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Abstract

In this paper we suggest a new type of decomposition of the unequal mass scattering amplitude. We introduce two non-commuting , non-disjunct Poincaré groups, P^+ and P^-, both being sugroups of the $P_1 \otimes P_2$ group where P_i is the Poincaré group of the i-th particle; the P^+ group is identical to the group of the two-particle Poincaré transformations in the direct channel. This decomposition of the scattering amplitude is a double one, it runs in terms of the P^+ and P^- groups simultaneously. The end-formula is valid for any s and t; in the equal mass case at $t=0$ it gives back the usual one for Lorentz-pole contributions. This formalism seems to be adequate for understanding the meaning of the spectrum generating group and for the treating the breakdown of the symmetry. The variables of the expansion functions are defined unambiguously by the kinematical variables and have branchpoints only at thresholds and pseudo-threshold, in opposition to other approaches.

1. Introduction

During the last few years many attempts were done to eliminate the singularity developing at $u=o$ in the unequal mass scattering amplitude when expanded in terms of Regge-poles. As to the root of the problem two observations appeared to be important:

1. The contraction of the little group of the two-particle four-momentum at zero energy,

2. The center of mass system turns out to be meaningless at that point on the mass-shell.

The first phenomenon is unavoidable and expresses the fact that lightlike Poincaré representations are essentially different from timelike or spacelike ones. The second point means that the four-momentum of two particles having different masses can never be equal. This way, the partial wave expansion of the scattering amplitude in center of mass system can not be used for analytic continuation to zero energy, and the singularity found there is not a singularity of the scattering amplitude, being outside of the possible domain of the four-momenta.

The solution of the problem is to suppose essentially the same situation as for equal mass scattering, namely, there exist families of Regge-poles gathered in irreducible representations of the SL/2,C/ group. This was first noticed by Freedman and Wang, [1] a detailed analysis from group theoretical point of view was made by Domokos et al. [2,3] and by Toller and coworkers [4]. Another way to get rid of the singularity was found by di Vecchia et al. [5] with the help of analytical method.

Several attempts were made to use the notion of Lorentz pole at $u\neq o$. Delbourgo, Salam and Strathdee published the first paper on it [6], a different approach was elaborated by Domokos and Surányi [7], and later by Toller [8]. The analytical methods are poverful enough for this case as well [5].

Let alone the analytical approach, in the others either off-mass shell amplitudes were necessary or there were problems with momentum-conservation. Here we present a method based on group theory without these defects.

The new idea in our method is that in the space of two-particle states we introduce a P^- Poincaré group that acts on both particles and that can step between the representations of the group of the well-known two-particle

Poincaré transformations /P^+ group/. The scattering amplitude can be expanded in terms of these groups simultaneously at any s, t values, staying on mass-shell. This way we find the meaning of the spectrum generating group and can see the connection between equal and unequal mass case, concerning the "additional symmetry" of the scattering amplitude at zero energy.

2.1. The two-particle states

First we give a detailed description of two-particle states from the point of view of Poincaré representations. The two-particle states are the elements of a linear space, defined as the direct product space of one--particle states. The most usual and simplest way to enumerate the vectors of the direct product space is to enumerate those for both particles, separately. A two-particle state, denoted as $|p_1\, s_1\, \lambda_1, p_2\, s_2\, \lambda_2 >$, has 12 indices, namely the four-momenta p_1 and p_2, the spins s_1 and s_2, the helicities λ_1 and λ_2. In this space the Poincaré transformations are generated by the twice ten operators $P'_\mu, M'_{\mu\nu}, P''_\mu, M''_{\mu\nu}$./ P'_μ and $M'_{\mu\nu}$ are the four-momentum operator and angular momentum tensor for particle 1, the double primed operators are the same for particle 2./ the former representation diagonalizes the P'_μ and P''_μ operators. The indices s_1 and s_2 are the eigenvalues of the Casimir-operators $W'_\mu W'_\mu$ and $W''_\mu W''_\mu$, where

$$W^1_\mu = \frac{1}{2}\, \varepsilon_{\mu\nu\rho\kappa}\, M^1_{\nu\rho}\, P^1_\kappa \qquad /2.1/$$

The helicities λ_1 and λ_2 are the eigenvalues of W'_o and W''_o. The irreducibility of the representation space manifests itself in the fact that beside s_1 and s_2 the eigenvalues of the other two Casimir-operators $P'_\mu P'_\mu$ and $P''_\mu P''_\mu$, are fixed: $p_1^2 = m_1^2$ and $p_2^2 = m_2^2$. There is one more Poincaré invariant quantity: the sign of the eigenvalues of P_o. For both particles we choose this sign to be positive, by convention. However, this set of quantum numbers is not a practical one when we want to exploit the Poincaré invariance. Inhomogeneous Lorentz transformations are primarily the transformations of space-time, consequently, the transformations of the two-particle system as a whole, generated by the operators

$$P^+_\mu = P'_\mu + P''_\mu \quad , \qquad M^+_{\mu\nu} = M'_{\mu\nu} + M''_{\mu\nu} \qquad /2.2/$$

They form a Poincaré subalgebra of the direct sum algebra of $P'_\mu, P''_\mu, M'_{\mu\nu}, M''_{\mu\nu}$. Having no simple transformation properties under the transformations of this Poincaré subgroup, p_1 and p_2 are usually changed to $s, \underline{P}, W, \bar{m}$, to the eigenvalues of the operators $P^+_\mu P^+_\mu$, P^+_i , $W^+_\mu W^+_\mu$, W^+_o or W^+_3 , respectively. The usefulness of s and W are obvious; $\underline{P}$ and $\bar{m}$ were chosen for getting explicitly translation invariant basis vectors, but as it will be shown, this choice is not the most adventageous one in some cases, it is better to take the eigenvalues of the operators

$$\frac{1}{4}\, \varepsilon_{\mu\nu\rho\kappa} M^+_{\mu\nu} M^+_{\rho\kappa} \,, \quad \frac{1}{2} M^+_{\mu\nu} M^+_{\mu\nu} \,, \quad M^+_i M^+_i \,, \quad M^+_3 \qquad /2.3/$$

where

$$M_i = \frac{1}{2}\, \varepsilon_{ijk} M_{jk}$$

instead of $\underline{P}$ and $\bar{m}$. This choice is built upon the SL/2,C/ part of the Poincaré subalgebra. We write the appropriate set of quantum numbers as j_o, σ, j, m ; the eigenvalues of the operators enumerated in /2.3/ are $ij_o\, \sigma$, $j_o^2 - \sigma^2 + 1$, $j(j + 1)$, m respectively. One more generalization can be done in choosing the set /2.3/, namely, we can use the Casimir operator of other subgroups of SL/2,C/ instead of $M^+_i M^+_i$ c.f. Appendix A.

One problem arises from the change of $\underline{P}^+$, W^+_o to the set /2.3/: W'_o and W''_o do not commute with them. However, the operators $(W'_\mu + W''_\mu)(W'_\mu + W''_\mu)$, $P^+_\mu (W'_\mu - W''_\mu)$ make complete again the set of the 12 commuting operators being necessary to describe the two-particle states. We shall come back later to these two operators. We note here only that they define "Lorentz invariant quatum numbers", that is to say they commute with the generators $M^+_{\mu\nu}$. We shall denote the eigenvalues of $(W'_\mu + W''_\mu)(W'_\mu + W''_\mu)$ and $P^+_\mu (W'_\mu - W''_\mu)$ as Σ^+ adn Λ^+ .

