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ABSTRACT

The Quantum Chromodynamics with massless quarks and infinite number of colors is represented as a theory of the noninteracting mesons which lie on the rising Regge-trajectories. The perturbation theory for these trajectories is developed. The expansion parameter /effective coupling/ is calculated and appears to be about 1/2. The expansion coefficients can also be calculated analytically as functions of spin and other quantum numbers. The calculations are carried through to the end in the zeroth and first order. The resulting trajectories look reasonable and are in qualitative agreement with experiment. The corrections from finite number of colors and from quark masses can also be found, but are not considered here.

АННОТАЦИЯ

Кзантовая Хромодинамика с кварками нулевой массы и бесконечного числа цветов представлена как теория невзаимодействующих мезонов, лежащих на возрастающих Редже-траекториях. Разработана теория возмущения для этих траекторий. Вычеслен параметр разложения /эффективная связь/ который оказывается быть около 1/2. Коэффициенты разложения тоже могут быть вычислены аналитически как функции от спина и других квантовых чисел. Вычисления проделаны до конца в нулевом и первом порядке. Получающиеся траектории оказываются довольно реальными и качественно совпадают результатами экспериментов. Поправки из-за конечного числа цветом и конечной массы кварков также могут быть найдены, но мы не рассматриваем эту задачу.

KIVONAT

Nulla tömegü és végtelen sok szinü kvarkot tartalmazó Kvantum Chromodinamikát növekvő Regge trajektóriákon fekvő egymással kölcsön nem ható mezonok segitségével irunk le. Ezen trajektóriákra perturbációszámitást dolgozunk ki. A sorfejtési paraméter /effektiv csatolás/ 1/2 körülien adódik. A sorfejtési együtthatók is analitikusan számithatók, mint a spin és más kvantumszámok függvényei. A számitásokat nulla-és első rendben végeztük el. Az ereményképpen adódó trajektóriák megfelelőek és kvalitativ egyezésben vannak a kisérletekkel. A véges sok szinből és véges kvarktömegből adódó korrekciók szintén számithatók, de ezt nem vizsgáljuk.

INTRODUCTION

There are good reasons to believe that the hadronic world is described by Quantum Chromodynamics /QCD/ - tricolored quark - gluon gange theory in gauge invariant phase /without Higgs phenomenon/.

Thus it is urgent to find a method of approximate solution of this theory. Below we modify the method which was originally proposed in [1] . This paper should be considered as a final version of [1].

To simplify the problem, let us neglect the quark masses - for u , d and s quarks it is reasonable, and the heavy c-quark do not participate in the low energy phenomena, which we are going to describe. Afterwards the quark masses can be taken into account perturbatively.

The more considerable simplification which also makes sence is to tend to infinity the number n_c of colors. The calculation of diagrams in the ultraviolet domain indicates, that the n_c-dependence is rather smooth - e.g. the coefficients of β -function vary within 2o-3o % when n_c varies from 3 to ∞ , and expansion of these coefficients in n_c^{-1} converges at $n_c=3$ rather fast.

As it was pointed out by G.'t Hooft [2] , at $n_c \to \infty$ the gauge invariant correlation functions /only these functions make sense in a gauge invariant phase/ can be expanded in n_c^{-1} , the expansion coefficients depending on $g^2 n_c$, where g is the coupling constant.

Only planar diagrams with the minimal number of quark loops /see fig. 1/, contribute to the leading order in n_c^{-1} . In the higher orders the blocks of planar diagrams are combined into nonplanar ones.

The basic problem is to sum up the planar diagrams, i.e. to solve the theory at $n_c = \infty$, $g^2 n_c$ fixed. The $1/n_c$ corrections would then readily be found.

In the multicolor limit we expect to obtain some kind of dual resonance theory, as it was discussed in [2], [3]. The correlation functions are expected to have only poles, but not cuts.

We are not going to repeat all the arguments of these papers. The major argument is that since there are no internal quark loops, the mesons, which are $\bar{q}q$ bound states, cannot decay into mesons. The decay amplitudes are down by some powers of n_c and so are the scattering amplitudes. The possible purely gluonic states /glueballs/ are also stable and do not interact at $n_c = \infty$.

In other words, the gauge theory at $n_c = \infty$ is a theory of free colorless mesons.

It would be true, however, only if the color is confined, i.e. if these mesons do not decay into quarks and gluons. In fact in any order in $g^2 n_c$, i.e. in each planar diagram of fig. 1 such a decay takes place, but still we believe that color is confined.

The renormalization group analysis shows that there is no contradiction between the apparent decay thresholds of the diagrams and the color confinement. Perturbation theory do not apply near the thresholds because of the increase of effective coupling in the infrared domain. The perturbative cuts might appear to be seguences of poles at closer examination.

The qualitative behaviour of the theory in the infrared domain was studied in [4] within the framework of the recursion equations. If one may trust these equations /they were shown to work within 2o-3o% for the known cases of phase transitions [5] / there is effectively linear rizing potential between colored objects, so that the color is confined. The confining force depends exponentially on $-1/n_c g^2$, and thus cannot be seen in perturbation theory. However, it appeared that the domain of the weak and strong coupling overlap, so that the parameters of the solution in the strong coupling domain can be found by matching with the perturbation theory in the crossover domain of distances of the order of exp $(1/n_c g^2)$

With this picture in mind we turn to the planar diagrams of multicolor QCD.

I. Duality

In this section we try to find the precise formulation of the overlap of the infrared and ultraviolet domains and come to the Duality.

Let us take a simplest example of the 2-point function

$$\mathcal{D} = \frac{1}{n_c} \int d^4x \, e^{ikx} \langle T^* \bar{\Psi}\Psi(x) \; \bar{\Psi}\Psi(0) \rangle \qquad /1/$$

The renormalizability implies that up to normalization factor this is a universal function of single variable

$$t = \frac{k^2}{\mu^2} \varphi(\lambda) \qquad /2/$$

where μ is the normalization point, k is the momentum in Minkovsci space,

$$\lambda = \frac{n_c g^2}{8\pi^2} \qquad /3/$$

is the renormalized coupling, and

$$\varphi(\lambda) = \lambda^{\frac{102}{121}} \exp\left(\frac{6}{11\lambda}\right) \qquad /4/$$

In the last equation we omitted the factor $(1 + O(\lambda))$. One can always redefine λ by adding $O(\lambda^2)$ in such a way, as to make this factor equal to 1 .

This definition of λ corresponds to the following β -function

$$\beta(\lambda) = \frac{\varphi(\lambda)}{\varphi'(\lambda)} = -\frac{11}{6}\lambda^2\left(1 - \frac{17}{11}\lambda\right)^{-1} \qquad /5/$$

There is no profound meaning in this choice of renormalization prescribtion, but it is convenient to diminish the number of unknown functions.[x]

Anyhow, from /2/ and /4/ we see that perturbation theory corresponds to asymptotic expansion at large t . The expansion parameter can be chosen to be $\bar{\lambda}$, where

$$\varphi(\bar{\lambda}) = -t \qquad /6/$$

At large $-t$ the effective coupling $\bar{\lambda}$ tends to zero /asymptotic freedom/

$$\bar{\lambda} \to \left(\frac{11}{6} \ln(-t) + \frac{17}{11} \ln \ln(-t)\right)^{-1} \qquad /7/$$

At $-t$ =1,61 there is a spurious singularity which indicates that some other definition of effective coupling should be used at small t/see fig. 2/.

From the general grounds we expect that the 2-point function do not have any branch points in t , but rather is mezomorphic:

$$\mathcal{D} = \sum_j \gamma_j^2/(t_j - t) + const + const \cdot t \qquad /8/$$

The poles correspond to meson masses

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x I am gratefull to K. Wilson and G.t Hooft for stimulating discussions concerning this point.

$$M_j^2 = t_j \mu^2 / \varphi(\lambda) \qquad /9/$$

and the residues - to the couplings γ_j of these mesons with the field

Now, is it possible to find the masses and the residues, if we know only the coefficients of asymptotic expansion in terms of $\bar{\lambda}$?

This problem do not have a unique solution. We may always add one more pole term to /8/, and the asymptotic expansion would not change, since the pole term behaves at $-t \to \infty$ as

$$t^{-1} \sim \exp\left(-\frac{6}{11\bar{\lambda}}\right) \qquad /1o/$$

The general solution for the 2-point function consists thus at sertain minimal solution plus arbitrary pole terms. The minimal solution can be fixed by requirement of duality, i.e. the fastest convergence to the perturbation theory at $-t \to \infty$.

Below we construct such a minimal solution - it approaches perturbation theory as

$$\exp(-\mathrm{const}\sqrt{-t}) \qquad /11/$$

The additional pole terms with positive residues would spoit the duality.

But why should we require the duality? We can only give a heuristic argument. Imagine that we started from the functional integral formulation of massive QCD at finite number of colors. Then, as is now well-known [6] appart from the powers of effective coupling, there would also be nonanalytice contributions

$$t \exp(-n_c k/\bar{\lambda}) \sim (-t)^{-\frac{11}{6} n_c k} \cdot t \qquad /12/$$

due to gauge fields with topological quantum number $k = ., 2, \ldots$

We see, that the Mellin transform $\tilde{\mathcal{D}}(\omega)$ of 2-point function

$$\mathcal{D}(t) = \int_{0-i\infty}^{0+i\infty} d\omega \, (-t)^{\omega+1} \tilde{\mathcal{D}}(\omega) / 2\pi i \qquad /13/$$

has singularities at

$$\omega = -11 n_c k/6 \qquad /14/$$

At $n_c \to \infty$ all these "instanton" singularities move to infinity and it is likely, that only the singularity at $\omega = 0$ remains. This is the mathematical formulation of duality.

At the same time we see that instantons are lost in $1/n_c$ -expansion.

We expect, that nevertheless the color is confined within $1/n_c$ expansion-due to quantum fluctuations at zero topological charge. But of course, the instanton contributions should be added to $1/n_c$ expansion and it is not yet clear, how to do it.

