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TRANSFORMATION OF TRIDIAGONAL COEFFICIENT MATRICES OF FIRST ORDER LINEAR SYSTEMS WITH UNIFORM TIME-DEPENDENCE IN THE SUBDIAGONALS INTO MATRICES HAVING CONSTANT EIGENVALUES

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ABSTRACT

A transformation method is given by which the time-dependent tridiagonal coefficient matrix of a system of first order linear differential equations can be transformed into an asymmetric time-deperdent tridiagonal matrix with constant eigenvalues. The method is applicable, if the subdiagonal elements of the coefficient matrix A (t) fulfill the condition
$A_{i,i-1}(t).A_{i-1,i}(t) = c_{i-1}\nu(t) \geq m > 0.$

АННОТАЦИЯ

В настоящей работе описан способ преобразования зависящей от времени тридиагональной матрицы A(t) коэффициентов системы линейных дифференциальных уравнений первого порядка в зависящую от времени несимметрическую тридиготальную матрицу, имеющую постоянные собственные значения в том случае, когда недиагональные элементы исходящей матрицы A(t) удовлетворяют условие
$A_{i,i-1}(t).A_{i-1,i}(t) = c_{i-1}\nu(t) \geq m > 0.$

KIVONAT

Jelen munkában egy eljárást ismertetünk, amelynek segitségével elsőrendü lineáris differenciálegyenlet rendszerek időfüggőnek feltételezett A(t) tridiagonális együtthatómátrixa olyan aszimmetrikus,időfüggő tridiagonális mátrixszá transzformálható, amelynek valamennyi sajátértéke állandó. A módszer abban az esetben alkalmazható, ha az A(t) mátrix mellékdiagonálisaiban álló elemek kielégitik az
$A_{i,i-1}(t).A_{i-1,i}(t) = c_{i-1}\nu(t) \geq m > 0$ feltételt.

Definitions

Let us consider a system of homogenous, first order differential equations for N unknown functions $\vec{P}(t) = P_1(t), \dots P_N(t)$ in the closed or on the right opened interval $[t_1, t_2]$ of the independent variable t, expressed as

$$\frac{d}{dt}\vec{P}(t) = A(t)\vec{P}(t) \tag{1.1}$$

The coefficient matrix $A(t)$ is supposed to be the function of the independent variable t with the following restrictions imposed on the matrix elements:

1°. All the elements of $A(t)$ are bounded real functions having continous derivatives in t. The off-mean and off-first subdiagonal elements of the matrix identically disappear, as

$$A_{jk}(t) \equiv 0 \quad if \quad |j-k| > 1 \quad j,k=1,\dots N \tag{1.2}$$

and there exists a positive constant upper bound M for which

$$A_{jk}(t) \leq M \qquad j,k=1,\dots N \qquad t \in [t_1, t_2]$$

2°. Under the above restrictive conditions the functions in the mean diagonal of $A(t)$ are optional. Further restrictions imposed on the non-diagonal elements are that the products of elements in symmetrical positions have a positive lower bound

m and that the products differ only in a constant factor given as

$$A_{j,j-1}(t).A_{j-1,j}(t) = a_j \nu(t) \geq m > 0 \qquad (1.3)$$

Theorem 1. The matrix $A(t)$ for all $t \in [t_1, t_2]$ has N different real eigenvalues; in other words $A(t)$ has simple structure.

Proof. The eigenvalues of $A(t)$ are the roots of its characteristic equation $(A(t) - \lambda E)\vec{u} = \vec{0}$. The characteristic equation can be solved if and only if the characteristic determinant $det(A(t) - \lambda E) = 0$. The characteristic determinant can be expressed as the N-th term of the recursion formula /for constant matrices see *F.R.Gantmacher and M.G.Krein* [1], Chapt. II. §1./:

$$\phi_0(\lambda, t) = 1$$

$$\phi_1(\lambda, t) = A_{11}(t) - \lambda$$

$$\phi_k(\lambda, t) = (A_{kk}(t) - \lambda)\phi_{k-1}(\lambda, t) - A_{k,k-1}(t) \times$$
$$\times A_{k-1,k}(t)\, \phi_{k-2}(\lambda, t) \qquad (1.4)$$

ϕ_k-s defined above are polynomials of k-th order in λ and satisfy the first two conditions of a Sturm series /see *O.Perron,* [2], Chapter I., §3./ for all $t \in [t_1, t_2]$.