Now let Λ^+ be an element of the homogeneous Lorentz group generated by the $M^+_{\mu\nu}$ -s, and $U(\Lambda^+)$ the operator which represents Λ^+ on the space spanned by the basis vectors $|m_1\, m_2\, s_1\, s_2\, s\, W^+ j_o \sigma j m,\, \Sigma^+ \Lambda^+\rangle$ /m_1, m_2, s_1, s_2 later will be suppressed/. Obviously

$$U(\Lambda^+)|\ldots;\, j_o\sigma jm\ldots\rangle = \sum_{j'm'} D^{j_o\sigma}_{j'm'jm}(\Lambda^+)|\ldots;\, j_o\sigma j'm'\ldots\rangle \qquad /2.4/$$

where $D^{j_o\sigma}_{j'm'jm}$ is an SL/2,C/ representation matrix element, well-known from the literature [14, 15, 16].

There exists another possible choice of quantum numbers and it is based upon the fact that one can pick out a Poincaré subgroup different from /2.2/:

$$P_o^- = P_o' - P_o'' , \quad P_i^- = P_i' + P_i'' , \quad N_i^- + N_i' - N_i'' , \quad M_i^- = M_i' + M_i'' \qquad /2.5/$$

Here $N_i = M_{oi}$. We shall call this subgroup the P^- group, contrary to the previous P^+ group. An element of its homogeneous part, generated by the M_i^- and N_i^- operators, will be denoted by Λ^- , it is evident we can define the analogons of the former (+) type commuting operators changing the (+) type generators to (-) type ones. Denoting the new quantum numbers as $\tau, W^-, l_o \rho l \mu, \Sigma^- \Lambda^-$, we can write the analogue of /2-4/ as follows:

$$U(\Lambda^-)|\ldots; l_o\rho l\mu\ldots\rangle = \sum_{l'\mu'} D^{l_o\rho}_{l'\mu', l\mu}(\Lambda^-)|\ldots; l_o\rho l'\mu'\ldots\rangle \qquad /2.6/$$

Obviously, the representation functions $D^{l_o\rho}_{l'\mu' l\mu}$ are the same as the $D^{j_o\sigma}$ functions being the differential equations and boundary conditions defining them quite the same.

Now we summarize the commuting operator sets and quantum numbers we have spoken about

a/ $P'_\mu P'_\mu$, $W'_\mu W'_\mu$, $P''_\mu P''_\mu$, $W''_\mu W''_\mu$; $\underline{P}'$, W'_o, $\underline{P}''$, W''_o
m_1 s_1 m_2 s_2 $\underline{p}_1$ λ_1 $\underline{p}_2$ λ_2

b/ $P^+_\mu P^+_\mu$, $W^+_\mu W^+_\mu$, $\underline{P}^+$, W^+_o ; W'_o, W''_o
s W^+ $\underline{p}$ $\bar{m}$ λ_1 λ_2

c/ $P^-_\mu P^-_\mu$, $W^-_\mu W^-_\mu$, $\underline{P}^-$, W^-_o ; W'_o, W''_o
τ W^- $\underline{p}$ λ_1 λ_2

d/ $P^+_\mu P^+_\mu$, $W^+_\mu W^+_\mu$, $\underline{M}^{+2}-\underline{N}^{+2}$, $\underline{M}^+\underline{N}^+$, $\underline{M}^{+2}$, M^+_3 ; $(W'_\mu+W''_\mu)^2$, $P^+_\mu(W'_\mu-W''_\mu)$
s W^+ (j_o,σ) j m Σ^+ Λ^+

e/ $P^-_\mu P^-_\mu$, $W^-_\mu W^-_\mu$, $\underline{M}^{-2}-\underline{N}^{-2}$ $\underline{M}^-\underline{N}^-$ $\underline{M}^{-2}$ M^-_3 ; $(W'_\mu+g_{\mu\nu}W''_\nu)^2$, $P^-_\mu(W'_\mu-g_{\mu\nu}W''_\nu)$
τ W^- (l_o,ρ) l μ Σ^- Λ^-

As to the quantum numbers, we make some remarks:

1. As a consequence of the fact that the P^+ and P^- algebras are not disjoint, being $\underline{P}^+=\underline{P}^-$, $\underline{M}^+=\underline{M}^-$, the quantum numbers $\underline{p}^+$, $\underline{p}^-$ and /j,m/, /l,μ/ are the same in the sets b/, c/ and d/, e/ respectively. Naturally, working with subgroups of SL/2,C/ being different from that of M_i-s , the appropriate indices loose this property.

2. A Λ^+ transformation leaves invariant the form $(p_o'+p_o'')^2 - (\underline{p}' + \underline{p}'')^2$ whereas Λ^- does the same for $(p_o' - p_o'')^2 - (\underline{p}' + \underline{p}'')^2$. The notions "vector", "tensor" etc. are different for P^+ and P^- groups.

3. In the sets a/, b/, c/ instead of λ_1 and λ_2 one can use the quantum numbers Σ^+, Λ^+ or Σ^-, Λ^- respectively.

4. It is a highly delicate question to ask for the transformation from one type of basis system to another one. This problem will be treated in the Appendices.

To conclude the discussion of quantum numbers we deal with the eigenvalues of $(W_\mu' + W_\mu'')(W_\mu' + W_\mu'')$ and $P_\mu^+(W_\mu' + W_\mu'')$. First it can be written on momentum states:

$$W_\mu \, |ps\lambda\rangle = \frac{1}{2}\,\varepsilon_{\mu\nu\rho\kappa}\, M_{\nu\rho}\, p_\kappa \, |ps\lambda\rangle = S_\mu(p)\,|ps\lambda\rangle \qquad /2.7/$$

The transformation property of the operator $S_\mu(p)$ under Lorentz-transformations:

$$U(\Lambda)\; S_\mu(p)\; U^{-1}(\Lambda) = L_{\mu\nu}(\Lambda)\; S_\nu(\Lambda p) \qquad /2.8/$$

The notation is obvious. Being $(W_\mu' + W_\mu'')^2$ and $P_\mu^+(W_\mu' - W_\mu'')$ covariant operators we confine ourselves to their eigenvalues on "equal velocity states" /the particles have the same velocity of opposite direction; see Section 3./. We use the phase convention of Jacob and Wick for two-particle states [9] and write:

$$|p_1 s_1 \lambda_1, \, p_2 s_2 \lambda_2\rangle = B_1(\alpha)\; R_2(\pi,\pi,0)\; B_2(\alpha)\,|m_1 s_1 \lambda_1\rangle \otimes (-1)^{s_2-\lambda_2}\,|m_2 s_2 \lambda_2\rangle$$

Here B and R are boost and rotation operators acting on one-particle states. Let us introduce the following linear combination:

$$|p_1 s_1, \, p_2 s_2; \sigma\lambda\rangle = \sum_{\lambda_1 \lambda_2} C^{\sigma\lambda}_{s_1\lambda_1 s_2-\lambda_2}\; |p_1 s_1 \lambda_1; \, p_2 s_2 \lambda_2\rangle \qquad /2.9/$$

The linear combination is made by the Clebsch-Gordan coefficients of the rotation group. These states are eigenstates of the operator $P^+_\mu (W'_\mu - W''_\mu)$ Using eqs. /2.7/ and /2.8/ we find:

$$P^+_\mu \left(W'_\mu - W''_\mu\right)|p_1 s_1,\ p_2 s_2;\sigma\lambda> = \frac{\lambda}{2}\left|s^2-2s\left(m_1^2+m_2^2\right) + \left(m_1^2-m_2^2\right)^2\right|^{1/2}|p_1 s_1,\ p_2 s_2,\sigma\lambda>$$

The situation is a bit complicated in the case of $\left(W'_\mu + W''_\mu\right)^2$. After some straightforward calculation we get the result :

$$\left(W'_\mu + W''_\mu\right)^2 |p_1 s_1,\ p_2 s_2,\ \sigma\lambda> = \{ |m_1(m_1-m_2)\ s_1\ (s_1+1) -$$

$$- m_2\ (m_1-m_2)\ s_2\ (s_2+1) + m_1 m_2 \sigma(\sigma+1)|\ \delta_{\sigma\sigma'} +$$

$$+ \left|2m_1 m_2 - \frac{\Delta\left(s,m_1^2,m_2^2\right)}{2s}\right| \sum_{\lambda_1\lambda_2} C^{\sigma\lambda}_{s_1\lambda_1 s_2-\lambda_2}\ C^{\sigma'\lambda}_{s_1\lambda_1 s_2-\lambda_2}\ \lambda_1\lambda_2\ |p_1 s_1, p_2 s_2, \sigma'\lambda>$$