The following comment might also be usefull to understand the meaning of duality.

The duality leads to analytic relation between the hadron masses and quantum numbers - as it is discussed below, there are rizing Regge - trajectories, which, as a matter of fact appear to be almost linear. This analytic behaviour would be spoiled, if we add several pole terms.

Thus, the duality seems to be a reasonable physical principle to be added to the rules of perturbation theory in order to obtain a unique solution.

In the next sections we modify the perturbation theory as to incorporate duality.

II. Meson Wave Function

Let us proceed with the quantum mechanical analysis of color confinement in QCD at $n_c = \infty$. We define in this section the basic quantity - the relativistic wave function of meson.

Consider, say, vector mesons. As it was argued in [3], [2], they consist of $\bar{q}q$ + gluons without additional quark pairs and without gluon pairs.

It is plausible to conjecture, that the meson state is a superposition of gauge invariant string states

$$\frac{1}{\sqrt{n_c}} \bar{\Psi}_f(x) \gamma_\mu T_c \exp\left(ig \int_x^y B_\mu dx^\mu\right) \Psi_{f'}(y) \, |0\rangle \qquad /15/$$

where f and f' =1,2,3 are flavors, C is the contour, connecting spacelike points x and y , and T_C orders the matrices of gluon fields

$$B_\mu = \tau^a B_\mu^a \qquad /16/$$

along the contour C . Apparantely the string state depends on the whole contour, and the gluon field strength

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + ig[B_\mu B_\nu] \qquad /17/$$

is a measure of this dependence. In principle we could have considered the multiconnected contour - with separate closed parts. At finite n_c these contours would also contribute to meson state, but at $n_c = \infty$ they are absent.

The connected parts describe <u>closed strings</u>

$$\frac{1}{n_c} \mathrm{Tr}\ T_c \exp(ig \oint B_\mu dx^\mu) \qquad /18/$$

According to [3] the mixture of one closed string in meson is of the order of n_c^{-1} .

Here we confine ourselves only to mesons. The dynamical problem is to find the weights of various contours C of the string in the meson state. The confinement corresponds to supression of the long strings. The average length, as well as the average amplitude of fluctuations is expected to come out $\sim \sqrt{\varphi(\lambda)}\,\mu^{-1}$ - the only scale of our theory.

The bare strings, however, are infinitely thin, and here comes the problem of ultraviolet divergences. One way to deal with this problem is to use a lattice gauge theory [7]. Actually, the string states were first considered within the framework of the lattice gauge theory [8].

But there is another way, which looks better, since it preserves the Lorentz invariance and locality.

Namely, we may introduce another equivalent basis instead of /15/-the <u>tensor states</u>

$$V_{\mu_1 \dots \mu_n}(x) \; |0\rangle \qquad /19/$$

$$V_{\mu_1 \dots \mu_n}(x) = \frac{1}{\sqrt{n_c}} \bar{\Psi}_f(x) \gamma_{\mu_1} \nabla_{\mu_2} \cdots \nabla_{\mu_n} \Psi_{f'}(x) \qquad /20/$$

where

$$\nabla_\mu = \partial_\mu + i g B_\mu \qquad /21/$$

is the covariant derivative.

The relation between tensor and string states is as follows: the state

$$\bar{\Psi}(x) \gamma_\mu \left(1 + dx^{(1)}_\alpha \nabla^\alpha\right) \left(1 + dx^{(2)}_\beta \nabla^\beta\right) \dots \Psi(x) \qquad /22/$$

which is superposition of tensor states, corresponds to a string, composed from elements $dx(1)$, $dx(2)$ ...

The C-dependence of the string state appears here because the tensor states are not symmetric in Lorentz indices. This asymmetry is related to the field strength

$$[\nabla_\mu \nabla_\nu] = ig F_{\mu\nu} \qquad /23/$$

Neither are the tensor states traceless. Altogether this means that tensor states are reducible with respect to Lorentz group.

The Lorentz irreducible tensor states can be obtained by substracting traces in all possible ways and by symmetrization and alternation.

Notice, that there is much more states, than in the free quark model, where tensor states were traceless and symmetric from the beginning. The additional states are related to the exitations of string.

In general we should also add the other Dirac matrices:

$$I, \gamma_5, \gamma_5\gamma_\mu, \sigma_{\mu\nu}$$

for completeness of our basis. But the corresponding S, P, A, T operators do not mix with V operators in the massless theory untill the γ_5-invariance is broken. We postpone the discussion of this breaking.

The Lorentz-irreducible fields, transforming according to representation

$$\left(\frac{p}{2}, \frac{p}{2}+q\right)$$

has $p+2q$ indices:

$$\mu_1 \ldots \mu_p, \quad \alpha_1\beta_1, \ldots \alpha_q\beta_q$$

It is symmetric in all μ

$$\mu_i \leftrightarrow \mu_j$$

in all pairs $\alpha\beta$

$$\alpha_i\beta_i \leftrightarrow \alpha_j\beta_j$$

and antisymmetric with respect to interchange inside pairs

$$\alpha_i \leftrightarrow \beta_i$$

If yields zero, when contracted with e_{abcd} in any three indices and is traceless in any two indices.

It is understood that some number 2τ of indices in /2o/ was contracted before symmetrization so that initially there was

$$n = p + 2q + 2\tau$$

indices. Notice, that there are many ways to contract 2τ indices from n, and the resulting fields are different. For brevity, we do not indicate this difference, and denote the irreducible field by

$$V^{i}_{\ldots}(x) \qquad /24/$$

Now we are in a position to discuss the divergences and renormalizations since the irreducible fields renormalize <u>multiplicatively</u>

$$\dot{V}^{i}_{\dots}(x) \to \sqrt{Z_i}\, \dot{V}^{i}_{\dots}(x) \qquad /25/$$

In principle, the renormalization constant Z_i is a matrix in a subspace of fields with given $pq\tau$. It goes without saying, that the fields are chosen so, that this matrix become diagonal.

After renormalizations our basis

$$\dot{V}^{i}_{\dots}(x)|0\rangle \qquad /26/$$

is free of ultaviolet divergences.

All the 2-point functions

$$\mathcal{D}^{ij} = \int d^4x\, e^{ikx} \langle 0| T^* \dot{V}^{i}_{\dots}(x)\, \dot{V}^{j}_{\dots}(0)|0\rangle \qquad /27/$$

depend only on scaling variable t in /2/ /not counting the trivial dependence on direction of K_μ, which is discussed later/.

As for the connected many-point functions, they are equal to zero at $n_c=\infty$, since the factors $1/\sqrt{n_c}$ in /2o/

are cancelled by the powers of n_c, coming from the quark loops only in the disconnected diagrams /see [3] for more details/.

The absence of connected many-point functions reflects the conservation laws, characteristic for a free field theory. Here it is the theory of infinite number of types of free mesons. The number of mesons of each type is conserved. The tensor basis /26/ is thus complete, in one-meson sector. The meson states are obtained by orthogonalization of tensor basis.

We may introduce the <u>meson wave function</u>, as a set of transition amplitudes from various irreducible tensor states to the meson state

$$\langle 0 | V^{i}_{\dots}(x) | k, j\{\gamma\}\rangle = e^{ikx} \Phi^{j,\{\gamma\}}_{i,\dots}(k) \qquad /28/$$

The meson state $|k, j \dots\rangle$ is described by 4-momentum on the mass shell

$$k_0 = \sqrt{\vec{k}^2 + M_j^2} \qquad /29/$$

by tensor indices

$$\{\gamma\} = \gamma_1 \cdots \gamma_s \qquad /30/$$

by spin S and by internal quantum numbers j. We use the standard representation for a particle with spin S : the wave function is symmetric and traceless with respect to $\gamma_1 \cdots \gamma_s$ and it is also transverse

$$K_{\gamma_i} \Phi^{\cdot\cdot\gamma_i\cdot\cdot}_{\cdots} = 0 \qquad /31/$$

to become irreducible $O(3)$-tensor in the rest frame.

Together with Lorentz covariance it fixes the K - dependence of the wave function

$$\Phi^{j\{\gamma\}}_{i\cdots}(K) = Y^{\{\gamma\}}_{\cdots}\left(\frac{K}{M}\right) g^{j}_{i} \qquad /32/$$

Here $Y(\xi_\mu)$ are standard orthonormalized polynomials in ξ_μ which are given by the group theory. We do not need their explicit form so far.

The problem is to find the reduced wave function g^j_i and the mass spectrum.

The input information is given by the rules of perturbation theory for the 2-point functions and by the concept of duality.

III. The Wave Operator

The relation between the meson wave function and the 2-point functions is given by the spectr l representation

$$\mathrm{Im}\, \mathcal{D}^{ii'} = \sum_{j\{\gamma\}} \Phi^{j\{\gamma\}}_{i\dots}(k)\, \bar{\Phi}^{j\{\gamma\}}_{i'\dots}\, \pi\, \delta(M_j^2 - k^2) \qquad /33/$$

It would be convenient to use the partial wave expansion

$$\mathcal{D}^{ii'} = \sum_{s,\{\gamma\}} Y^{\{\gamma\}}_{\dots}\left(\frac{k}{|k|}\right) \bar{Y}^{\{\gamma\}}_{\dots}\left(\frac{k}{|k|}\right) \cdot \cdot (|k|)^{-|\Delta - \Delta'|}\, f^{s}_{ii'}(k^2) \qquad /34/$$

here Δ and Δ' are the **dimensions of operators** V^{i} and $V^{i'}$ respectively. The factor

$$|k|^{-|\Delta-\Delta'|}, \qquad |k| \equiv \sqrt{k^2} \qquad /35/$$

removes the kinematic singularities from the partial waves.[x] The partial functions

$$f^{s}_{ii'}(k^2) \equiv \left(\hat{f}(k^2)\right)_{ii'} \qquad /36/$$

are supposed to be analytic at $k^2 = 0$.

[x] It will also play another role /see below/.

