{out}\langle\tilde{0}|\tilde{a}(out)\dots\tilde{b}(out)\dots\tilde{c}(out)\dots\tilde{S}C^+(in)\dots b^+(in)\dots a^+(in)\dots|0\rangle_{in}; \tag{6}$$

$$\{\tilde{a}^+(out), \tilde{a}(out), \dots \tilde{c}(out)\} = \tilde{\mathcal{U}}^{-1}\{a^+(out), a(out), \dots c(out)\}\tilde{\mathcal{U}},$$
$${}_{out}\langle\tilde{0}| = {}_{out}\langle 0|\tilde{\mathcal{U}}. \tag{7}$$

The matrix element (6) differs from the corresponding matrix elements of QED without the external current and with the initial and final states of the type (I) in that the creation and annihilation operators as well as the vacuum vectors which stand to the right and left of the $\tilde{S}$- matrix are different. Therefore the direct application of the Wick's normal ordering technique with respect to the one vacuum proves to be nonefficient when calculating such matrix elements. In appendix A the calculation technique of matrix elements of such a type is suggested. Now we use the results presented there.

It follows from the structure of the operator $\tilde{\mathcal{U}}$ and the nature of the connections between the in- and out- creation and annihilation operators that the operators $\tilde{c}^+(out)$, $\tilde{c}(out)$, and $c^+(in)$, $c(in)$ are related by the linear canonical transformation which is a shift. Such transformation always admits a transition to the generalized normal form with respect to the vacua ${}_{out}\langle\tilde{0}|$ and $|0\rangle_{in}$. The operators $\tilde{a}^+(out)$, $\tilde{a}(out)$, $\tilde{b}^+(out)$, $\tilde{b}(out)$ are connected with the operators $a^+(in)$, $a(in)$, $b^+(in)$, $b(in)$ by the linear similarity transformation. Consider the case when the latter admits a transition to the generalized normal form with respect to the vacua ${}_{out}\langle\tilde{0}|$ and $|0\rangle_{in}$. (The explicit form of the corresponding conditions will be obtained below.) Then, owing to the linearity of the operators $\tilde{\psi}(x)$, $\tilde{\bar{\psi}}(x)$, $\tilde{\underline{A}}(x)$ with respect to the creation and annihilation operators of in- type, the former may be represented in the following way:

$$\tilde{\psi}(x)=\tilde{\psi}^{(-)}(x)+\tilde{\psi}^{(+)}(x),\quad \tilde{\bar{\psi}}(x)=\tilde{\bar{\psi}}^{(-)}(x)+\tilde{\bar{\psi}}^{(+)}(x),$$

$$\tilde{\underline{A}}(x)=\tilde{\underline{A}}^{(-)}(x)+\tilde{\underline{A}}^{(+)}(x)+\tilde{\underline{A}}^{(0)}(x), \tag{8}$$

$$\tilde{\psi}^{(-)}(x)|0\rangle_{in}=\tilde{\bar{\psi}}^{(-)}(x)|0\rangle_{in}=\tilde{\underline{A}}^{(-)}(x)|0\rangle_{in}=0,$$

$${}_{out}\langle\tilde{0}|\tilde{\psi}^{(+)}(x)={}_{out}\langle\tilde{0}|\tilde{\bar{\psi}}^{(+)}(x)={}_{out}\langle\tilde{0}|\tilde{\underline{A}}^{(+)}(x)=0,$$

$$\tilde{\underline{A}}^{(0)}(x)={}_{out}\langle\tilde{0}|\tilde{\underline{A}}(x)|0\rangle_{in}\cdot C_0^{-1}; \tag{9}$$

$$C_0={}_{out}\langle\tilde{0}|0\rangle_{in}={}_{out}\langle 0|\tilde{\mathcal{U}}|0\rangle_{in}, \tag{10}$$

where C_0 is the probability amplitude for the vacuum to remain vacuum to the zeroth order with respect to the radiative interaction and when the external current $\mathcal{J}(x)$ and the external field $A^{q}(x)$ are present. To reduce the operator $\tilde{S}$ to the generalized normal form with respect to the vacua ${}_{out}\langle 0|$ and

$|0\rangle_{in}$ and to represent after that the perturbation expansion for the matrix element (6) by diagrams one needs to find:

- generalized chronological coupling of the operators $\underline{\tilde{A}}(x)$;
- anticommutators of the operators $\tilde{\psi}^{+}(x)$, $\tilde{\bar{\psi}}^{(+)}(x)$ with $\tilde{a}(out)$, $\tilde{b}(out)$ and of the operators $\tilde{\psi}^{(-)}(x)$, $\tilde{\bar{\psi}}^{(-)}(x)$ with $a^{+}(in)$, $b^{+}(in)$;
- commutators of the operators $\underline{\tilde{A}}(x)$ with $\tilde{C}(out)$ and of the operators $\underline{\tilde{A}}^{(-)}(x)$ with $C^{+}(in)$;
- generalized chronological coupling of the operators $\tilde{\psi}(x)$ and $\tilde{\bar{\psi}}(y)$;
- the amplitude C_0, the field $A^{\mathcal{P}}(x)$;
- the amplitudes of relative probabilities of processes in the presence of the external current $J(x)$ and the external field $A^{\mathcal{P}}(x)$ to the zeroth order with respect to the radiative interaction:

$$w(\overset{+}{m}\ldots\bar{s}\ldots\vec{k}\lambda\ldots|\vec{x}\nu\ldots\bar{n}\ldots\overset{+}{l}\ldots) = {}_{out}\langle\tilde{0}|\tilde{a}_m(out)\ldots b_s(out)\ldots \cdot\tilde{C}_{\vec{k}\lambda}(out)\ldots C^{+}_{\vec{x}\nu}(in)\ldots b^{+}_{n}(in)\ldots a^{+}_{l}(in)\ldots|0\rangle_{in}\cdot C_0^{-1}. \tag{II}$$

Let us accomplish this program in the listed order.

By using the explicit form of the operator $\mathcal{U}$ and the expressions for the in- and out- photon creation and annihilation operators one can find

$$\underline{\tilde{A}}^{(-)}(x) = \sum_{\vec{k}\lambda}(2V\kappa)^{-\frac{1}{2}}e^{-i\kappa(t-t_{in})+i\vec{k}\vec{x}}\,C_{\vec{k}\lambda}(in)e_{\vec{k}\lambda},$$
$$\underline{\tilde{A}}^{(+)}(x) = \sum_{\vec{k}\lambda}(2V\kappa)^{-\frac{1}{2}}e^{i\kappa(t-t_{in})}\,\tilde{C}^{+}_{\vec{k}\lambda}(out)e_{\vec{k}\lambda}. \tag{I2}$$

The operators $C_{\vec{k}\lambda}(in)$, $\tilde{C}^{+}_{\vec{k}\lambda}(out)$ differ from the free operators $C_{\vec{k}\lambda}$, $C^{+}_{\vec{k}\lambda}$ only in the C- numerical shifts. Therefore

$$[\underline{\tilde{A}}^{(-)}(x), \underline{\tilde{A}}^{(+)}(y)]_{-} = i D_0^{(-)}(x-y). \tag{I3}$$

Demand

$$\underline{\tilde{A}}^{(0)}(x) = 0, \tag{I4}$$

by choosing the auxiliary field $A^{\mathcal{P}}(x)$ from the condition

$$A^{\mathcal{P}}(x) = {}_{out}\langle\tilde{0}|\tilde{A}(x)|0\rangle_{in}\cdot C_0^{-1} = \frac{\delta\, i\ln C_0}{\delta J(x)} \tag{I5}$$

Then the generalized chronological coupling of the operators $\underline{\tilde{A}}(x)$ which are in the operator $\tilde{S}$ will coincide, due to (A.I2), (I3), (I4), with the free chronological coupling

$$\overbrace{\underline{\tilde{A}}(x)\,\underline{\tilde{A}}(y)} = {}_{out}\langle\tilde{0}|T\underline{\tilde{A}}(x)\underline{\tilde{A}}(y)|0\rangle_{in}\cdot C_0^{-1} = i\,D_0^c(x-y). \tag{I6}$$

From (I2) it follows that

$$[\tilde{C}_{\vec{\kappa}\lambda}(out),\,\underline{\tilde{A}}^{(+)}(x)]_- = (2V\kappa)^{-\frac{1}{2}}e^{i\kappa(t-t_{in})-i\vec{\kappa}\vec{x}}e_{\vec{\kappa}\lambda}, \tag{I7}$$

$$[\underline{\tilde{A}}^{(-)}(x),\,C^+_{\vec{\kappa}\lambda}(in)]_- = (2V\kappa)^{-\frac{1}{2}}e^{-i\kappa(t-t_{in})+i\vec{\kappa}\vec{x}}e_{\vec{\kappa}\lambda},\quad \lambda=1,2.$$

Consider the function $\tilde{G}(x,x')$ which is the $\vec{x}$ representation matrix element of the evolution operator of the Dirac equation with an arbitrary complex vector potential $A^{\mathcal{P}}(x)$. The function $\tilde{G}(x,x')$ statisfies the Dirac equation and the condition

$$\tilde{G}(x,x')\Big|_{t=t'} = \delta(\vec{x}-\vec{x}').$$

For it the relations

$$\int\tilde{G}(xy)\tilde{G}(yx')d\vec{y} = \tilde{G}(x,x'), \tag{I8}$$

$$\tilde{G}^+_*(x,x') = \tilde{G}(x',x), \tag{I9}$$

hold where the lower case asterisk indicates that the corresponding quantity is taken for the complex conjugated potential $A^{\mathcal{P}*}(x)$. The function $\tilde{G}(x,x')$ may be built using any set of solutions $\{\tilde{\varphi}_\kappa(x)\}$ of the Dirac equation in the "external field" $A^{\mathcal{P}}(x)$ if this set is complete and orthonormal at the time t':

$$\tilde{G}(x,x') = \sum_\kappa \tilde{\varphi}_\kappa(x)\,\tilde{\varphi}^+_\kappa(x'). \tag{20}$$

The properties of the function $\tilde{G}(x,x')$ imply that the operators $\tilde{\Psi}(x)$, $\bar{\tilde{\Psi}}(x)$ obeying the equations (5) are connected for different time-moments by means of the function $\tilde{G}(x,x')$

$$\tilde{\Psi}(x) = \int\tilde{G}(x,x')\tilde{\Psi}(x')d\vec{x}',\quad \bar{\tilde{\Psi}}(x) = \int\bar{\tilde{\Psi}}(x')\gamma^0\tilde{G}(x',x)\gamma^0 d\vec{x}'. \tag{2I}$$

Eqs. (2I) allow us to find the connection between the operators $\{\tilde{a}^+(out),\,\tilde{a}(out),\,\tilde{b}^+(out),\,\tilde{b}(out)\}$ and $\{a^+(in),\,a(in),\,b^+(in),\,b(in)\}$. Put $t = t_{out}$, $t'=t_{in}$ in (2I)write the l.-h. sides with the aid of the representation (4) and substitute the decompositions (I.3I) into them, while the decompositions (I.22) substitute in the r.-h. sides. This yields:

$$\tilde{a}(out) = \tilde{G}(+|_+)a(in) + \tilde{G}(+|_-)b^+(in),$$
$$\tilde{b}^+(out) = \tilde{G}(-|_+)a(in) + \tilde{G}(-|_-)b^+(in), \tag{22}$$
$$\tilde{a}^+(out) = a^+(in)\tilde{G}(_+|^+) + b(in)\tilde{G}(_-|^+),$$
$$\tilde{b}(out) = a^+(in)\tilde{G}(_+|^-) + b(in)\tilde{G}(_-|^-),$$
$$\tilde{G}(^\pm|_\pm)_{mn} = \int {}^\pm\varphi_m^+(\vec{x})\tilde{G}(\vec{x}t_{out}, \vec{x}'t_{in})_\pm\varphi_n(\vec{x}')d\vec{x}d\vec{x}',$$
$$\tilde{G}(_\pm|^\pm)_{nm} = \int {}_\pm\varphi_n^+(\vec{x}')\tilde{G}(\vec{x}'t_{in}, \vec{x}t_{out})^\pm\varphi_m(\vec{x})d\vec{x}d\vec{x}'. \tag{23}$$

Put $t = t_{in}$, $t' = t_{out}$ in (2I), write the r.-h. sides with the aid of the representation (4) and substitute the decompositions (I.3I) into the r.-h. sides and (I.22) into the l.-h. sides. This yields

$$a(in) = \tilde{G}(_+|^+)\tilde{a}(out) + \tilde{G}(_+|^-)\tilde{b}^+(out),$$
$$b^+(in) = \tilde{G}(_-|^+)\tilde{a}(out) + \tilde{G}(_-|^-)\tilde{b}^+(out), \tag{24}$$
$$a^+(in) = \tilde{a}^+(out)\tilde{G}(^+|_+) + \tilde{b}(out)\tilde{G}(^-|_+),$$
$$b(in) = \tilde{a}^+(out)\tilde{G}(^+|_-) + \tilde{b}(out)\tilde{G}(^-|_-).$$

The matrices $\tilde{G}(^\pm|_\pm)$ and $\tilde{G}(_\pm|^\pm)$ statisfy the completeness and orthogonality relations which are consequences of the (I.20)-type relations for the functions ${}_\pm\varphi_n(\vec{x})$ and ${}^\pm\varphi_m(\vec{x})$, the property (I8) of the function $\tilde{G}(x,x')$:

$$\tilde{G}(^\pm|_+)\tilde{G}(_+|^\pm) + \tilde{G}(^\pm|_-)\tilde{G}(_-|^\pm) = I, \quad \tilde{G}(^\pm|_+)\tilde{G}(_+|^\mp) + \tilde{G}(^\pm|_-)\tilde{G}(_-|^\mp) = 0,$$
$$\tilde{G}(_\pm|^+)\tilde{G}(^+|_\pm) + \tilde{G}(_\pm|^-)\tilde{G}(^-|_\pm) = I, \quad \tilde{G}(_\pm|^+)\tilde{G}(^+|_\mp) + \tilde{G}(_\pm|^-)\tilde{G}(^-|_\mp) = 0.$$

From (I9) it follows that $\tilde{G}(^\pm|_\pm)^+ = \tilde{G}(_\pm|^\pm)$. By applying eqs. (22), (24) one may find the amplitudes (II) of elementary processes into which charged particles are involved: scattering, annihilation and pair creation.

$$w(\overset{+}{m}|\overset{+}{n}) = \tilde{G}^{-1}(_+|^+)_{mn}, \quad w(\bar{m}|\bar{n}) = \tilde{G}^{-1}(^-|_-)_{nm},$$
$$w(0|\overset{+}{n}\bar{e}) = \{\tilde{G}(_-|^+)\tilde{G}^{-1}(_+|^+)\}_{ne} = -\{\tilde{G}^{-1}(^-|_-)\tilde{G}(^-|_+)\}_{ne}, \tag{25}$$
$$w(\overset{+}{m}\bar{s}|0) = \{\tilde{G}^{-1}(_+|^+)\tilde{G}(_+|^-)\}_{ms} = -\{\tilde{G}(^+|_-)\tilde{G}^{-1}(^-|_-)\}_{ms}.$$

From (22), (24) and (25) it follows that

$$\tilde a_m(out) = \sum_n w(\overset{+}{m}|\overset{+}{n})\, a_n(in) - \sum_s w(\overset{+}{m}\bar s|0)\,\tilde b_s^+(out), \qquad (26)$$

$$\tilde b_m(out) = \sum_n w(\bar m|\bar n)\, b_n(in) + \sum_s w(\overset{+}{s}\bar m|0)\,\tilde a_s^+(out),$$

$$a_n^+(in) = \sum_m w(\overset{+}{m}|\overset{+}{n})\,\tilde a_m^+(out) - \sum_l w(0|\bar l\overset{+}{n})\, b_l(in),$$

$$b_n^+(in) = \sum_m w(\overset{+}{m}|\bar n)\,\tilde b_m^+(out) + \sum_l w(0|\bar n\overset{+}{l})\, a_l(in).$$

Relations (26) are the specification of the general representation (A.3) for the case under consideration. It is seen now that the transformation (22) admits transition to the generalized normal form with respect to the vacua ${}_{out}\langle\tilde 0|$ and $|0\rangle_{in}$ if the inverse matrices $\tilde G^{-1}(+|^+)$ and $\tilde G^{-1}(^-|_-)$ exist, in full accordance with the general requirement (A.4). With the aid of (26) one can find and explicit form of the representation (8) for the operators $\tilde\Psi(x)$ and $\tilde{\bar\Psi}(x)$. By setting $t' = t_{in}$ in (2I) and using the decompositions (I.22) in the r.-h. sides we obtain

$$\tilde\Psi(x) = \sum_n \{a_n(in)\,{}_+\tilde\varphi_n(x) + b_n^+(in)\,{}_-\tilde\varphi_n(x)\}, \qquad (27)$$

$$\tilde{\bar\Psi}(x) = \sum_n \{a_n^+(in)\,{}_+\tilde{\bar\varphi}_n(x) + b_n(in)\,{}_-\tilde{\bar\varphi}_n(x)\},$$

where

$${}_\pm\tilde\varphi_n(x) = \int \tilde G(x, \vec x' t_{in})\,{}_\pm\varphi_n(\vec x')\,d\vec x', \qquad (28)$$

$${}_\pm\tilde{\bar\varphi}_n(x) = \int {}_\pm\varphi_n^+(\vec x')\,\tilde G(\vec x' t_{in}, x)\,\gamma^0 d\vec x'.$$

By setting $t' = t_{out}$ in (2I) and using the representation (4) in the r.-h. sides and the decompositions (I.3I) we obtain

$$\tilde\Psi(x) = \sum_m \{\tilde a_m(out)\,{}^+\tilde\varphi_m(x) + \tilde b_m^+(out)\,{}^-\tilde\varphi_m(x)\}, \qquad (29)$$

$$\tilde{\bar\Psi}(x) = \sum_m \{\tilde a_m^+(out)\,{}^+\tilde{\bar\varphi}_m(x) + \tilde b_m(out)\,{}^-\tilde{\bar\varphi}_m(x)\},$$

where

$${}^\pm\tilde\varphi_m(x) = \int \tilde G(x, \vec x' t_{out})\,{}^\pm\varphi_m(\vec x')\,d\vec x',$$

$${}^\pm\tilde{\bar\varphi}_m(x) = \int {}^\pm\varphi_m^+(\vec x')\,\tilde G(\vec x' t_{out}, x)\,\gamma^0 d\vec x'. \qquad (30)$$

In our notations ${}_\pm\tilde{\bar\varphi}_n$ and ${}^\pm\tilde{\bar\varphi}_m$ are not Dirac-conjugated to ${}_\pm\tilde\varphi_n$ and ${}^\pm\tilde\varphi_m$ in the general case of complex potential $A^\rho(x)$ By combining eqs. (26), (27) and (29) we find

$$\tilde{\psi}^{(-)}(x)=\sum_n {}_+\tilde{\varphi}_n(x)\,a_n(in),\quad \tilde{\psi}^{(+)}(x)=\sum_m {}^-\tilde{\varphi}_m(x)\,\tilde{b}_m^+(out),$$

$$\tilde{\tilde{\psi}}^{(-)}(x)=\sum_n {}_-\tilde{\tilde{\varphi}}_n(x)\,b_n(in),\quad \tilde{\tilde{\psi}}^{(+)}(x)=\sum_m {}^+\tilde{\tilde{\varphi}}_m(x)\,\tilde{a}_m^+(out),$$

$${}_+\tilde{\varphi}_n(x)={}_+\varphi_n(x)+\sum_\ell w(0|\bar{\ell}\overset{+}{n})\cdot{}_-\tilde{\varphi}_\ell(x)=\sum_m w(\overset{+}{m}|\overset{+}{n})\,{}^+\tilde{\varphi}_m(x), \tag{31}$$

$${}^-\tilde{\varphi}_m(x)={}^-\varphi_m(x)-\sum_s w(\overset{+}{s}\,\bar{m}|0)\cdot{}^+\tilde{\varphi}_s(x)=\sum_n w(\bar{m}|\bar{n})\,{}_-\tilde{\varphi}_n(x),$$

$${}_-\tilde{\tilde{\varphi}}_n(x)={}_-\tilde{\tilde{\varphi}}_n(x)-\sum_\ell w(0|\bar{n}\overset{+}{\ell})\,{}_+\tilde{\tilde{\varphi}}_\ell(x)=\sum_m w(\bar{m}|\bar{n})\,{}^-\tilde{\tilde{\varphi}}_m(x),$$

$${}^+\tilde{\tilde{\varphi}}_m(x)={}^+\tilde{\tilde{\varphi}}_m(x)+\sum_s w(\overset{+}{m}\bar{s}|0)\,{}^-\tilde{\tilde{\varphi}}_s(x)=\sum_n w(\overset{+}{m}|\overset{+}{n})\,{}_+\tilde{\tilde{\varphi}}_n(x).$$

Consequently the following anticommutators are different from zero:

$$[\tilde{a}_m(out),\tilde{\tilde{\psi}}^{(+)}(x)]_+={}^+\tilde{\tilde{\varphi}}_m(x),\quad [\tilde{b}_m(out),\tilde{\psi}^{(+)}(x)]_+={}^-\tilde{\varphi}_m(x), \tag{32}$$
$$[\tilde{\psi}^{(-)}(x),a_n^+(in)]_+={}_+\tilde{\varphi}_n(x),\quad [\tilde{\tilde{\psi}}^{(-)}(x),b_n^+(in)]_+={}_-\tilde{\tilde{\varphi}}_n(x).$$

The generalized chronological coupling of the spinor field operators has, due to (A.I2), (27) and (29), the form

$$\widetilde{\tilde{\psi}(x)\,\tilde{\tilde{\psi}}(y)}={}_{out}\langle\tilde{0}|T\tilde{\psi}(x)\tilde{\tilde{\psi}}(y)|0\rangle_{in}\cdot C_0^{-1}=-i\tilde{S}^c(x,y), \tag{33}$$

$$\tilde{S}^c(x,y)=\begin{cases}\tilde{S}^{(-)}(x,y), & x^0>y^0,\\ -\tilde{S}^+(x,y), & x^0<y^0;\end{cases} \tag{34}$$

$$\tilde{S}^{(-)}(x,y)=i[\tilde{\psi}^{(-)}(x),\tilde{\tilde{\psi}}^{(+)}(y)]_+=i\sum_{nm}{}^+\tilde{\varphi}_m(x)\,w(\overset{+}{m}|\overset{+}{n})\,{}_+\tilde{\tilde{\varphi}}_n(y),$$

$$\tilde{S}^{(+)}(x,y)=i[\tilde{\psi}^{(+)}(x),\tilde{\tilde{\psi}}^{(-)}(y)]_+=i\sum_{nm}{}_-\tilde{\varphi}_n(x)\,w(\bar{m}|\bar{n})\,{}^-\tilde{\tilde{\varphi}}_m(y).$$