This result says that there is a one-to-one correspondence between the "total spin" values σ and the eigenvalues of the operator $\left(W'_\mu + W''_\mu\right)^2$, and its eigenstates are linear combinations of the states defined by eq. /2.9/:

$$|p_1 s_1,\ p_2 s_2,\ \Sigma\Lambda> = \sum_\sigma A_{\Sigma\sigma}\ |p_1 s_1,\ p_2 s_2,\ \sigma\lambda> \quad ,$$

$$\left(W'_\mu + W''_\mu\right)^2\ |p_1 s_1,\ p_2 s_2,\ \Sigma\Lambda> \sim |p_1 s_1,\ p_2 s_2\ \Sigma\Lambda> \qquad /2.11/$$

We mention that in practical cases /$s_1=0$, s_2 arbitrary; $s_1=s_2=1/2$; $p_1=p_2=o$/ no diagonalization is necessary.

Similar results can be found repeating the calculations for the (-) type operators $\left(W'_\mu + g_{\mu\nu}\ W''_\nu\right)^2$, $P^-_\mu\left(W'_\mu - g_{\mu\nu}\ W''_\nu\right)$. In the following, either we are working with the (+) type set of quantum numbers or with the (-) type one, we shall use the symbols Σ and Λ, omitting $\pm$ indices. We think this will not lead to confusion, however it do not mean at all that there is a diagonality between Σ^+ and Σ^- values, but Λ^+ and Λ^- are essentially the same, c.f. App.C.

3.1. The expansion of the scattering amplitude.

After these preliminary steps now we concentrate to our very problem, to the expansion of the scattering amplitude. Let us consider the scattering process drawn in Fig.1. We should like to treat it at high values of s supposing exchanged poles in the t-channel. Since we have the crossing relation between s- and t-channel CM scattering amplitude [10]:

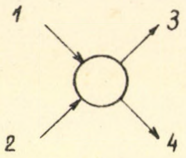

Fig.1.

$$s = (p_1+p_2)^2$$

$$t = (p_1-p_3)^2$$

$$\langle p_1\lambda_1,\ p_2\lambda_2|T|p_3\lambda_3,\ p_4\lambda_4\rangle = \sum_{\lambda_i'} d^{s_1}_{\lambda_1'\lambda_1}\, d^{s_2}_{\lambda_2'\lambda_2}\, d^{s_3}_{\lambda_3'\lambda_3}\, d^{s_4}_{\lambda_4'\lambda_4}$$

$$\langle p_1^c\lambda_1',\ p_3^c\lambda_3'|T|\ p_2^c\lambda_2',\ p_4^c\lambda_4'\rangle$$

where
$$(p_1+p_2)^2 = s \qquad (p_1^c + p_3^c)^2 = t$$
$$\underline{p}_1+\underline{p}_2 = 0\ ; \qquad \underline{p}_1^c + \underline{p}_3^c = 0 \qquad /3.1.1/$$
$$(p_1-p_3)^2 = t \qquad (p_1^c - p_2^c)^2 = s$$

Instead of expanding $\langle p_1,p_2|T|p_3,p_4\rangle = f^s$ in the crossed channel we shall do the same with $\langle p_1^c,p_3^c|T|\ p_3^c,p_4^c\rangle = f^t$ in the direct one. First we shall take f^t in its physical domain, and than we continue it analytically in s and t to the physical domain of f^s.

At the end of this process on both side of T in the twoparticle states one particle is negative timelike. Toller showed [4] that such a functional does not exist everywhere in the crossed domain though it is dense in it. This way, in a more rigorous treatment our formulae ought to be considered as means to define a functional, the domain of definition of what can be extended to the whole region in question. However we note, our way of speaking is accepted in physical literature, see e.g. [11] and many others.

During the analytic continuation the kinematical singularities could cause trouble so we get rid of them, multiplying f^t with an appropriate $K(s,t)$ function. Its form for any definite process is well known [12].

$$\bar{f}^t = K(s,t)\, f^t \qquad /3.1.2/$$

In what follows we shall speak only about the right-hand-side ket of f^t to save place but we always mean the whole amplitude.

The r.h.s. ket can be written as follows:

$$|p_4\lambda_4, p_2\lambda_2>_{CM} = L' p_4 |s_4\lambda_4> \circ L'' p_2 \; -1^{s_2-\lambda_2} |s_2\lambda_2> = e^{-i\phi M_3^+} e^{-i\Theta M_2^+} \cdot$$

$$\cdot\, e^{-i\alpha_4 N_3' - i\alpha_2 N_3''} e^{-i\pi M_2} e^{i\phi M_3^+} |s_4\lambda_4, s_2\lambda_2> \qquad /3.1.3/$$

The $-1^{s_2-\lambda_2}$ phase factor is introduced for convenience./This agrees with the Jacob-Wick phase convention [9]. /We introduce the notation

$$e^{-i\pi M_2''} e^{i\phi M_3^+} (-1)^{s_2-\lambda_2} |s_4\lambda_4, s_2\lambda_2> = e^{i\phi(\lambda_2+\lambda_4)} |s_4\lambda_4, s_2-\lambda_2> = |R> \qquad /3.1.4/$$

In eq. /3.1.3/ we can write

$$\exp\left(-i\alpha_4 N_3'\right) \exp\left(-i\alpha_2 N_3''\right) = \exp\left(-i\alpha N_3^+\right) \exp\left(-i\beta N_3^-\right) \qquad /3.1.5/$$

Inserting the last equation into eq. /3.1.3/, we get

$$|p_4\lambda_4, p_2\lambda_2>_{CM} = e^{-i\phi M_3^+} e^{-i\Theta M_2^+} e^{-i\alpha N_3^+} e^{-i\beta N_3^-} |R> \qquad /3.1.6/$$

Since any general two-particle state appears as $\Lambda^+|p_1\lambda_1, p_2\lambda_2>_{CM}$ /c.f.eq. 3.1.6./, it has the form $\Lambda^+\Lambda^-|R>$ as well, where Λ^+ is a general homogeneous Lorentz transformation of 6 parameters /the last rotation gives only a phase, 5 parameters are essential/, Λ^- is a (-) type boost of one parameter, along the z-axis. /Wigner rotations are suppressed/. This way, the $\exp\left(-i\beta N_3^-\right)|R>$ is a good basis state for two-particle states in that sence that any other state can be obtained by Lorentz transformation; nay, a better base than the CM-states because we have trouble with the latter at t=o. Another aspect of eq. 3.1.6.: any two-particle state is a function over both the P^+ and P^- group. Our expansion of the scattering amplitude is nothing else but a simultaneous expansion in terms of these groups.

One can easily check that in eq. 3.1.6.

$$\mathrm{ch}\beta = \frac{1}{2\sqrt{m_2 m_4}} \sqrt{t - (m_2-m_4)^2} \quad , \qquad \mathrm{ch}\alpha = \frac{m_2+m_4}{2\sqrt{t m_2 m_4}} \sqrt{t - (m_2-m_4)^2} \qquad /3.1.7/$$

Hence

$$\beta = \ln \frac{1}{2\sqrt{m_2 m_4}} \left| \sqrt{t - (m_2-m_4)^2} + \sqrt{t - (m_2+m_4)^2} \right| \qquad /3.1.8/$$

and there is a similar expression for α . Now we continue analytically in t. The way of continuation is drawn in Fig.2.