There remains in /34/ the multiple pole at $k^2=0$

$$\left(\frac{1}{k^2}\right)^{q+q'+\frac{1}{2}(p+p')+\frac{1}{2}|\Delta-\Delta'|} \qquad /37/$$

The partial functions with different S should obey certain conspiracy relations to cancell the poles at $k^2=0$ in the D-function. Since the Feynman diagrams for partial functions are determined up to addition of an arbitrary polynomial in k^2, we may always satisfy these conspiracy relations without altering the singular parts of the partial functions.

In the following we always consider only singular parts and do not care about conspiracy.

In accordance with the analysis of the previous sections we look for the solution for f-matrix in a mezomorphic form

$$\hat{f}(k^2) = \hat{Q}^{-1}(k^2)\,\hat{P}(k^2) \qquad /38/$$

where $\hat{Q}$ and $\hat{P}$ are some entire matrix functions, of k^2. The singularities of $\hat{f}$ are the poles, coming from zeros of $\det\hat{Q}$. There is an obvious relation between Q-matrix and residues of these poles

$$\hat{Q}(k^2)\,\mathrm{Im}\,\hat{f}(k^2)\Big|_{pole} = 0 \qquad /39/$$

In terms of reduced wave function it reads

$$\sum_i Q_{i'i}(M_j^2)\, M_j^{|\Delta'-\Delta|}\, g_i^j = 0 \qquad /4o/$$

Q-matrix thus has a meaning of the wave operator and /4o/ is the wave equation for our theory of free mesons. The P-matrix governs the normalization of wave function.

Now let us try to construct the perturbation theory for Q and P matrices.

This is not straightforward, since the original perturbation theory for $\hat{f}$-matrix do not have a mezomorphic form.

As it was discussed above, the original perturbation theory is valid far enough from the positive real axis in complex k^2-plane /fig. 3/.

The width of the nonperturbative strip is about

$$\mu^2/\varphi(\lambda) \qquad /41/$$

Our aim is to f nd an analytic continuation of perturbation expansion into this strip, which continuation would preserve the mezomorphic form.

We realize already that this continuation is not unique, and look for a minimal solution in a spirit of duality.

It will be convenient to proceed with an ordinary perturbation expansion in λ rather then in $\bar{\lambda}$. Let us take λ very small - this means that our normalization point

$$k^2 = -\mu^2 \qquad /42/$$

is very far from the forbidden strip.

We may calculate some number of derivatives of f - matrix with respect to k^2 at normalization point, using perturbation theory. When λ goes to zero, the number of derivatives which can be calculated perturbatively goes to infinity. This phenomenon will be discussed later.

Anyhow, suppose that we calculated $2N$ derivatives and let us construct the Påde- approximant , the rational matrix function

$$[N/N] = \hat{Q}_N^{-1}(k^2)\ \hat{P}_N(k^2) \qquad /43/$$

which reproduces $2N+1$ Taylor terms of $\hat{f}$ -matrix near normalization point

$$\hat{f}(k^2) - [N/N] = O\left(1 + \frac{k^2}{\mu^2}\right)^{2N+1} \qquad /44/$$

The numerator $\hat{P}_N$ and denominator $\hat{Q}_N$ are Nth degree matrix polynomials in K^2 to be determined from Pade-equations

$$\left(\frac{d}{dk^2}\right)^{\ell}\left(\hat{Q}_N\hat{f}-\hat{P}_N\right)_{k^2=-\mu^2}=0 \qquad /45/$$

The last N equations determine Q_N since P_N do not contribute. Then the first $N+1$ equations give an explicit expression for P_N. In terms of dispersion integral

$$\hat{P}_N=\hat{Q}_N\hat{f}-\int_0^\infty\frac{dK_1^2}{\pi(K_1^2-K^2)}\left(\frac{\mu^2+K^2}{\mu^2+K_1^2}\right)^{N+1}\hat{Q}_N(K_1^2)\,\mathrm{Im}\hat{f}(K_1^2) \qquad /46/$$

The integral cancels Taylor terms of $\hat{Q}_N\hat{f}$ starting from $N+1$th. Equation for Q_N reads

$$0=\int_0^\infty dk^2(\mu^2+k^2)^{m-2N-2}\hat{Q}_N(k^2)\,\mathrm{Im}\,\hat{f}(k^2) \qquad m=0,\dots N-1 \qquad /47/$$

This equation implies, that $\hat{Q}_N(k^2)$ is the orthogonal polynomial with respect to matrix measure

$$\mathrm{Im}\,\hat{f}(k^2)\left(k^2+\mu^2\right)^{-2N-2} \qquad /49/$$

Suppose, that we substituted into /47/, /46/ several terms of perturbation expansion for $\hat{f}$ -matrix

$$\hat{f}=\hat{f}^{(0)}+\lambda\hat{f}^{(1)}+\dots \qquad /50/$$

and found several terms of perturbation expansion for $\hat{Q}_N$ and $\hat{P}_N$

$$\hat{Q}_N = \hat{Q}_N^{(0)} + \lambda \hat{Q}_N^{(1)} + \dots \tag{/51/}$$

$$\hat{P}_N = \hat{P}_N^{(0)} + \lambda \hat{P}_N^{(1)} + \dots \tag{/52/}$$

At finite N this is some kind of phenomenological theory with two parameters: N and λ. Apparantely, we should tend N to infinity, and then the approximant would hopefully converge to a mezomorphic matrix function /38/.

$$\hat{Q}_\infty(k^2) = \hat{Q}(k^2) \tag{/53/}$$

$$\hat{P}_\infty(k^2) = \hat{P}(k^2) \tag{/54/}$$

This limit is independent on λ, but of course the critical N, after which Q and P approach the limit, do depend on λ.

As we show later

$$N_{crit} = \sqrt{\varphi(\lambda)} \tag{/55/}$$

The delicate problem of extrapolation to $N > N_{crit}$ is considered in the Section VII.

Prior to that we derive the rules of perturbation theory for $\hat{Q}_N$ and $\hat{P}_N$.

IV. Perturbation Theory For the Wave Operator

From the mathematical point of view we have the following problem: to construct the perturbation theory for the orthogonal polynomial, given the perturbation theory for the measure.

There is no general theory for that, but in our case such a theory can be developed.

Let us first describe the ordinary perturbation theory for the f-matrix.

In the zeroth order in λ the theory is conformally invariant . The 2-point functions of conformal tensors have the standard form

$$\langle T^* V^{i}_{\dots}(x) V^{i'}_{\dots}(0) \rangle = Z_i \delta_{ii'} X^{-2\Delta_i} \left\{ M_{..}(x) \cdots M_{..}(x) - \text{traces} \right\}_{sym} \qquad /56/$$

with

$$M_{\mu\nu}(x) = g_{\mu\nu} - 2 x_\mu x_\nu x^{-2} \qquad /57/$$

$$\Delta_i = n + 2 + 2\tau \qquad /58/$$

being the normal dimension of operator V^i

$$n = p + 2q \qquad /59/$$

being the total number of indices.

The substraction of traces and symmetrization /alternation/ is performed in the same way as for the operator V^i.

The normalization constans Z_i are irrelevant for our purposes; we may use the normalization

$$Z_i = \mu^{4-2\Delta_i} \qquad /60/$$

The conformal invariance is not at all trivial. If we simply take the operator, say the symmetric one

$$\left\{ \bar{\Psi} \gamma_0 \partial_0 \cdots \partial_0 \Psi - \text{traces} \right\}_{sym} \qquad /61/$$

and calculate the 2-point function /one quark loop at $\lambda = 0$/, it would not have the form /56/.

The conformal tensor is obtained from /61/ by adding total derivatives, namely

$$\sum_K (-)^K \binom{n-1}{K}^2 \left\{ \bar{\Psi} \gamma_0 \underbrace{\overleftarrow{\partial}_0 \cdots \overleftarrow{\partial}_0}_{K} \underbrace{\overrightarrow{\partial}_0 \cdots \overrightarrow{\partial}_0}_{n-1-K} \Psi - \text{traces} \right\}_{sym} \qquad /62/$$

This relation was found by A. B. Zamolodckikov /unpublished/.

In principle one may always find the conformal tensor, starting from its expression at zero total momentum and applying the conformal transformation. This transformation would yield the derivative terms, like /62/. We are not going to discuss the details here, since the explicit form of conformal tensors would not be requized in the first order calculations, which are described below.

It is natural to expect that in the first order in λ the conformal invariance will not break, but only the normal dimension /58/ would be shifted by anomalous dimension

$$\Delta_i(\lambda) = n + 2 + 2\tau + \lambda \gamma_i \qquad /63/$$

This conjecture can be verified by means of conformal Ward identities, but we prefer the following heuristic argument.

Consider the theory with the large number n_f of flavors. As it is well-known, at

$$\frac{n_f}{n_c} \to \frac{11}{2} - 0 \qquad /64/$$

where n_c is the number of colors, the first coefficient of β -function goes to zero, while the second coefficient has the opposite sign, so that there exists the fixed point

$$\lambda_* = \frac{4}{53}\left(11 - \frac{2n_f}{n_c}\right) + O(\lambda_*^2) \qquad /65/$$

At the fixed point the theory is conformally invariant. For the gauge-variant fields Ψ, B_μ the conformal transformations should in general be accompanied by the nonlinear gauge transformations, but the gauge invariant fields transforms by the linear law

$$i[\hat{K}_\mu V^i] = 2\Delta_i^* x_\mu V^i + 2x_\nu i[\hat{\Sigma}_{\mu\nu} V^i] + (x^2 g_{\mu\nu} - 2x_\mu x_\nu)\partial_\nu V^i \qquad /66/$$

where $\hat{K}_\mu$ is the generator of conformal transformations, $\hat{\Sigma}_{\mu\nu}$ is the generator of Lorentz transformations and Δ_i^* is the renormalized dimension

$$\Delta_i^* = n + 2 + 2\tau + \lambda_* \bar{\gamma}_i + O(\lambda_*^2) \qquad /67/$$

The conformal invariance of the 2-point function

$$\langle T^* [\hat{K}_\lambda V^i] V^j \rangle + \langle T^* V^i [\hat{K}_\mu V^j] \rangle = 0 \qquad /68/$$

determines its structure /56/.