Here $\tilde{S}^c(x,y)$ is the Green function for the Dirac equation in the external field with the vector potential $A^{\mathcal{P}}(x)$

$$(i\hat{\partial}-e\hat{A}^{\mathcal{P}}(x)-m)\tilde{S}^c(x,y)=-\delta(x-y). \tag{35}$$

The anticommutators (32) can be expressed, as usual, in terms of $\tilde{S}^c(x,y)$:

$${}_{+}\tilde{\Psi}_n(x) = -i\int \tilde{S}^c(x, \vec{x}'t)\gamma^0 {}_{+}\varphi_n(\vec{x}')d\vec{x}';$$

$${}^{-}\tilde{\Psi}_n(x) = i\int \tilde{S}^c(x, \vec{x}'t_{out})\gamma^0 {}^{-}\varphi_n(\vec{x}')d\vec{x}';$$

$${}_{-}\tilde{\bar{\Psi}}_n(x) = i\int {}_{-}\varphi_n^+(\vec{x}')\tilde{S}^c(\vec{x}'t_{in}, x)d\vec{x}';$$

$${}^{+}\tilde{\bar{\Psi}}_n(x) = -i\int {}^{+}\varphi_n^+(\vec{x}')\tilde{S}^c(\vec{x}'t_{out}, x)d\vec{x}'.$$

Now we are going to find C_o. Considering $\mathcal{J}(x)$ and $A^{\mathcal{P}}(x)$ as independent functional arguments in the expression (IO) we get

$$\frac{\delta i \ln C_o}{\delta A^{\mathcal{P}}(x)} = {}_{out}\langle \tilde{0}|\tilde{j}(x)|0\rangle_{in} \cdot C_o^{-1} = ie\,tr\,\gamma\,\tilde{S}^c(x,x) =$$

$$= -i\frac{\delta}{\delta A^{\mathcal{P}}(x)} \cdot Tr \ln \tilde{S}^c,$$

$$\frac{\delta^2 i \ln C_o}{\delta \mathcal{J}(x)\delta \mathcal{J}(y)} = -i\, \overbrace{\underline{\tilde{A}}(x)\underline{\tilde{A}}(y)} = D_o^c(x-y),$$

where the operation Tr includes also the coordinate integration. These equations and the condition

$$C_o(\mathcal{J} = A^{\mathcal{P}} = 0) = {}_{out}\langle 0|U_o|0\rangle_{in},$$

where U_o is the evolution operator, corresponding to the free Hamiltonian $\mathcal{H}_o = \mathcal{H}_e + \mathcal{H}_\gamma$ allow us to write[I)]

$$C_o = {}_{out}\langle 0|U_o|0\rangle_{in} \cdot exp\left[-Tr \ln\left\{\frac{\tilde{S}^c}{\tilde{S}^c(\mathcal{J}=A^{\mathcal{P}}=0)}\right\} - \right. \tag{36}$$

$$\left. -\frac{i}{2}\mathcal{J}D_o^c\mathcal{J} - iA_o^{\mathcal{P}}\mathcal{J}\right].$$

From (I5) and (36) it follows that

$$A^{\mathcal{P}}(x) = \int D_o^c(x-y)\mathcal{J}(y)dy + A_o^{\mathcal{P}}(x), \tag{37}$$

$$A_o^{\mathcal{P}}(x) = \sum_{\vec{k}\lambda}(2V\kappa)^{-\frac{1}{2}}\left\{z_{\vec{k}\lambda}(in,t)e^{i\vec{k}\vec{x}} + z^*_{\vec{k}\lambda}(out,t)e^{-i\vec{k}\vec{x}}\right\}e_{\vec{k}\lambda},$$

where

$$z_{\vec{k}\lambda}\left({}^{in}_{out}, t\right) = \frac{g_{\lambda\lambda}}{\sqrt{2V\kappa}}\int \left(e_{\vec{k}\lambda}, \left[\kappa A^{\substack{in\\out}}(x) + i\dot{A}^{\substack{in\\out}}(x)\right]\right)e^{-i\vec{k}\vec{x}}d\vec{x}, \tag{38}$$

I) Here the abbreviations of the type $\int \mathcal{J}(x)A(x)dx = \mathcal{J}A$ are used

$$\Box A^{in}_{out}(x)=0,$$

$$A^{in}_{out}(x)\Big|_{\substack{t_{in}\\ t_{out}}} = A^{in}_{out}(\vec{x}), \quad \dot{A}^{in}_{out}(x)\Big|_{\substack{t_{in}\\ t_{out}}} = \dot{A}^{in}_{out}(\vec{x}),$$

$$z_{\vec{\kappa}\lambda}\left({}^{in}_{out}, t\right) = z_{\vec{\kappa}\lambda}\left({}^{in}_{out}\right) e^{-i\kappa(t-t_{\substack{in\\ out}})}$$

The fields $A^{\mathcal{P}}(x)$ and $A_0^{\mathcal{P}}(x)$ obey the equations

$$\Box A^{\mathcal{P}}(x) = \mathcal{J}(x), \quad \Box A_0^{\mathcal{P}}(x)=0.$$

One can see, that the amplitude C_0 factorizes

$$C_0 = C_0^e C_0^\gamma,$$

$$C_0^e = {}^{e}_{out}\langle 0|\mathcal{U}_e|0\rangle^e_{in} \cdot exp\{-Tr\, ln \frac{\tilde{S}^c}{\tilde{S}^c(\mathcal{J}=A^{\mathcal{P}}=0)}\}, \tag{39}$$

$$C_0^\gamma = {}_{out}^{\gamma}\langle 0|\mathcal{U}_\gamma|0\rangle^\gamma_{in} \cdot exp\{-\frac{i}{2}\mathcal{J}\mathcal{D}_0^c\mathcal{J} - iA_0^{\mathcal{P}}\mathcal{J}\},$$

where the operators $\mathcal{U}_e$ and $\mathcal{U}_\gamma$ correspond to the Hamiltonians $\mathcal{H}_e$ and $\mathcal{H}$. The matrix elements ${}_{out}^{\gamma}\langle 0|\mathcal{U}_\gamma|0\rangle^\gamma_{in}$ and ${}_{out}^{e}\langle 0|\mathcal{U}_e|0\rangle^e_{in}$ may be calculated:

$${}_{out}^{\gamma}\langle 0|\mathcal{U}_\gamma|0\rangle^\gamma_{in} = exp\sum_{\vec{\kappa},\lambda=1,2}\Big\{-\frac{|z_{\vec{\kappa}\lambda}(out)|^2}{2} - \frac{|z_{\vec{\kappa}\lambda}(in)|^2}{2} +$$

$$+ z^*_{\vec{\kappa}\lambda}(out)\, z_{\vec{\kappa}\lambda}(in, t_{out})\Big\}, \tag{40}$$

$${}_{out}^{e}\langle 0|\mathcal{U}_e|0\rangle^e_{in} = det\,\tilde{G}(-|-)\Big|_{\mathcal{J}=A^{\mathcal{P}}=0}.$$

In the latter case we have used the results obtained in (Gitman, 1977) (see also Sec.I, Chapter I).

To determine the amplitudes (II) suffice it, in accordance with the formula (A.I7), to find all the generalized couplings of the operators $\tilde{a}(out)$, $\tilde{b}(out)$, $\tilde{c}(out)$, $a^+(in)$, $b^+(in)$, $c^+(in)$ and the C- numerical parts of the operators $\tilde{c}(out)$ and $c^+(in)$ in the sense of the decomposition (A.5). The generalized couplings of the electron-positron operators are the amplitudes (II) of the elementary processes in which electrons and positrons are involved. Those of them, which are nonzero, are already determined by the expressions (25). The C- numerical parts of the electron-positron operators are equal to zero.

The generalized couplings of the operators $\tilde{C}(out)$, $\tilde{C}(out)$ and $C^+(in)$, $C^+(in)$ are equal to zero since the in- and out-electromagnetic field operators differ from free ones only in c- numbers. Due to the same reason and in accordance with the formulae (A.II), (A.I8) we have

$$\underbrace{\tilde{C}_{\vec{k}\lambda}(out)C^+_{\vec{\varkappa}\nu}}(in)=\delta_{\vec{k}\lambda,\vec{\varkappa}\lambda}+w(\vec{k}\lambda|0)\,w(0|\vec{\varkappa}\nu)\,,\quad \lambda,\nu=1,2\,. \tag{4I}$$

The quantities $w(\vec{k}\lambda|0)$ and $w(0|\vec{\varkappa}\nu)$ $(\lambda,\nu=1,2)$ are the relative probability amplitudes for the creation and annihilation of the transverse photon. In accordance with the definition (II) they can be calculated by using the connections (I.I3), (I.29) and the well-known property of the coherent states of transverse photons

$$C^+|z\rangle=\left(\frac{z}{2}^*+\frac{\partial}{\partial z}\right)|z\rangle,\quad \langle z|C=\left(\frac{z}{2}+\frac{\partial}{\partial z^*}\right)\langle z|\,.$$

Then

$$w(\vec{k}\lambda|0)=-\frac{z_{\vec{k}\lambda}(out)}{2}+\frac{\delta\ln C_0}{\delta z^*_{\vec{k}\lambda}(out)}=$$

$$=z_{\vec{k}\lambda}(in,t_{out})-z_{\vec{k}\lambda}(out)-\frac{i}{\sqrt{2V\kappa}}\int J(x)e^{i\kappa(t-t_{out})-i\vec{k}\vec{x}}e_{\vec{k}\lambda}\,d\vec{x}\,; \tag{42}$$

$$w(0|\vec{k}\lambda)=-\frac{z^*_{\vec{k}\lambda}(in)}{2}+\frac{\delta\ln C_0}{\delta z^*_{\vec{k}\lambda}(in)}=$$

$$=z^*_{\vec{k}\lambda}(out,t_{in})-z^*_{\vec{k}\lambda}(in)-\frac{i}{\sqrt{2V\kappa}}\int J(x)e^{-i\kappa(t-t_{in})+i\vec{k}\vec{x}}e_{\vec{k}\lambda}\,d\vec{x}\,. \tag{43}$$

The matrix elements (39) may be represented by the usual Feynman diagrams. For establishing the rules of correspondence one should represent the $\tilde{S}$- matrix in the generalized normal form with respect to the vacua ${}_{out}\langle\tilde{0}|$ and $|0\rangle_{in}$. This can be done with the aid of the usual Wick theorem for the T- products if instead of the normal products and chronological couplings their generalized counterparts are taken. It is useful to represent beforehand the operator of the current in the generalized normal form:

$$\tilde{j}(x)=e\tilde{N}\,\tilde{\bar{\psi}}(x)\gamma\tilde{\psi}(x)+\tilde{J}(x),$$

$$\tilde{J}(x)={}_{out}\langle\tilde{0}|\,\tilde{j}(x)|0\rangle_{in}\cdot C_0^{-1}=ie\,tr\,\gamma\,\tilde{S}^c(x,x).$$

Thus, the problem reduces to calculating the matrix elements of the generalized normal products of the following form

$_{out}\langle \tilde{0} | \tilde{a}(out) \dots \tilde{b}(out) \dots \tilde{c}(out) \dots \tilde{N}(\dots) c^+(in) \dots b^+(in) \cdots a^+(in) \dots | 0 \rangle_{in}$.

It is evident that this matrix element is different from zero if the sum of numbers of particles of each field in the initial and final states is greater than or equal to the number of operator functions of the given field in the generalized normal product.

Consider the case when for each field operator $\tilde{\psi}(x)$, $\tilde{\bar{\psi}}(x)$, $\underset{\sim}{A}(x)$ taken from the generalized normal product there may be found a corresponding operator $a^+(in)$, $b^+(in)$, $c^+(in)$ from the initial state or $\tilde{a}(out)$, $\tilde{b}(out)$, $\tilde{c}(out)$ from the final state which will cancel it after the commutation. Such a matrix element can be represented by Feynman diagrams with the following rules of correspondence:

I. Electron in the initial (final) state with the quantum number $n(m)$ is represented by the factor $_+\tilde{\psi}_n(x)$ $({}^+\tilde{\bar{\psi}}_m(x))$.

2. Position in the initial (final) state with the quantum number $n(m)$ is represented by the factor $_-\tilde{\bar{\psi}}_n(x)$ $({}^-\tilde{\psi}_m(x))$.

3. Internal electron line directed from the point x' into the point x is represented by the generalized coupling $-i\,\tilde{S}^c(x,x')$.

4. To the closed electron line the generalized vacuum current $\tilde{J}(x)$ is put into correspondence.

5. Contribution of every diagram contains the amplitude C_0 of probability for the vacuum to remain vacuum as a factor.

The rest of the rules of correspondence are the same as those in the standard QED (Bogoliubov, Shirkov, I959). Consider now the case when the total number of operators in the initial and final states exceeds the one needed for the compensation of the generalized normal product. Such a matrix element is equal to the sum of products of contributions coming from the Feynman graphs for which the "interaction" of the generalized normal product with the operators from the initial and final states is responsible by the amplitudes $w(\overset{+}{m} \dots \bar{s} \dots ; k\lambda \dots | \boldsymbol{x}\nu \dots ; \bar{n} \dots \overset{+}{\ell} \dots)$ coming from the noncompensated creation and annihilation operators of such states.

Let us introduce the exact Green functions, connected with the coefficient functions of the scattering matrix $\tilde{S}$ which is reduced to the generalized normal form with respect to the vacua $_{out}\langle \tilde{0}|$ and $|0\rangle_{in}$

$$\mathcal{G}^{\mathcal{P}}_{n\nu}(xyz) = {}_{out}\langle 0|\mathcal{U}_y \check{T}[\check{\psi}(x_1)\ldots\check{\psi}(x_n)\check{\bar{\psi}}(y_1)\ldots\check{\bar{\psi}}(y_n)\check{A}(z_1)\ldots\check{A}(z_\nu)]|0\rangle_{in}\cdot C^{-1} =$$

$$= {}_{out}\langle\tilde{0}|T\,\tilde{\psi}(x_1)\ldots\tilde{\psi}(x_n)\tilde{\bar{\psi}}(y_1)\ldots\tilde{\bar{\psi}}(y_n)\tilde{A}(z_1)\ldots\tilde{A}(z_\nu)\tilde{S}|0\rangle_{in}\cdot C^{-1},$$

$$C = {}_{out}\langle 0|\mathcal{U}_y|0\rangle_{in} = {}_{out}\langle\tilde{0}|\tilde{S}|0\rangle_{in}\,. \tag{44}$$

The reduction formulae one can get by substituting the expression

$$\tilde{S} = \sum_{n\nu}\tilde{S}_{n\nu}(xyz)\tilde{N}[\tilde{\bar{\psi}}(x_1)\ldots\tilde{\bar{\psi}}(x_n)\tilde{\psi}(y_1)\ldots\tilde{\psi}(y_n)\underline{\tilde{A}}(z_1)\ldots\underline{\tilde{A}}(z_\nu)]\,dx\,dy\,dz$$

into the expression (44). For example

$$\mathcal{G}^{\mathcal{P}}_{01}(x) = A^{\mathcal{P}}(x) + i\int D^c_0(x-x')\tilde{S}_{01}(x')dx'\cdot\tilde{S}^{-1}_{00}, \quad \tilde{S}_{00} = C\cdot C_0^{-1},$$

$$\mathcal{G}^{\mathcal{P}}_{02}(xy) = A^{\mathcal{P}}(x)A^{\mathcal{P}}(y) + iD^c_0(x-y) - 2\int D^c_0(x-x')\tilde{S}_{02}(x'y')D^c_0(y'-y)dx'dy'\cdot\tilde{S}^{-1}_{00},$$

$$\mathcal{G}^{\mathcal{P}}_{10}(xy) = -i\tilde{S}^c(xy) - \int\tilde{S}^c(xx')\tilde{S}_{10}(x'y')\tilde{S}^c(y'y)dx'dy'\cdot\tilde{S}^{-1}_{00}\,.$$

The Green functions (44) are the functional derivatives of the generating functional $Z^{\mathcal{P}}$

$$Z^{\mathcal{P}} = {}_{out}\langle 0|\mathcal{U}'_y(I,\bar{\eta},\eta)|0\rangle_{in}\,, \quad Z^{\mathcal{P}}\Big|_{I=\bar{\eta}=\eta=0} = C,$$

$$\mathcal{G}^{\mathcal{P}}_{n\nu}(x_1,\ldots x_n,\,y_1\ldots y_n,\,z_1\ldots z_\nu) = \tag{45}$$

$$= \frac{(i)^\nu}{Z^{\mathcal{P}}}\,\frac{\vec{\delta}^{\,n+n+\nu}Z^{\mathcal{P}}}{\delta\bar{\eta}(x_1)\ldots\delta\bar{\eta}(x_n)\delta\eta(y_1)\ldots\delta\eta(y_n)\delta I(z_1)\ldots\delta I(z_\nu)}\Bigg|_{I=\bar{\eta}=\eta=0}\,.$$

For the functional $Z^{\mathcal{P}}$ one can get convenient representations which are equivalent to the perturbation theory with respect to the radiative interaction.

I. Write

$$\mathcal{U}'_y = \tilde{\mathcal{U}}\tilde{S}', \quad Z^{\mathcal{P}} = {}_{out}\langle\tilde{0}|\tilde{S}'|0\rangle_{in}\,,$$

$$\tilde{S}' = exp(-iIA^{\mathcal{P}})T\,exp\{-i(\tilde{j}\underline{\tilde{A}} + I\underline{\tilde{A}} + \tilde{\bar{\psi}}\eta + \bar{\eta}\tilde{\psi})\}\,. \tag{46}$$

By using the formula (A.I6) and the expressions for the generalized couplings (I3), (33) we find

$$Z^{\varphi}=exp(-iIA^{\varphi})\,exp\,e\frac{\vec{\delta}}{\delta\eta}\gamma\frac{\vec{\delta}}{\delta\bar{\eta}}\frac{\delta}{\delta I}exp(-\tfrac{i}{2}ID_0^cI+i\bar{\eta}\tilde{S}^c\eta)C_0\,. \tag{47}$$

II. By using the representation (46) for Z^{φ} and the formulae (A.I4), (A.I6) we write

$$Z^{\varphi}=exp(-iIA^{\varphi})\,exp(\tfrac{i}{2}\frac{\delta}{\delta A}D_0^c\frac{\delta}{\delta A})\,exp(-iIA)\cdot$$

$$\cdot\,{}_{out}\langle\tilde{0}|T exp\{-i(\tilde{j}A+\tilde{\bar{\Psi}}\eta+\bar{\eta}\tilde{\Psi})\}|0\rangle_{in}\Big|_{A=0}\,,$$

where the quantities A are the classical functional arguments. One may establish that

$${}_{out}\langle\tilde{0}|T exp\{-i(\tilde{j}A+\tilde{\bar{\Psi}}\eta+\bar{\eta}\tilde{\Psi})\}|0\rangle_{in}=$$

$$={}^{A^{\varphi}+A}_{out}\langle\tilde{0}|T exp\{-i(\tilde{\bar{\Psi}}_{A^{\varphi}+A}\cdot\eta+\bar{\eta}\tilde{\Psi}_{A^{\varphi}+A})\}|0\rangle_{in}\,,$$

where the quantities with the sub- or supercripts $A^{\varphi}+A$ are given by the formulae (4), (7), where $A^{\varphi}+A$ must be taken as an external field. Let us transform the right-hand side of the latter relation in accordance with the formula (A.I6), then it acquires the form

$$C_0(A^{\varphi}+A)\,exp\,i\bar{\eta}\tilde{S}^c(A^{\varphi}+A)\eta\,.$$

By replacing A by $\frac{i\delta}{\delta I}$ as applied to $exp(-iIA)$ and taking the explicit expression (36) for C_0 into account we will obtain

$$Z^{\varphi}={}_{out}\langle 0|U_0|0\rangle_{in}\,exp\Big(-\tfrac{i}{2}JD_0^cJ-iJA_0^{\varphi}-Tr\,ln\,\frac{\tilde{S}^c(A^{\varphi}+i\frac{\delta}{\delta I})}{\tilde{S}^c(A^{\varphi}=0)}\Big)\times$$

$$\times\,exp(i\bar{\eta}\tilde{S}^c(A^{\varphi}+i\frac{\delta}{\delta I})\eta)\,exp(-\tfrac{i}{2}ID_0^cI)\,. \tag{48}$$

The formulae (47) and (48) are analogous to the corresponding formulae in the theory with the external field (Fradkin, I965a, b; Batalin, Fradkin, I970). Note, that the derivation presented here enables to write at once the explicit form of the Green functions in the external field, which are in the expressions obtained.

The functional Z^{φ} satisfies the functional equations (2.I7) under $\lambda=1$. One can get from these equations in the usual way the equations for the Green functions both in the differential functional and in the integral form. To do so the following quantities should be introduced

$$W^{\varphi}=i\ln Z^{\varphi},$$

$$\left.\frac{\delta W^{\varphi}}{\delta I(x)}\right|_{\bar\eta=\eta=0}=\alpha(x),\quad \left.\frac{\delta^2 W^{\varphi}}{\delta I(x)\delta I(y)}\right|_{\bar\eta=\eta=0}=D(xy), \tag{49}$$

$$\left.\frac{\delta^2 W^{\varphi}}{\delta\bar\eta(x)\delta\eta(y)}\right|_{\bar\eta=\eta=0}=S(xy),$$

which make sense under $I(x)=0$:

$$\alpha(x)\big|_{I=0}=\langle A(x)\rangle^{\varphi}={}_{out}\langle\tilde 0|T\tilde A(x)\tilde S|0\rangle_{in}\cdot C^{-1},$$

$$D(xy)\big|_{I=0}={}_{out}\langle\tilde 0|T\underline{\tilde A}(x)\underline{\tilde A}(y)\tilde S|0\rangle_{in}\cdot C^{-1}- \tag{50}$$

$$-{}_{out}\langle\tilde 0|T\underline{\tilde A}(x)\tilde S|0\rangle_{in}\cdot{}_{out}\langle\tilde 0|T\underline{\tilde A}(y)\tilde S|0\rangle_{in}\cdot C^{-2},$$

$$S(xy)\big|_{I=0}={}_{out}\langle\tilde 0|T\tilde\Psi(x)\bar{\tilde\Psi}(y)\tilde S|0\rangle_{in}\cdot C^{-1}.$$

The corresponding equation for these quantities will have, formally, the form (2.20), (2.2I) if one puts there $\lambda=1$, does away with the matrix indicies and replaces $\alpha_\lambda(x)$, $D_{\lambda\beta}(x,y)$, $S_{\lambda\beta}(x,y)$ by $\alpha(x), D(x,y), S(x,y)$ respectively. The iterationof the set of equations starting with the bare quantities $A^{\varphi}(x)$, $\tilde S^c(x,y), D^c_0(x,y)$ where $\tilde S^c(x,y)$ is the Green function in the field $A^{\varphi}(x)$ leads to the correct perturbation expansions for the corresponding exact quantities. Thus, the complex field $\alpha(x)$ parametrizes the Green functions $D(x,y)$, $S(x,y)$.

