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ABSTRACT

It is shown that on the basis of semiclassical calculations the charge number dependence of the Lamb shift differs from the experimentally observed one.

АННОТАЦИЯ

На основании семиклассической теории была определена зависимость смещения Лемба от порядкого номера. Результат отличается от экспериментальных данных.

KIVONAT

Megmutatjuk, hogy a szemiklasszikus számitások alapján a Lamb eltolódás töltésszámfüggésére a kisérletileg mért eredményektől eltérőt kapunk.

The first attempt to calculate the Lamb shift in a semiclassical way, i.e. without the canonical quantization of the electromagnetic field, was made by Crisp and Jaynes[1]. Later it was shown by van den Doel and Kokkedee[2] that the inclusion of the relativistic spin current in the semiclassical theory produced level shifts which are in strong disagreement with experiment. Recently Barwick[3] has dealt with Lamb shift calculations within the framework of a classical theory i.e. he ignored the probability interpretation of the wave function and he treated the wave-mechanical expressions of the charge and current as classical charge and current densities. Contrary to refs. 1 and 2, Barwick has obtained a very accurate value of the hydrogenic Lamb shift, but his work is not without inconsistencies[4]. The common feature of these semiclassical theories is that the calculated Lamb shift is caused by the electron current density. Since in the last twenty years several Lamb shift measurements have been carried out on hydrogenics of different Z (Z is the charge number), and the quantum electrodynamics is in very accurate agreement with these experimental data[5], it is reasonable to expect that a seriously considered semiclassical theory should give the right Z dependence for the Lamb shift. With this in mind the Z dependence of the Lamb shift caused by the electron current is investigated here.

The current density of a stationary electron state is

$$\underline{i} = -\Psi^{+}\underline{\alpha}\Psi e \ , \tag{1}$$

where the vector $\underline{\alpha}$ is constructed from the Dirac matrices α_1, α_2 and α_3, and Ψ is a bispinor. In the $nS_{1/2}$ and $nP_{1/2}$ states of a hydrogenic of charge number Z, the current density has only a ϕ component and this has the form

$$i_{\phi} = \frac{e}{2\pi} f_n(r) g_n(r) \sin\theta \ , \tag{2}$$

where r, Θ and Φ are the spherical coordinates and $f_n(r)$ and $g_n(r)$ are the ordinary normalized radial Dirac eigenfunctions[6] of an $S_{1/2}$ or a $P_{1/2}$ state of principal quantum number n. In this case the stationary Maxwell equation takes the form

$$(\Delta - \frac{1}{r^2 \sin^2\Theta}) A_\Phi = - 4\pi \, i_\phi \tag{3}$$

The Θ dependence of i_ϕ means that A_Φ can be written as

$$A_\phi = a(r) \sin\Theta \ , \tag{4}$$

and as a consequence of Eq.(4) the equation for a(r) is

$$(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{2}{r^2}) \, a(r) = - 4\pi i(r), \tag{5}$$

where

$$i(r) = \frac{e}{2\pi} f_n(r) \, g_n(r). \tag{6}$$

If we introduce the new variable $x = \lambda r$ and the new function $u(x) = x \, a(x)$, Eq.(5) takes the form

$$(\frac{d^2}{dx^2} - \frac{2}{x^2}) \, u(x) = - \frac{4\pi}{\lambda^2} x \, i(x) \tag{7}$$

The general solution of this second order inhomogeneous differential equation is[7]

$$u = u_2(x) \int_0^x u_1(\omega) \frac{f(\omega)}{D(\omega)} d\omega - u_1(x) \int_0^x u_2(\omega) \frac{f(\omega)}{D(\omega)} d\omega + \tag{8}$$

$$+ k_1 u_1(x) + k_2 u_2(x) \ ,$$

where $u_1(x)$ and $u_2(x)$ are linearly independent solutions of the homogeneous equation, D is the Wromskian of $u_1(x)$ and $u_2(x)$

$$D = u_1(x) \frac{d}{dx} u_2(x) - u_2(x) \frac{d}{dx} u_1(x), \tag{9}$$

$$f = - 4\pi x \, i(x)/\lambda^2 \ , \tag{10}$$

and k_1 and k_2 are constants determined by the boundary conditions of the vector-potential