Sertainly, the conformal tensor is not the same as at $\lambda = 0$. If contains the counter terms, which are proportional to λ_* at small λ_*. These counter terms remove divergences from the first order diagramms /see fig. 4/ for the symmetric tensor/, and make these diagrams conformally invariant.

But the first order diagrams do not depend on ratio n_f/n_c, since there are no internal quark loops. The external quark loop only gives overall factor n_f, which is removed after normalization.

This means, that the same counter terms, which make the 2-point functions invariant up to λ_*^2 at the particular ratio n_f/n_c, would make them invariant up to λ^2 at any ratio, and in particular, at

$$n_f/n_c = 0$$

which is our case.[x/]

It would be important to know the explicit form of the counter terms, but for the first order calculation it is sufficient to know the anomalous dimensions $\bar{\gamma}_i$.

Those can be extracted from the vertices

$$\langle T^* V^i \psi \bar{\psi} \rangle$$

at zero momentum where the derivative terms are absent, and the loop integrals simplify. For the symmetric tensor V^p the anomalous dimensions were calculated by Gross and Wilczek [9]

$$\bar{\gamma}_p = \frac{1}{2} - \frac{1}{p(p+1)} + 2\sum_{\ell=1}^{\infty} \frac{(p-1)}{(\ell+p)(\ell+1)} \qquad /69/$$

Similar expressions can be obtained for the other anomalous dimensions.

Let us now turn to the f-matrix.

Up to λ^2, untill there is conformal invariance, the f-matrix is diagonal and have the following dependence on k^2

[x/] Finally we are interested in the theory with 4 flavors and 3 colors, but now we expand in n_c^{-1} at $n_f=3$ / charmed quark is neglected, being heavy/.

$$f_{ij} = \delta_{ij} \frac{C_i(S,\lambda)}{\sin \pi \Delta_i(\lambda)} \left\{ \left(\frac{-k^2}{\mu^2}\right)^{\Delta_i(\lambda)-2} - \left(\frac{-k^2}{\mu^2}\right)^{\Delta_i-2} \right\} \qquad /70/$$

The coefficients $C_i(S,\lambda)$ can be found explicitely from /56/ after transforming into momentum space and projecting to the spin S. Fortunately we do not need these complicated expressions.

The second term in /7o/ with the normal dimension Δ_i is analytic in k^2 and do not contribute to the Fourier integral at $x \neq 0$. We added this irrelevant term for convencience - otherwise singularity would appear at $\lambda = 0$.

The imaginary part of f_{ij} which enters in equation for Q-matrix, has the simpler form

$$\operatorname{Im} f_{ij}(k^2 + i0) = \delta_{ij} \, C_i(S,\lambda) \left(\frac{k^2}{\mu^2}\right)^{\Delta_i(\lambda)-2} \qquad /71/$$

Thus we should find the orthogonal polynomial to the measure

$$\delta_{ij} \, (k^2)^{\Delta_i(\lambda)-2} (\mu^2 + k^2)^{-2N-2} \, \theta(k^2) \qquad /72/$$

Notice that s-dependence disappeared from the problem at this order in λ.

The corresponding polynomial reduces to Jacobi polynomial $P_N^{(\alpha,\beta)}(z)$ [10]

$$Q_{ij,N} = \delta_{ij} \, N^{-\nu} P_N^{(\nu,-\nu)}\left(\frac{\mu^2 - k^2}{\mu^2 + k^2}\right)\left(1 + \frac{k^2}{\mu^2}\right)^N \qquad /73/$$

with

$$\nu = \Delta_i(\lambda) - 2 \qquad /74/$$

The normalization factor here is a matter of convenience.

We are interested in large N and $k^2 << \mu^2$ i.e. $z \to 1$. In this limit the Jacobi polynomial reduces to the Bessel function [10]

$$Q_{ij,N}(k^2) \to \delta_{ij} \Lambda_\nu (k^2 N^2/\mu^2) \qquad /75/$$

where

$$\Lambda_\nu(u) = u^{-\nu/2} J_\nu(2\sqrt{u}) = (-u)^{-\nu/2} I_\nu(2\sqrt{-u}) \qquad /76/$$

This is very interesting phenomenon: the original scale μ is renormalized, so that at large N only μ/N enters. As we see later, this will be also true in the higher orders.

The calculation of numerator $\hat{P}_N$ is now straight-forward, and we find the approximant

$$[N/N] \to \delta_{ij} \frac{C_i(s,\lambda)}{\sin \pi\nu} \left\{ \left(\frac{-k^2}{\mu^2}\right)^{\nu} \frac{I_{-\nu}(2\sqrt{-u})}{I_\nu(2\sqrt{-u})} - \left(\frac{-k^2}{\mu^2}\right)^{\Delta-2} \right\} \qquad /77/$$

This is a mezomorphic function of K^2 with positive poles at

$$k^2 = \frac{\mu^2}{N^2} u(\nu) \qquad /78/$$

The branches of $u(\nu)$ are shown schematically at Fig 5.

According to general theory of Pade-approximants [11] the residues in these poles are also positive at $\nu > -1$ /one may verity it numerically/.

At $N \to \infty$ the scale N/μ tends to infinity and the poles condence. The approximant /72/ approaches the original function /7o/ exponentially, since

$$\frac{I_{-\nu}(2\sqrt{-u})}{I_{\nu}(2\sqrt{-u})} \underset{u\to\infty}{\longrightarrow} 1 + O(e^{-4\sqrt{-u}}) \qquad /79/$$

At this order the critical value of N do not yet display itself.

V. The Higher Orders

In the higher orders the f-matrix will have the structure

$$f_{ij} = \delta_{ij}\, f_i + \tilde{f}_{ij} \qquad /8o/$$

where f_i is the part which is conformally invariant and $\tilde{f}_{ij}$ is the part which breaks conformal invariance. The breaking starts from λ^2 as discussed above.

There would be two types of breaking terms

i/ Transitions between the operators with the same number of fields $\bar{\Psi}, \Psi, B$.

ii/ Transitions between the operators with the different number of fields.

The diagrams of the type i/ are indicabed at Fig 6 . They exist also in the first order, but in the first order they were diagonal in $\dot{i}\dot{j}$, while in the higher orders the nondiagonal terms would appear. In general these terms would exist for arbitrary difference between $p_i q_i$ and $p_j q_j$.

The diagrams of the type ii/ are indicated at fig. 7 We observe, that each additional gluon line yields coupling constant $\sqrt{\lambda}$, since it should be absorbed by the quark line.

The minimal number of gluon lines in the vertex is equal to

$$q + \tau \qquad /81/$$

Number q counts the number of commutators

$$[\nabla_\mu \nabla_\nu] = ig F_{\mu\nu}$$

Number τ counts the number traces. Each trace gives at least one gluon line, since

$$\nabla_\alpha \nabla_\alpha \Psi = i \sigma_{\mu\nu} F_{\mu\nu} \Psi \qquad /82/$$

according to equation of motion

$$\gamma_\mu \nabla_\mu \Psi = 0 \qquad /83/$$

If the numbers of lines are different in the left and right vertices, then the additional lines would yield $\sqrt{\lambda}$ and the amplitude would be proportional at least to

$$\left(\sqrt{\lambda}\right)^{|q_i + \tau_i - q_j - \tau_j|} \qquad /84/$$

Thus, if we start from the oparators with $q=\tau=0$, they will mix with high operators only in high orders.

As usual, the Feynman integrals will give only powers of K^2 and $1/d-4$, if the dimensional regularization will be used. The problem is to calculate numbers in front of these powers of K^2 and $1/d-4$ and to check, how these poles are removed by renormalizations.

Presumably one can do it by hand in the second order and in the higher orders the computer calculations are requized.

Suppose that we calculated the f-matrix, to some order in λ.

Then the problem is to find the wave operator to the same order.

Let us write

$$Q_{ij} = \delta_{ij}\Lambda_{\nu_i}(K^2N^2/\mu^2) + \tilde{Q}_{ij} \qquad /85/$$

where Λ_ν is determined by /75/ with the corresponding index ν, and let us substitute it together with /8o/ into Pade-equation /47/. The product of diagonal terms drops and we are left with

$$\int_0^\infty \frac{dK^2}{(K^2+\mu^2)^{L+1}}\left[\tilde{Q}_{ij}(K^2/\mu^2)^{\nu_j} + \operatorname{Im} R_{ij}\right] = 0 \qquad /86/$$

where $L = N+1, \dots 2N$,

$$R_{ij} = \sum_K Q_{iK}\,\tilde{f}_{Kj}\,C_j^{-1} \qquad /87/$$

$$\mathrm{Im}\, f_j = c_j (k^2/\mu^2)^{\nu_j} \qquad /88/$$

In the Appendix this equation is reduced to the form

$$\tilde{Q}_{ij} = \frac{N^2}{\mu^2} \int_0^\infty dk_1^2\, G_{\nu_j}(k_1^2, k^2)\, \mathrm{Im} \left\{ R_{ij}(k_1^2) \left(\frac{-\mu^2}{k_1^2} \right)^{\nu_j} \right\} \qquad /89/$$

/no summation over j / where $G_\nu(k_1^2, k^2)$ is the polinomial of N th degree in k^2. The explicit expression for G_ν is given in the Appendix. At large N this expression simlifies

$$G_\nu = \int_0^\infty dw\, \Lambda_\nu(u+w)\, \Lambda_{2-\nu}(w+v) \qquad /90/$$

Here

$$u = k^2 N^2/\mu^2 \qquad /91/$$

$$v = k_1^2 N^2/\mu^2 \qquad /92/$$

We may now write the following equation for Q -matrix

$$Q_{ij}(u) = \delta_{ij} \Lambda_{\nu_j}(u) +$$

$$+ \sum_k \iint_0^\infty dv\, dw\, Q_{ik}(v) \rho_{kj}(v) \Lambda_{\nu_j}(u+w) \Lambda_{2-\nu_j}(v+w) \qquad /93/$$

where

$$\rho_{kj} = c_j^{-1}\, \mathrm{Im} \left\{ \tilde{f}_{kj} \cdot (-N^2/v)^{\nu_j} \right\} \qquad /94/$$

This ρ still depends on N , but the powerlike dependence cancells so that only logarithmic dependence remains.