Let us construct, with the aid of the Legendre transformation of the functional W^{φ}, the effective action $\Gamma^{\varphi}(\alpha)$ (Later on we will put everywhere the sources $\bar\eta$ and η equal to zero.)

$$\Gamma^{\varphi}(\alpha)=I\alpha-W^{\varphi},$$

where the source I in the right-hand side should be expressed through α with the aid of (49). From $\frac{\delta\Gamma^{\varphi}}{\delta\alpha}=I$ and the first relation (50) it follows that the field $\langle A(x)\rangle^{\varphi}$ gives the extremum to the functional $\Gamma^{\varphi}(\alpha)$. Thus the finding of the functional $\Gamma^{\varphi}(\alpha)$ is equivalent to the establishing of the closed equation for the field $\langle A(x)\rangle^{\varphi}$. The explicit form of the functional $\Gamma^{\mathcal{P}}(\alpha)$ one can obtain in the similar way as in Sec.2. When doing so we will keep exactly the interaction with the field $A^{\mathcal{P}}$ or with the field α. The final result has the form:

$$\Gamma^{\mathcal{P}}(\alpha)=\frac{1}{2}\bar{\alpha}\Box\bar{\alpha}-i\ln C_0 \quad - \quad \text{single-ind. part } \Delta\overline{W}^{\mathcal{P}}(J=\Box\bar{\alpha})=$$

$$=\frac{1}{2}(\alpha-A^{\mathcal{P}})\Box(\alpha-A^{\mathcal{P}})-i\ln C_0+Tr\ln\frac{\tilde{S}^c(\alpha)}{\tilde{S}^c(A^{\mathcal{P}})}-$$

-single-indecomposable vacuum diagrams $\Delta W^{\mathcal{P}}(\tilde{S}^c(\alpha))$,

where

$$\bar{\alpha}=\alpha-A^{\mathcal{P}},\quad \Delta\overline{W}^{\mathcal{P}}=W^{\mathcal{P}}-JA^{\mathcal{P}}-i\ln C_0-\frac{1}{2}ID_0^cI,$$

and C_0 is fixed by the expressions (36), (39), (40).

Let us investigate, at last, how the above suggested approach to QED with the intense mean field is related to the Furry approach to quantum electrodynamics with the external field (Chapter I).

Consider the perturbation expansions with respect to the radiative interaction for the matrix elements (I), (6) in the approximation in which the mean field at the final time-moment is taken to the zeroth order with respect to the radiative interaction (Sec.2), that is

$$\langle A(x)\rangle^{M}\rightarrow A^{M}(x)=\int D^{ret}(x-x')\mathcal{J}(x')dx'+A^{in}(x). \tag{5I}$$

Having the specific expression for the mean field available, one can obtain all the needed in the perturbation theory components. By setting $t=t_{out}$ in (5I) and using the formulae (28), (30) we will get the quantities $\overset{\circ}{z}_{\vec{k}\lambda}(out)$. (Here and elsewhere zero on top means that the corresponding quantity is determined by the mean field (5I).)

$$\overset{\circ}{z}_{\vec{k}\lambda}(out)=z_{\vec{k}\lambda}(in)e^{-i\kappa(t_{out}-t_{in})}+$$
$$+\frac{ig_{\lambda\lambda}}{\sqrt{2V\kappa}}\int(e_{\vec{k}\lambda},\mathcal{J}(x))e^{-i\kappa(t_{out}-t)-i\vec{k}\vec{x}}d\vec{x}. \tag{52}$$

By substituting (52) into the formula (37) we get

$$\overset{\circ}{A}{}^{\mathcal{P}}_0(x)=A^{in}(x)+\int D_0^{+}(x-x')\mathcal{J}(x')dx'.$$

By taking into account that $D_0^c(x-x')+D_0^+(x-x')=D^{ret}(x-x')$, (Bogoliubov, Shirkov, 1959) we obtain

$$\overset{\circ}{A}_0{}^{\varphi}(x)=\int D^{ret}(x-x')\mathcal{J}(x')dx'+A^{in}(x)=A^{M}(x). \tag{53}$$

Thus, the auxiliary classical field $\overset{\circ}{A}{}^{\varphi}(x)$ is real in this case and coincides with the mean field of the zeroth order approximation.

The amplitude C_0^{γ} (see (39), (40)) may be in this case evaluated explicitly, their modulus is equal to unity (the photon vacuum turns, to the zeroth order with respect to the radiative interaction, again into the vacuum during the evolution).

$$\overset{\circ}{C}{}_0^{\gamma}=\exp-\frac{i}{2}\int\mathcal{J}(x)A^{M}(x)dx. \tag{54}$$

By substituting (52) into the expressions (42), (43) one can make sure that the relative probabilities $w(\vec{k}\lambda|0)$ and $w(0|\vec{æ}\nu)$ of creation and annihilation of transverse photons are equal to zero. Therefore the coupling (41) has the form

$$\underbrace{\tilde{C}_{\vec{k}\lambda}(out)C^+_{\vec{æ}\nu}(in)}=\delta_{\vec{k}\lambda,\vec{æ}\nu}. \tag{55}$$

If the quantities $A^{\varphi}(x)$, C_0^{γ}, $\underbrace{\tilde{C}(out)C^+(in)}$ have the form (53-55), the perturbation expansions for the matrix elements of the process (I) coincide completely with the corresponding expansions for the matrix elements (I.I.16) in Furry approach to QED with the external field (51). Thus, from the viewpoint of the treatment, which is based on the mean field conception and discussed in this Section, QED with an external field describes the calculation of matrix elements of transitions to the zeroth order with respect to radiative corrections, which determine the changing of the mean field (51) at the final time-moment. This statement may be the basis of one of the possible interpretations of QED with the external field.

It is useful to represent the matrix element (I) in the following way:

$$M_{in\to out}=\sum_{\substack{K,S,M\\ \{\vec{k}'\lambda',\bar{s}',\overset{+}{m}'\}}} Q(\overset{+}{m}\ldots\bar{s}\ldots\vec{k}\lambda\ldots|\underbrace{\vec{k}'\lambda'\ldots}_{K}\ \underbrace{\bar{s}'\ldots}_{S}\ \underbrace{\overset{+}{m}'\ldots}_{M})\cdot$$

$$\cdot{}_{out}\langle\overset{\circ}{0}|\overset{\circ}{a}_{m'}(out)\ldots\overset{\circ}{b}_{s'}(out)\ldots\overset{\circ}{C}_{\vec{k}'\lambda'}(out)\,U_{\gamma}C^+_{\vec{æ}\nu}(in)\ldots b^+_n(in)\ldots a^+_{\ell}(in)\ldots|0\rangle_{in}, \tag{56}$$

$$Q(\overset{+}{m}\ldots\bar{s}\ldots\vec{k}\lambda\ldots|\vec{k}'\lambda'\ldots\bar{s}'\ldots\overset{+}{m}'\ldots)=$$

(57)

$$= \frac{1}{K!S!M!}\ {}_{out}\langle 0|a_m(out)\ldots b_s(out)\ldots C_{\vec{k}\lambda}(out)\ldots \mathring{C}^{+}_{\vec{k}'\lambda'}(out)\ldots \mathring{b}^{+}_{s'}(out)\ldots \mathring{a}^{+}_{m'}(out)\times$$
$$\times \ldots |\mathring{0}\rangle_{out}.$$

In the form (56) it is represented by the sum of products of the coefficients Q which take into account the difference between the real mean field at the final time-moment and the field (5I), by the matrix elements of QED with the external field (5I).

The coefficients Q may be calculated with the aid of the formulae (A.I7), where the nonzero components are

$$\underbrace{a_m(out)\,\mathring{a}^{+}_{m'}(out)} = \mathring{G}^{-1}(+|^{+})_{mm'},$$

$$\underbrace{b_s(out)\,\mathring{b}^{+}_{s'}(out)} = \mathring{G}^{-1}(-|-)_{s's},$$

$$\underbrace{\mathring{b}^{+}_s(out)\,\mathring{a}^{+}_m(out)} = \{\mathring{G}(-|^{+})\mathring{G}^{-1}(+|^{+})\}_{sm},$$

$$\underbrace{a_m(out)\,b_s(out)} = \{\mathring{G}^{-1}(+|^{+})\,\mathring{G}(+|^{-})\}_{ms},$$

$$\mathring{G}(\mp|^{\pm})_{mn} = \int {}^{\mp}\mathring{\varphi}^{+}_m(\vec{x})^{\pm}\varphi_n(\vec{x})d\vec{x},$$

$$\mathring{G}(^{\pm}|_{\mp})_{mn} = \int {}^{\pm}\varphi^{+}_m(\vec{x})^{\mp}\mathring{\varphi}_n(\vec{x})d\vec{x},$$

$${}_{out}\langle 0|C_{\vec{k}\lambda}(out)|\mathring{0}\rangle_{out}\cdot {}_{out}\langle 0|\mathring{0}\rangle^{-1}_{out} = \mathring{z}_{\vec{k}\lambda}(out) - z_{\vec{k}\lambda}(out),$$

$${}_{out}\langle 0|\mathring{C}^{+}_{\vec{k}\lambda}(out)|\mathring{0}\rangle_{out}\cdot {}_{out}\langle 0|\mathring{0}\rangle^{-1}_{out} = z^{*}_{\vec{k}\lambda}(out) - \mathring{z}^{*}_{\vec{k}\lambda}(out),$$

$${}_{out}\langle 0|\mathring{0}\rangle_{out} = \det \mathring{G}(-|-)\exp\sum_{\vec{k},\lambda=1,2}\Big\{-\frac{|z_{\vec{k}\lambda}(out)|^2}{2} -$$
$$- \frac{|\mathring{z}_{\vec{k}\lambda}(out)|^2}{2} + z^{*}_{\vec{k}\lambda}(out)\,\mathring{z}_{\vec{k}\lambda}(out)\Big\}.$$

APPENDIX A. GENERALIZATION OF THE WICK TECHNIQUE TO UNSTABLE VACUUM

In a number of cases when the vacuum vector is unstable with respect to the creation of particles it appears that the matrix elements of processes and Green functions of quantum field theory have the following typical form

$$\langle\tilde{0}|\tilde{a}\ldots TF(\varphi)a^+\ldots|0\rangle, \tag{1}$$

where $\{a^+, a\}$ is a complete set of creation and annihilation operators, $|0\rangle$ is the corresponding vacuum vector $(a|0\rangle = 0)$, $\varphi(x)$ are the field operators in some representation where they are linear with respect to the operators $\{a^+, a\}$, $F(\varphi)$ is an arbitrary operator functional admitting the series expansion in powers of φ, $\{\tilde{a}^+, \tilde{a}\}$ is a set of operators related with the operators $\{a^+, a\}$ by a similarity transformation of linear type

$$\begin{aligned} \tilde{a} &= V^{-1} a V = \Phi_1 a + \Psi_1 a^+ + f_1, \\ \tilde{a}^+ &= V^{-1} a^+ V = \Psi_2 a + \Phi_2 a^+ + f_2, \end{aligned} \tag{2}$$

$$[\tilde{a}, \tilde{a}^+]_{-\varepsilon} = 1, \quad [\tilde{a}, \tilde{a}]_{-\varepsilon} = [\tilde{a}^+, \tilde{a}^+]_{-\varepsilon} = 0, \qquad \varepsilon = \begin{cases} 1 & \text{Bose} \\ -1 & \text{Fermi} \end{cases}$$

$\langle\tilde{0}| = \langle 0|V$, and V is, generally, a nonunitary operator. Consider here the generalization of the Wick technique which makes it possible to calculate efficiently such matrix elements.

We say that the transformation (2) admits a transition to a generalized normal form with respect to the vacua $\langle\tilde{0}|$ and $|0\rangle$ if the explicit and single-valued representation

$$\begin{aligned} \tilde{a} &= Aa + B\tilde{a}^+ + \gamma, \\ a^+ &= Ca + \mathcal{E}\tilde{a}^+ + \delta \end{aligned} \tag{3}$$

is possible. To make it possible the existence of the matrix Φ_2^{-1} is enough. Indeed, if one consideres equations (2) as a set of linear equations for $\{\tilde{a}, a^+\}$ one sees that the solution may exist if

$$\det \begin{pmatrix} I & -\Psi_1 \\ 0 & \Phi_2 \end{pmatrix} = \det \Phi_2 \neq 0. \tag{4}$$

If the condition (4) holds then

$$A = \Phi_1 - \Psi_1 \Phi_2^{-1} \Psi_2, \quad B = \Psi_1 \Phi_2^{-1}, \quad \gamma = -\Psi_1 \Phi_2^{-1} f_2 + f_1,$$
$$C = -\Phi_2^{-1} \Psi_2, \quad \mathcal{E} = \Phi_2^{-1}, \quad \delta = -\Phi_2^{-1} f_2 .$$

In the case the transformation (2) is canonical the Bose-type matrix $\Phi_2^{-1} = (\Phi_1^*)^{-1}$ always exists (Berezin, 1965). It may be not the case, however, for a Fermi-type canonical transformation.

If the linear similarity transformation (2) admits a transition to a generalized normal form with respect to the vacua $\langle \tilde{0}|$ and $|0\rangle$ then any field operator $\varphi(x)$ linear in $\{a^+, a\}$ may be represented in the form

$$\varphi(x) = \varphi^{(-)}(x) + \varphi^{(+)}(x) + \varphi^{(0)}(x), \tag{5}$$

where

$$\varphi^{(-)}(x)|0\rangle = \langle \tilde{0}|\varphi^{(+)}(x) = 0, \quad \varphi^{(0)}(x) = \langle \tilde{0}|\varphi(x)|0\rangle \langle \tilde{0}|0\rangle^{-1}. \tag{6}$$

At the same time

$$\begin{aligned} [\varphi^{(-)}(x), \varphi^{(-)}(x')]_{-\varepsilon} &= [\varphi^{(+)}(x), \varphi^{(+)}(x')]_{-\varepsilon} = 0 \\ [\varphi^{(-)}(x), \varphi^{(+)}(x')]_{-\varepsilon} &= C - \text{number} \end{aligned} \tag{7}$$

(The quantities $f_1, f_2, \gamma, \delta, \varphi^{(0)}(x)$ in the Fermi case belong to the Grassmann algebra).

The proof of representation (5)-(6) is based on the remark that with the aid of eqs. (3) any operator linear in $\{a^+, a\}$ can be linearly expressed only in terms of the operators a and $\tilde{a}^+$ the commutators or anticommutators of the operators a and $\tilde{a}^+$ being C- numbers due to (2). It is evident that $\varphi^{(-)}(x)$ is the part of the operator $\varphi(x)$ that contains only the operators a and $\varphi^{(+)}(x)$ is its part, containing only $\tilde{a}^+$. The second relation in (6) is obtained by averaging (5) between different vacua with the usage of the first two properties in (6).

The form whose every term has all its operators $\varphi^{(+)}(x)$ placed to the left of all its operators $\varphi^{(-)}(x)$ is called the generalized normal form of the operator functional $F(\varphi)$.

The product of operators $\varphi(x)$ is called the generalized normal product if it is reduced to the generalized normal form with all the commutators or anticommutators between $\varphi^{(-)}(x)$, $\varphi^{(+)}(x)$ being considered zero while the reduction is being performed. The generalized normal product will be denoted by the symbol $\tilde{N}(\dots)$ It is evident that

$$\langle\tilde{0}|\tilde{N}\prod_{i=1}^{n}\varphi(x_i)|0\rangle=\langle\tilde{0}|0\rangle\cdot\prod_{i=1}^{n}\varphi^{(0)}(x_i). \tag{8}$$

The functions

$$\underbrace{\varphi(x)\varphi(y)}=\varphi(x)\varphi(y)-\tilde{N}\varphi(x)\varphi(y), \tag{9}$$

$$\overbrace{\varphi(x)\varphi(y)}=T\varphi(x)\varphi(y)-\tilde{N}\varphi(x)\varphi(y), \tag{10}$$

will be called, respectively, the generalized coupling and generalized chronological coupling. Relations (7) indicate that the couplings (9), (10) are C- numbers and may thus be represented as follows

$$\underbrace{\varphi(x)\varphi(y)}=\langle\tilde{0}|\varphi(x)\varphi(y)|0\rangle\langle\tilde{0}|0\rangle^{-1}-\varphi^{(0)}(x)\varphi^{(0)}(y), \tag{11}$$

$$\overbrace{\varphi(x)\varphi(y)}=\langle\tilde{0}|T\varphi(x)\varphi(y)|0\rangle\langle\tilde{0}|0\rangle^{-1}-\varphi^{(0)}(x)\varphi^{0}(y). \tag{12}$$

Evidently one can use, by replacing the normal products and couplings by their generalized analogues, the conventional versions of Wick's theorem in order to reduce an arbitrary operator functional $F(\varphi)$ to the generalized normal form. The corresponding functional formulations are similar to the usual formulations and have the form

$$Sym\,F(\varphi)=\tilde{N}exp\frac{\varepsilon}{2}\frac{\overrightarrow{\delta}}{\delta\varphi}\underbrace{\varphi\varphi}\frac{\overrightarrow{\delta}}{\delta\varphi}\cdot F(\varphi), \tag{13}$$

$$TF(\varphi)=\tilde{N}exp\frac{\varepsilon}{2}\frac{\overrightarrow{\delta}}{\delta\varphi}\overbrace{\varphi\varphi}\frac{\overrightarrow{\delta}}{\delta\varphi}F(\varphi). \tag{14}$$

By combining (8) with (13), (14) we obtain

$$\langle\tilde{0}|Sym\,F(\varphi)|0\rangle=\langle\tilde{0}|0\rangle exp\frac{\varepsilon}{2}\frac{\overrightarrow{\delta}}{\delta\varphi}\underbrace{\varphi\varphi}\frac{\overrightarrow{\delta}}{\delta\varphi}F(\varphi)\Big|_{\varphi=\varphi^{(0)}}, \tag{15}$$

$$\langle\tilde{0}|TF(\varphi)|0\rangle=\langle\tilde{0}|0\rangle exp\frac{\varepsilon}{2}\frac{\overrightarrow{\delta}}{\delta\varphi}\overbrace{\varphi\varphi}\frac{\overrightarrow{\delta}}{\delta\varphi}F(\varphi)\Big|_{\varphi=\varphi^{(0)}}. \tag{16}$$

The formulae obtained allow us to calculate the matrix elements which have the form (1). To do so one should, evidently, reduce $TF(\varphi)$ to the generalized normal form with the aid of (14), substitute it into (1) and perform, as usual, corresponding commutations. As a result the problem reduces to the calculation of matrix elements (1) where $F(\varphi)\equiv 1$. They also may be calculated by reducing the product of the operators $\tilde{a} \ldots a^{+}\ldots$ to the generalized normal form. By using the suitable functional formulation of Wick's theorem (Vasil'ev, 1976) we will get the formula

$$\langle\tilde{0}|\tilde{a}(\kappa_1)\ldots\tilde{a}(\kappa_n)a^{+}(\varkappa_{n+1})\ldots a^{+}(\varkappa_{n+m})|0\rangle=$$

$$=\langle\tilde{0}|0\rangle\prod_{i<j}\Big(1+\frac{\overleftarrow{\delta}}{\delta\tilde{a}_i}\underbrace{\tilde{a}\tilde{a}}\frac{\overleftarrow{\delta}}{\delta\tilde{a}_j}+\frac{\overleftarrow{\delta}}{\delta\tilde{a}_i}\underbrace{\tilde{a}a^{+}}\frac{\overleftarrow{\delta}}{\delta a_j^{+}}+ \tag{17}$$

$$+\frac{\overleftarrow{\delta}}{\delta a_i^{+}}\underbrace{a^{+}a^{+}}\frac{\overleftarrow{\delta}}{\delta a_j^{+}}\Big)\tilde{a}_1(\kappa_1)\ldots a^{+}_{n+m}(\varkappa_{n+m})\Bigg|\begin{array}{l}\tilde{a}_i(\kappa_i)=\langle\tilde{0}|\tilde{a}(\kappa_i)|0\rangle,\\ a_i^{+}(\varkappa_i)=\langle\tilde{0}|a^{+}(\varkappa_i)|0\rangle.\end{array}$$

Thus the indicated matrix elements may be expressed in terms of couplings of the operators $\tilde{a}$, a^{+} and their C- numerical parts, in the sense of the decomposition (5), only. All these values may be expressed in terms of the similarity transformation coefficients (3):

$$\begin{aligned}
&\underbrace{\tilde{a}\tilde{a}}=(\Phi_1-\Psi_1\Phi_2^{-1}\Psi_2)\Psi_1^{T};\\
&\underbrace{a^{+}a^{+}}=-\Phi_2^{-1}\Psi_2,\\
&\underbrace{\tilde{a}a^{+}}=\Phi_1-\Psi_1\Phi_2^{-1}\Psi_2,\\
&\langle\tilde{0}|\tilde{a}|0\rangle=-\Psi_1\Phi_2^{-1}f_2+f_1,\\
&\langle\tilde{0}|a^{+}|0\rangle=-\Phi_2^{-1}f_2.
\end{aligned} \tag{18}$$

APPENDIX B. GREEN FUNCTIONS IN AN EXTERNAL ELECTROMAGNETIC FIELD*)

§ I. Introduction

In the present paper all the Green functions mentioned in chapters I, II are derived explicitly in an external field, being the combination of a constant and uniform field and a plane wave field. A brief review of the well-known works, as well as the definition of the scalar Green functions in the quantum field theory frame and their representations over the solutions of the relativistic wave equations are given in Sec.I. In Sec.2 the Green functions for the scalar QED with a constant and uniform electric field are obtained. This simple example illustrates the calculation method of the Green functions. The Green functions for the spinor QED with a constant and uniform electric field are obtained in Sec.3. In Sec.4 a complete and orthonormal set of the solutions of the Klein-Gordon and Dirac equations for a field, being the combination of a constant and uniform field and a plane wave field, is derived. With it's aid the corresponding Green functions are obtained in Sec.5. The results are given in the form of proper time contour integrals. In Sec.6 the operator representations for the calculated Green functions are given.