$$\lim_{x \to \infty} a(x) = 0 \tag{11}$$

$$\lim_{x \to 0} \underline{B}(x) = \text{non-singular}, \tag{12}$$

where the vector $\underline{B}$ has only B_r and B_Θ components.
Using Eqs.(6) and (10) and the concrete forms of the $2S_{1/2}$ and $2P_{1/2}$ radial eigenfunctions[6] we get

$$f = K e^{-x} \sum_{i=1}^{3} b_i x^{2\varepsilon + i - 2} \tag{13}$$

where the coefficients b_i are

$$b_1 = \frac{(2\varepsilon+1)(N-2k)}{2(N-k)} \tag{14}$$

$$b_2 = - \frac{(N-k)}{N}$$

$$b_3 = \frac{N-k}{2N(2\varepsilon+1)}, \tag{16}$$

and $k = -1$ for the $2S_{1/2}$ state and $k = 1$ for the $2P_{1/2}$ state, furthermore $\varepsilon = \sqrt{1-(\alpha Z)^2}$, $N = \sqrt{2(1+\varepsilon)}$, $\lambda = 2Z/(N a_o)$ (α is the fine structure constant and a_o is the Bohr radius) and

$$K = \frac{e\lambda}{N\Gamma(2\varepsilon+1)} (-\alpha Z k). \tag{17}$$

The solution of Eq.(5) can be obtained from Eq.(8) with the aid of Eq.(13), using the definition of u(x); the constants k_1 and k_2 are restricted by boundary conditions (11) and (12). The final form of a(x) is

$$a(x) = - \frac{K}{3} \sum_{i=1}^{3} b_i \left[\frac{1}{x^2} \gamma(2\varepsilon+i+1,x) + x\Gamma\ (2\varepsilon+i-2,x) \right] \tag{18}$$

The sum of the magnetic field energy produced by the electron current and the corresponding $-\underline{i}\,\underline{A}$ interaction term in the stationary case[8] is

$$W = -\frac{1}{2}\int \underline{i}\,\underline{A}\,d^3x \,. \tag{19}$$

Using Eqs.(2), (4) and (10), W has the form

$$W = \frac{1}{3\gamma}\int f(x)\,a(x)\,xdx \,. \tag{20}$$

Substituting Eqs.(13) and (18) into Eq.(20) and using the integral formulae of the incomplete gamma functions[9], one gets

$$W = -\frac{2}{9}A(\alpha Z)^2 \sum_{i,j=1}^{3} b_i b_j c_{ij} \frac{{}_2F_1(1,4\varepsilon+i+j-1;2\varepsilon+j+2;1/2)}{(2\varepsilon+j+1)} \tag{21}$$

where

$$A = RZ/\left[N^3\varepsilon^2\Gamma^2(2\varepsilon)\right], \tag{22}$$

$$c_{ij} = \Gamma(4\varepsilon+i+j-1)/\ 2^{4\varepsilon+i+j-1} \,, \tag{23}$$

${}_2F_1$ denotes the hypergeometrical function and $R = me^4/2\hbar^2$. As it can be seen from Eqs.(21),(22) and (23) and from the definition of ε and N that W has a rather complicated Z dependence, we give the numerical values of the $W_{2S_{1/2}}-W_{2P_{1/2}}$ energy differences and the corresponding experimentally observed Lamb shifts[5] as a function of the nuclear charge number, in *Table 1.*

It can be seen from *Table 1.* that the Lamb shift produced only by the electron current has quite a different Z dependence from the experimentally observed one, thus we can conclude that these semiclassical theories need modification so that the Lamb shift can be explained correctly[10]. It is hoped that this note will help to clarify some of the problems connected with semiclassical ideas.

Table 1.

The experimentally observed Lamb shifts L and the $W_{2S_{1/2}}-W_{2P_{1/2}}$ magnetic energy differences computed from Eq.(21), as a function of Z in GHz units

Z	L	$W_{2S_{1/2}} - W_{2P_{1/2}}$
1	1.057	0.355
2	14.045	2.840
3	63.030	9.585
6	780.1	76.78
8	2203	182.3
9	3339	259.8
18	38000	2102

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[10] A possible modification giving satisfactory Z dependence of the Lamb shift is discussed in ref.8.

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