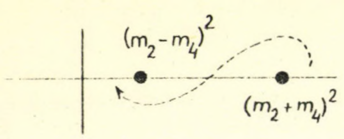

Fig.2.

we continue from $t'+i\varepsilon$, $t' > (m_2+m_4)^2$ to $t''-i\varepsilon$, $0 < t'' < (m_2-m_4)^2$ $\varepsilon > 0$. For the time being t" can not be smaller than zero, because at t=o singularities appear in ch α .During the continuation t somewhere is real, we choose this point between the threshold and pseudo-threshold [10]. A glance at eq. 3.1.8. shows that during this continuation we cross the cut of the ln-function and we cross the cut of one of the square-roots, hence their relative sign alters. This way, at the and of the continuation β goes over to $\beta'+i\pi/2$ where $\beta'+i\pi/2$

$$\mathrm{ch}\beta' = \frac{1}{2\sqrt{m_2 m_4}} \sqrt{(m_2+m_4)^2 - t} \qquad /3.1.9/$$

Similarly for α : it goes over to $\alpha'+i\pi/2$ where

$$\mathrm{ch}\alpha' = \frac{m_2-m_4}{2\sqrt{t m_2 m_4}} \sqrt{(m_2+m_4)^2 - t} \qquad /3.1.10/$$

We shift the effect of the $i\pi/2$ angles to $|R>$ let alone a phase factor it will alter the sign ot the mass of the crossed particle and does nothing else. Denoting $\exp i\pi N_3'' \; |R> = |\bar{R}>$ we are left with the following expression for f^t :

$$f^t_{\lambda_1\lambda_2\lambda_3\lambda_4} = <\bar{R}'| \, e^{i\beta'' N_3^-} e^{i\alpha'' N_3^+} \, T \, e^{-i\Theta M_2^+} e^{-i\alpha' N_3^+} e^{-i\beta N_3^-} \, |\bar{R}> \qquad /3.1.11/$$

where β'' and α'' can be got from β' and α' if $m_1 \leftrightarrow m_4$, $m_2 \leftrightarrow m_3$, and

$$\cos\Theta = \frac{t(s-u) + \left(m_1^2-m_3^2\right)\left(m_2^2-m_4^2\right)}{\left\{ \left|t-(m_1-m_3)^2\right| \left|t-(m_3+m_1)^2\right| \left|t-(m_2-m_4)^2\right| \left|t-(m_2+m_4)^2\right|\right\}^{1/2}}$$

Supposing $[T, M^{+}_{\mu\nu}]=0$, the (+) type transformations can be added [14]:

$$e^{i\alpha'' N_3} e^{-i\Theta M_2} e^{-i\alpha' N_3} = e^{-i\psi M_2} e^{-i\xi N_3} e^{-i\chi M_2}$$

where

$$\mathrm{ch}\xi = \left\{ s(m_1m_2+m_3m_4) - u(m_1m_3+m_2m_4) + (m_1-m_3)(m_2-m_4)(m_1m_3+m_2m_4)\right\} \cdot \left\{4m_1m_2m_3m_4 \; |(m_1+m_3)^2-t|\,|(m_2+m_4)^2-t|\right\}^{-1/2}$$

$$\cos\psi = -\mathrm{sh}^{-1}\xi\,|\mathrm{ch}\alpha' \,\mathrm{sh}\alpha'' - \cos\Theta\,\mathrm{sh}\alpha'\,\mathrm{ch}\alpha''|$$

$$\cos\chi = \mathrm{sh}^{-1}\xi\,|\mathrm{ch}\alpha''\,\mathrm{sh}\alpha' - \cos\Theta\,\mathrm{sh}\alpha''\,\mathrm{ch}\alpha'| \qquad /3.1.12/$$

As we see, the ugly singularity of $t=0$ has gone away.

However it worth examining whether the supposition $[T, M^{+}_{\mu\nu}] = 0$ below the pseudothreshold is just a consequence of Lorentz invariance, or something more. To see it, we make again the continuation without introducing α and β. We begin with the following state above threshold:

$$e^{-i\alpha_4 N_3'} e^{-i\alpha_2 N_3''} |R\rangle \qquad /3.1.13/$$

where

$$\mathrm{ch}_4 = \frac{t + m_4^2 - m_2^2}{2\sqrt{t}\, m_4}, \qquad \mathrm{ch}_2 = \frac{t + m_2^2 - m_4^2}{2\sqrt{t}\, m_2}$$

During the same continuation as before $\alpha_4 \to -\alpha_4$, $\alpha_2 \to -\alpha_2 + i\pi$
We again shift the effect of π onto $|R\rangle$ that makes the sign of m_2 changed. Since this causes a change in the sign of $\underline{p}_2$ as well, if we want to maintain the condition $\underline{p}_4+\underline{p}_2=0$, we either have to introduce an exp $-i\pi M_2''$

factor or tot change the sign of N". We choose the latter, but this means that under the continuation the (+) type and (-) type generators change role. In other words: above threshold the $[T, M^{+}_{\mu\nu}] = 0$ condition means Lorentz invariance, but below pseudo-threshold it is a new condition that we forced to make to maintain the analyticity of the scattering amplitude. /Analyticity would require it only between lightlike two-particle states, however we suppose it to be true as operator relation./ The fact that the P^{+} and P^{-} groups change role is not so surprising. If t is below the pseudo-threshold, the f^{s} amplitude is in its physical domain, there the Lorentz transformations are generated by $M^{1}_{\mu\nu} \circ M^{3}_{\mu\nu}$ and $M^{2}_{\mu\nu} \circ M^{4}_{\mu\nu}$. In the amplitude f^{t} the 2. and 3. particles are crossed and as we have shown this causes a change in the sign of the N_i and P_o generators.

After having added the (+) type transformations in eq. 3.1.11. we insert full systems of $|+\rangle$ and $|-\rangle$ vectors, and from λ_1, λ_2 we go over to Σ, Λ quantum numbers in $|\bar{R}\rangle$, $|\bar{R}'\rangle$:

$$f_{\Sigma\Lambda\Sigma^*\Lambda^*}{}^{s,t} = \sum \langle m_1 s_1\, p_1=0,\ -m_3 s_3\, p_3=0,\ \Sigma\Lambda|-\rangle \langle -|e^{i\beta'' N_3^-}|-\rangle \,.$$

$$\langle -|+\rangle \langle +|e^{-i\psi M_2^+}\, e^{-i\xi N_3^+}\, e^{-i\chi M_2^+}|+\rangle \langle +|T|+\rangle \langle +|-\rangle \,.$$

$$\langle +|-\rangle \,.\langle -|e^{-i\beta' N_3^-}|-\rangle \langle -|m_4 s_4\, p_4=0;\ -m_2 s_2\, p_2=0,\ \Sigma^*\Lambda^*\rangle =$$

$$= \sum \langle m_1 s_1 p_1=0,\ -m_3 s_3 p_3=0,\ \Sigma\Lambda|(m_1+m_3)^2,\ l_o\rho\ l\mu,\ W^-_,\ \Sigma\Lambda\rangle \,.$$

$$.\ d^{l_o\rho}_{l\mu l'}(\beta'') \langle (m_1+m_3)^2,\ l_o\rho\ l'\mu,\ W^-_,\Sigma\Lambda|\ t; j_o\sigma jm,\ W^+,\Sigma'\Lambda'\rangle \,.$$

$$.\ D^{j_o\sigma}_{jmj'm'}(\psi,\xi,\chi)\ T_{\Sigma'\Lambda',\Sigma''\Lambda''}(t;\ j_o,\sigma,\ W) \,.$$

$$.\ \langle t,\ j_o\sigma j'm',\ W^+,\ \Sigma''\Lambda''|(m_2+m_4)^2,\ l'_o\rho'' l''\mu'',\ W^{-\prime},\ \Sigma^*\Lambda^*\rangle \,.$$

$$.\ d^{l'_o\rho'}_{l''\mu l''}(\beta') \langle (m_2+m_4)^2;\ l'_o\rho' l''\mu',\ W^{-\prime},\Sigma^*\Lambda^*|m_4 s_4 p_4=0,$$

$$-m_2 s_2 p_2=0,\ \Sigma^*\Lambda^*\rangle$$

In eq. 3.1.14. the summation /if necessary, integration/ runs over both the unitary and non-unitary representations of the (+) and (-) type Lorentz group, since the scattering amplitude is not syuare-integrable in the crossed channel. The Chlebsch-Gordan coefficients and the overlapping function between the (+) and (-) type vectors are given in the Appendices. The reduced matrix element of T does not depend on j , m because of the Wigner-Eckart theorem.