Some of the obtained here Green functions were calculated earlier for specific external fields. Thus, the explicit expression for the anticommutator function $\tilde{G}(x,x')$ in a constant and uniform electromagnetic field was obtained in (Fock, I937) by the proper time method[I]. The same result was obtained in (Geheniau, I950; Geheniau, Demeur, I95I; Demeur, I95I) by solving of the equations for $\tilde{D}(x,x')$ and $\tilde{G}(x,x')$ with a given initial condition and also in (Belov, I975) by the Maslov canonical operator method. The Green function $\tilde{S}^c(x,x')$ in a constant field, and also in a field of a plane electromagnetic field was obtained by Schwinger (Schwinger, I95I) by the proper time method. The functions $\tilde{S}^c(x,x')$ and $\tilde{D}^c(x,x')$ in a constant

* The work is carried out in common with Gavrilov S.P., Shwartsman Sh.M., Wolfengaut J.J. (Dept. of Math. Analysis, Pedagogical Inst., 634044 Tomsk, USSR)

I) Notations for Green functions see in I,II.

field were calculated in (Geheniau, 1950; Geheniau, Demeur 1951; Demeur, 1951) by using the explicit form of $\tilde{G}(x,x')$ and $\tilde{D}(x,x')$, and in (Nikishov, 1969; Narozhny, Nikishov, 1976) by the summing over the corresponding solutions of the Dirac and Klein-Gordon equations. The causal Green function $\tilde{D}^c(x,x')$ of the scalar field in a plane wave field was obtained in (Fradkin, 1965; Barbashow, 1965) by functional methods. The general solution for the Green function $\tilde{S}^c(x,x')$ of the Dirac equation in an arbitrary external field was obtained for the first time in (Fradkin, 1966) in the form of the path integral over the classical trajectories and Grassmann variables. With it's aid the explicit expression for the Green function $\tilde{S}^c(x,x')$ of the Dirac equation in a plane wave field was calculated in (Fradkin, 1966). The Green functions $\tilde{S}^c(x,x')$ and $\tilde{D}^c(x,x')$ in a plane wave field were obtained also in (Reiss, Eberly, 1966) by the straightforward solving of the nonhomogeneous equations. In (Oleinik, 1968; 1969) the function $\tilde{S}^c(x,x')$ is calculated in a uniform magnetic field and a plane wave field, propagating along the magnetic field. The Green functions $\tilde{D}^c(x,x')$ and $\tilde{S}^c(x,x')$ in an external field, being a superposition of a constant and uniform electromagnetic field and a plane wave field, are obtained in (Batalin, Fradkin, 1970) by the functional method. In (Narozhny, Nikishov, 1976) the functions $\tilde{D}^c(x,x')$ and $\tilde{S}^c(x,x')$ in a constant and uniform electric field combined with a plane wave field are obtained by the summing over the solutions of the corresponding relativistic wave equations.

The Green functions of the spinor QED are defined by the relations (I.1.43), (I.2.6), (I.1.27), (II.2.14), and they may be expressed in terms of the solutions of the Dirac equation (I.1.44), (I.2.7), (I.1.27), (II.2.14).

The Green functions of the scalar QED are defined like in the spinor case:

$$\tilde{D}^c(x,x') = i\,{}_{out}\langle\tilde{0}|T\tilde{\psi}(x)\tilde{\psi}^+(x')|0\rangle_{in}\cdot C_v^{-1}, \tag{1}$$

$$\tilde{D}^-(x,x') = i\,{}_{out}\langle\tilde{0}|\tilde{\psi}(x)\tilde{\psi}^+(x')|0\rangle_{in}\cdot C_v^{-1}, \quad \tilde{D}^+(x,x') = -i\,{}_{out}\langle\tilde{0}|\tilde{\psi}^+(x')\tilde{\psi}(x)|0\rangle_{in}\cdot C_v^{-1},$$

$$\tilde{D}(x,x') = [\tilde{\psi}(x), \tilde{\psi}^+(x')], \tag{2}$$

$$\tilde{\tilde{D}}^-(x,x') = i\,{}_{in}\langle 0|\tilde{\psi}(x)\tilde{\psi}^+(x')|0\rangle_{in}, \quad \tilde{\tilde{D}}^+(x,x') = -i\,{}_{in}\langle 0|\tilde{\psi}^+(x')\tilde{\psi}(x)|0\rangle_{in},$$

$$\tilde{\tilde{D}}^c(x,x') = i\,{}_{in}\langle 0|T\tilde{\psi}(x)\tilde{\psi}^+(x')|0\rangle_{in}, \tag{3}$$

$$\tilde{\tilde{D}}^{\bar{c}}(x,x') = i\,{}_{in}\langle 0|\bar{T}\tilde{\varphi}(x)\tilde{\varphi}^{+}(x')|0\rangle_{in} = \tilde{\tilde{D}}^{c}(x,x') - i\varepsilon(x_0 - x_0')\tilde{D}(x,x'). \qquad (4)$$

Here $\tilde{\varphi}(x)$ is the scalar field operator satisfying the Klein-Gordon equation in the external electromagnetic field $A_\mu^{ext}(x)$ and the commutation relations

$$[\tilde{\varphi}(x), \tilde{\varphi}^{+}(x')]_{x_0=x_0'} = 0, \quad \left[\frac{\partial\tilde{\varphi}(x)}{\partial x_0}, \tilde{\varphi}^{+}(x')\right]_{x_0=x_0'} = -i\delta^{(3)}(\vec{x}-\vec{x}'),$$

$|\tilde{0}\rangle_{out} = \tilde{U}^{-1}|0\rangle_{out}$, $\tilde{U}$ is the evolution operator of the scalar field interacting with the external field, $|0\rangle_{out}$, $|0\rangle_{in}$ are the vacuum vectors of out- and in-particles, $C_v = {}_{out}\langle\tilde{0}|0\rangle_{in}$.

The functions $\tilde{D}^c(x,x')$, $\tilde{\tilde{D}}^c(x,x')$ and $\tilde{\tilde{D}}^{\bar{c}}(x,x')$ are the partial solutions of the nonhomogeneous Klein-Gordon equation

$$(\mathcal{P}^2(x)-m^2)D(x,x') = -\delta^{(4)}(x-x'), \quad D(x,x') = (\tilde{D}^c(x,x'), \tilde{\tilde{D}}^c(x,x')); \qquad (5a)$$

$$(\mathcal{P}^2(x)-m^2)\tilde{\tilde{D}}^{\bar{c}}(x,x') = \delta^{(4)}(x-x'). \qquad (5b)$$

The rest functions satisfy the homogeneous Klein-Gordon equation. The function $\tilde{D}(x,x')$ satisfies the initial condition

$$\tilde{D}(x,x')\Big|_{x_0=x_0'} = 0, \quad \frac{\partial\tilde{D}(x,x')}{\partial x_0}\Big|_{x_0=x_0'} = -i\delta^{(3)}(\vec{x}-\vec{x}'). \qquad (6)$$

Likewise the spinor case, the Green functions (1) - (4) may be expressed in terms of the solutions of the Klein-Gordon equation in an external electromagnetic field. To do so we will use the decomposition of the scalar field operators $\tilde{\varphi}(x)$ over the complete and orthonormal set of solutions $\{{}^{\pm}\varphi_n(x)\}$ $(\{{}_{\pm}\varphi_n(x)\})$, describing particles (+) and antiparticles (-) under $x_0 \to +\infty$, $(x_0 \to -\infty)$:

$$({}^{\pm}\varphi_n, {}^{\pm}\varphi_{n'}) = \pm\delta_{n,n'}, \quad ({}^{\pm}\varphi_n, {}^{\mp}\varphi_{n'}) = 0,$$

$$(\varphi_n, \varphi_{n'}) = \int \varphi_n^*(x)\left(i\overleftrightarrow{\frac{\partial}{\partial x_0}} - 2eA_0(x)\right)\varphi_{n'}(x)\,d\vec{x},$$

$$\sum_n \{{}^{+}\varphi_n(x)\,{}^{+}\varphi_n^*(x') - {}^{-}\varphi_n(x)\,{}^{-}\varphi_n^*(x')\}_{x_0=x_0'} = 0,$$

$$\sum_n \{\partial_0\,{}^{+}\varphi_n(x)\,{}^{+}\varphi_n^*(x') - \partial_0\,{}^{-}\varphi_n(x)\,{}^{-}\varphi_n^*(x')\}_{x_0=x_0'} = -i\delta^{(3)}(\vec{x}-\vec{x}').$$

Similar conditions are valid also for $\{{}_{\pm}\varphi_n(x)\}$. Thus we get

$$\tilde{D}^c(x,x') = \theta(y_0)\tilde{D}^{-}(x,x') - \theta(-y_0)\tilde{D}^{+}(x,x'), \qquad (7)$$

$$\tilde{D}^{-}(x,x') = i\sum_{n,\kappa} {}^{+}\varphi_n(x)\,\omega(\overset{+}{n}|\overset{+}{\kappa})_{+}\varphi_{\kappa}^{-}(x'),$$

$$\tilde{D}^{+}(x,x') = -i\sum_{n,\kappa} {}_{-}\varphi_{\kappa}(x)\,\omega(\bar{n}|\bar{\kappa})^{-}\varphi_n^{*}(x'), \tag{8}$$

$$\tilde{D}(x,x') = \sum_n \left\{ {}_{+}\varphi_n(x)_{+}\varphi_n^{*}(x') - {}_{-}\varphi_n(x)_{-}\varphi_n^{*}(x') \right\}, \tag{9}$$

$$\tilde{\tilde{D}}^{c}(x,x') = \theta(y_0)\tilde{\tilde{D}}^{-}(x,x') - \theta(-y_0)\tilde{\tilde{D}}^{+}(x,x'), \tag{10}$$

$$\tilde{\tilde{D}}^{\mp}(x,x') = \pm i\sum_n {}_{\pm}\varphi_n(x)_{\pm}\varphi_n^{*}(x'), \tag{11}$$

$$\tilde{\tilde{D}}^{\bar{c}}(x,x') = \theta(-y_0)\tilde{\tilde{D}}^{-}(x,x') - \theta(y_0)\tilde{\tilde{D}}^{+}(x,x'), \tag{12}$$

$$\tilde{D}^{\mp}(x,x') = \tilde{\tilde{D}}^{\mp}(x,x') \pm \tilde{D}^{a}(x,x'), \tag{13}$$

$$\tilde{D}^{a}(x,x') = i\sum_{n,\kappa} {}_{-}\varphi_n(x)\,\omega(0|\bar{n}\overset{+}{\kappa})_{+}\varphi_{\kappa}^{*}(x'), \tag{14}$$

$$\tilde{D}^{c}(x,x') = \tilde{\tilde{D}}^{c}(x,x') + \tilde{D}^{a}(x,x'). \tag{15}$$

Here $\omega(\overset{\pm}{n}|\overset{\pm}{\kappa})$, $\omega(0|\bar{n}\overset{+}{\kappa})$ are the matrix elements for the processes of the single-particle scattering and annihilation of two scalar particles in an external field.

§2. Constant electric field. Scalar case

By calculating the Green functions of the scalar QED in a constant and uniform electric field we will trace the train of reasoning, which enables to get by using the complete sets of the solutions of the K-G (Klein-Gordon) and Dirac equations the proper time integral representations for the Green functions.

Let us choose the vector potential as $A_0^{ext} = A_3^{ext} = \frac{Ex_-}{2}$, $A_1^{ext} = A_2^{ext} = 0$. For our purposes the most convenient are the complete sets of the solutions of the K-G equation which have a semiclassical form (Narozhny, Nikishov, 1976):

$$ {}^{-}_{+}\varphi_{p_-,p_1,p_2}(x) = \frac{1}{\sqrt{2}}(2\pi)^{-\frac{3}{2}}(eE)^{-\frac{1}{4}}\exp\left\{-i\,\frac{p_-x_+}{2} + \nu\ln(\mp\tilde{\pi}_-) - ip_\perp x\right\}, \tag{1}$$

$$ {}^{+}\varphi_{p_-,p_1,p_2}(x) = \theta(\tilde{\pi}_-)C_{p_\perp^2}\,{}_{+}\varphi_{p_-,p_1,p_2}(x);\quad {}_{-}\varphi_{p_-,p_1,p_2}(x) = \theta(-\tilde{\pi}_-)C_{p_\perp^2}\,{}^{-}\varphi_{p_-,p_1,p_2}(x), \tag{2}$$

$$C_{p_\perp^2} = \sqrt{2\pi}\,\Gamma^{-1}\left(\frac{1+i\lambda}{2}\right)e^{-\frac{\pi}{4}(\lambda+i)},\ \ln(\pm\tilde{\pi}_-) = \ln|\tilde{\pi}_-| + i\pi\theta(\pm\tilde{\pi}_-),\ \tilde{\pi}_- = \frac{\pi_-}{\sqrt{eE}};$$

$$\pi_- = p_- - eEx_-,\ x_\pm = x^0 \pm x^3,\ \nu = \frac{i\lambda - 1}{2},\ \lambda = \frac{m^2 - p_\perp^2}{eE},\ p_\perp^\mu = (0, p_\perp^1, p^2, 0).$$

For definiteness we will consider $eE > 0$ In this case the signes "$\pm$" in (I), (2) correspond to the sign of the kinetic momentum π_- under $x_- \to \pm\infty$.

In (Narozhny,Nikishov,I976) it is shown, that the solutions (I),(2) one can express in terms of the solutions of the other form

$$
\begin{gathered}
{}^{\pm}_{\pm}\varphi_{p_-, p_1, p_2}(x) = (2\pi eE)^{-\frac{1}{2}} \int_{-\infty}^{\infty} M^*(p_3, p_-)\, {}^{\pm}_{\pm}\varphi_{\vec p}(x)\, dp_3, \\
M(p_3, p_-) = exp\left\{ -i\, \frac{(p_- - 2p_3)^2 - 2p_3^2}{4eE} \right\},
\end{gathered} \tag{3}
$$

$$
\begin{gathered}
{}^{-}_{+}\varphi_{\vec p}(x) = B e^{i\vec p \vec x} D_\nu[\pm(1-i)\xi];\quad {}^{+}_{-}\varphi_{\vec p}(x) = B e^{i\vec p \vec x} D_{\nu^*}[\pm(1+i)\xi], \\
B = (2\pi)^{-\frac{3}{2}} (2eE)^{-\frac{1}{4}} exp\left\{ -\frac{\pi\lambda}{8} - \frac{3\pi i}{8} + i\frac{\lambda}{4}\ln 2 - i\frac{eE}{2}\left(\frac{x_0^2}{2} - x_3^2\right) \right\}, \\
\xi = (eE)^{-\frac{1}{2}} (eEx_0 - p_3)
\end{gathered} \tag{4}
$$

The solutions (4) can be got (Narozhny,I968) in the slow changing electric field limit from the asymptotically $(x_0 \to \pm\infty)$ free solutions, what enables to establish their classification by the energy sign under $x_0 \to \pm\infty$. Thus, follow (Narozhny, Nikishov, I976) we satisfy ourselves that the classification of the solutions (I), (2) into the particles and antiparticles is equivalent to the usual classification.

The solutions (4) are orthonormal, and since

$$\int_{-\infty}^{\infty} M^*(p_3, p_-') M(p_3, p_-)\, dp_3 = 2\pi eE\, \delta(p_- - p_-'), \tag{5}$$

from (3) follows that the solutions (I), (2) are orthonormal too:

$$\left({}^{+}_{+}\varphi_{p_- p_1 p_2},\ {}^{+}_{+}\varphi_{p_-' p_1' p_2'} \right) = \delta(p_- - p_-')\, \delta^{(2)}(p_\perp - p_\perp'), \tag{6a}$$

$$\left({}^{-}_{-}\varphi_{p_- p_1 p_2},\ {}^{-}_{-}\varphi_{p_-' p_1' p_2'} \right) = -\delta(p_- - p_-')\, \delta^{(2)}(p_\perp - p_\perp'), \tag{6b}$$

$$\left({}^{+}_{+}\varphi_{p_- p_1 p_2},\ {}^{-}_{-}\varphi_{p_-' p_1' p_2'} \right) = 0.$$

On making sure that the solutions (I), (2) form the two complete and orthonormal sets $\{{}_\pm\varphi_n(x)\}$, $\{{}^\pm\varphi_n(x)\}$ we will use them to construct the functions (I.8), (I.I4), taking into account that (Narozhny, Nikishov, I976)

$$w(p_- \overset{\pm}{p_1} p_2 \,|\, p_-' \overset{\pm}{p_1'} p_2') = \delta(p_- - p_-')\, \delta^{(2)}(p_\perp - p_\perp')\, C_{p_\perp^2}^{-1},$$

$$\omega(0|P_- \bar{P}_1 P_2, P_-' \overset{+}{P_1'} P_2') = \delta(P_- - P_-')\delta^{(2)}(P_1 - P_1') C_{P_1}^{-1} e^{-\frac{\pi}{2}(\lambda + i)} .$$

Then

$$\tilde{D}^-(x,x') = \int_{eEx_-}^{+\infty} {}_{+}\tilde{f}_E(x,x',P_-)dP_-, \quad \tilde{D}^+(x,x') = -\int_{-\infty}^{eEx_-} {}^{-}\tilde{f}_E(x,x',P_-)dP_-, \tag{7}$$

$$\tilde{D}^a(x,x') = -\int_{-\infty}^{eEx_-} {}_{+}\tilde{f}_E(x,x',P_-)dP_-, \tag{8}$$

$$ {}^{-}_{+}\tilde{f}_E(x,x',P_-) = i\int_{-\infty}^{+\infty} dP_1 dP_2 \, {}^{-}_{+}\varphi_{P_- P_1 P_2}(x) \, {}^{-}_{+}\varphi^*_{P_- P_1 P_2}(x'). \tag{9}$$

We will get the proper time integral representations for the functions (7), (8) by transforming the P_- integration into integration over S

$$S = \frac{1}{2eE}\left[(\ln(\mp\tilde{\pi}_-'))^* - \ln(\mp\tilde{\pi}_-)\right], \quad \tilde{\pi}_-' = \tilde{\pi}_- + \sqrt{eE}\,y_-, \tag{I0}$$
$$y_\mu = x_\mu - x_\mu',$$

and doing the P_1 and P_2 integrations.

Let us consider first the functions $\tilde{D}^\pm(x,x')$ Suppose $\mp y_- > 0$, then $S = \frac{1}{2eE}\ln\left|\frac{\tilde{\pi}_-'}{\tilde{\pi}_-}\right|$ and then we get from (7)

$$\mp\tilde{D}^\pm(x,x') = \int_{\Gamma^c} f_E(x,x',s)ds, \quad \pm y_- < 0, \quad \text{(see fig.1)} \tag{II}$$

where

$$f_E(x,x',s) = \frac{e^{ie\chi_{II}}}{(4\pi)^2} eEs^{-1}sh^{-1}eEs\, exp\{-im^2 s -$$
$$-i\frac{eE}{4}(y_0^2 - y_3^2)\, cth\, eEs - i\frac{y_\perp^2}{4s}\}, \quad \chi_{II} = -\frac{E}{4}y_+(x_- + x_-'). \tag{I2}$$

For $\mp y_- < 0$, $S = \frac{1}{2eE}\left(\ln\left|\frac{\tilde{\pi}_-'}{\tilde{\pi}_-}\right| - i\pi\theta(\pm\tilde{\pi}_-')\right)$ and

$$\pm\tilde{D}^\pm(x,x') = \int_{C_1+C_2} f_E(x,x',s)ds, \quad \mp y_- < 0 \quad \text{(see fig.2)} \tag{I3}$$

By combining (II), (I3) and replacing the contour $C_1 + C_2$ with the $\Gamma^c - \Gamma$ (see fig.1 and fig.3) we get

$$\mp\tilde{D}^\pm(x,x') = \int_{\Gamma^c} f_E(x,x',s)ds - \theta(\pm y_-)\int_{\Gamma} f_E(x,x',s)ds . \tag{I4}$$

Let us now consider the function $\tilde{D}^a(x,x')$. If $y_- < 0$, then $S = \frac{1}{2eE}\left(\ln\left|\frac{\tilde{\pi}_-'}{\tilde{\pi}_-}\right| - 2i\pi\right)$ and then we obtain from (8)

$$\tilde{D}^a(x,x') = \int_{\Gamma^a} f_E(x,x',s)ds, \quad y_- < 0 \quad \text{(see fig.1)} \tag{I5}$$

For $y_- > 0$, $S = \frac{1}{2eE}\left(\ln\left|\frac{\pi_-'}{\pi_-}\right| - i\pi - i\pi\theta(-\pi_-')\right)$ then

$$\tilde{D}^a(x,x') = \int_{C_2+C_3} f_E(x,x',s)\,ds, \quad y_- > 0 \qquad \text{(see fig.2)} \qquad (16)$$

By combining (15), (16) and replacing the contour $C_2 + C_3$ with the $\Gamma^a + \Gamma_1^a$ (see fig.1 and fig.3) we get (see also (26))

$$\tilde{D}^a(x,x') = \int_{\Gamma^a} f_E(x,x',s)\,ds + \theta(y_3)\int_{\Gamma_1^a} f_E(x,x',s)\,ds. \qquad (17)$$

By evaluating the functions (14), (17) we supposed $y_- \neq 0$, therefore we should make sure that these functions with arbitrary x, x' really satisfy the K-G equation.