To see this property, let us recall, that the matrix elements of f-matrix are proportional to

$$|K|^{\nu_i^0+\nu_j^0+|\nu_i^0-\nu_j^0|} = (K^2)^{max(\nu_i^0,\nu_j^0)} \qquad /95/$$

where ν_i^0 corresponds to the normal dimension of operator V_i . The perturbation theory gives also the corrections

$$\sim (K^2)^{max(\nu_i^0,\nu_j^0)} \lambda^{n_1}\left(ln\left(\frac{-K^2}{\mu^2}\right)\right)^{n_2} \qquad /96/$$

Altogether we may write

$$\tilde{f}_{Kj} = \left(\frac{-v}{N^2}\right)^{max(\nu_K,\nu_j)} F_{Kj}\left(\lambda, ln\left(\frac{-v}{N^2}\right)\right) \qquad /97/$$

and we observe, that for

$$\nu_K \leq \nu_j \qquad /98/$$

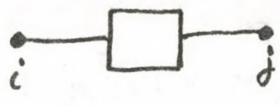
the powers of N cancell in /94/, while for $\nu_K > \nu_j$ the negative power remains, so that these matrix elements of ρ tend to zero. /Now we see the importance of the factor $|K|^{-|\Delta-\Delta'|}$ in definition of partial amplitudes/.

The surviving matrix elements of ρ depend on λ and $ln(v/N^2)$. We may iterate /93/ in terms of ρ and thus find the corrections to the wave operator.

It is convenient to use diagrams, shown at Fig. 8 The wave operator Q is represented as a line with a circle

$$= Q_{ij}(u) \qquad /99/$$

The line with the square represents the ρ matrix



$$= \theta(\nu_j - \nu_i)\, \rho_{ij}(u) \qquad /100/$$

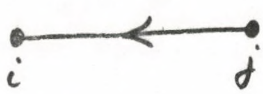
The line without arrow represents the diagonal term

$$= \delta_{ij}\, \Lambda_{\nu_j}(u) \qquad /101/$$

The line with arrow from left to right corresponds to

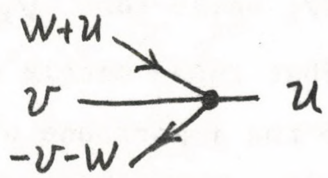
$$= \Lambda_{\nu_j}(u) \qquad /102/$$

The line with an arrow from right to left corresponds to



$$= \Lambda_{2-\nu_j}(-u) \qquad /103/$$

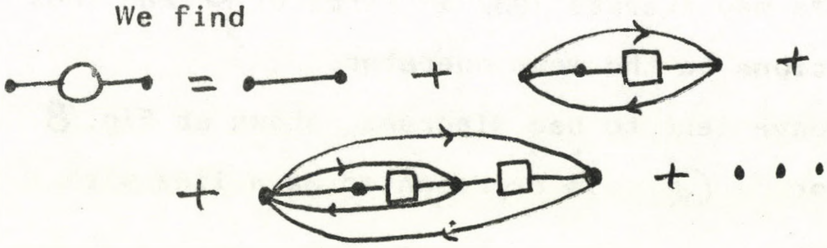
The u -variables are conserved in the vertices



/104/

and integration from zero to infinity is performed over free variables v, w.

We find



/105/

The arising integrals contain Bessel functions and powers of

$$\ln u$$

coming from the ρ -matrix. The oscillations of Bessel functions provide the convergence of the integrals, and the Θ -functions in /1oo/ restrict the summation over intermediate states, so that there is no sourse of divergeness.

As a matter of fact, these Θ -functions leads to asymmetry of the wave operator

$$Q_{ij} = 0 \quad \text{if} \quad \nu_i > \nu_j \qquad /1o6/$$

However, the theorems of Pade-theory [11] guarantee the symmetry of the approximant, since the original perturbative f -matrix is symmetric.

$$\text{if} \quad f_{ij} = f_{ji} \quad \text{then} \quad [N/N]_{ij} = [N/N]_{ji} \qquad /1o7/$$

Now let us discuss the perturbation theory for hadronic spectrum and for the wave functions.

If we substitute /85/ into wave equation /4o/, and take into account /1o6/, we come to the equation

$$\Lambda_{\nu_i}(u)\, \chi_i^{(j)} + \sum_{i'} \tilde{Q}_{ii'}(u)\, \chi_{i'}^{(j)} = 0 \qquad /1o8/$$

where $u = M_j^2 N^2/\mu^2$,

$$\chi_i^{(j)} = (M_j)^{\nu_i} g_i^{(j)} \qquad /1o9/$$

and M_j is the mass of meson. The sum over i' includes only states with

$$\nu_{i'} \geq \nu_i \qquad /110/$$

Without additional term $\tilde{Q}$ the wave function is simply

$$\chi_i^{(j)} = \delta_{ij} \qquad /111/$$

up to irrelevant normalization.

We should apply the standard perturbation theory to diagonalise Q -operator, i.e. to solve /108/ with the r.H.S.

$$E_j(u)\,\chi_i^{(j)} \qquad /112/$$

Then we should found the spectrum from

$$E_j(M_j^2 N^2/\mu^2) = 0 \qquad /113/$$

In the first order in $\tilde{Q}$ we find

$$E_j = \Lambda_{\nu_j}(u) + \tilde{Q}_{jj}(u) \qquad /114/$$

or

$$\frac{M_j^2 N^2}{\mu^2} = u(\nu_j) - \frac{\tilde{Q}_{jj}(u(\nu_j))}{\Lambda'_{\nu_j}(u(\nu_j))} \qquad /115/$$

We see, that up to λ^2 the mesons are described by the same quantum numbers, as the tensor operators. The mixing appears in the second order in λ

$$\chi_i^{(j)} = \delta_{ij} - (1 - \delta_{ij}) \frac{\tilde{Q}_{ij}(\mathcal{U}(\nu_j))}{\Lambda_{\nu_i}(\mathcal{U}(\nu_j))} \qquad /116/$$

Let us summarize this section as follows: the main problem is to calculate the diagrams of the ordinary perturbation theory for f-matrix - the corresponding terms in the wave functions and the spectrum would then readily be found.

VI. The Physical Coupling

So far we were unable to put

$$N = \infty \qquad /117/$$

Now we are going to rearrange the perturbation theory in such a way, that this limit will exist in any order in new expansion parameter.

The N-dependence of the terms of perturbation theory is only logarithmic and it is natural to try to renormalize the coupling constant λ as to eliminate this dependence.

It is possible due to the following important property of the wave operator: it depends only on two variables

$$Q_{ij} = Q_{ij}(\mathcal{U}, \lambda_N) \qquad /118/$$

where λ_N is the effective coupling, defined by the equation

$$\varphi(\lambda_N) = \varphi(\lambda)/N^2 \qquad /119/$$

At small λ

$$\lambda_N \to \frac{\lambda}{1-\frac{11}{3}\lambda \ln N} \qquad /12o/$$

When N tends to infinity the effective coupling λ_N increases and at

$$\lambda_N = \frac{11}{17} \;, \quad N^2 = \varphi(\lambda)/1.61 \qquad /121/$$

there is singularity. The problem of extrapolation to larger N will be considered below.

First let us prove the scaling property /118/. To this end it is convenient to use the equation /93/ where only the zeroth order term is included in ν , i.e.

$$\nu = \nu^0 = p + 2q + 2\tau \qquad /122/$$

The corresponding ρ -matrix then starts from the first order in λ and can be written as

$$\rho_{kj} = \frac{Im\,(f_{kj} - f^0_{jj}\delta_{kj})}{Im\, f^0_{jj}} \qquad /123/$$

where f_{kj} is the full f -matrix, and f^0 correspond to the zeroth term in it

$$Im\, f^0_{ij} = \left(\frac{k^2}{\mu^2}\right)^{\nu^0_i} C_j(s,0)\,\delta_{ij} \qquad /124/$$

Now, the renormalizability implies, that this ρ -matrix depends only on scaling variable (2) , i.e.

$$\rho_{kj} = \rho_{kj}\left(k^2\varphi(\lambda)/\mu^2\right) = \rho_{kj}\left(\nu\,\varphi(\lambda_N)\right) \qquad /125/$$

Since there is no other dependence on λ or N in the equation /93/ with $\nu=\nu^0$, we conclude that the wave operator depends only on $\mathcal{U}$ and λ_N.

In order to obtain the expansion in terms of λ_N one might have chosen the new normalization point for ρ - matrix:

$$\mu' = \mu/N \qquad /126/$$

The coupling in this point coincides with λ_N.

In practice, however, it would be easier to use the expansion in λ as discussed in the previous section and then substitute

$$\lambda = \lambda_N + \frac{11}{3}\lambda_N^2 \ln N + \cdots \qquad /127/$$

The terms with $\ln N$ should cancell in any order in λ_N due to renormalizability.

Thus we find

$$Q_{ij} = \delta_{ij}\left(\Lambda_{\nu_j}(\mathcal{U}) + \lambda_N \bar{\gamma}_j \frac{\partial}{\partial \nu}\Lambda_{\nu_j}(\mathcal{U})\right) + \cdots \qquad /128/$$

$$M_j^2 = \frac{\mu^2}{N^2}\left[\mathcal{U}(\nu_j^0) + \lambda_N \bar{\gamma}_j\, \mathcal{U}'(\nu_j^0) + \cdots\right] \qquad /129/$$

The mass spectrum has the structure

$$M_j^2 = \frac{\mu^2}{N^2}\, \Phi_j\left(\frac{\varphi(\lambda)}{N^2}\right) \qquad /13o/$$

The perturbative expansion /129/ corresponds to asymptotic expension of $\Phi_j(x)$ at large argument.