3.2. Behaviour at t=o, Lorentz-poles.

First we rewrite eq. 3.1.14. using the results of Appendix B and we separate the unitary and non-unitary representations of Λ^+

$$f^t_{\Sigma\Lambda\Sigma'\Lambda'} = \sum_{j_o} \int d\sigma \mathcal{N}^{j_o\sigma} \sum_{\substack{l_o\rho l \\ l'_o\rho' l'}} \mathcal{N}^{l_o\rho} \mathcal{N}^{l'_o\rho'} d^{l_o\rho}_{\Sigma\Lambda l}(\alpha'') \cdot$$

$$\langle m_1+m_3{}^2, l_o\rho l\lambda, .. | t, j_o\sigma jm .. \rangle D^{j_o\sigma}_{jmj'm'}(\psi,\xi,\chi) \langle t, j_o\sigma j'm' .. | m_2+m_4{}^2 l'_o\rho' l'\lambda' .. \rangle .$$

$$d^{l'_o\rho'}_{l'\Sigma'\Lambda'}(\alpha') T^{j_o\sigma}_{s\lambda s'\lambda'}(t,W^2) + \text{non unitary terms}$$

Now let it be t=o and $m_1=m_3$, $m_2=m_4$. As one can see from eq. 3.1.lo. $\alpha' = \alpha''=0$, hence we can sum over the $|->$ type vectors. The eigenvalues of the operators W^2_μ, $P_\mu(W'_\mu-W''_\mu)$ are zero since the components of the total four-momentum are zero.

$$F^t_{\Sigma\Lambda\Sigma'\Lambda'}(t=0, m_1=m_3, m_2=m_4) = \sum_{j_o} \int d\sigma \mathcal{N}^{j_o\sigma} D^{j_o\sigma}_{\Sigma\Lambda\Sigma'\Lambda'} T^{j_o\sigma}_{\Sigma\Sigma'} +$$

$$+ \text{non unitary terms}$$

The assumption of the Regge theory is that if $s\to\infty$ some non-unitary representations give the main contribution to the scattering amplitude and everything else can be dropped. The fact what kind of representations does appear, depends on t continuosly. This way, in eq. 3.2.1. we retain only some non unitary term. One of these at t=o gives a sequence of Regge poles, grouped in a Lorentz-family, as we know well from the "classical" papers. Maybe this introduction of the Regge poles seems a bit artificial at first sight, however we think it is just that, what everybody expected.

At t=o we know the interrelation between Regge- and Lorentz-poles, but we do not know what to say if $t \neq o$. However, the most important observation is that we need not say anything! If we pick up a Regge-pole at t=o, this pole through the overlapping function and through the $d^{l_o \rho}$ functions defines a sequence of $|\ldots l_o \rho \ldots\rangle$ vectors. These vectors do not depend on t! But from these t independent vectors through t dependent $d^{l_o \rho}$ functions and overlapping functions we get a lot of interfering Lorentz-poles:

$$\sum_{\substack{l_o \rho l \mu \\ l' s}} \langle t=0,\ j_o \sigma j m \ldots | (m_1+m_2)^2\ l_o \rho l \mu \ldots\rangle\ d^{l_o \rho}_{l \mu s}\big(\alpha(t=0)\big)\ d^{l_o \rho}_{s \mu l'}\big(\alpha(t)\big)\ .$$

$$\langle (m_1+m_2)^2,\ l_o \rho\ l' \mu \ldots |\ t, j_o' \sigma' j' m' \ldots\rangle = f\big(j_o \sigma, j' \sigma', j, j', t \ldots\big) \qquad /3.2.2/$$

This is not surprising, from the model of Domokos and Surányi we have got something similar: to a pole of j_o, σ quantum number in the first order of t poles with $j_o \pm 1, \sigma$; $j_o, \sigma \pm 1$ quantum numbers are mixed. A detailed evaluation of eq. 3.2.2. and further applications of the model will be the topic of a forthcoming paper.

Finally, we make a short remark on the expansion of the scattering amplitude. The technique developed in the Appendices for working with the IG /interpolating group, c.f. App.A./ makes us able to give a form for Lorentz-poles at t=o which is simpler, at least from a certain point of wiew, than the previous one.

This alternative method for the expansion is based on the observation of the former discussion that any two-particle state can be written as

$$|p_1 s_1 \lambda_1;\ p_2 s_2 \lambda_2\rangle = \Lambda^+ B^-\ |m_1 s_1 \lambda_1;\ m_2 s_2 \lambda_2\rangle = \Lambda^+\ |m_1 s_1 \lambda_1,\ m_2 s_2 \lambda_2;\ v\rangle$$

We shall refer to the ket $|m_1 s_1 \lambda_1,\ m_2 s_2 \lambda_2;\ v\rangle$ as "equal velocity state". The individual paricles of this state have the same velocity, determined by the boost B^-. The states of this kind have total four-momentum $p_o/1,0,0,v/$, consequently, they can be expanded immediately using the IG of "v-velocity" as subgroup of the homogeneous Lorentz-group:

$$|m_1 s_1 \lambda_1;\ m_2 s_2 \lambda_2;\ v\rangle \sim \sum_{j_o \sigma l \mu} \mathcal{N}^{j_o \sigma}\ \delta_{l \Sigma}\ \delta_{\Lambda \mu}\ \delta_{\Lambda, \lambda_1 - \lambda_2}\ \delta_{wl}\ |s, W, j_o \sigma l \mu,\ \Sigma \Lambda,\ v\rangle$$

Here the quantum number W is defined as

$$w^2 = p_o^2\ (1-v^2)\ j\ (j+1) = p_o^2 \left(l^2 - \tfrac{1}{4}\ (1-v^2)\right)$$

Now we are able to write the expansion of the two-particle state in the form:

$$|p_1 s_1 \lambda_1;\ p_2 s_2 \lambda_2\rangle \sim \sum_{\substack{j_o \sigma l \mu \\ l'\mu' l''\mu''}} \langle j_o \sigma l'\mu';0|j_o \sigma l \mu;v\rangle\, D^{j_o \sigma}_{l''\mu'' l'\mu'}(\Lambda^+)\,|\ldots j_o \sigma l''\mu'';0;\ldots\rangle \,.$$

$$\delta_{l\Sigma}\ \delta_{\Lambda\mu}\ \delta_{\Lambda\ \lambda_1-\lambda_2}\ \delta_{wl}$$

where $\langle j_o \sigma l \mu;0|j_o \sigma l'\mu' v\rangle$ stands for the overlapping integral between the states of SL/2,C/ representations with v=o and the actual v. In the t=o case it is the well-known E/2/ΛT/2/↔SU/1,1/ function. Now the Lorentz-pole part of the scattering amplitude can be written at t=o as follows:

$$\langle|T|\rangle_{pole} \sim \sum \langle j_o \sigma l'\mu';0|j_o \sigma l \mu;1\rangle\, T^{j_o \sigma}$$

$$D^{j_o \sigma}_{l'\mu' l''\mu''}\ \langle j_o \sigma l \mu;1|j_o \sigma l''\mu'';0\rangle$$

It can be seen again that there is not any singularity in the pole-contributions at t=o. For other values of t the form /3.2.2/ must be preferred because of the presence of the fixed $l_o \rho$ values.