The function $f_E(x,x',s)$ (12) coincides with the well-known (Fock, 1937; Schwinger, 1951) solution of the K-G equation with the proper time:

$$i\partial_s f_E(x,x',s) = -(\mathcal{P}^2(x) - m^2) f_E(x,x',s) \qquad (18)$$

Then by using the representations of the delta functions

$$\lim_{s\to\pm 0} \frac{1}{\sqrt{s}} e^{\frac{i(x_\mu - x_\mu')^2}{as}} = \sqrt{a\pi}\, e^{\pm i\frac{\pi}{4}} \delta(x_\mu - x_\mu'), \quad a > 0 \qquad (19)$$

we get

$$(\mathcal{P}^2(x) - m^2)\int_{\Gamma^c} f_E(x,x',s)\,ds = -\delta^{(4)}(x-x'), \qquad (20)$$

$$(\mathcal{P}^2(x) - m^2)\int_{\Gamma} f_E(x,x',s)\,ds = -2\delta^{(4)}(x-x'), \qquad (21)$$

$$(\mathcal{P}^2(x) - m^2)\int_{\Gamma^a} f_E(x,x',s)\,ds = 2i\delta(y_-)\delta(y_+) f_E^{\perp}(x,x') = \qquad (22)$$

$$= i\delta(y_0)\delta(y_3) f_E^{\perp}(x,x'),$$

$$(\mathcal{P}^2(x) - m^2)\int_{\Gamma_1^a} f_E(x,x',s)\,ds = -4i\,\delta(y_-)\delta(y_+) f_E^{\perp}(x,x') = \qquad (23)$$

$$= -2i\,\delta(y_0)\delta(y_3) f_E^{\perp}(x,x'),$$

$$f_E^{\perp}(x,x') = -i\frac{eE}{4\pi^2}\exp\left\{-\pi\frac{m^2}{eE} + \frac{eE}{4\pi} y_\perp^2\right\} \qquad (24)$$

By evaluating (21)-(23) we separated the singularities in the integrals over the contours Γ and Γ_1^a explicitly:

$$\int_{\Gamma} f_E(x,x',s)\,ds = \theta(y^2) R(x,x') + \frac{e^{iex_{\shortparallel}}}{2\pi}\delta(y^2), \qquad (25)$$

$$\int_{\Gamma_1^a} f_E(x,x',s)ds = \theta(y_3^2-y_0^2)\left[\theta(y_0-0)+\theta(-y_0-0)\right]\int_{\Gamma_R^a} f_E(x,x',s)ds$$

where (26)

$$R(x,x') = \int_{\Gamma_R} f_E(x,x',s)ds \tag{27}$$

is the Riemann function in the Fock representation (Fock, 1937) and contours Γ_R, Γ_R^a are given in fig.4. The equality (25) is valid obviously for $y^2 \neq 0$ and equalities (26) are for $y_+ \neq 0$, $y_- \neq 0$. Moreover, for $y_0 \to \pm 0$ we have

$$\begin{aligned} \theta(y^2) &= \frac{4\pi}{3}|y_0|^3\delta^{(3)}(\vec{y}), \\ \delta(y^2) &= 2\pi\left[\theta(y_0-0)+\theta(-y_0-0)\right]|y_0|\delta^{(3)}(\vec{y}). \end{aligned} \tag{28}$$

By taking (25) (26) into account, we represent (14) (17) in the form

$$\mp\tilde{D}^{\pm}(x,x') = \int_{\Gamma^c} f_E(x,x',s)ds - \theta(\pm y_0)\int_{\Gamma} f_E(x,x',s)ds. \tag{29}$$

Now it is easy to show, by using the relations (20)-(23) and (26), that the functions (17), (29) satisfy the K-G equation for arbitrary x, x' and therefore they are just the functions we have introduced in (7), (8).

In conclusion of this section we will note, that by using the representations (29) one can obtain the commutator function

$$\tilde{D}(x,x') = \tilde{D}^-(x,x') + \tilde{D}^+(x,x') = \varepsilon(y_0)\int_{\Gamma} f_E(x,x',s)ds \tag{30}$$

which, as it is clear from (25), (28), satisfies the conditions (1.6), and the Green function $\tilde{D}^c(x,x')$

$$\tilde{D}^c(x,x') = \int_{\Gamma^c} f_E(x,x',s)ds \tag{31}$$

which satisfies the equation (1.5a). The representations (30), (31) coincide with Fock (Fock, 1937) and Schwinger (Schwinger, 1951) representations respectively. These results have been obtained in (Gavrilov, Gitman, Shwartsman, 1979a).

§3. Constant electric field. Spinor case

Let us show that the evaluation of the Green functions in QED with a constant and uniform electric field reduces on the whole to actions we have already performed in the scalar case.

The complete sets of the solutions of the Dirac equation with the vector potential $A_0^{ext} = A_3^{ext} = \frac{eEx_-}{2}$, $A_1^{ext} = A_2^{ext} = 0$ which are similar to (2.1), (2.2) and have a semiclassical form, are:

$$ {}_{+}^{-}\varphi_{p_-p_1p_2\tau}(x) = [\gamma^0\, {}_{+}^{-}\Phi^{-1}_{p_-p_1p_2}(x) \mp (\gamma_\perp p_\perp + m)\, {}_{+}^{-}\Phi^{+1}_{p_-p_1p_2}(x)]\, u_\tau , \tag{1}$$

$$ {}_{-}^{+}\varphi_{p_-p_1p_2\tau}(x) = [\gamma^0\, {}_{-}^{+}\Phi^{-1}_{p_-p_1p_2}(x) \pm (\gamma_\perp p_\perp + m)\, {}_{-}^{+}\Phi^{+1}_{p_-p_1p_2}(x)]\, u_\tau , \tag{2}$$

$$ {}^{+}\Phi^{\ell}_{p_-p_1p_2}(x) = \theta(\pi_-)\, d_{p_\perp^2}\, {}_{+}\Phi^{\ell}_{p_-p_1p_2}(x), \quad {}_{-}\Phi^{\ell}_{p_-p_1p_2}(x) = \theta(-\pi_-)\, d_{p_\perp^2}\, {}^{-}\Phi^{\ell}_{p_-p_1p_2}(x), $$

$$ {}_{+}^{-}\Phi^{\ell}_{p_-p_1p_2}(x) = \frac{1}{\sqrt{2}}(2\pi)^{-3/2}(eE)^{-\frac{1+\ell}{4}} \exp\left\{-i\frac{p_-x_+}{2} + \nu_\ell \ln(\mp\tilde{\pi}_-) - ip_\perp x_\perp\right\}, \tag{3}$$

$$ \nu_\ell = i\frac{\lambda}{2} - \frac{1+\ell}{2}, \quad d_{p_\perp^2} = \sqrt{2\pi}\,\Gamma^{-1}\left(\frac{i\lambda}{2}\right) e^{-\frac{\pi\lambda}{4}} \left(\frac{\lambda}{2}\right)^{-\frac{1}{2}}, \quad \ell = \pm 1, \ \tau = \pm 1, $$

$$ u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \quad u_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}. $$

In (Narozhny, Nikishov, 1976) it is shown, that the following conditions are valid

$$ {}_{\pm}^{\pm}\Phi^{\ell}_{\vec{p}}(x) = (2\pi eE)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} M(p_3, p_-)\, {}_{\pm}^{\pm}\Phi^{\ell}_{p_-p_1p_2}(x)\, dp_- , \tag{4}$$

$$ {}_{+}^{-}\Phi^{\ell}_{\vec{p}}(x) = {}_{+}^{-}B_\ell\, e^{i\vec{p}\vec{x}} \cdot D_{\nu_\ell}[\pm(1-i)\xi], \quad {}_{-}^{+}\Phi^{\ell}_{\vec{p}}(x) = {}_{-}^{+}B_\ell\, e^{i\vec{p}\vec{x}} D_{-1-\nu_\ell}[\pm(1+i)\xi], \tag{5}$$

$$ {}_{+}^{-}B_\ell = i(2eE)^{-\frac{\ell}{4}} e^{-i\frac{\pi\ell}{8}} B, \quad {}_{-}^{+}B_\ell = i(2eE)^{-\frac{\ell}{4}} e^{-i\frac{\pi\ell}{8}} \left(\frac{\lambda}{2}\right)^{-\frac{\ell}{2}} B. $$

By using (2.8) we get from (4) the inverse relation, similar to (2.3)

$$ {}_{\pm}^{\pm}\Phi^{\ell}_{p_-p_1p_2}(x) = (2\pi eE)^{-\frac{1}{2}} \int_{-\infty}^{\infty} M^*(p_3, p_-)\, {}_{\pm}^{\pm}\Phi^{\ell}_{\vec{p}}(x)\, dp_3 . \tag{6}$$

The solutions of the Dirac equation

$$ \begin{aligned} {}_{+}^{-}\varphi_{\vec{p}\tau}(x) &= [\gamma^0\, {}_{+}^{-}\Phi^{-1}_{\vec{p}}(x) \mp (\gamma_\perp p_\perp + m)\, {}_{+}^{-}\Phi^{+1}_{\vec{p}}(x)]\, u_\tau , \\ {}_{-}^{+}\varphi_{\vec{p}\tau}(x) &= [\gamma^0\, {}_{-}^{+}\Phi^{-1}_{\vec{p}}(x) \pm (p_\perp\gamma_\perp + m)\, {}_{-}^{+}\Phi^{+1}_{\vec{p}}(x)]\, u_\tau . \end{aligned} \tag{7}$$

are classified (Narozhny, 1968) with respect to the energy sign of asymptotically free solutions, which in the slow changing field limit coincide with (7). Thus in the spinor case the classification (1) with respect to the sign of the kinetic momentum $\tilde{\pi}_-$ under $x_- \to \pm\infty$ is equivalent to the usual classification. Since the solutions (7) are orthonormal, we get from (6), (2.5) that the solutions (1), (2) are orthonormal too:

$$\left({}^{\pm}_{\pm}\varphi_{p_- p_1 p_2 r},\ {}_{\pm}^{\pm}\varphi_{p_-' p_1' p_2' r'}\right)_D = \delta(p_- - p_-')\delta^{(2)}(p_\perp - p_\perp')\delta_{rr'}, \tag{8a}$$

$$\left({}^{+}_{+}\varphi_{p_- p_1 p_2 r},\ {}^{-}_{-}\varphi_{p_-' p_1' p_2' r'}\right)_D = 0 . \tag{8b}$$

Taking into account (Nikishov, I969)

$$\omega(p_- \overset{\pm}{p_1} p_2 r | p_-' \overset{\pm}{p_1'} p_2' r') = \delta(p_- - p_-')\delta^{(2)}(p_\perp - p_\perp')\delta_{rr'} d^{-1}_{p_\perp^2},$$

$$\omega(0 | p_- \overset{-}{p_1} p_2 r, p_-' \overset{+}{p_1'} p_2' r') = -\delta(p_- - p_-')\delta^{(2)}(p_\perp - p_\perp')\delta_{rr'} d^{-1}_{p_\perp^2}\, e^{-\frac{\pi\lambda}{2}}$$

for the Green functions (I.1.44), (I.2.II), which are constructed with the aid of the solutions (1), (2), we will get:

$$\tilde{S}^-(x,x') = \int_{eEx_-}^{+\infty} {}_+\tilde{g}_E(x,x',p_-)dp_-; \quad \tilde{S}^+(x,x') = \int_{-\infty}^{eEx_-} {}^-\tilde{g}_E(x,x',p_-)dp_-, \tag{9}$$

$$\tilde{S}^a(x,x') = -\int_{-\infty}^{eEx_-} {}_+\tilde{g}_E(x,x',p_-)dp_-, \tag{I0}$$

$${}^-_+\tilde{g}_E(x,x',p_-) = i\sum_r \int_{-\infty}^{+\infty} dp_1 dp_2\ {}^-_+\varphi_{p_- p_1 p_2 r}(x)\ {}^-_+\bar{\varphi}_{p_- p_1 p_2 r}(x'). \tag{II}$$

Let us calculate the sum in (II)

$$\sum_r u_r u_r^+ = \frac{1+\alpha_3}{2}, \tag{I2}$$

and note that

$$[\gamma^0 + (\gamma_\perp p_\perp + m)\frac{1}{\tilde{\pi}_-}]e^{i\frac{\lambda}{2}\ln(\pm\tilde{\pi}_-)}\frac{1+\alpha_3}{2}(m - \gamma_\perp p_\perp)\gamma^0 =$$
$$= [\gamma^0 2i\frac{\partial}{\partial x_-} + (\gamma_\perp p_\perp + m)]\frac{1-\alpha_3}{2}e^{i\frac{\lambda}{2}\ln(\pm\tilde{\pi}_-)}. \tag{I3}$$

Since

$$\gamma^0 \mathcal{P}_0(x) + \gamma^3\mathcal{P}_3(x) = \frac{\gamma^0+\gamma^3}{2}(2i\frac{\partial}{\partial x_+} - eEx_-) + \frac{\gamma^0-\gamma^3}{2}2i\frac{\partial}{\partial x_-}, \tag{I4}$$

one can express (11) through (2.9) by using (12), (13):

$$ {}^{-}_{+}\tilde{g}_E(x,x',p_-) = \mp(\hat{\mathcal{P}}(x)+m)\Big[\frac{1+\alpha_3}{2}e^{{}^{-}_{+}aeE} + \frac{1-\alpha_3}{2}e^{-{}^{-}_{+}aeE}\Big]{}^{-}_{+}\tilde{f}_E(x,x',p_-), \tag{15}$$

$$ {}^{-}_{+}a = \frac{1}{2eE}\left[(\ln(\mp\tilde{\pi}'_-))^* - \ln(\mp\tilde{\pi}_-)\right]. \tag{16}$$

By making the substitution of variable $s = {}^{-}_{+}a$ and integrating over p_1 and p_2 one can use completely the results of the preceding section:

$$\begin{aligned} &\tilde{S}^{\pm}(x,x') = (\hat{\mathcal{P}}(x)+m)\tilde{\Delta}^{\pm}(x,x'),\\ &\mp\tilde{\Delta}^{\pm}(x,x') = \int_{\Gamma^c} g_E(x,x',s)ds - \theta(\pm y_0)\int_{\Gamma} g_E(x,x',s)ds, \end{aligned} \tag{17}$$

$$\begin{aligned} &\tilde{S}^{a}(x,x') = (\hat{\mathcal{P}}(x)+m)\tilde{\Delta}^{a}(x,x'),\\ &\tilde{\Delta}^{a}(x,x') = \int_{\Gamma^a} g_E(x,x',s)ds + \theta(y_3)\int_{\Gamma^a_1} g_E(x,x',s)ds, \end{aligned} \tag{18}$$

$$ g_E(x,x',s) = e^{eEs\alpha_3} f_E(x,x',s). \tag{19}$$

Here $g_E(x,x',s)$ coincides with the well-known (Fock, 1937; Schwinger, 1951) solution of the Dirac equation with the proper time:

$$ i\partial_s g_E(x,x',s) = -(\hat{\mathcal{P}}^2(x) - m^2)g_E(x,x',s). \tag{20}$$

Since the differential part in $\hat{\mathcal{P}}^2(x) = \mathcal{P}^2(x) - ieE\alpha_3$ is the same as in the scalar case, all the reasons prooving the validity of the representations (17), (18) for arbitrary x, x' are perfectly the same.

The anticommutator function

$$ \tilde{G}(x,x') = -i(\hat{\mathcal{P}}(x)+m)\tilde{\Delta}(x,x'), \quad \tilde{\Delta}(x,x') = \varepsilon(y_0)\int_{\Gamma} g_E(x,x',s)ds \tag{21}$$

constructed by means of the representations (17), (18) satisfies the condition (I.1.25), and the Green function

$$ \tilde{S}^{c}(x,x') = (\hat{\mathcal{P}}(x)+m)\tilde{\Delta}^{c}(x,x'), \quad \tilde{\Delta}^{c}(x,x') = \int_{\Gamma^c} g_E(x,x',s)ds, \tag{22}$$

satisfies the equation (I.1.45).

§4. Combination of a constant field and a plane wave field. Solutions of the Klein-Gordon and Dirac equations

Let us obtain two complete and orthonormal sets $\{{}_{\pm}\varphi_\kappa(x)\}$, $\{{}^{\pm}\varphi_\kappa(x)\}$ of semiclassical form solutions of the K-G and Dirac equations in the field

$$\mathcal{F}_{\mu\nu}(x) = F_{\mu\nu} + f_{\mu\nu}(nx), \qquad (1)$$

where F is a constant and uniform field, moreover the invariants $\mathcal{F} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ and $\mathcal{G} = -\frac{1}{4} F^*_{\mu\nu} F^{\mu\nu}$ $(F^*_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta})$ do not vanish together, and n is the isotropic vector $(n^2 = 0)$ satisfying the equation $F_{\mu\nu}n^\nu = \mathcal{E} n_\mu$, $\mathcal{E} = [\sqrt{\mathcal{F}^2 + \mathcal{G}^2} - \mathcal{F}]^{\frac{1}{2}}$ and $f(nx)$ is the transverse plane wave field: $n^\mu f_{\mu\nu}(nx) = f_{\mu\nu}(nx) n^\nu = 0$.

Let us choose the reference frame in which the electric and the magnetic fields from F are collinear, then they have the same direction as the space part of the propagation vector $\vec{n}$; we will choose this direction as the direction of the unit vector $\vec{e}_3$:

$$F_{\mu\nu} = F^{\perp}_{\mu\nu} + F^{\parallel}_{\mu\nu}, \quad F^{\perp}_{\mu\nu} = H(\delta^2_\mu \delta^1_\nu - \delta^1_\mu \delta^2_\nu),$$
$$F^{\parallel}_{\mu\nu} = E(\delta^0_\mu \delta^3_\nu - \delta^3_\mu \delta^0_\nu), \quad f_{\mu\nu}(nx) = \sum_{\kappa=1}^{2} f^{\kappa}_{\mu\nu} \dot{A}_\kappa(x_-), \qquad (2)$$
$$f^{\kappa}_{\mu\nu} = n_\mu \delta^\kappa_\nu - n_\nu \delta^\kappa_\mu, \quad \dot{A}_\kappa = \frac{dA_\kappa(x_-)}{dx_-}, \quad (nx) = x_- .$$

The solutions of the semiclassical form of the K-G and Dirac equations in the field (2) with the vector potential

$$A^{ext}_\mu(x) = A^E_\mu(x) + A^H_\mu(x) + A_\mu(x_-),$$
$$A^E_\mu(x) = \frac{Ex_-}{2}(1, 0, 0, 1), \quad A^H_\mu(x) = Hx_2 g_{\mu 1}, \qquad (3)$$
$$A_\mu(x_-) = (0, A_1(x_-), A_2(x_-), 0)$$

are obtained in (Bagrov, Gitman, Jushin, 1976) (see also (Borgardt, Karpenko, 1974)). We represent them in the following form: the solutions of the K-G equation are[2)]

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2) Here and elsewhere we will often omit the matrix indices what corresponds to the matrix notation, for example

$$A(x_-) F K(x_-) = A^\mu(x_-) F_{\mu\nu} K^\nu(x_-)$$

$$\mathop{\varphi}\limits^{-}_{+}{}_{p_-p_1n}(x)=(eE)^{-\frac{1}{4}}e^{-\frac{1}{2}\ln(\mp\tilde{\pi}_-)}\mathop{\psi}\limits^{-}_{+}{}_{p_-n}(x)\psi_{p_1n}(x_\perp), \tag{4a}$$

and the solutions of the Dirac equation are

$$\mathop{\psi}\limits^{-}_{+}{}_{p_-p_1n\tau}(x)=[\gamma^0+(\gamma_\perp\mathcal{P}_\perp(x)+m)\frac{1}{\tilde{\pi}_-}]e^{-\frac{i}{2}\frac{H\tau}{E}\ln(\mp\tilde{\pi}_-)}u_\tau\times$$

$$\times\mathop{\psi}\limits^{-}_{+}{}_{p_-n}(x)\psi_{p_1n}(x_\perp),$$

$$\mathop{\psi}\limits^{-}_{+}{}_{p_-n}(x)=(4\pi)^{-\frac{1}{2}}\exp\{-i\frac{p_-x_-}{2}+\frac{i\lambda_n}{2}\ln(\mp\tilde{\pi}_-)+i\mathop{J}\limits^{-}_{+}(x_-)-i\mathop{K}\limits^{-}_{+}(x_-)\pi_\perp\}, \tag{4b}$$

$$\psi_{p_1n}(x_\perp)=\left(\frac{\sqrt{|eH|}}{2^{n+1}\pi^{3/2}n!}\right)^{\frac{1}{2}}e^{-ip_1x^1}\exp\{-\frac{|eH|}{2}(x_2-\frac{p_1}{eH})^2\}H_n\left[\sqrt{|eH|}\left(\frac{p^1}{eH}+x^2\right)\right],$$

$$\pi_\mu^\perp=i\partial_\mu^\perp-eA_\mu^H(x),\quad \mathcal{P}_\mu^\perp(x)=\pi_\mu^\perp-eA_\mu(x_-),\quad \lambda_n=\frac{m^2+|eH|(2n+1)}{eE},$$

$$\mathop{J}\limits^{-}_{+}(x_-)=-\frac{1}{2eE}\int_{\mp\infty}^{\tilde{\pi}_-}eA\left(\frac{p_--\tau}{eE}\right)\left[eA\left(\frac{p_--\tau}{eE}\right)+eF\mathop{K}\limits^{-}_{+}\left(\frac{p_--\tau}{eE}\right)\right]\tau^{-1}d\tau,$$

$$\mathop{K}\limits^{-}_{+}(x_-)=-\frac{1}{eE}\int_{\mp\infty}^{\tilde{\pi}_-}\exp\left\{-\frac{F}{E}\ln\frac{\tilde{\pi}_-}{\tau}\right\}eA\left(\frac{p_--\tau}{eE}\right)\tau^{-1}d\tau.$$

Here p_-, p_1, n are the eigenvalues of operators which are integrals of the motion, moreover $-\infty<p_-,\ p_1<+\infty$, $n=0,1,2,\ldots$ The signes "$\pm$" in (4) correspond to the sign of the kinetic momentum $\tilde{\pi}_-$ under $x_-\to\pm\infty$ $(eE\gtrless 0)$. We define the solutions (4) as analytic functions of x_- what is valid if $A(x_-)$ is a piecewise continuous function and the integral over τ converges uniformly. Since we always suppose a rather intense decay of the plane wave under $x_-\to\pm\infty$, these conditions hold. We will choose the principal value of the logarithmic function in the same way as in the preceding sections, that is so that $\ln(\mp\tilde{\pi}_-)=\ln|\tilde{\pi}_-|+i\pi\theta(\pm\tilde{\pi}_-)$. The contours in the integrals over τ and the arguments $\tilde{\pi}_-$ are shown in fig.5.