We are interested in the opposite limit where we expect

$$\Phi_j(x) \underset{x\to 0}{\longrightarrow} t_j/x \qquad /131/$$

$$M_j^2 \underset{N\to\infty}{\longrightarrow} t_j\,\mu^2/\varphi(\lambda) \qquad /132/$$

In principle one may try to check this behaviour by extrapolating sufficiently large number of terms of expansion. In practice, however, it is sufficient to consider the ratios of masses, since the overall scale is unknown.

We fix the mass M_ρ of the lighest ρ -meson and consider the perturbation theory for the ratios

$$\frac{M_j^2}{M_\rho^2} = \frac{\mathcal{U}(\nu_j^o)}{\mathcal{U}_{min}(1)} + \lambda_N \bar{\gamma}_j \frac{\mathcal{U}'(\nu_j^o)}{\mathcal{U}_{min}(1)} \qquad /133/$$

We used the fact that ρ -meson is coupled to vector current

$$\bar{\Psi}\gamma_\mu\Psi \qquad /134/$$

which has $\nu^o = 1$ and

$$\bar{\gamma}_\rho = 0 \qquad /135/$$

The anomalous dimension is anbsent, because vector current is conserved. One can see it directly from /69/ at $p=1$.

The wave operator in /128/ may also be expressed in terms of M_ρ, λ_N since

$$\mathcal{U} = \frac{k^2}{M_\rho^2}\left\{\mathcal{U}_{min}(1) + O(\lambda_N^2)\right\} \qquad /136/$$

Now the N -dependence enters only in the effective coupling λ_N.

But this coupling is singular at $N \to \infty$. It would be reasonable to express λ_N it terms of some physical quantity which coincides with λ_N at small λ_N, but do not have singularity at $N \to \infty$.

In other words, the physical coupling constant should be introduced instead of λ_N. The author apologices, that the physical coupling was not introduced from the very beginning, but in the beginning there, were no physical quantities to define the physical coupling. Now we have the spectrum and may try to define through the properties of the spectrum.

There are various possibilities.

One may define the physical coupling λ_ρ as an effective coupling, corresponding to the scale M_ρ, i.e.

$$\varphi(\lambda_\rho) \equiv \frac{M_\rho^2}{\mu^2}\varphi(\lambda) \qquad /137/$$

This λ_ρ can be related to λ_N

$$\varphi(\lambda_\rho) = \varphi(\lambda_N)\,\Phi_\rho(\varphi(\lambda_N)) = \\ = \varphi(\lambda_N)\left[\mathcal{U}_{min}(1) + O(\lambda_N^2)\right] \qquad /138/$$

or

$$\lambda_N = \lambda_\rho + \frac{11}{6}\lambda_\rho^2 \ln \mathcal{U}_{min}(1) + O(\lambda_\rho^3) \qquad /139/$$

One may expect, that at $N \to \infty$ this λ_ρ tends to constant. According to /132/

$$\varphi(\lambda_\rho) \rightarrow t_\rho \qquad /140/$$

This physical coupling is calculable in principle, but in practice it will remain as a phenomenological parameter. This definition was used in earlier version of this paper /1/. .

But there is another possible definition of physical couplings when it can be calculated explicitely at $N=\infty$.

Namely, let us consider the slope α'_ρ of ρ-trajectory at ρ -mass. It can be calculated perturbatively in terms of λ_N from /129/, /69/

$$\alpha'_\rho = \left(\frac{dM_\rho^2}{dP}\right)^{-1}_{P=1} = \frac{N^2}{\mu^2}\left(u'_{min}(1) + \lambda_N \bar{\delta}'(1)\, u'_{min}(1) + \ldots\right)^{-1} =$$

$$= \frac{N^2}{\mu^2 u'_{min}(1)}\left(1 - \lambda_N \bar{\delta}'(1) + O(\lambda_N^2)\right) \qquad /141/$$

$$\bar{\delta}'(1) = \frac{\pi^2}{3} - \frac{5}{4} \qquad /142/$$

There are also higher terms in λ_N in /141/, which can be calculated from the higher terms in /129/.

Now we may convert the expansion /141/ as follows

$$\lambda_N = \lambda_\rho + O(\lambda_\rho^2) \qquad /143/$$

where

$$\lambda_\rho \equiv \frac{1}{\bar{\delta}'(1)}\left[1 - \frac{\alpha'_\rho \mu^2 u'_{min}(1)}{N^2}\right] \qquad /144/$$

This definition of λ_ρ is convenient for extrapolation to $N \to \infty$ since <u>if</u> the slope α'_ρ tends to finite limit at $N \to \infty$ then the physical coupling tends to calculable number

$$\lambda_\rho \to \frac{1}{\bar{\gamma}'(1)} = 0.490 \qquad /145/$$

The qualitative dependence of λ_ρ of N is shown at Fig.9.

We may now expend the wave operator and the mass spectrum in λ_ρ , it we substitute /143/ into /133/. At $N = \infty$ we find finally

$$\frac{M_j^2}{M_\rho^2} = \frac{u(\nu_j^0)}{u_{min}(1)} + \frac{\bar{\gamma}_j}{\bar{\gamma}'(1)} \frac{u'(\nu_j^0)}{u_{min}(1)} + O(\lambda_\rho^2) \qquad /146/$$

This is a numerical expansion, which might appear to be asymptotic. The experience with ε-expension shows that sometimes the few first terms yield the reasonable approximation espessially it the Pade-Borel transformation is applied.

Anyhow, one should calculate the next term in /146/ and see, what happens. If it will come out to be smaller then the first term, than our approach would be reasonable.

VII. Discussion

Let us discuss the properties of the spectrum which we obtain in the first order in physical coupling constant.

First of all, where is dependence of the mass on spin S ? The first order formula /146/ depends only on quantum numbers of the operator V^j but not on spin of meson.

However, at given number

$$m_j = p_j + 2q_j \qquad /147/$$

of components of tensor V^j the spin S cannot exeed $p+q$

$$q_j \leq S \leq S_{max} = p_j + q_j \qquad /148/$$

For larger values of S the Y-polynomials are equal to zero.

It means that each mass M_j is degenerate - it corresponds to daughter mesons with spins /148/.

In the second order this degeneracy will split, since the ρ-matrix in /93/ depends on spin S .

Next, we observe that in the zeroth order there is additional degeneracy - the mass depends only on

$$\nu_j^0 = m_j + 2\tau_j \qquad /149/$$

This degeneracy is split already in the first order, since the anomalous dimensions γ_j depend on p_j, q_j, τ_j separately.

The formulas of the first order coincide with those of previous paper [1]. The procedure of higher order calculations which was proposed in [1], had several flaws, which we improved here. There was no cancellation of factors of N in higher orders in [1] and no restriction of summation over intermediate states. Here it was achieved by incorporating the factor $|K|^{|\Delta-\Delta'|}$ in the partial amplitudes. The treatment of Goldstone paeticles was also incorrect in [1]. This problem will be discussed in separate publication. Also in [1] the number of colors n_c was baken to be 3 rather then infinity. As it was discussed above, we expect the mezomorphicity of 2-point functions only for $n_c = \infty$. The $1/n_c$ corrections come both from ordinary diagrams /which was baken into account in [1] /, and from the decays of mesons to mesons. The second corrections will be discussed elsewhere.

Finally the coupling constant was fitted to experiment in [1], whereas here we propose to use the different definition which leads to the value /145/ at $N = \infty$.

The best fit of [1] corresponds in our notations to

$$(\lambda_\rho)_{fit} = .83 \qquad /150/.$$

This is almost twice larger then /145/ but the trajectory is not so sensitive to λ_ρ so that our value is also not bad.

The Fig. 10 shows the first order ρ -trajectories at two values of λ_ρ

$$\lambda_\rho = \left\{ \begin{matrix} 0.33 \\ 0.83 \end{matrix} \right\} \qquad /151/$$

The trajectory for our value /145/ lies in between. The lower branches corresponds to the second roots of Bessel function.

The other trajectories were also considered in [1] and the agreement to experiment was achieved at $\lambda_\rho = .83$ but we do not reproduce these results here, since the second order corrections in λ_ρ may change the the situation.

The only thing which can be said now is that the qualitative properties of trajectories are correct. The quantitative predictions can be given only after calculation the corrections from λ_ρ, $1/n_c$ and quark masses.

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Appendix

Green function of Pade equation

Here we solve the equation for self-energy part $\tilde{Q}$

$$\int_0^\infty \frac{dt \; (\tilde{Q}_{ij}(t)\, t^{\nu_j} + \operatorname{Im} R_{ij}(t+i0))}{(t+1)^{L+1}} = 0$$ /A.1/

$$L = M+N,\; M+N-1, \ldots M+1$$

The function

$$R_{ij}(t) = \sum_k Q_{ik} \tilde{f}_{kj} G_j^{-1} \quad , \; t = \frac{k^2}{\mu^2}$$ /A.2/

is supposed to be known.

Here we do not assume that M=N.

We return to a contour integral equation

$$\int_C \frac{dt}{2i \sin \pi\nu} \; \frac{\tilde{Q}(t)(-t)^\nu + R(t) \sin \pi\nu}{(t+1)^{L+1}} = 0$$ /A.3/

and look for a Green's function $G_\nu(t',t)$ which satisfies the equation

$$\int_C \frac{dt \; (-t)^\nu}{2i \sin \pi\nu} \; \frac{G_\nu(t',t)}{(1+t)^{L+1}} = - \frac{(-t')^\nu}{(1+t')^{L+1}}$$ /A.4/

In terms of the Green's function the solution for $\tilde{Q}$ has the form

$$\tilde{Q}_{ij}(t) = \int_{C} \frac{dt'}{2i} \frac{R_{ij}(t')\, G_{\nu_j}(t',t)}{(-t')^{\nu_j}} \qquad /A.5/$$

We understand, that the solution of homogeneous equation is included in Λ_ν in (85).