4. Discussion.

In this paper we have given an expansion of the scattering amplitude in terms of the Lorentz-group, without any restriction for the external masses and for the invariant variables s,t. The expansion was made possible by introducing the P^+ and P^- groups and appropriate basis vectors, and by the supposition $[T, M^+_{\mu\nu}] = 0$. We suggest on the basis of eq. /3.2.2/. that in our formalism the daughter trajectories are not parallel, although, for the time being we have not verified it. A nice feature of the angles in eq. /3.1.9-10/ that they have singularity only at thresholds and pseudothresholds, in opposition to the "old" formalism. The factorizability of residua and that the amplitude in the form /3.1.14/ met the kinematical constraints need further investigations. We have not introduced the signature factor into eq. 3.1.14. To do it, we should have followed the well-known way. [22]

At last we should like to enlighten why we have introduced the IG. To find the overlap function $\langle +|-\rangle$ we needed the expansion of the "equal velocity" two-particle state in terms of its little group. At that point where the group structure changes much care should be administered to get faithful representation, i.e. to avoid singularity in the representation functions. Appendix A is essentially the description of this method both for unitary and nonunitary representations. In the unitary case the dimension of the representations changes, this altering j value has no physical meaning in our treatment. Had not we worked this way, a singularity would have developed in $<+|->$ at t=o.

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Appendix A

1. The interpolating group.

It was proved in the former paragraphs that when investigating the scattering amplitude it is extremely useful to work with other little groups, that is with other subgroups of the homogeneous Lorentz-group, than the usual rotation group. Now we give a detailed account for these "unusual" subgroups and representations of the Lorentz-group, parametrized making use of them.

As it is well-known, the generators of any little group can be found if the operators

$$W_\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho\kappa} M_{\nu\rho} P_\kappa \qquad /A1/$$

are taken on states having the special four-momentum which we want to concern ourselves with. Specially, we want p to be:

$$P = p_o/1,0,0,v/, \qquad /A2/$$

where $p_o > o$ fixed, and $o \leq v < \infty$.

After dividing by p_o we get the three independent operators /since $P_\mu W_\mu = o$/:

$$s_1(v) = M_1 + vN_2, \quad s_2(v) = M_2 - vN_1, \quad s_3(v) = M_3 \qquad /A3/$$

They can be used for generating the little group. Their commutation relations read:

$$[s_1, s_2] = i(1-v^2)s_3, \quad [s_1, s_3] = -is_2, \quad [s_2, s_3] = is_1 \qquad /A4/$$

It is obvious from eqs. /A3/,/A4/ that S_i'-s form subalgebra of the Lorentz-algebra. At the points v=o, 1 and $\sqrt{2}$ we have the well-known SU/2/, E/2/ T/2/ and SU/1,1/ algebras, and for different values of v the little groups turn smoothly into one another. We shall call the group generated by S_i - s interpolating group /IG/, because it "interpolates" between the timelike, lightlike and spacelike little groups.

The general form for the group-elements is:

$$G = \exp \; -i \sum_{i=1}^{3} \alpha_i \, s_i(v)$$

The ranges of the parameters will be discussed later. It is easy to show that the group can be parametrized in the Eulerian way, as it is usual for the special cases v=o,1,$\sqrt{2}$:

$$G = e^{-i\phi s_3(v)} \, e^{-i\theta s_2(v)} \, e^{-i\phi' s_3(v)} \qquad /A5/$$

Namely, after simple calculations for $v < 1$ we get:

$$e^{-i\Sigma\alpha_i s_i} = e^{-its_3(v)} \, e^{-ius_2(v)} \, e^{-i\gamma s_3(v)} \, e^{+ius_2(v)} \, e^{its_3(v)} \qquad /A6/$$

where

$$t = \mathrm{arctg}\,\alpha_2/\alpha_1 \quad , \quad \gamma = \left| (1-v^2)\left(\alpha_1^2+\alpha_2^2\right) + \alpha_3^2 \right|^{1/2}$$

$$u = \frac{1}{\sqrt{1-v^2}} \, \mathrm{arctg}\,\alpha_3^{-1} \left| (1-v^2)\left(\alpha_1^2+\alpha_2^2\right) \right|^{1/2} \qquad /A7/$$

Now the composition rule is necessary for writing the elements /A6/ into the compact form /A5/. It will be clear from the following that this rule is the same as the one for the rotation group. In the case $v > 1$ the calculations can be carried out similarly.

2. The representations of the $S_i(v)$ algebra.

Before examining the representations of the IG, we discuss the hermitian representations of the Lie-algebra of the generators $S_i(v)$ It will be a useful guide to get the representations of the IG. First we suppose $o \le v \le 1$ and take the well-known Lie-algebra of SU/2/:

$$|J_i, J_j| = i\,\varepsilon_{ijk}\,J_k \qquad /A8/$$

Next, we subject the J's to a transformation:

$$J_1^\lambda = \lambda J_1\,,\quad J_2^\lambda = \lambda J_2\,,\quad J_3^\lambda = J_3 \qquad /A9/$$

here $\lambda = \sqrt{1-v^2}$. The algebra of the operators J_i^λ is the same as that of the $S_i(v)$'s. Being the transformation /A9/ real and nonsingular except the point $\lambda=0$, we conclude that the algebra of $S_i(v)$'s for $o < v < 1$ has representations of the same kind as it has at v=o. Namely, all the hermitian irreducible representations are finite dimensional. The linear space, on which the representation is based, can be spanned by the eigenstates of the operator as basis system:

$$s_3(v) = |v;\ jm> = m\ |v;\ jm> \qquad /A10/$$

where m is integer or half-integer. The different irreducible representations have different maximal weight, j=max/m/. We write also v as index for the basis vectors, denoting the actual value of $\vec{v}$ for which we want to represent the algebra of S_i' s. The eigenvalue of the Casimir operator $s_1^2(v)+s_2^2(v) + (1-v^2)\ s_3^2(v)$, as usual, can be used as characteristic quantum number for irreducible representations:

$$\left[s_1^2 + s_2^2 + (1-v^2)\ s_3^2\right] |v;\ jm> = (1-v^2)\ j\ (j+1)\ |v;\ jm> \qquad /A11/$$

Writing like this the eigenvalue of the Casimir-operator, we lay stress on the connection of the $S_i(v)$ algebra with the one at v=o. It is obvious from the commutation relations that the operators $S_\pm(v) = S_1(v) \pm iS_2(v)$ are the raising and lowering ones. The matrix form of the $S_i(v)'$ s in the $|v,\ jm>$ basis is:

$$<v;jm|S_1|v,jm'> = i\ \frac{\sqrt{1-v^2}}{2}\left[\sqrt{j(j+1)-m(m+1)}\ \delta_{m',m+1} + \sqrt{j(j+1)-m(m-1)}\ \delta_{m',m-1}\right]$$

$$<v;jm|S_2|v,jm'> = -i\ \frac{\sqrt{1-v^2}}{2}\left[\sqrt{j(j+1)-m(m-1)}\ \delta_{m',m-1} - \sqrt{j(j+1)-m(m+1)}\ \delta_{m',m+1}\right]$$

$$<v,jm|S_3|v,jm'> = m\delta_{mm'} \qquad /A12/$$

Now we turn to the case v=1. It is highly an exceptional point being the algebra /A4/ the not semi-simple E/2 ΛT/2/ algebra. This break in the structure of the Lie-algebra is strongly correlated with the fact, that the trausformation /A9/ turns to be singular. Although the commutation relations for the J_i^o's are formally the same as those for the $S_i(1)$s, we cannot get the representations of the $S_i(1)$ algebra identifying them with those of the J_i^o's , that is to say, writing simply v=1 in formulae /A12/. This phenomenon is called contraction; in particular we have seen the E/2/ΛT/2/ algebra to be the contraction of the SU/2/ algebra. The problem arising from the fact of contraction is that the simple limit at v=1 gives unfaithful representations of the "contradted" algebra being not zero operator only $J^o{}_3$, the representative of the subalgebra with respect to which the SU/2/ algebra was contracted, For achieving faithful representations a standard method is to choose a divergent sequence of j's, too, when going to v=1. Namely, taken