The solutions (4) may be expressed through the solutions of the other form like in the case we were concerned with the electric field only:

$$\mathop{\varphi}\limits^{-}_{+}{}_{p_-}(x)=(2\pi eE)^{-\frac{1}{2}}\int_{-\infty}^{+\infty}M^*(p_3,p_-)\mathop{\psi}\limits^{-}_{+}{}_{p_3}(x)dp_3, \tag{5}$$

where

$$ {}_{+}^{-}\varphi_{p_3}(x) = (2\pi eE)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} M(p_3, p_-)\, {}_{+}^{-}\varphi_{p_-}(x)\, dp_- \, . \tag{6} $$

Here the quantum numbers indices $p_1 n$, $p_1 n \tau$ are omitted to emphasize the fact that the integral transformations (5), (6) are the same both for the solutions of the K-G equation and the Dirac equation. The saddle points $\tilde{\pi}_- = -2\xi$ give the main contribution to the integrals (6) under $x_0 \to \pm\infty$ (compare with (Narozhny, Nikishov, 1976)). Since the plane wave vanishies from ${}_{+}^{-}\varphi_{p_-}(x)$ under $\tilde{\pi}_- \to \pm\infty$ the solutions ${}_{+}^{-}\varphi_{p_3}(x)$ reduce to the already studied (2.4), (3.7) under $x_0 \to \pm\infty$ [3]. Thus ${}_{+}\varphi_{p_-}(x)$ $({}^{-}\varphi_{p_-}(x))$ from (4) do describe the particle (the antiparticle) under $x_0 \to -\infty$ $(x_0 \to +\infty)$. The normalizing integrals for the solutions (4) do not depend on the x_0 and therefore they may be evaluated in the same way as the normalizing integrals for the solutions (2,1), (3.1):

$$ \begin{gathered} \left({}_{+}^{-}\varphi_{p_- p_1 n}, {}_{+}^{-}\varphi_{p_-' p_1' n'} \right)_{K-\Gamma} = \mp \delta(p_- - p_-')\delta(p_1 - p_1')\delta_{nn'}, \\ \left({}_{+}^{-}\varphi_{p_- p_1 n \tau}, {}_{+}^{-}\varphi_{p_-' p_1' n' \tau'} \right)_{D} = \delta(p_- - p_-')\delta(p_1 - p_1')\delta_{nn'}\delta_{\tau\tau'} . \end{gathered} \tag{7} $$

In the electric field the solutions (2.1), (3.1) form a complete set: ${}^{-}\varphi_{p_-}(x)$ if $\tilde{\pi}_- < 0$ and ${}_{+}\varphi_{p_-}(x)$ if $\tilde{\pi}_- > 0$ that is for the solutions (2.2), (3.2) we have

$$ {}^{+}\varphi_{p_-}(x) = 0, \quad \tilde{\pi}_- < 0\, ; \quad {}_{-}\varphi_{p_-}(x) = 0, \quad \tilde{\pi}_- > 0 . \tag{8} $$

The plane wave and the magnetic field are not capable to violate this characteristic of the solutions of the semiclassical form, therefore we will assume that the solutions (2.2), (3.2) describing the particle under $x_0 \to +\infty$ and the antiparticle under $x_0 \to -\infty$ in the field (2) also satisfy the conditions (8). We could verify this assumption by straightforward calculations with the functions (4), but it is unnecessary at this stage since the corresponding proof will be given in the frame of the general proof of the completeness of the sets being used, after the Green functions $\tilde{D}(x,x')$, $\tilde{G}(x,x')$ are constructed. To define completely the functions $\{{}_{\pm}\varphi_{\kappa}(x)\}$, $\{{}^{\pm}\varphi_{\kappa}(x)\}$ we will use the relations,

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3) The presence of the magnetic field together with the electric field results in the substitution $(2\pi)^{-1} e^{-ip_1 x_1} \to \varphi_{p_1 n}(x_\perp)$, $\gamma_1 p_1 \to \gamma_\perp \pi_\perp$, $\lambda \to \lambda_{n\tau} = \frac{m^2 + |eH|(2n+1) - eH\tau}{eE}$, where $\tau = \pm 1$ for the solutions of the Dirac equation and $\tau = 0$ for the solutions of the K-G equation.

which are valid for the complete and orthonormal sets of solutions

$$ {}_+\varphi(x) = {}^+\varphi(x)G({}^+|_+) + {}^-\varphi(x)G({}^-|_+), \quad {}^+\varphi(x) = {}_+\varphi(x)G({}_+|^+) + \eta\,{}_-\varphi(x)G({}_-|^+), \tag{9} $$

$$ {}_-\varphi(x) = {}^+\varphi(x)G({}^+|_-) + {}^-\varphi(x)G({}^-|_-), \quad {}^-\varphi(x) = \eta\,{}_+\varphi(x)G({}_+|^-) + {}_-\varphi(x)G({}_-|^-), $$

where $\eta = +1$ for the solutions of the Dirac equation, $\eta = -1$ for the solutions of the K-G equation, and the coefficient matrices $G({}^\pm|_\pm) = G^+({}_\pm|^\pm)$ satisfy the conditions

$$ G(\pm|^+)G({}^+|_\pm) + \eta G(\pm|^-)G({}^-|_\pm) = (\pm 1)^{\frac{1-\eta}{2}} I, $$

$$ G({}^\pm|_+)G({}_+|^\pm) + \eta G({}^\pm|_-)G({}_-|^\pm) = (\pm 1)^{\frac{1-\eta}{2}} I, \tag{10} $$

$$ G({}_+|^+)G({}^+|_-) + \eta G({}_+|^-)G({}^-|_-) = G({}^+|_+)G({}_+|^-) + \eta G({}^+|_-)G({}_-|^-) = 0 . $$

It is obvious from the explicit form of the functions (4) that the conditions (9) are diagonal at any rate with respect to the quantum number p_- what makes it possible by using (8) and (9) to get the solutions of the K-G equation

$$ {}^+\varphi_{p_-p_1n}(x) = \theta(\pi_-)[{}_+\varphi(x)G({}_+|^+)]_{p_-p_1n}\,; \; {}_-\varphi_{p_-p_1n}(x) = \theta(-\pi_-)[{}^-\varphi(x)G({}^-|_-)]_{p_-p_1n}\,, \tag{11a} $$

and the solutions of the Dirac equation

$$ {}^+\varphi_{p_-p_1n\tau}(x) = \theta(\pi_-)[{}_+\varphi(x)\,G({}_+|^+)]_{p_-p_1n\tau}\,, \tag{11b} $$

$$ {}_-\varphi_{p_-p_1n\tau}(x) = \theta(-\pi_-)[{}^-\varphi(x)\,G({}^-|_-)]_{p_-p_1n\tau} . $$

Since below we will make sure that the solutions ${}^+\varphi_{p_-}$ (${}_-\varphi_{p_-}$) form a complete set over the $x_0 = const$ together with the solutions ${}^-\varphi_{p_-}$ (${}_+\varphi_{p_-}$) and together with them only, the orthonormality relations are valid:

$$ ({}^+_+\varphi_{p_-p_1n},\, {}^-_-\varphi_{p'_-p'_1n'})_{K-G} = 0, \quad ({}^+_+\varphi_{p_-p_1n\tau},\, {}^-_-\varphi_{p'_-p'_1n'\tau'})_D = 0 \tag{12} $$

The orthonormality of the solutions (11) follows even from the properties of the coefficients $G({}^\pm|_\pm)$ (10) by taking into account (9), (7), (12):

$$ ({}^+_-\varphi_{p_-p_1n},\, {}^+_-\varphi_{p'_-p'_1n'})_{K-G} = \pm\delta(p_- - p'_-)\delta(p_1 - p'_1)\delta_{nn'}, \tag{13} $$

$$ ({}^+_-\varphi_{p_-p_1n\tau},\, {}^+_-\varphi_{p'_-p'_1n'\tau'})_D = \delta(p_- - p'_-)\delta(p_1 - p'_1)\delta_{nn'}\delta_{\tau\tau'} . $$

§5. Combination of a constant field and a plane wave field. Green functions

By using the complete sets of the solutions of the K-G and Dirac equations, which are obtained in Sec.4, we will get all the Green functions for the scalar and the spinor QED, which were mentioned in Sec.1 for the field (4.1).

The calculation method of the Green functions in this field does not differ from the preceding case. Therefore we will give the final results only:

$$\tilde{S}^{\pm}(x,x') = (\hat{\mathcal{P}}(x)+m)\tilde{\Delta}^{\pm}(x,x'),$$
$$\mp\tilde{\Delta}^{\pm}(x,x') = \int_{\Gamma^c} g(x,x',s)ds - \theta(\pm y_0)\int_{\Gamma} g(x,x',s)ds\,; \tag{1}$$

$$\tilde{S}^{c}(x,x') = (\hat{\mathcal{P}}(x)+m)\tilde{\Delta}^{c}(x,x'),$$
$$\tilde{\Delta}^{c}(x,x') = \int_{\Gamma^c} g(x,x',s)ds\,; \tag{2}$$

$$\tilde{S}^{a}(x,x') = (\hat{\mathcal{P}}(x)+m)\hat{\Delta}^{a}(x,x'), \tag{3}$$
$$\tilde{\Delta}^{a}(x,x') = \int_{\Gamma_{in}} g(x,x',s)ds + \theta(y_3)\int_{\Gamma_1^a} g(x,x',s)ds;$$

$$\tilde{\tilde{S}}^{\pm}(x,x') = (\hat{\mathcal{P}}(x)+m)\tilde{\tilde{\Delta}}^{\pm}(x,x'), \tag{4}$$
$$\mp\tilde{\Delta}^{\pm}(x,x') = \int_{\Gamma_{in}} g(x,x',s)ds - \theta(\pm y_0)\int_{\Gamma} g(x,x',s)ds - \theta(y_3)\int_{\Gamma_1^a} g(x,x',s)ds;$$

$$\tilde{\tilde{S}}^{c}(x,x') = (\hat{\mathcal{P}}(x)+m)\tilde{\tilde{\Delta}}^{c}(x,x'), \tag{5}$$
$$\tilde{\tilde{\Delta}}^{c}(x,x') = \int_{\Gamma_{in}} g(x,x',s)ds - \theta(y_3)\int_{\Gamma_1^a} g(x,x',s)ds;$$

$$\tilde{\tilde{S}}^{\bar{c}}(x,x') = (\hat{\mathcal{P}}(x)+m)\tilde{\tilde{\Delta}}^{\bar{c}}(x,x'), \tag{6}$$
$$\tilde{\tilde{\Delta}}^{\bar{c}}(x,x') = \int_{\Gamma_{in}} g(x,x',s)ds - \theta(y_3)\int_{\Gamma_1^a} g(x,x',s) - \int_{\Gamma} g(x,x',s)ds;$$

$$\tilde{G}(x,x') = -i(\hat{\mathcal{P}}(x)+m)\tilde{\Delta}(x,x'), \quad \tilde{\Delta}(x,x') = \varepsilon(y_0)\int_{\Gamma} g(x,x',s)ds\,; \tag{7}$$

where

$$g(x,x',s) = \Big[\exp\{\tfrac{1}{2}eF_{\mu\nu}\sigma^{\mu\nu}s\} + (\gamma^0-\gamma^3)\gamma e^{eFs} \times$$
$$\times \int_0^s e^{-2eFu} e \frac{dA(x_-(u))}{du} du \cdot \frac{\operatorname{sh} eEs}{eEy_-}\Big] f(x,x',s); \tag{8}$$

$$f(x,x',s)=\frac{e^{ie\chi}}{(4\pi)^2}\cdot\frac{e^2EH}{\sin(eHs)\,\mathrm{sh}\,eEs}\cdot\exp\{-im^2s+i\Phi(s)+$$

$$+\frac{i}{2}y_\perp eFi(s)-i\frac{eE}{4}(y_0^2-y_3^2)\,\mathrm{cth}\,eEs-i\frac{eH}{4}(y_\perp+i(s))^2\,\mathrm{ctg}\,eHs\};\tag{9}$$

$$\chi=-\frac{H}{2}y_1(x_2+x_2')-\frac{E}{4}y_+(x_-+x_-'),\quad \sigma^{\mu\nu}=\frac{1}{2}[\gamma^\mu,\gamma^\nu],$$

$$x_-(u)=x_-'+y_-\frac{1-e^{-2eEu}}{1-e^{-2eEs}};\quad i(s')=2\int_0^{s'}e^{2eF(s'-u)}eA(x_-(u))du;\tag{10}$$

$$\Phi(s)=\int_0^s eA(x_-(u))[eA(x_-(u))+eFi(u)]du;\quad \Gamma_1^a=C_2+C_3-\Gamma^a,$$

and the contour Γ_{in} see in fig.5.

The function $g(x,x',s)$ satisfies, as it will be shown in Sec.6, the Dirac equation with the proper time

$$i\partial_s g(x,x',s)=-(\hat{\mathcal{P}}^2(x)-m^2)g(x,x',s);$$

the function $f(x,x',s)$ satisfies the K-G equation with the proper time

$$i\partial_s f(x,x',s)=-(\mathcal{P}^2(x)-m^2)f(x,x',s).$$

The Green functions for the scalar QED one can obtain from (1)-(7) by means of the substitutions

$$\hat{\mathcal{P}}(x)+m\to 1,\quad g(x,x',s)\to f(x,x',s)$$

Note also, that the relations (2.25)-(2.27) for the functions (1)-(7) in the field (4.1) are valid too.

The function (7) satisfies the usual conditions over the arbitrary space-like surface. This verifies completeness of the set of the solutions (4.4), (4.9) over the arbitrary space-like surface.

By eliminating the plane wave from (7) we will get the expression obtained by Fock (Fock, 1937). By eliminating the plane wave or the constant field from (2) we will get the Schwinger result (Schwinger, 1951). It is also interesting to note, that

$$\theta(y_0)\tilde{S}^-(x,x')-\theta(-y_0)\tilde{S}^+(x,x')=\theta(y_-)\tilde{S}^-(x,x')-\theta(-y_-)\tilde{S}^+(x,x'),$$

that is the function $\hat{S}^c(x,x')$ in the field (4.1) is the same for the usual QED and for the QED on a null-plane (Rohrlich,1970; Kogut, Soper, 1970; Bjorken, Kogut, Soper, 1971; Neville, Rohrlich, 1971).

It is easy to represent the expressions (8), (9) in the invariant form:

$$g(x,x';s)=[\exp\{\tfrac{1}{2}F_{\mu\nu}\sigma^{\mu\nu}s\}+(n\gamma)\gamma e^{eFs}\times$$
$$\times\int_0^s e^{-2eFu}\, e\,\frac{dA(nx(u))}{du}du\cdot\frac{\operatorname{sh}e\mathcal{E}s}{e\mathcal{E}(ny)}]\, f(x,x';s); \tag{11}$$

$$f(x,x';s)=\frac{e^{ie\chi}e^2\mathcal{H}\mathcal{E}}{(4\pi)^2\operatorname{sh}e\mathcal{E}s\cdot\sin e\mathcal{H}s}\exp\{-im^2s+\frac{i}{2}yeFi(s)-$$
$$-\frac{i}{4}(y+i(s))eF\operatorname{cth}eFs(y+i(s))+i\Phi(s)\}; \tag{12}$$
$$\Phi(s)=\int_0^s eA(nx(u))[eA(nx(u))+eFi(u)]du;$$
$$i(u)=2\int_0^u e^{2e(u-u')F}eA(nx(u'))du',\quad nx(u)=nx'+ny\frac{1-e^{-2e\mathcal{E}u}}{1-e^{-2e\mathcal{E}s}};$$
$$\mathcal{E}=[\sqrt{\mathcal{F}^2+\mathcal{G}^2}-\mathcal{F}]^{1/2}\quad \mathcal{H}=[\sqrt{\mathcal{F}^2+\mathcal{G}^2}+\mathcal{F}]^{1/2},$$

where χ is the gauge function of the vector potential $A^F_\mu(x)$: $\chi=\int_{x'}^{x}A^F_\mu(x)dx^\mu$ the integral being taken along a straight line. Note, that in (Batalin, Fradkin, 1970,1971) the Green function $\tilde{S}^c(x,x')$ in the field (4.1) was obtained first in the form of a proper time integral of the function which in fact coincides with (11).

To write the Green functions (1)-(7) in the invariant form one should replace the function $g(x,x';s)$ by the expressions (11), (12), and in all the contours E by $\mathcal{E}$

§6. Operator representation of the Green functions

Let us show that the function $g(x,x';s)$ (5.8) coincides with the transformation function $i\langle x|\exp\{iS(\hat{\mathcal{P}}^2-m^2)\}|x'\rangle$, which for the field (4.2) may be obtained by the Schwinger method (Schwinger, 1951).

Consider the matrix element

$$\langle x|U(s)|x'\rangle=\langle x(s)|x'(0)\rangle,\quad U(s)=e^{is\hat{\mathcal{P}}^2}, \tag{1}$$

where the operators of momentum $\mathcal{P}_\mu$ and of coordinates X_ν satisfy the usual commutation relations:

$$[\mathcal{P}_\mu,X_\nu]=ig_{\mu\nu},\quad [\mathcal{P}_\mu,\mathcal{P}_\nu]=-ie\mathcal{F}_{\mu\nu},$$

and $|x'\rangle$ is the normalized eigenvector of X_μ:

$$X_\mu|x'\rangle=x'_\mu|x'\rangle,\quad \langle x|x'\rangle=\delta^{(4)}(x-x').$$

Let us introduce operators of momentum and coordinates in the Heisenberg picture:

$$X^{\mu}(s)=U^{-1}(s)X^{\mu}U(s),\quad X^{\mu}(0)=X^{\mu},\quad X_{\pm}=X^{0}\pm X^{3},\quad X_{\mu}^{\perp}=(0,X_1,X_2,0),$$
$$\mathcal{P}^{\mu}(s)=U^{-1}(s)\mathcal{P}^{\mu}U(s),\quad \mathcal{P}^{\mu}(0)=\mathcal{P}^{\mu},\quad \mathcal{P}_{\pm}=\mathcal{P}^{0}\pm\mathcal{P}^{3},\quad \mathcal{P}_{\mu}^{\perp}=(0,\mathcal{P}_1,\mathcal{P}_2,0),$$

which satisfy the equations

$$\frac{dX^{\mu}}{ds}=2\mathcal{P}^{\mu}(s), \tag{2}$$

$$\frac{d\mathcal{P}_{\mu}(s)}{ds}=2e\mathcal{F}_{\mu\nu}(s)\mathcal{P}^{\nu}(s)+i(\gamma^{0}(s)-\gamma^{3}(s))\gamma_{\nu}(s)\frac{\partial}{\partial X^{\mu}(s)}e\dot{A}^{\nu}(X_{-}(s)), \tag{3}$$

where $\mathcal{F}_{\mu\nu}(s)=F_{\mu\nu}+f_{\mu\nu}(X_{-}(s))$, $\gamma_{\mu}(s)=U^{-1}(s)\gamma_{\mu}U(s)$, $[\mathcal{P}^{\nu}(s),\mathcal{F}_{\nu\mu}(s)]=0$.
The function (1) satisfies the Schrödinger equation

$$i\partial_{s}\langle x(s)|x'(0)\rangle=-\langle x(s)|\hat{\mathcal{P}}^{2}|x'(0)\rangle \tag{4}$$

and the equations, connecting the operator $\mathcal{P}_{\mu}$ with it's coordinate representation:

$$\begin{aligned}&\Big(i\frac{\partial}{\partial x^{\mu}}-eA_{\mu}^{e}(x)\Big)\langle x(s)|x'(0)\rangle=\langle x(s)|\mathcal{P}_{\mu}(s)|x'(0)\rangle,\\&\Big(-i\frac{\partial}{\partial x'^{\mu}}-eA_{\mu}^{e}(x')\Big)\langle x(s)|x'(0)\rangle=\langle x(s)|\mathcal{P}_{\mu}(0)|x'(0)\rangle,\end{aligned} \tag{5}$$

and also the boundary condition

$$\lim_{s\to+0}\langle x(s)|x'(0)\rangle=\delta^{(4)}(x-x'), \tag{6}$$

where one should proceed to the limit along the real axis.

By solving of theequations (3) we get

$$\mathcal{P}_{-}(s)=e^{-2eEs}\mathcal{P}_{-}(0), \tag{7}$$

$$\mathcal{P}_{\perp}(s)=e^{2eFs}\Big[\mathcal{P}_{\perp}(0)-\int_{0}^{s}e^{-2eFu}e\,\frac{dA(X_{-}(u))}{du}du\Big], \tag{8}$$

$$\begin{aligned}\mathcal{P}_{+}(s)=e^{2eEs}\Big[\mathcal{P}_{+}(0)+\mathcal{P}_{-}^{-1}(0)\int_{0}^{s}\Big\{&2e\frac{dA(X_{-}(u))}{du}\mathcal{P}_{1}(u)+\\&+i(\gamma^{0}-\gamma^{3})\gamma e^{-2(eF+eE)u}\cdot e\frac{d\dot{A}(X_{-}(u))}{du}\Big\}du\Big],\end{aligned} \tag{9}$$

and by substituting the expressions(7-9) into the equations (2) we obtain

$$X_{-}(s)-X_{-}(0)=\frac{1}{eE}(1-e^{-2eEs})\mathcal{P}_{-}(0), \tag{10}$$

$$X_\perp(s)-X_\perp(0)=\frac{1}{eF}(e^{2eFs}-1)\mathcal{P}_\perp(0)-2\int_0^s du\int_0^u du' e^{2eF(u-u')} e\frac{dA(X_-(u'))}{du'}, \tag{11}$$

$$X_+(s)-X_+(0)=\frac{1}{eE}(e^{2eEs}-1)\mathcal{P}_+(0)+\mathcal{P}_-^{-1}(0)\int_0^s\frac{1}{eE}(e^{2eEs}-e^{2eEu})\times$$
$$\times\Big[2e\frac{dA(X_-(u))}{du}\mathcal{P}_\perp(u)+i(\gamma^0-\gamma^3)\gamma e^{-2(eF+eE)u}e\frac{d\dot{A}(X_-(u))}{du}du\Big], \tag{12}$$

from where come the commutation relations

$$[X_-(s),\, X_-(0)]=[X_-(s),\, \mathcal{P}_-(0)]=[X_-(0),\, \mathcal{P}_-(0)]=0,$$
$$[X_+(s),\, X_-(0)]=\frac{2i}{eE}(e^{2eEs}-1), \tag{13}$$
$$[X_\mu^\perp(s),\, X_\nu^\perp(0)]=i\Big[\frac{1}{eF^\perp}(e^{2eF^\perp s}-1)\Big]_{\mu\nu}.$$

If we write the operators $\mathcal{P}_\mu(0)$ in terms of coordinates $X_\mu(s)$, $X_\mu(0)$ with the aid of relations (10)-(12), we will get, by using the commutation relations (13) the following

$$\langle x(s)|\,\hat{\mathcal{P}}^2|x'(0)\rangle=\langle x(s)|x'(0)\rangle\cdot\Big\{\frac{e^2E^2}{4sh^2eEs}(y_0^2-y_3^2)+$$
$$+\frac{e^2H^2}{4\sin^2eHs}(y_\perp+\tilde{i}(s))^2+\mathcal{Z}_1(s)+\mathcal{Z}_2(s)+ieE\,cth\,eEs+ieH\,ctg\,eHs+$$
$$+eH\Sigma^3-ieE\alpha^3-i(\gamma^0-\gamma^3)\gamma e\dot{A}(x_-')\Big\}, \tag{14}$$

$$\mathcal{Z}_1(s)=2\int_0^s\Big(\frac{e^{2eEu}-1}{e^{2eEs}-1}-1\Big)e\frac{dA(x_-(u))}{du}\Big[(y_\perp+\tilde{i}(s)\frac{eFe^{2eF(s-u)}}{e^{2eFs}-1}-$$
$$-\int_0^u e^{2eF(u-u')}e\cdot\frac{dA(x_-(u'))}{du'}du'\Big]du,$$

$$\mathcal{Z}_2(s)=i(\gamma^0-\gamma^3)\int_0^s\Big(\frac{e^{2eEu}-1}{e^{2eEs}-1}-1\Big)\gamma e^{-2(eF+eE)u}e\frac{d\dot{A}(x_-(u))}{du}du,$$

$$\tilde{i}(s)=i(s)+\frac{1}{eF}(1-e^{2eFs})eA(x_-'),$$

where the functions $i(s)$, $x_-(u)$ are defined in (5.10).