Let us check, that the solution for the Green's function is given by the following Mellin-Berns integral

$$G = (-1)^{\nu} \int_{C} \frac{dz}{2\pi i} \int_{C'} \frac{dz'}{2\pi i} \frac{1}{z-z'} \frac{f(z)\,(1+t)^{z}}{f(z')(1+t')^{z'+1}} \qquad /A.6/$$

where

$$f(z) = \frac{\Gamma(M+N+1-z)\,\Gamma(-z)}{\Gamma(M+1-\nu-z)\,\Gamma(N+1-z)} \qquad /A.7/$$

Contour C encloses the poles of $f(z)$ at $z = 0, 1, \ldots$ N, while the contour C' encloses the poles of $1/f(z')$ at $z' = M+1-\nu,\ M+2-\nu,\ \ldots +\infty$. When we substitute /A.6/ into /A.4/ we integrate first over t

$$\int \frac{dt\,(-t)^{\nu}}{2i\sin\pi\nu} (1+t)^{z-L-1} = B(L-\nu-z, \nu+1) \qquad /A.8/$$

Then the integral over z has the form

$$\int_{C} \frac{dz}{2\pi i} \frac{f(z)\, B(L-\nu-z, \nu+1)}{z-z'} \qquad /A.9/$$

The function $f(z)$ in /A.7/ was chosen in such a way that the integrand in /A.9/ reduces to the rational function

$$\frac{\Gamma(\nu+1)}{z-z'} \prod_{k=L+1}^{M+N} (k-z) \prod_{\ell=M+1}^{L-1} (\ell-\nu-z) \prod_{n=0}^{N} (z-n)^{-1} \qquad /A.10/$$

This function decreases as z^{-3} at infinity and has only one pole $z=z'$ outside the contour C. Thus the integral in /A.9/ is given by the residue at this pole

$$-f(z')\, B(L-\nu-z', \nu+1) \qquad /A.11/$$

Then $f(z')$ cancells and the remaining integral over z' reduces to

$$-(-1)^{\nu} \int_{C'} \frac{dz'}{2\pi i} (1+t')^{-z'-1} B(L-\nu-z', \nu+1) =$$

$$= -(1+t')^{-L-1} (-t')^{\nu} \qquad /A.12/$$

which is the r.h.s. of /A.4/. Thus /A.6/ is the solution. As it should be both $G_{\nu}(t',t)$ and $\tilde{Q}(t)$ in /A.5/ are polynomials in t of N th degree.

Let us now investigate the solution in more detail. If we represent $(z'-z)^{-1}$ in /A.6/as a integral

$$(z-z')^{-1} = -\int_0^{\infty} ds\, (1+s)^{z-z'-1} \qquad /A.13/$$

then the Green's function reduces to the following

$$G_{\nu}(t',t) = \int_0^{\infty} ds\, F((1+t)(1+s))\, \Phi((1+t')(1+s)) \qquad /A.14/$$

$$F(x) = \int_C \frac{dz}{2\pi i} f(z)\, x^{z} \qquad /A.15/$$

$$\Phi(x) = -\int_{C'} \frac{dz'}{2\pi i} \frac{1}{f(z')} \frac{1}{x^{z'+1}} (-1)^{\nu} \qquad /A.16/$$

These function may be expressed in terms of hypergeometric function F(a,b,c,z)

$$F(x) = \frac{\Gamma(M+N+1)}{\Gamma(N+1)\Gamma(M-\nu-1)} F(-N, \nu-M, -M-N, x)$$

/A.17/

$$\Phi(x) = (-1)^{\nu} x^{\nu-2-M} \frac{\Gamma(\nu-M+N)}{\Gamma(N+\nu)\Gamma(\nu-M-1)} \cdot$$

$$\cdot F(M+2-\nu, 1-N-\nu, 1+M-N-\nu, 1/x) \qquad /A.18/$$

Now, let us tend M, N to infinity. Then we may use in /A.15/, /A.16/ the asymptotic form of f

$$f(z) \to (-z)^{\nu-1} \exp\left(-\frac{MN}{z}\right)$$

/A.19/

which reduces F and Φ to the Bessel functions

$$F \to \left(\frac{MN}{t+s}\right)^{\nu/2} J_{\nu}\left(2\sqrt{MN(t+s)}\right) \qquad /A.20/$$

$$\Phi \to \left(\frac{MN}{t'+s}\right)^{1-\nu/2} J_{2-\nu}\left(2\sqrt{MN(t'+s)}\right) \qquad /A.21/$$

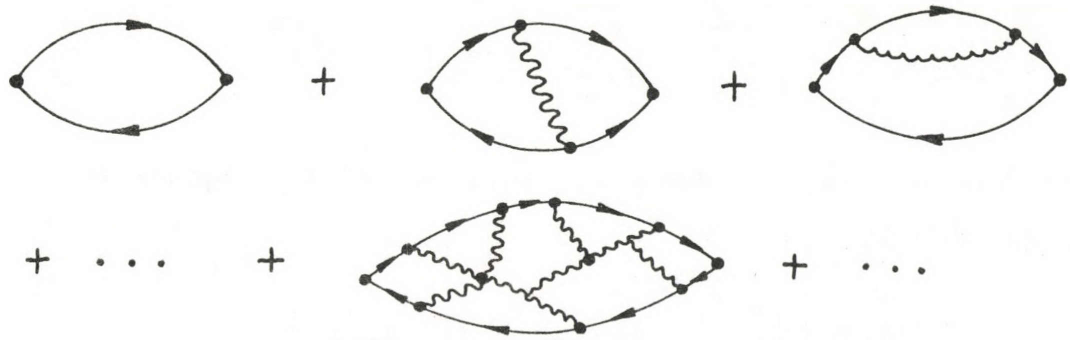
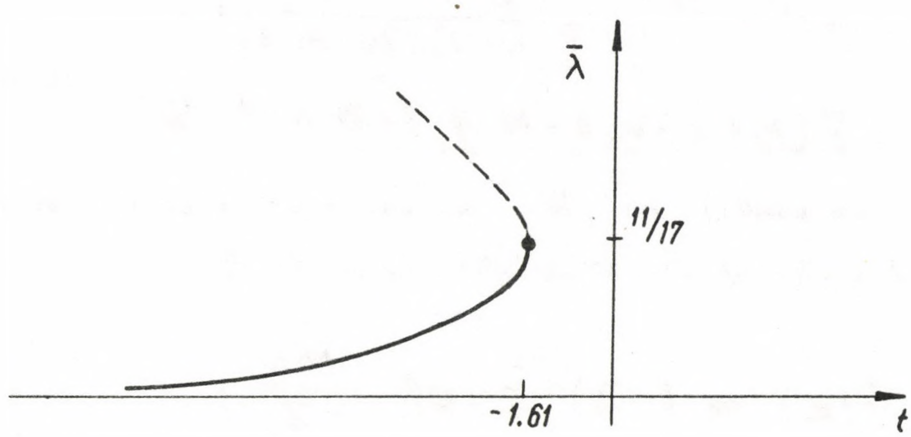


Fig. 1



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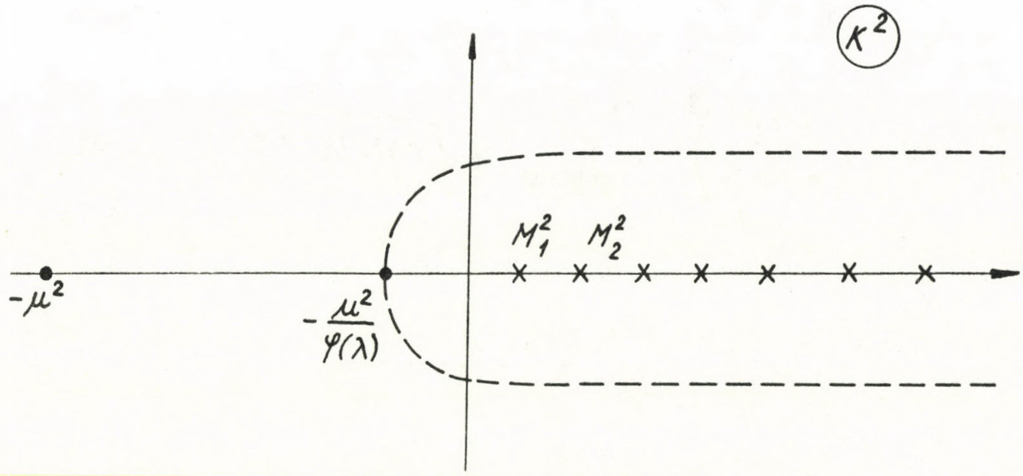