3°. $\phi_0(\lambda, t)$ is positive and independent of λ in $[t_1, t_2]$.

4^{o}. Taking $t = t_o \in [t_1, t_2]$, and if there exists a $\lambda = \lambda_o$ such that $\phi_{k-1}(\lambda_o, t_o) = 0$, then, because of 2^{o} its neighbouring terms $\phi_k(\lambda_o, t_o)$ and $\phi_{k-2}(\lambda_o, t_o)$ have opposite signs.

For any $t_o \in [t_1, t_2]$ the matrix $A(t_o)$ can be treated as a constant matrix for which Gantmacher and Krein have proved in their - previously cited - work /*Gantmacher and Krein*, [1]/, that under the conditions 3^{o}, 4^{o} each ϕ_k $k = 1, \dots N$ has k separate real roots $\lambda_1, \dots \lambda_k$. Since the first two conditions of a Sturm series i.e. 3^{o} and 4^{o} hold for ϕ_k-s in the whole $[t_1, t_2]$, ϕ_N for all t has N real, separate roots $\lambda_1^A, \dots \lambda_N^A$, and so $A(t)$ has simple structure in $[t_1, t_2]$.

Theorem 2. The eigenvalues of $A(t)$ are continously differentiable implicit functions of t in $[t_1, t_2]$ defined by the characteristic equation of $A(t)$.

Proof. Each λ^A exists as an implicit function of t in a certain neighbourhood of t_o, if $\phi_N(\lambda^A, t_o) = 0$, the partial derivatives $\left.\frac{\partial \phi_N}{\partial t}\right|_{t_o, \lambda^A}$ and $\left.\frac{\partial \phi_N}{\partial \lambda^A}\right|_{t_o, \lambda^A}$ exist and $\left.\frac{\partial \phi_N}{\partial \lambda_A}\right|_{t_o, \lambda^A}$ is nonzero.

The first condition that λ^A is an implicit function of t around t_o has been proved in the proof of the Theorem 1; the further two requirements are also met as it can be easily shown. $\frac{\partial \phi_N}{\partial t}$ can be formally expressed by indirect derivation with respect to the matrix elements as

$$\frac{\partial \phi_N}{\partial t} = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial \phi_N}{\partial A_{ij}} \cdot \frac{\partial A_{ij}}{\partial t} \tag{2.1}$$

Any determinant can continuously differentiated with respect to its elements, further because of condition 1° the time derivative of the matrix /=determinant/ elements also exists and is bounded and continuous so the right--hand side of the expression (2.1) exists and defines a bounded and continous partial time derivative $\frac{\partial \phi_N}{\partial t}$.

In order to prove the existence of $\frac{\partial \phi_N}{\partial \lambda}$ let us formulate the factorized form of the characteristic determinant ϕ_N as

$$\phi_N(\lambda, t) = \prod_{j=1}^{N} (\lambda - \lambda_j^A(t)) \tag{2.2}$$

Its partial derivative in λ has the form

$$\frac{\partial \phi_N}{\partial \lambda} = \sum_{i=1}^{N} \prod_{j=1 \neq i}^{N} (\lambda - \lambda_j^A(t)) \tag{2.3}$$

For any $\lambda = \lambda_k^A(t)$ $k = 1, \dots N$ satisfying the characteristic equation, the sum on the right-hand side of (2.3) contains one finite, non-zero term, namely the product which does not contain λ_k^A. In other words, a nonzero partial derivative $\frac{\partial \phi_N}{\partial \lambda}$ exists at the definition point t_o of $\lambda_k^A(t_o)$. The existence and the behaviour of the partial derivatives imply that $\lambda_k^A = \lambda_k^A(t)$ $k = 1, \dots N$ in a certain neighbourhood of

$t_o \in [t_1, t_2]$ and that λ_k^A has continuous derivatives in t which can be expressed in terms of the above partial derivatives as

$$\left.\frac{d\lambda_k^A}{dt}\right|_{t_o} = - \frac{\left.\dfrac{\partial \phi_N}{\partial t}\right|_{t_o,\lambda_k^A}}{\left.\dfrac{\partial \phi_N}{\partial \lambda}\right|_{t_o,\lambda_k^A}} \tag{2.4}$$

<u>Theorem 3.</u> There exists a non-singular matrix $D(t)$ for any coefficient matrix $A(t)$ defined by (1.1) transforming the functions $\vec{P}(t)$ into $\vec{Q}(t) = D^{-1}(t)\vec{P}(t)$ and the coefficient matrix into a symmetric matrix $B(t)$.