One can verify that the following identities are valid:

$$\frac{e^2H^2}{4\sin^2eHs}(y_\perp+\tilde{i}(s))^2+\mathcal{Z}_1(s)=$$
$$\frac{d}{ds}\Big[-\frac{eH}{4}(y_\perp+i(s))^2ctg\,eHs+\frac{1}{2}yeFi(s)+\Phi(s)\Big], \tag{15}$$

where $\Phi(s)$ is defined in (5.11), and

$$-\omega(s)\big[eH\Sigma^3-ieE\alpha^3-i(\gamma^0-\gamma^3)\gamma e\dot{A}(x_-')+\mathcal{Z}_2(s)\big]=i\frac{d\omega(s)}{ds},$$
$$\omega(s)=exp\{(eE\alpha^3+ieH\Sigma^3)s\}+(\gamma^0-\gamma^3)\gamma e^{eFs}\times \tag{16}$$

$$\times \int_0^s e^{-2eFu} e \frac{dA(x_-(u))}{du} du \cdot \frac{sh\, eEs}{eEy_-}.$$

By substituting the expressions (15), (16) into (14) we will obtain the solution of the equation (4):

$$\langle x(s)|x'(0)\rangle = C(x,x')\,w(s)\exp\{-i\frac{eE}{4}(y_0^2-y_3^2)\,cth\, eEs -$$
$$-i\frac{eH}{4}(y_\perp+i(s))^2 ctg\, eHs + \frac{i}{2} yeFi(s) + i\,\Phi(s) - \ln(sh\, eEs\, \sin eHs)\}, \quad (17)$$

where $C(x,x')$ is an arbitrary function of x, x' which we will determine by satisfying the equations (5) and the boundary condition (6):

$$C(x,x') = -i\frac{e^2EH}{(4\pi)^2} e^{ie\chi}. \quad (18)$$

Here the function χ under an arbitrary gauge of the vector potential $A^F(x)$ of the constant field is represented as the integral along a straight line: $\chi = -\int_{x'}^{x} A^F(x)dx$ which results in the expression $\chi = \chi_\parallel + \chi_\perp$ in the case of the vector potentials $A^E(x)$, $A^H(x)$. Thus we have obtained

$$g(x,x',s) = i\langle x|\exp\{is[\hat{\mathcal{P}}^2 - m^2]\}|x'\rangle. \quad (19)$$

To calculate the radiative corrections it is often convenient to use the operator technique (Schwinger, 1973; Baier, Katkov, Strakhovenko, 1974, 1975). The starting-point of this calculation method is the representation of the Green functions as matrix elements of an operator, where the states are numbered by the space-time coordinates (Schwinger, 1951)[4]:

$$S(x,x') = \langle x|S|x'\rangle,$$
$$S = (\breve{S}^c, \tilde{\tilde{S}}^c, \tilde{\tilde{S}}^{\bar{c}}, \breve{S}^{\mp}, \breve{S}^a, \tilde{\tilde{S}}^{\mp}). \quad (20)$$

Here we will obtain the operators S for the Green functions calculated in this paper. Note, that all the obtained Green functions have the following structure

$$S(x,x') = (\hat{\mathcal{P}}(x)+m)\Delta(x,x'),$$
$$\Delta(x,x') = \sum_j \beta_j(y)\int_{\gamma_j} g(x,x',s)ds, \quad (21)$$

where β_j and γ_j are the given functions and contours, and

4) The notation of the operator has here the same indices as the corresponding Green function.

$g(x,x',s)$ is, as it has been shown, the matrix element of the operator $W(s)=exp\{is(\hat{\mathcal{P}}^2-m^2)\}$. Then the corresponding operators are

$$S=(\hat{\mathcal{P}}+m)\Delta\,,\quad \Delta=i\sum_j\int_{\gamma_j}ds\int_{-\infty}^{\infty}\beta_j(\kappa)e^{i\kappa X}W(s)e^{-i\kappa X}d^4\kappa,$$
$$\beta_j(\kappa)=(2\pi)^{-4}\int\beta_j(y)e^{-i\kappa y}d^4y. \tag{22}$$

By evaluating $\beta_j(\kappa)$ for the obtained here Green functions we will get

$$\tilde{\Delta}^c=i\int_{\Gamma^c}W(s)ds\,; \tag{23}$$

$$\tilde{\tilde{\Delta}}^c=i\int_{\Gamma_{in}}W(s)ds-\frac{1}{2\pi}\int_{\Gamma_1^a}ds\int_{-\infty}^{\infty}\frac{d\tau}{\tau-i\varepsilon}e^{i\tau\bar{n}X}W(s)\,e^{-i\tau\bar{n}X},\quad \bar{n}^\mu=(0,0,0,1) \tag{24}$$

$$\tilde{\tilde{\Delta}}^{\bar{c}}=i\int_{\Gamma_{in}}W(s)ds-\frac{1}{2\pi}\int_{\Gamma_1^a}ds\int_{-\infty}^{\infty}\frac{d\tau}{\tau-i\varepsilon}e^{i\tau\bar{n}X}W(s)e^{-i\tau\bar{n}X}-i\int_{\Gamma}W(s)ds, \tag{25}$$

$$\mp\tilde{\Delta}^{\pm}=i\int_{\Gamma^c}W(s)ds-\frac{1}{2\pi}\int_{\Gamma}ds\int_{-\infty}^{\infty}\frac{d\tau}{\tau-i\varepsilon}e^{\pm i\tau X_0}W(s)e^{\mp i\tau X_0}, \tag{26}$$

$$\tilde{\Delta}=\frac{1}{\pi}\int_{\Gamma}ds\fint_{-\infty}^{+\infty}\frac{d\tau}{\tau}\,e^{i\tau X_0}W(s)e^{-i\tau X_0}, \tag{27}$$

$$\tilde{\Delta}^a=i\int_{\Gamma^a}W(s)ds+\frac{1}{2\pi}\int_{\Gamma_1^a}ds\int_{-\infty}^{\infty}\frac{d\tau}{\tau-i\varepsilon}e^{i\tau\bar{n}X}W(s)\,e^{-i\tau\bar{n}X}, \tag{28}$$

$$\mp\tilde{\tilde{\Delta}}^{\pm}=i\int_{\Gamma_{in}}W(s)ds-\frac{1}{2\pi}\int_{\Gamma_1^a}ds\int_{-\infty}^{\infty}\frac{d\tau}{\tau-i\varepsilon}e^{i\tau\bar{n}X}W(s)e^{-i\tau\bar{n}X}-$$
$$-\frac{1}{2\pi}\int_{\Gamma}ds\int_{-\infty}^{+\infty}\frac{d\tau}{\tau-i\varepsilon}\,e^{\pm i\tau X_0}W(s)e^{\mp i\tau X_0};\qquad \Gamma_1^a=C_2+C_3-\Gamma^a. \tag{29}$$

The operators (23)-(25) are inverse to the operator $\hat{\mathcal{P}}^2-m^2$ and the operators (26)-(29) are orthogonal to it. The operator $\tilde{\Delta}^c$ (23) coincides with the Schwinger representation (Schwinger 1951) of the inverse operator $[\hat{\mathcal{P}}^2-m^2]^{-1}$.

The operator $\tilde{\Delta}^a$ (28) contains the factor $e^{-\pi\frac{m^2}{e\mathcal{E}}}$ and therefore vanishes when eliminating $(\mathcal{Y}=0)$ the electric field. As one can represent the operator $\tilde{\tilde{\Delta}}^c$ (24) as $\check{\Delta}^c=\tilde{\Delta}^c-\tilde{\Delta}^a$ then it coincides when eliminating the electric field with $\tilde{\Delta}^c$.

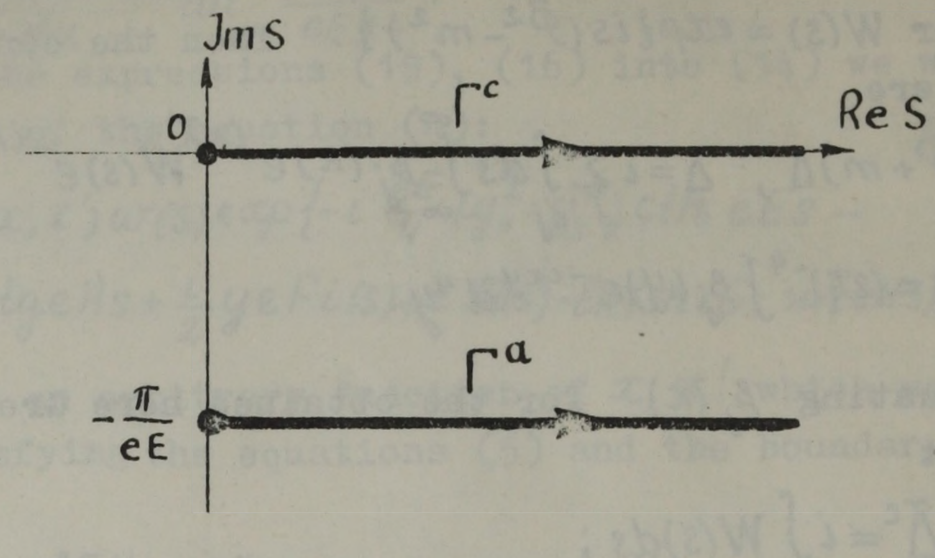


Fig.I

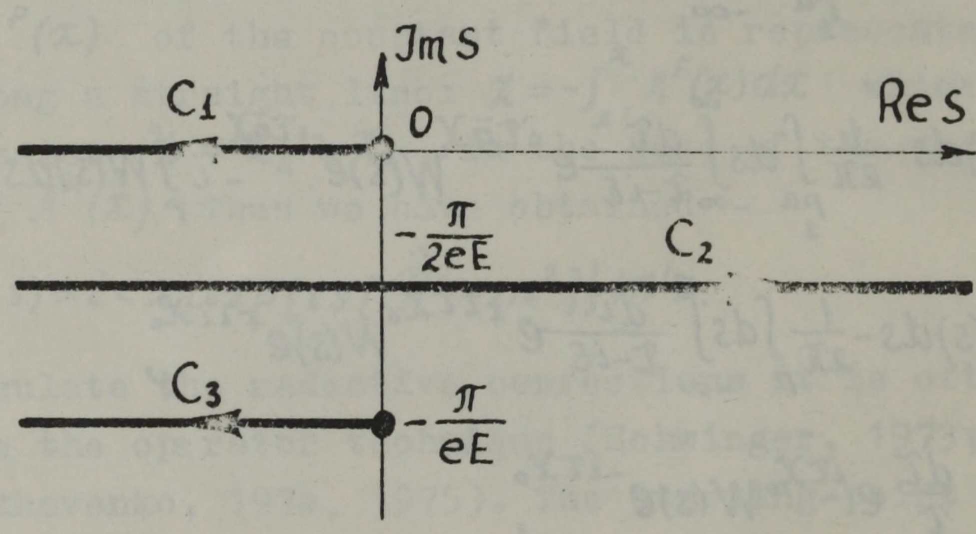


Fig.2

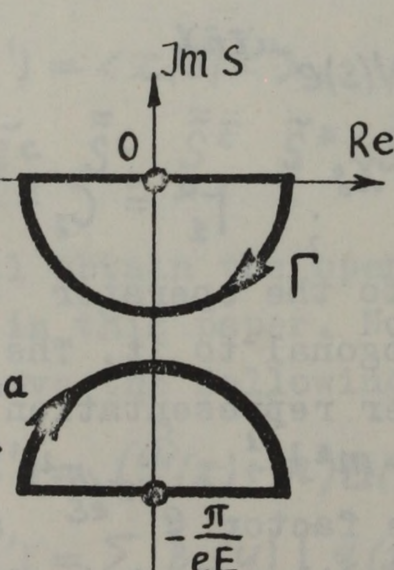


Fig.3

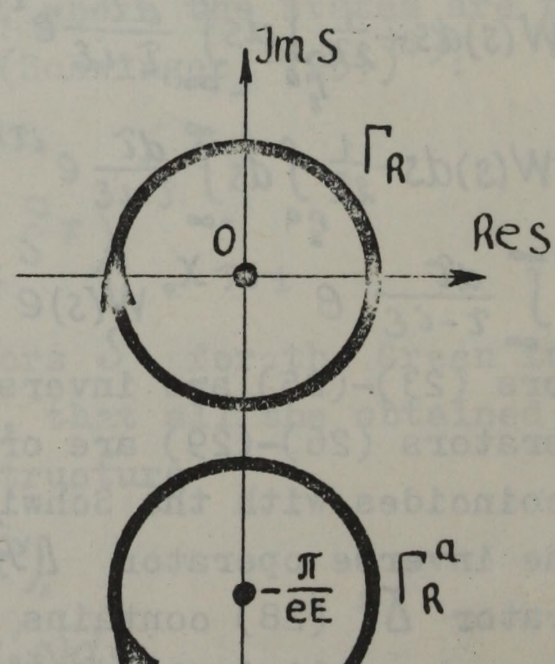


Fig.4

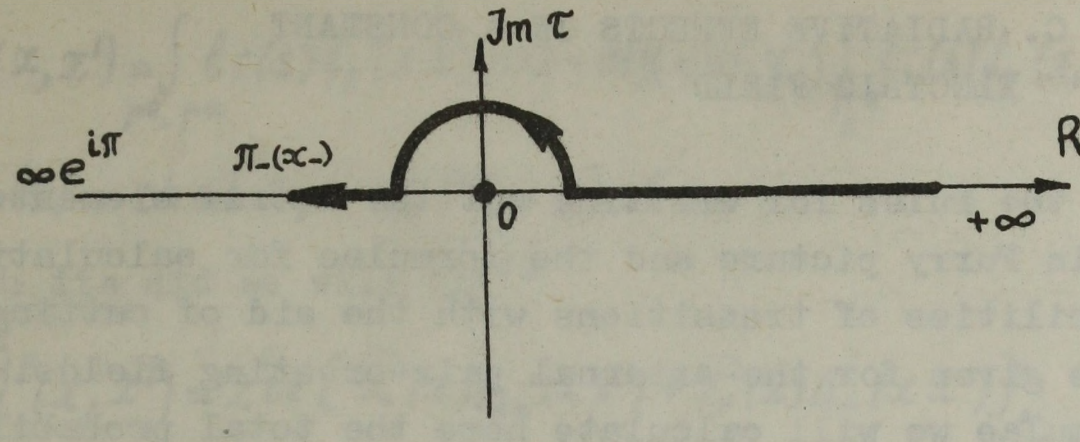


Fig.5a Contour for $_{+}K(x_{-})$, $_{+}J(x_{-})$

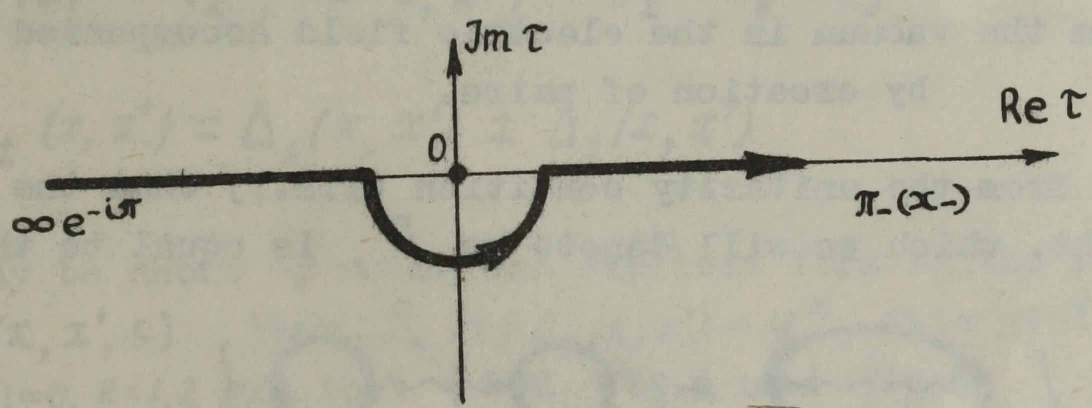


Fig.5b Contour for $^{-}K(x_{-})$, $^{-}J(x_{-})$

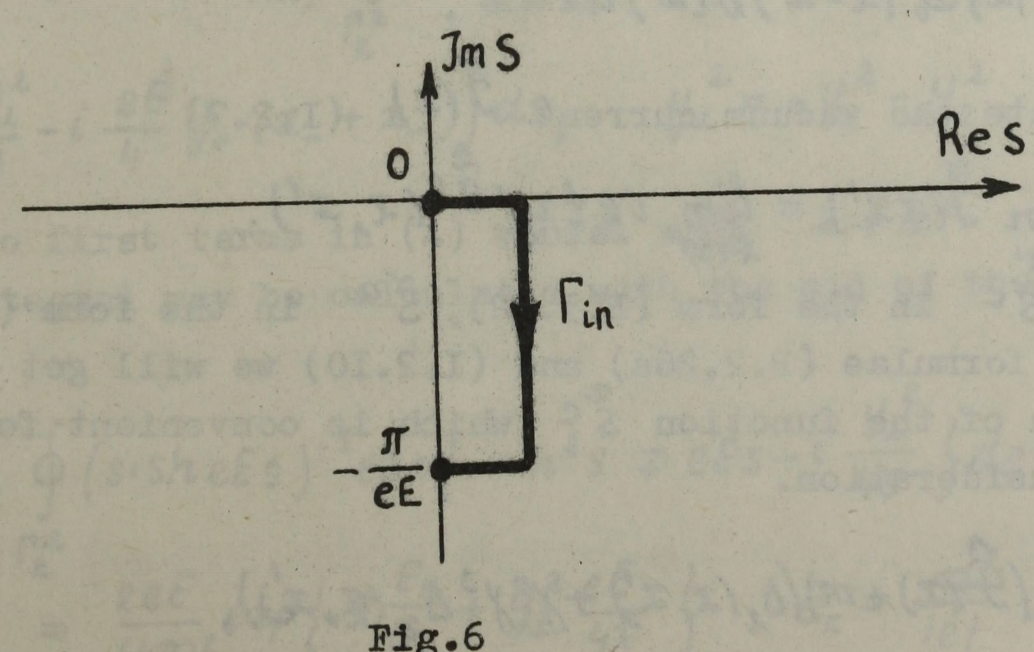


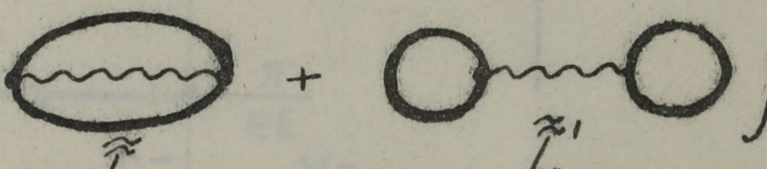
Fig.6

APPENDIX C. RADIATIVE EFFECTS IN A CONSTANT ELECTRIC FIELD*)

In Chapter I the rules for writting out the matrix elements of transitions in Furry picture and the formulae for calculating the total probabilities of transitions with the aid of cutting the diagrams are given for the external pair-creating fields. By using these formulae we will calculate here the total probability of the photon irradiation from the vacuum accompanied by creation of pairs and the total probability of transition from a single-electron state accompanied by the photon irradiation and creation of pairs for a constant electric field.

a) The total probability of the photon irradiation from the vacuum in the electric field accompanied by creation of pairs.

It follows from the unitarity condition (I.2.7) that the quantity sought, which we will denote by P, is equal to the following

$P = 2\,\mathrm{Im}\,\{$  $\}$, (I)

$$\tilde{\tilde{L}} = -\frac{e^2}{2}\int D_0^c(x-x')\,\mathrm{tr}\,\gamma\,\tilde{\tilde{S}}^c(xx')\,\gamma\,\tilde{\tilde{S}}^c(x'x)\,dx\,dx',$$

$$\tilde{\tilde{L}}' = -\frac{1}{2}\int \tilde{\tilde{J}}(x)\,D_0^c(x-x')\,\tilde{\tilde{J}}(x')\,dx\,dx'. \quad (2)$$

Let us calculate the vacuum current $\tilde{\tilde{J}}(x)$ (I.2.7)

$$\tilde{\tilde{J}}(x) = \lim_{x\to x'} \tilde{\tilde{J}}(xx') = \lim_{x\to x'} ie\,\mathrm{tr}\,\gamma\,\tilde{\tilde{S}}^c(x,x').$$

By choosing $\tilde{S}^c$ in the form (B.3.22), $\tilde{S}^a$ in the form (B.3.I8) and using the formulae (B.2.26a) and (I.2.IO) we will get the representation of the function $\tilde{\tilde{S}}^c$ which is convenient for the case under consideration.

$$\tilde{\tilde{S}}^c(x,x') = (\hat{\mathcal{P}}(x)+m)(\Delta_1(x,x')+\gamma^0\gamma^3\Delta_2(x,x')), \quad (3)$$

---

*) The work is carried out by D.M.Gitman, S.P. Gavrilov, Schwartsman Sh.M. and Wolfengaut J.J. (Dept. of Math. Analysis, Pedagogical Inst., 634044 Tomsk, USSR)

$$\Delta_i(x,x') = \int\limits_{\Gamma^c_- \Gamma^a} l_i(s) f_E(x x' s) ds - \theta(y_-)\theta(-y_+) \int\limits_{\Gamma^a_R} l_i(s) f_E(x,x's) ds,$$

$$l_1(s) = ch eEs, \quad l_2(s) = sh eEs.$$

With its aid we will get[I)]

$$\tilde{\tilde{J}}^0(x,x') = 2ie[\mathcal{P}_-(x)\Delta_+(xx') + \mathcal{P}_+(x)\Delta_-(xx')],$$

$$\tilde{\tilde{J}}^K(x,x') = -4ie\,\mathcal{P}_K(x)\Delta_1(x,x'), \quad K=1,2,$$

$$\tilde{\tilde{J}}^3(x,x') = -2ie\,[\mathcal{P}_-(x)\Delta_+(x,x') - \mathcal{P}_+(x)\Delta_-(x,x')],$$

$$\mathcal{P}_K(x) = i\partial_K, \quad K=1,2; \quad \mathcal{P}_\pm = \mathcal{P}_0 \mp \mathcal{P}_3,$$

$$\Delta_\pm(x,x') = \Delta_1(x,x') \pm \Delta_2(x,x').$$

It may be shown by using the explicit form of the function $f_E(x,x',s)$ that $\mathcal{P}_K(x)\Delta_1(x,x') \sim y^K$, this yields $\tilde{\tilde{J}}^K(x)=0$, $K=1,2$ and that (see fig.1 and fig.4)

$$\mathcal{P}_\mp(x)\Delta_\pm(x) = \frac{eEy_\mp}{2}\left[\int\limits_{\Gamma^c_- \Gamma^a} \frac{f_E(xx's)ds}{sh eEs} - \theta(y_-)\theta(-y_+)\oint\limits_{\Gamma^a_R} \frac{f_E(xx's)ds}{sh eEs}\right] +$$

$$\pm 2i\theta(\pm y_\mp)\delta(y_\pm)\frac{eE}{(4\pi)^2}\oint\limits_{\Gamma^a_R}(s\cdot sh eEs)^{-1}\exp\Big\{-im^2 s \mp eEs - \tag{4}$$

$$-i\frac{y_\perp^2}{4s} - i\frac{eE}{4}y_+(x_- + x'_-)\Big\}ds, \quad y_\perp^2 = -y_1^2 - y_2^2.$$

The two first terms in (4) vanish when $x \to x'$ and the latter integral may be calculated with the aid of the residue method:

$$\frac{eE}{(4\pi)^2}\oint\limits_{\Gamma^a_R}(s\cdot sh eEs)^{-1}\exp\Big\{-im^2 s \mp eEs - i\frac{y_\perp^2}{4s}\Big\}ds =$$

$$= \frac{2eE}{(4\pi)^2}\exp\Big\{-\pi\frac{E_K}{E} + y_\perp^2\frac{eE}{4\pi}\Big\}, \quad E_K = \frac{m^2}{|e|}.$$

I) The notations and definitions of appendix B are used.