Fig. 3

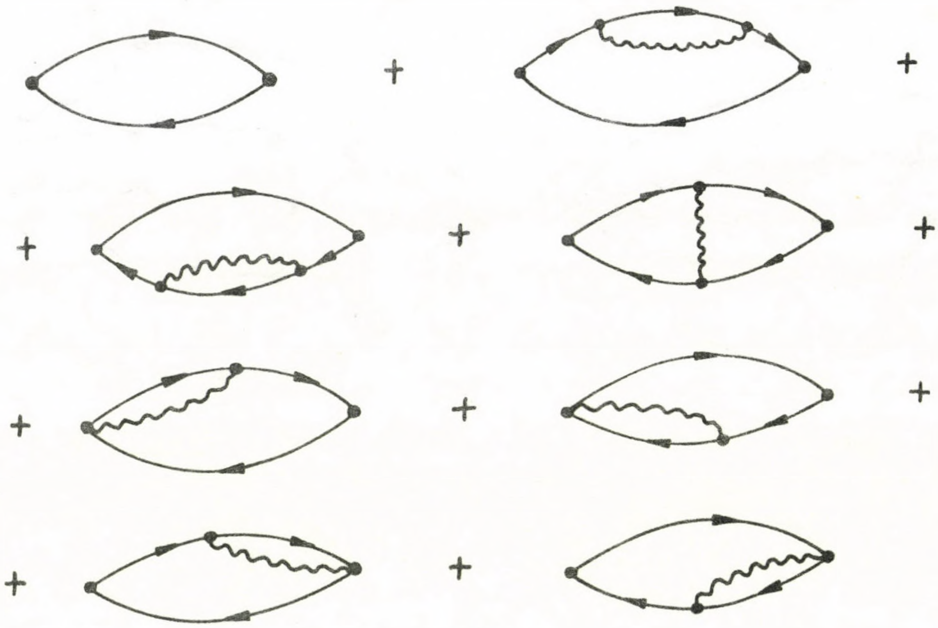

Fig. 4

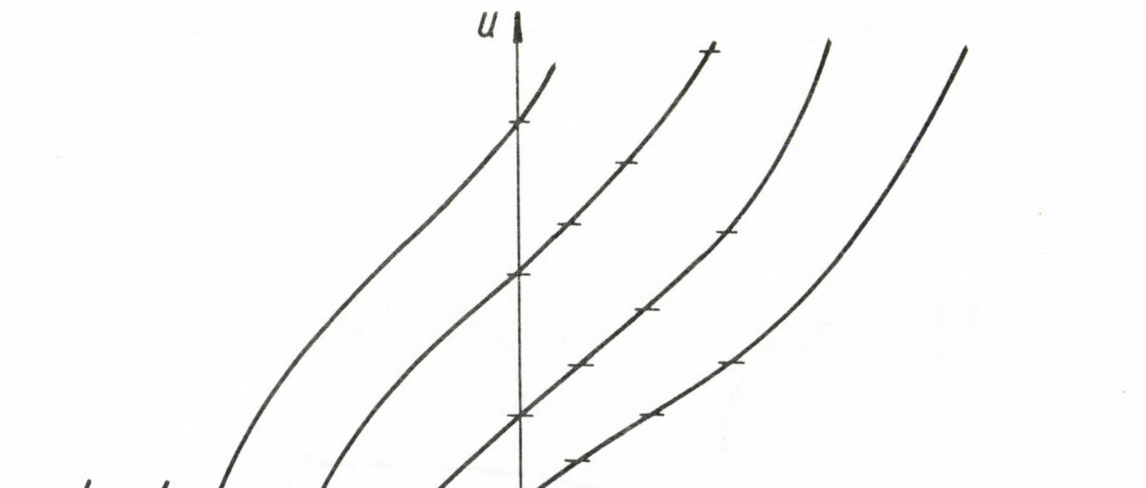

Fig. 5

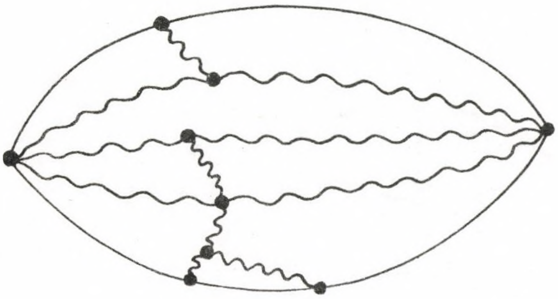

Fig. 6

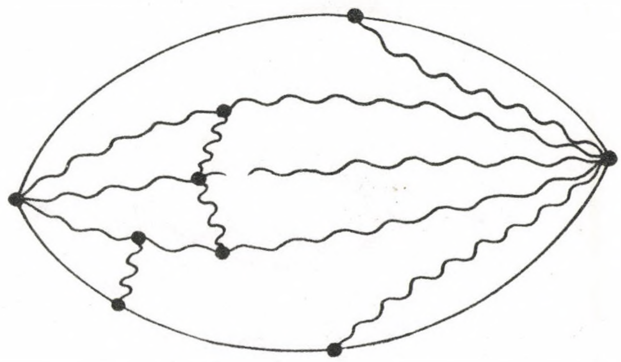

Fig. 7

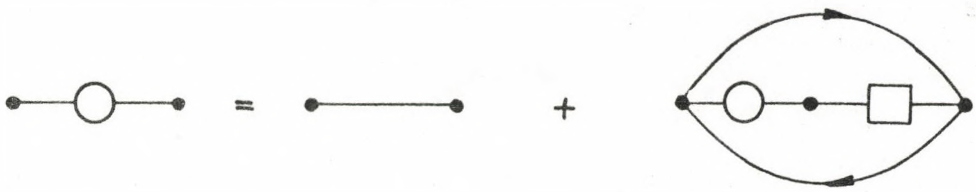

Fig. 8

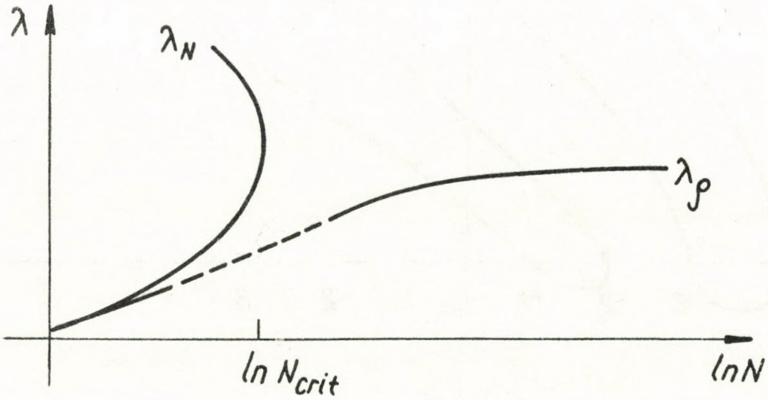

Fig. 9

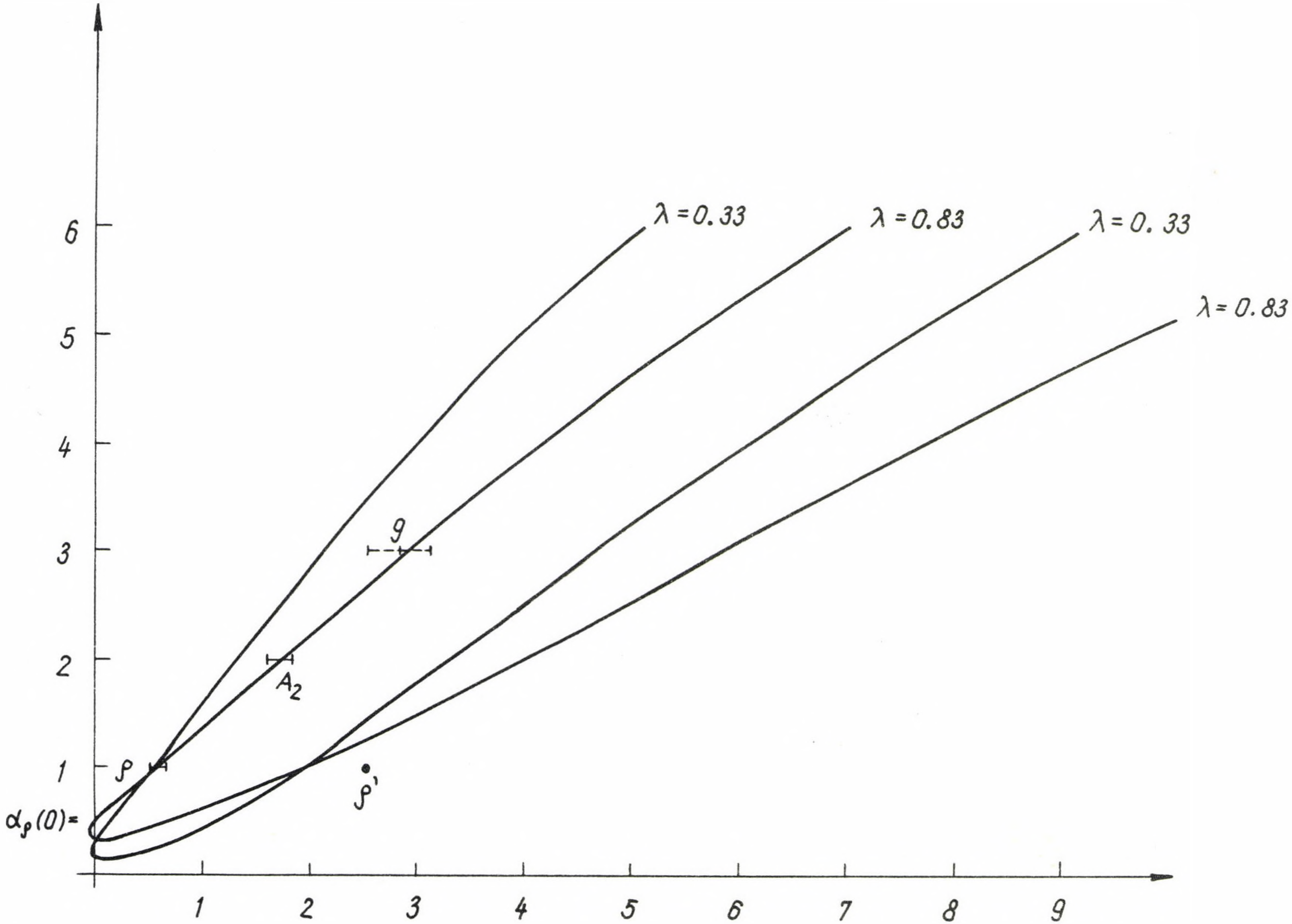

Fig. 10

Kiadja a Központi Fizikai Kutató Intézet
Felelős kiadó: Vasvári Béla igazgató
Szakmai lektor: Kuti Gyula
Nyelvi lektor: Forgács Gábor
Példányszám: 395 Törzsszám: 77-287
Készült a KFKI sokszorosító üzemében
Budapest, 1977. március hó