<u>Proof.</u> The formal substitution of $D(t)\vec{Q}(t)$ for $\vec{P}(t)$ leads after the derivation $\frac{d\vec{P}(t)}{dt} = \frac{dD(t)}{dt}\vec{Q}(t) + D(t)\frac{d\vec{Q}(t)}{dt}$ to the equation for $\vec{Q}(t)$

$$\begin{aligned}\frac{d}{dt}\vec{Q}(t) &= \left(D^{-1}(t)A(t)D(t) - D^{-1}(t)\frac{d}{dt}D(t)\right)\vec{Q}(t) \\ &= B(t)\vec{Q}(t)\end{aligned} \tag{3.1}$$

The symmetry condition of the matrix $B(t)$ can be expressed in terms of the off-diagonal elements of $A(t)$ as

$$\frac{D_k(t)}{D_{k-1}(t)} A_{k-1,k}(t) = \frac{D_{k-1}(t)}{D_k(t)} A_{k,k-1}(t) \tag{3.2}$$

Choosing $D_1(t) = 1$, the k-th element of the symmetrizing diagonal matrix can be written as

$$D_k(t) = \prod_{j=2}^{k} \left(\frac{A_{k,k-1}(t)}{A_{k-1,k}(t)} \right)^{1/2} \tag{3.3}$$

Taking into account, that $A(t)$ has non-zero subdiagonal elements if $t \in [t_1, t_2]$, the product in (3.3) always defines a non-zero and bounded quantity i.e. $D(t)$ can be expressed with the elements of the coefficient matrix and exists with its inverse.

The diagonal elements of matrix $B(t)$ are:

$$B_{kk}(t) = A_{kk}(t) - D_k^{-1}(t) \frac{d}{dt} D_k(t)$$

$$= A_{kk}(t) - \frac{1}{2} \prod_{j=2}^{k} \left(\frac{A_{j,j-1}(t)}{A_{j-1,j}(t)} \right)^{1/2}$$

$$\sum_{i=2}^{k} \frac{1}{\left(\frac{A_{i,i-1}(t)}{A_{i-1,i}(t)} \right)} \frac{d}{dt} \left(\frac{A_{i,i-1}(t)}{A_{i-1,i}(t)} \right) \tag{3.4.1}$$

and those in its subdiagonals are given by

$$B_{k,k-1}(t) = B_{k-1,k}(t) = \left(A_{k,k-1}(t) A_{k-1,k}(t) \right)^{1/2} =$$

$$= \left(a_k \, \nu(t) \right)^{1/2} \tag{3.4.2}$$

Theorem 4. In the recursion formula defining the characteristic determinant of $B(t)$, the coefficient of ϕ_{k-2}-s is the same, as that of these terms in the characteristic determinant of $A(t)$.

Proof. For any tridiagonal matrix satisfying the conditions 3° and 4° the characteristic determinant is defined by a recursion formula /see Theorem 1./, in which the k-th term is expressed with the k-1-th and k-2-th ones. In the expression for ϕ_k the coefficient of ϕ_{k-2} is the product of subdiagonal elements in symmetric positions. Comparing (3.4.2) with (1.4) these products are the same for $A(t)$ and $B(t)$.

Theorem 5. Introducing the new variable τ defined by the differential expression

$$d\tau = \nu(t)\,dt \tag{5.1}$$

the dependence on the variable τ can be eliminated from the subdiagonal elements of $B(t)$.

Proof. Because of condition 2° both side of equation (3.1) can be divided by $\nu(t)$ and thus the differentiation on the left hand side formally changes to $\frac{d}{\nu(t)\,dt}=\frac{d}{d\tau}$. On the right-hand side the diagonal elements of the new matrix $B(\tau)$ maintain their dependence on the variable τ, as

$$B_{kk}(\tau) = \left[\frac{B_{kk}(t)}{(\nu(t))^{1/2}}\right]_{t=\varphi(\tau)} \tag{5.2.1}$$

while the off-diagonal elements are written as

$$B_{k,k-1}(\tau) = B_{k-1,k}(\tau) = a_k \qquad (5.2.2)$$

The differential expression (5.1) - because of condition 1^o - defines such a mapping $\varphi(\tau)$ between t and τ, that the continuity and differentiability of the matrix $B(t)$ is not affected by the use of the new variable τ.