Finally we will get

$$\tilde{\tilde{J}}^{\mu}(x) = \frac{em^2}{16\pi^3}\cdot\frac{|E|T}{E_\kappa}\cdot e^{-\pi\frac{E_\kappa}{|E|}}\delta_3^{\mu}. \tag{5}$$

Here T is the time interval in the course of which the electric field acts. (We will, as usual, assume $2\pi\delta(x_0'-x_0)|_{x_0'\to x_0}=T$.

The expression (5) is the mean current of particles created by the electric field from the vacuum.

In view of the fact that $\tilde{\tilde{J}}(x)$ does not depend on the space-time variables there is no irradiation of the photons with $\vec{k}\neq 0$ by the vacuum current $\tilde{\tilde{J}}(x)$. This leads to the fact that the imaginary part of the diagram $\tilde{\tilde{L}}'$ is equal to zero. Therefore it is not necessary to calculate the diagram $\tilde{\tilde{L}}'$ in the case under consideration.

Consider the calculation of the diagram $\tilde{\tilde{L}}$. To do so we will use the representation of the function $\tilde{\tilde{S}}^c$ following from the relations (I.2.30), (B.3.22) and (B.3.I8)

$$\tilde{\tilde{S}}^c(x,x') = (\hat{\mathcal{P}}(x)+m)\left[G_1(x,x')+\gamma^0\gamma^3 G_2(x,x')\right], \tag{6}$$

$$G_i(x,x') = \int_{\Gamma^c} l_i(s) f_E(xx's) - \theta(y_3)\int_{C_2+C_3} l_i(s) f_E(x,x')\,ds -$$

$$-\theta(-y_3)\int_{\Gamma^a} l_i(s) f_E(xx's)\,ds. \quad \text{(see fig. 1 and fig. 2)} \tag{7}$$

The contours arising here and below are defined by the figures (I-6) of the appendix B.

We will get by calculating the trace, which appears in the expression (2) for $\tilde{\tilde{L}}$, the following

$$\Pi = tr\{\gamma\tilde{\tilde{S}}^c(xx')\gamma\tilde{\tilde{S}}^c(x'x)\} = \Pi_1+\Pi_2+\Pi_3, \tag{8}$$

$$\Pi_1 = 16m^2 G_1(xx')G_1(x'x) = 16m^2\int_{\Gamma^c,\Gamma^a} ds_1 \int_{\Gamma^c-C_2-C_3} ds_2\,\cdot$$

$$\cdot\, l_1(s_1) f_E(xx's_1)\, l_1(s_2) f_E(xx's_2),$$

$$\Pi_2 = 8\sum_{\kappa=1}^{2}\left[\mathcal{P}_\kappa(x)G_1(xx')\mathcal{P}_\kappa(x')G_1(x'x) - \mathcal{P}_3(x)G_2(xx')\cdot\right.$$

$$\cdot \mathcal{P}_k(x')G_2(x'x)] = y_1^2 \int\limits_{\Gamma^c_\varepsilon \Gamma^a} ds_1 \int\limits_{\Gamma^c_\varepsilon C_2 - C_3} ds_2 \cdot p \cdot f_E(xx's_1)\, f_E(x'xs_2),$$

$$\Pi_3 = -4\mathcal{P}_-(x)G_+(xx')\mathcal{P}_+(x')G_-(x'x) - 4\mathcal{P}_+(x)G_-(xx')\mathcal{P}_-(x')G_+(x'x) =$$

$$= (y_0^2 - y_3^2) \int\limits_{\Gamma^c_\varepsilon \Gamma^a} ds_1 \int\limits_{\Gamma^c_\varepsilon C_2 - C_3} ds_2\, q\, f_E(xx's_1)\, f_E(x'xs_2),$$

$$p = 2(s_1 s_2)^{-1} \operatorname{ch} eE(s_1 - s_2), \quad q = 2(\operatorname{sh} eEs_1 \cdot \operatorname{sh} eEs_2)^{-1}(eE)^2,$$

$$G_\pm(xx') = G_1(xx') \pm G_2(xx').$$

Note, that the expression (8) is even with respect to $x - x'$ and substitute $D_0^c(x - x')$ taken in the form $2\theta(x_0 - x_0')\cdot D^-(x - x')$ into (2). By performing x_- and x'_- integrations in (2) we will get

$$\tilde{\tilde{L}} = -2i \frac{e^2 VT(eE)^2}{(4\pi)^4} \int\limits_{\Gamma^c - \Gamma^a} ds_1 \int\limits_{\Gamma^c - C_2 - C_3} ds_2 \frac{\operatorname{sign} b \cdot \exp\{-im^2(s_1 + s_2)\}}{s_1 \cdot s_2 \cdot \operatorname{sh} eEs_1 \cdot \operatorname{sh} eEs_2}.$$

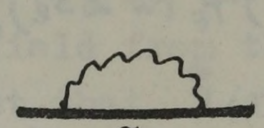
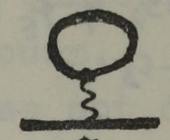
$$\cdot \left\{ 4m^2 \frac{\operatorname{ch} eEs_1 \operatorname{ch} eEs_2}{b - a} \ln \frac{b}{a} - i\left[\frac{q - p}{(b-a)^2} \ln \frac{b}{a} - \frac{qb^{-1} - pa^{-1}}{(b-a)} \right] \right\}$$

$$a = s_1^{-1} + s_2^{-1}, \quad b = eE(\operatorname{cth} eEs_1 + \operatorname{cth} eEs_2). \tag{9}$$

$$P = 2 \operatorname{Im} \tilde{\tilde{L}} \tag{10}$$

b) The total probability of transition from a single-electron state accompanied by the photon irradiation and creation of pairs.

It follows from the unitarity condition (I.2.8) that the quantity sought, which we will denote by P_ℓ, where ℓ is the total combination of quantum numbers of the initial electron in the electric field, the electron being described by the asymptotic expressions for the functions ${}_+\tilde{\varphi}_{p_-p_1p_2\tau}(x)$ (B.3.I) under $t\to-\infty$, is equal to the following:

$$P_\ell = 2\,\mathrm{Im}\left\{\tilde{\tilde{L}} + \tilde{\tilde{L}}' + \right.$$  $$+$$  $$\left.\right\} \qquad \text{(II)}$$

$$\tilde{\tilde{M}}_\ell \qquad\qquad \tilde{\tilde{M}}'_\ell$$

$$\tilde{\tilde{M}}_\ell = ie^2\int D_0^c(x-x')\{{}_+\bar{\tilde{\tilde{\varphi}}}_\ell(x)\gamma\,\tilde{\tilde{S}}^c(xx')\gamma\,{}_+\tilde{\varphi}_\ell(x')\}\,dx\,dx', \qquad \text{(I2)}$$

$$\tilde{\tilde{M}}'_\ell = \int {}_+\bar{\tilde{\tilde{\varphi}}}_\ell(x)\gamma\,{}_+\tilde{\varphi}_\ell(x)\,D_0^c(x-x')\,\tilde{\tilde{J}}(x')\,d^4x\,d^4x'.$$

The diagrams $\tilde{\tilde{L}}'$ and $\tilde{\tilde{M}}'$ do not contribute to P_ℓ due to the space-time invariance of the current $\tilde{\tilde{J}}(x)$ and the diagram $\tilde{\tilde{L}}$ has been calculated in the item a) of this appendix.

Let us calculate the expression within the curly brackets in the integrand in (I2). When doing so we will choose the functions $\tilde{\tilde{S}}^c$ in the form (6), and the functions ${}_+\tilde{\varphi}_\ell$ in the form (B.3.I).

$$M = {}_+\bar{\tilde{\tilde{\varphi}}}_\ell(x)\gamma\,\tilde{\tilde{S}}^c(x,x')\gamma\,{}_+\tilde{\varphi}_\ell(x') = M_1 + M_2 + M_3,$$

$$M_1 = -2\left[{}_+\Phi^{-1*}(x)\,{}_+\Phi^{-1}(x')\,\mathcal{P}_+(x)\,G_-(x,x') + (p_\perp^2+m^2)\,{}_+\Phi^{+1*}(x)\,{}_+\Phi^{+1}(x')\cdot\right.$$

$$\left.\cdot\,\mathcal{P}_-(x)\,G_+(x,x')\right],$$

$$M_2 = 2\left[(p_1+i\tau p_2)\,{}_+\Phi^{-1*}(x)\,{}_+\Phi^{+1}(x')\left(\mathcal{P}_1(x)-i\tau\mathcal{P}_2(x)\right)G_-(x,x')+\right.$$

$$\left.+(p_1-i\tau p_2)\,{}_+\Phi^{+1*}(x)\,{}_+\Phi^{-1}(x')\left(\mathcal{P}_1(x)+i\tau\mathcal{P}_2(x)\right)G_+(x,x')\right],$$

$$M_3 = 4m^2\left[{}_+\Phi^{-1*}(x)\,{}_+\Phi^{+1}(x') + {}_+\Phi^{+1*}(x)\,{}_+\Phi^{-1}(x')\right]G_1(x,x').$$

Here $\tau = \pm 1$ is the spin quantum number. The functions ${}_+\Phi^{\pm1}(x)$ are defined in (B.3.3) and depend on the quantum numbers p_-, p_1, p_2.

To do the x and x' integrations it is convenient to choose the proper time representation for $D_0^c(x-x')$ in the expression for $\tilde{\tilde{M}}$:

$$D_0^c(x-x') = \frac{1}{(4\pi)^2}\int_0^\infty \frac{dt}{t^2}\exp\{-i\mu^2 t - i\,\frac{(x-x')^2}{4t}\}$$

Here we have introduced the photon mass μ since the diagram $\tilde{\tilde{M}}$ contains an infrared divergence.

The final result has after the integrating the following form:

$$\tilde{\tilde{M}}_e = \quad m S\cdot(1-e^{-\pi\lambda})<\tilde{\tilde{M}}_e>, \tag{I3}$$

$$<\tilde{\tilde{M}}_e> = -\frac{e^2 eE}{8\pi^2}\Big[\int_{\Gamma^\xi,\Gamma^a} ds_1 \int_{\Gamma^c} ds_2 M(s_1,s_2) + \int_{C_3} ds_1 \int_{\Gamma^c} ds_2\,\cdot$$

$$\cdot\,\theta\Big(-s_2+\frac{\exp(-2eEs_1)-1}{2eE}\Big) M(s_1,s_2)\big(1-e^{-\pi\lambda}\big)^{-1}\Big].$$

$$\lambda = \frac{m^2+p_\perp^2}{eE},\quad M(s_1,s_2) = 2\Big[\tau_2+\tau_3-2eEs_2\frac{eE\lambda}{m^2}+\tau_3\frac{p_\perp^2}{m^2}\frac{s_2}{s_1+s_2}\Big]\cdot$$

$$\cdot\,(\tau_2^2-\tau_1^2)^{-1}\,\frac{z^{i\frac{\lambda}{2}}}{s_1+s_2}\exp\{-i\mu^2 s_2 - is_1(m^2+p_\perp^2\frac{s_2}{s_1+s_2})\}\},$$

$$\tau_1 = \operatorname{ch} 2eEs_1 - 1,\qquad \tau_2 = 2eEs_2 + \operatorname{sh} eEs_1,$$

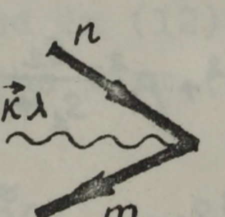
$$\tau_3 = 2eEs_2\operatorname{ch}2eEs_1 + \operatorname{sh}2eEs_1,\quad z = \frac{1+eEs_2(\operatorname{cth}eEs_1+1)}{1+eEs_2(\operatorname{cth}eEs_1-1)}\,.$$

$$P_e = P + 2\,\operatorname{Im}\tilde{\tilde{M}}\,. \tag{I4}$$

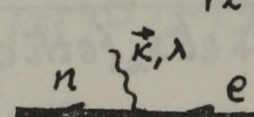
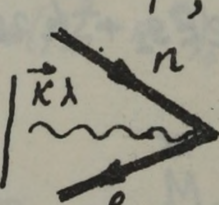
Here we denoted the divergent integral $\frac{m}{eE}\int_{-\infty}^{\infty}\frac{dk_3}{\sqrt{k_3^2+1}}$ by S. It may be shown (Nikishov, 1970) that from the conservation law for the creation of a pair accompanied by the photon irradiation it follows that the k_3- integration may be replaced by the π_-- integration, so that S is equal to $\frac{m}{eE}\int_0^\infty\frac{d\pi_-}{\pi_-}$, where $\pi_- = p_- - eEx_-$ (see appendix B). By studying the classical analogue of the quantity π_- one can make sure that S is the time interval, in the course of which the field acts, measured in the units of the proper time of an electron in the electric field.

c) Discussion

For the pair-creating fields the calculation of the total probabilities based on the unitarity conditions differs in principle from the case of the fields which do not product pairs. This difference is in that one should calculate the diagrams which are subject to cutting with the aid of the noncausal Green function $\tilde{\tilde{S}}^{c}$. Therefore it seemed to us to be of importance to check the results obtained in this way by the straightforward summing of the probabilities of transitions, that is by the straightforward calculation of the left-hand sides of the unitarity conditions (I.2.7) and (I.2.8). It is convenient to perform such checking after a certain transformation of the left-hand sides of the relations (I.2.7) and (I.2.8). Namely, let us express the functions ${}^{+}\tilde{\tilde{\psi}}$ and ${}_{+}^{-}\tilde{\psi}$ in terms of the functions ${}_{\pm}\tilde{\varphi}$ in accordance with the relations (I.I.4I) and use the completeness of the $\widetilde{out}$ - states. Then

$$P=\sum_{m,n,\vec{\kappa},\lambda}\Bigg|$$  $$\Bigg|^{2}= \tag{I5}$$

$$=\sum_{m,n,\vec{\kappa},\lambda}\left|-ie\int {}_{+}\tilde{\tilde{\varphi}}_{m}(x)\,\frac{\hat{e}(\vec{\kappa},\lambda)}{\sqrt{2V\kappa}}\,{}_{-}\tilde{\varphi}_{n}(x)e^{i\kappa x}dx\right|^{2},$$

$$P_{e}=P+\sum_{n,\vec{\kappa},\lambda}\Bigg|$$  $$\Bigg|^{2}-\sum_{n,\vec{\kappa},\lambda}\Bigg|$$  $$\Bigg|^{2}=$$

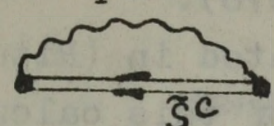
$$=P+\sum_{n,\kappa,\lambda}\Big\{\left|-ie\int {}_{+}\tilde{\tilde{\varphi}}_{n}(x)\,\frac{\hat{e}(\vec{\kappa},\lambda)}{\sqrt{2V\kappa}}\,{}_{+}\varphi_{e}(x)\,e^{i\kappa x}dx\right|^{2}- \tag{I6}$$

$$-\left|-ie\int {}_{+}\tilde{\tilde{\varphi}}_{e}(x)\,\frac{\hat{e}(\vec{\kappa},\lambda)}{\sqrt{2V\kappa}}\,{}_{-}\tilde{\varphi}_{n}(x)e^{i\kappa x}\,dx\right|^{2}\Big\}.$$

Here we have chosen the vacuum current $\tilde{\mathcal{J}}$ be equal to zero, what is valid for the constant electric field.

We have carried out the straightforward calculation of the quantities P and P_e by using the formulae (I5) and (I6). The results of this calculation coincide with the expressions obtained by using the unitarity conditions. This coincidence is the direct proof of the unitarity of the S- matrix to the first order of the perturbation expansion.

Note now, that in (Baier, Katkov, Strakhovenko, 1974; Ritus, 1978)[2] the mass operator

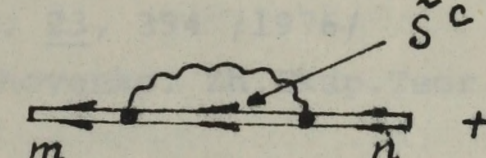
$\tilde{M} =$ 

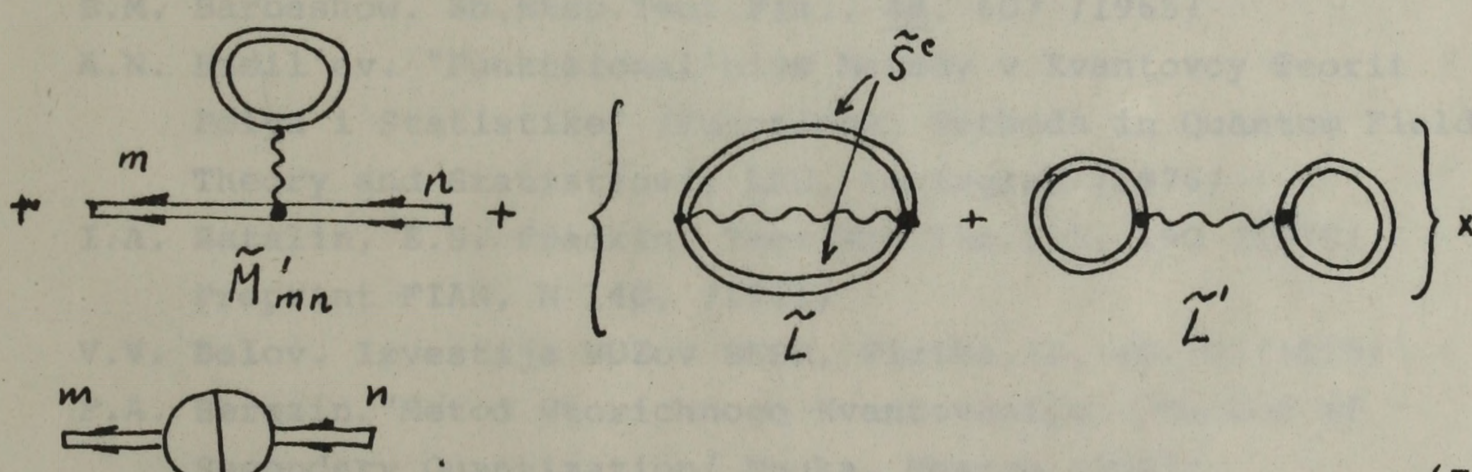
for a constant field was obtained, which was then averaged over some solutions of the Dirac equation, these solutions were eigenfunctions of the mass operator. The result obtained in this way has, when only the electric field is present, the following form:

$$\langle \tilde{M} \rangle = -\frac{e^2 eE}{8\pi^2} \int_{\Gamma^c} ds_1 \int_{\Gamma^c} ds_2 \, M(s_1 s_2) \tag{I8}$$

$(M(s_1, s_2)$ is defined by the formula (I4))

The expression (I8) is in the second-order radiative correction to the scattering of an electron in the electric field, which in accordance with the results obtained in Sec.I of Chapter I has the form

$${}_{out}\langle \tilde{0} | \tilde{a}_m(out)\, S^{(2)} \overset{+}{a}_n(in) | 0 \rangle_{in} =$$ 



(I9)

The vacuum current $\tilde{J}$ is equal to zero in the electric field, therefore the diagrams $\tilde{M}'_{mn}$ and $\tilde{L}'$ should not be considered. The diagram $\tilde{M}_{mn}$ can be expressed in the case under consideration in terms of $\langle \tilde{M} \rangle$ in the following way:

$$\tilde{M}_{mn} = \langle \tilde{M} \rangle \Psi_{mn} C_v,$$

where

$$\Psi_{mn} = \int {}^{+}\tilde{\bar{\psi}}_m(x)\, {}_{+}\tilde{\psi}_n \, dx = \frac{1}{2} S \cdot w(\overset{+}{m}|\overset{+}{n}),$$

and C_v and $w(\overset{+}{m}|\overset{+}{n})$ were calculated, for instance, in (Niki-

2) We do not touch here the papers related to the calculation of the radiative effects in the fields which do not creat pairs.

shov, 1969; Narozhny, Nikishov, 1976).

The diagram $\tilde{L}\, C_{\nu}^{-1}$ was calculated in (Ritus, 1975, 1977) for a constant field. The result of this calculation for the case, when only the electric field is present, is given by the formulae (9) where the S_1- and S_2- integrations are performed along the contours Γ^{c} only. Note also, that in the electric field $\tilde{L}\, C_{\nu}^{-1}$ is, in accordance with the results of Sec.3 of Chapter II, the second order correction to the effective Lagrangian for the field $\langle A \rangle^{\mathcal{P}}$.

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