$$j = \left[\rho/\sqrt{1-v^2}\right] \qquad \text{or} \qquad j = \left[\frac{\rho}{1-v^2}\right] + \frac{1}{2} \qquad /A13/$$

where ρ is a positive number and [c] denotes the integer part of number c, we get the matrices:

$$\lim_{\substack{v\to 1\\ j\to\infty}} \langle v;jm|S_1|v;jm'\rangle = \frac{1}{2}\,\rho\,\left(\delta_{m',m-1} + \delta_{m',m+1}\right) \quad ;$$

$$\lim_{\substack{v\to 1\\ j\to\infty}} \langle v;jm|S_2|v;jm'\rangle = \frac{i}{2}\,\rho\,\left(\delta_{m',m+1} - \delta_{m',m-1}\right) \; ; \quad \lim_{\substack{v\to 1\\ j\to\infty}} \langle v,jm|S_3|v;jm'\rangle = m\delta_{mm'}$$

These matrices are hermitian, commute like the elements of the E/2/ΛT/2/ Lie-algebra. Consequently, we reached the result: the $S_j(1)$ algebra is represented by hermitian operators on an infinite dimensiónal linear space spanned by $|\rho m\rangle$ basis vectors. For every value of ρ we have two kinds of representations depending on whether the eigenvalues of S_3 are integer or halfinteger. These representations are irreducible for any positive number. The Casimir-operator $S_1^2(1)+S_2^2(1)$ has the eigenvalue ρ^2 when acting ont these basis vectors. An alternative definition for the eigenvalue of the Casismir-operator instead of /A11/ is:

$$\left|S_1^2(v) + S_2^2(v) + \left(1-v^2\right) S_3^2(v)\right| \; |v;km\rangle = \left|k^2 - \frac{1}{4}\left(1-v^2\right)\right| \; |v;km\rangle \qquad /A15/$$

We changed here the index j in the basis vectors, too. The connection between j and k can be written as follows:

$$j = -\frac{1}{2} + \frac{1}{\sqrt{1-v^2}}\,k \qquad /A16/$$

The index k has its advantage in becoming a continuous one when going to v=1, being the distance of its possible values $\sqrt{1-v^2}$, and remains finite not like j does, being its limit the actual ρ we want to reach at v=1.

We do not repeat the discussion for the case $v \geq 1$, we write only the two main properties:

$$S_3(v)|v;jm\rangle = m|v;jm\rangle$$

$$\left[-S_1^2(v) - S_2^2(v) + (v^2-1)\,S_3^2(v)\right]|v,jm\rangle = (v^2-1)\,j\,(j+1)|v;jm\rangle \qquad /A17/$$

The possible values of j and m, the types of SU/1,1/ representations are well-known [14]. Attention must be paid again to the point v=1. Here we get the E/2/$\wedge$T/2/ algebra as the contraction of the SU/1,1/ algebra. The way for getting faithful representations is the same as in the former case. It is noteworthy that we cannot reach the same representation of E/2/$\wedge$T/2/ using different kinds of SU/1,1/ representations at the limiting procedure. Namely, we can get the principal series of E/2/$\wedge$T/2/ using that one of SU/1,1/, but we cannot get any representation of E/2/$\wedge$T/2/ from the discrete series of SU/1,1/. Writing again not j but $j=-\frac{1}{2}+\frac{1}{\sqrt{1-v^2}}k$ with continuous real parameter k, we see k is the most convenient parameter for distinguishing the irreducible representations.

3. The unitary representations of the IG.

We have worked with the algebra rather than the finite group elements. Now we apply the results for getting the unitary representations of the one-parameter group elements:

$$U(\Theta;\ v) = \exp -i\Theta S_2(v)$$

It is certainly true, that the unitary representations can be created on the linear spaces defined in the previous part. To do this we follow the standard way described e.g. in [11]. That is, we seek the solution, regular at λ=o of the differential equation:

$$\left[\frac{d^2}{d\Theta^2} + \lambda ctg\bar{\Theta}\frac{d}{d\Theta} - \frac{\lambda^2}{\sin^2\bar{\Theta}}\left(m^2+m'^2 - 2mm'\cos\bar{\Theta}\right) - \lambda^2\left(\frac{1}{4} - k^2\kappa^2\right)\right] d^{kv}_{mm'} = 0$$

Here and hence forth we use the notation : $\bar{\alpha} = \alpha\sqrt{1-v^2} = \alpha\lambda = \alpha/\kappa$
In eq.A.18. $d^{k,v}_{m,m'}$ stands for the matrix element:

$$\langle v;km|\ \exp\ -i\Theta S_2(v)\ |v;\ km'\rangle$$

The equation /A18/ can be solved easily displacing singularities to 0,1 and ∞ in the variable $z=\cos\bar{\Theta}$ when it casts hypergeometric form. The solution for the case $m \geq m'$ can be written as : /for the case $m \geqslant m'$ we must simply change m and m'/

$$d^{kv}_{mm'}(\Theta) = \mathcal{N}(k,v,m,m')\left(\frac{1+\cos\bar{\Theta}}{2}\right)^{\frac{1}{2}(m+m')}\left(\frac{1-\cos\bar{\Theta}}{2}\right)^{\frac{1}{2}(m-m')}$$

$$F\left(\frac{1}{2} + m' - k\kappa,\ \frac{1}{2} + m' + k\kappa\ ;\ m' - m + 1;\ \frac{1-\cos\bar{\Theta}}{2}\right) \qquad /A19/$$

Here $\mathcal{N}$/k,v;m,m'/ is a normalization factor defined to be 1 for m=m'. Obviously, /A19/ gives the well-known SU/2/ functions for v=o and the SU/1,1/ functions for $v=\sqrt{2}$ except for that our notation is j=1/2+k and j=1/2+ik, respectively. In the limit v=1 it can be written:

$$d^{kv}_{mm'}(\Theta) \underset{\substack{v\to 1\\ k\to\rho}}{\longrightarrow} \mathcal{N}(\rho,1,m,m')\ \frac{\bar{\Theta}}{2}^{m'-m}\ F\left(\frac{k}{\lambda},\ \frac{-k}{\lambda}\ ;\ m'-m+1;\ \frac{\bar{\Theta}}{4}\right) \qquad /A20/$$

and from Hansen's formula [21]

$$\lim_{a,b\to\infty} x^{\frac{1}{2}(c-1)}\ F\left(a,\ -b;\ c;\ \frac{x}{ab}\right) = \Gamma(c)\ J_{c-1}\left(2\sqrt{x}\right)$$

we get again, that

$$\lim_{\substack{v\to 1\\ k\to\rho}} d^{kv}_{mm'}(\Theta) = d^{\rho}_{mm'}(\Theta) \sim J_{m'-m}(\rho\Theta)$$

Here J_n/x/ denotes Bessel-function of the first kind. This result means that the solution /A19/ gives the E/2/ΛT/2/ functions, too. The normalization factor $\mathcal{N}$/k,v;m,m'/ is determined from /A12/:

$$\mathcal{N}(k,v,m,m') = \frac{1}{\Gamma(m'-m+1)} \left[\frac{\Gamma\left(\frac{1}{2}+m'+\frac{k}{\lambda}\right)\Gamma\left(\frac{1}{2}-m+\frac{k}{\lambda}\right)}{\Gamma\left(\frac{1}{2}+m+\frac{k}{\lambda}\right)\Gamma\left(\frac{1}{2}-m'+\frac{k}{\lambda}\right)} \right]^{1/2}$$