Theorem 6. There exists also a diagonal matrix $\tilde{D}(\tau)$ transforming the functions $\vec{Q}(\tau)$ into $\vec{R}(\tau) = \tilde{D}^{-1}(\tau)\,\vec{Q}(\tau)$ and the coefficient matrix $B(\tau)$ into a new coefficient matrix $C(\tau)$ having constant characteristic numbers.

Proof. Upon the substitutions and derivations, similar to those in the proof of Theorem 3, the coefficient matrix $C(\tau)$ of the equation for the functions $\vec{R}(\tau)$ can be expressed as

$$C(\tau) = \tilde{D}^{-1}(\tau)\, B(\tau)\,\tilde{D}(\tau) - \tilde{D}^{-1}(\tau)\,\frac{d}{d\tau}\,\tilde{D}(\tau) \qquad (6.1)$$

and given in detail as

$$C_{kk}(\tau) = B_{kk}(\tau) - \frac{d}{d\tau}\, ln\, \tilde{D}_k(\tau) \qquad (6.2.1)$$

$$C_{k,k-1}(\tau) = \frac{\tilde{D}_k(\tau)}{\tilde{D}_{k-1}(\tau)}\, B_{k,k-1}(\tau) = \frac{\tilde{D}_k(\tau)}{\tilde{D}_{k-1}(\tau)} \cdot a_k. \qquad (6.2.2)$$

$$C_{k-1,k}(\tau) = \frac{\tilde{D}_{k-1}(\tau)}{\tilde{D}_k(\tau)}\, B_{k-1,k}(\tau) = \frac{\tilde{D}_{k-1}(\tau)}{\tilde{D}_k(\tau)} \cdot a_k \qquad (6.2.3)$$

The recursion formula describing the characteristic determinant of $C(\tau)$ contains the dependence on the variable τ only in the coefficient of the first term:

$$\phi_o(\lambda^c, \tau) \equiv 1$$

$$\phi_1(\lambda^c, \tau) = B_{11}(\tau) - \frac{d}{d\tau} \ln \tilde{D}_1(\tau) - \lambda^c$$

$$\phi_k(\lambda^c, \tau) = \left(B_{kk}(\tau) - \frac{d}{d\tau} \ln \tilde{D}_k(\tau) - \lambda^c\right) \times$$
$$\times \phi_{k-1}(\lambda^c, \tau) - a_{k-1} \phi_{k-2}(\lambda^c, \tau) \qquad (6.3)$$

The terms of the recursion formula (6.3) become independent of τ if the elements of $\tilde{D}(\tau)$ are chosen such that the coefficient of ϕ_{k-1} in the k-th step will be constant. The coefficient of ϕ_{k-1} will be constant under the condition that

$$B_{kk}(\tau) - \frac{d}{d\tau} \ln \tilde{D}_k(\tau) = \gamma_k = const \qquad k=1,\dots N \qquad (6.4)$$

This condition defines N independent equations, having the solutions

$$D_k(\tau) = D_{ko} \exp\left\{\int_{\tau_1}^{\tau} \left(B_{kk}(\omega) - \gamma_k\right) d\omega\right\} \quad k=1,\dots N \qquad (6.5)$$

The exponential form of the k-th element $\tilde{D}_k(\tau)$ of the diagonal matrix $\tilde{D}(\tau)$ ensures that in the interval $[\tau_1, \tau_2] = [\varphi(t_1), \varphi(t_2)]$ $\tilde{D}(\tau)$ and its inverse exist, thus the transformation (6.1) can be carried out and the recursion formula (6.3) has no τ dependent element and it defines constant eigenvalues λ^c.

References

[1] *Gantmacher, F.R., Krein. M.G.:* Oszcillationsmatrizen, Oszcillationskerne und kleine Schwingungen Mechanischer Systeme. Akademie Verlag, Berlin, 1960.

[2] *Perron, O.:* Algebra I-II. Göschens Lehrbücherei. München, 1933.

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