We have not spoken about the ranges of the group parameters. Remembering to the procedure which gave connection between the representations of $S_i(v)$ s for various v's and those of $S_i(0)$'s or $S_i(\sqrt{2})$'s, it is evident that

a/. if $0 \leq v < 1$ $\quad -\pi\kappa \leq \alpha_i \leq \pi\kappa \; ; \; -\pi \leq \alpha_3 \leq \pi$

$\qquad\qquad i = 1,2$

b/. if $1 \leq v$ $\quad -\infty < \alpha_i < \infty \; ; \; -\pi \leq \alpha_3 \leq \pi$

In terms of Euler-parameters:

a/. if $0 \leq v < 1$ $\quad 0 \leq \theta \leq \pi\kappa \,, \quad 0 \leq \phi,\phi' < 2\pi$

b/. if $1 \leq v$ $\quad 0 \leq \theta < \infty \quad 0 \leq \phi,\phi' < 2\pi$

The properties of the SU/2/, SU/1,1/ and E/2/ $\wedge$ T/2/ functions are well-known. Being the functions given by /A19/, in a very simple connection with these special cases, it can be justified even directly, that the whole group is covered when taking parameters from the ranges specified, and it is covered only once.

The invariant measure for integration over the IG is:

$$d\mu_v = -\frac{1}{8\pi^2}\,\kappa\,\sin\bar{\theta}\;d\theta\;d\phi\;d\phi'$$

The normalization:

$$\int_{G(v)} d\mu_v = \begin{cases} \kappa^2 & \text{for} \quad 0 \leq v < 1 \\ \infty & \text{for} \quad v \geq 1 \end{cases}$$

This choice gives the usual measures in the SU/2/, SU/1,1/ and E/2/ T/2/ cases. The orthogonality relations are:

$$\int D^{jv}_{mm'}\left(\phi,\theta,\phi'\right)\,D^{j'v}_{\bar{m}\bar{m}'}\left(\phi,\theta,\phi'\right)\,d\mu_v = \kappa^2\,\frac{\delta_{jj'}}{2j+1}\,\delta_{m\bar{m}}\,\delta_{m'\bar{m}'}$$

if $0 < v < 1$, and

$$\int D^{jv}_{mm'}\left(\phi,\Theta,\phi'\right) D^{j'v}_{\bar{m}\bar{m}'}\left(\phi,\Theta,\phi'\right) d\mu_v = \kappa^2 \frac{\delta(ij-ij')}{i(2j+1)} \delta_{m\bar{m}} \delta_{m'\bar{m}'}$$

if $v > 1$ and we deal with the principal series.

4. The IG as subgroup of the Lorentz-group. The representations on four-vectors.

In the previous section we have discussed some of the properties of the IG. Now we turn our attention to the Lorentz-group. As it is known, its Lie-algebra is spanned by the 6 generators M_i, N_i commuting as:

$$\left[M_i, M_j\right] = -\left[N_i, N_j\right] = i\,\varepsilon_{ijk} M_k\,, \qquad \left[M_i, N_j\right] = i\,\varepsilon_{ijk} N_k \qquad i = 1,2,3 \qquad /A21/$$

If we introduce

$$S_1(v) = M_1 + vN_2\,, \quad S_2(v) = M_2 - vN_1\,, \quad S_3 = M_3 \qquad /A22/$$

we get just the same Lie-algebra we examined previously. With the help of eq. /A22/ we can write an IG element in the 4x4 representation:

$$\exp(-i\alpha S_3)\exp(-i\beta S_2)\exp(-i\gamma S_3) = S(\alpha,\beta,\gamma) =$$

$$\begin{bmatrix} \kappa^2\left(1-v^2\cos\bar{\beta}\right) & \kappa v\sin\bar{\beta}\cos\gamma & -\kappa v\sin\bar{\beta}\sin\gamma & -\kappa^2 v\left(1-\cos\bar{\beta}\right) \\ \kappa v\cos\alpha\sin\bar{\beta} & \cos\alpha\cos\bar{\beta}\cos\gamma-\sin\alpha\sin\gamma & -\cos\alpha\cos\bar{\beta}\sin\gamma-\sin\alpha\cos\gamma & -\kappa\cos\alpha\sin\bar{\beta} \\ \kappa v\sin\alpha\sin\bar{\beta} & \sin\alpha\cos\bar{\beta}\cos\gamma-\cos\alpha\sin\gamma & -\sin\alpha\cos\bar{\beta}\sin\gamma+\cos\alpha\cos\gamma & -\kappa\sin\alpha\sin\bar{\beta} \\ \kappa^2 v\left(1-\cos\bar{\beta}\right) & \kappa\sin\bar{\beta}\cos\gamma & -\kappa\sin\bar{\beta}\sin\gamma & \kappa^2\left(\cos\bar{\beta}-v^2\right) \end{bmatrix}$$

This representation is analytic for $v=1$:

$$\lim_{v\to 1} e^{-i\beta S_2} = \begin{bmatrix} 1+\frac{\beta^2}{2} & -\beta & 0 & -\frac{\beta^2}{2} \\ \beta & 1 & 0 & -\beta \\ 0 & 0 & 1 & 0 \\ \frac{\beta^2}{2} & \beta & 0 & 1-\frac{\beta^2}{2} \end{bmatrix} \qquad /A23,a/$$

In the previous section we have learned a method for getting faithful representations after contraction. The representations were unitary there, but their dimension altered with v. Here an other method is exhibited: we do not make the dimension changed, but we always take non-unitary representations. It can be immediately seeen, that the matrices /A23/ do not change the vector /1,0,0,v/.

The elementes of the Lorentz-group are usually given in the Euler-parametrized form:

$$\Lambda = e^{-i\alpha M_3} e^{-i\beta M_2} e^{-i\gamma N_1} e^{-i\delta M_3} e^{-i\varepsilon M_2} e^{-i\phi M_3} \qquad /A24/$$

/N_2 or N_3 are as good as N_1 in this formula In some cases we prefer N_3, in other ones N_1/. The question arises whether any other little group can be used for Euler-parametrization instead of the rotation group or not. The answer is affirmative. To see it you must take the normal form

$$\Lambda = \exp\left(-\, i\ \alpha_i M_i + \beta_i N_i\right) = \exp\left(-\, i\ \gamma_k G_k\right)$$

where $G_i = M_i$ and $G_{i+3} = N_i$ /i=1,2,3/. In the vector space G's we perform a non-singular transformation G'=UG, where

$$U = \left[\begin{array}{ccc|ccc} 1 & & & & & v \\ & 1 & & & -v & \\ & & 1 & & & \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right]$$

Comparing with eq. /A22/: $G'_i = S_i$, $G'_{i+3} = N_i$. As U is regular for any v, there is a one-to-one correspondence between the elements $\exp(-i\gamma G$ and $\exp(-i\gamma' G'$. The way to get the Euler-parametrized form is similar in both case. As a consequence we shall write the elements of the orthochronous Lorentz-group in the form:

$$\Lambda_v = e^{-i\phi S_3} e^{-i\Theta S_2} e^{-i\xi N_1} e^{-i\alpha S_3} e^{-i\beta S_2} e^{-i\gamma S_3}$$

$$0 \le \phi, \alpha, \gamma < 2\pi, \quad 0 \le \xi < \infty, \quad \begin{cases} 0 \le \Theta, \beta \le \pi\kappa^2 & \text{if} \quad v < 1 \\ 0 \le \Theta, \beta. < \infty & \text{if} \quad v \ge 1 \end{cases}$$

It will not be useless to classify the orbits for Λ_v /Orbit is the set of points $\Lambda \overset{o}{q}$ where $\overset{o}{q}$ is a fixed four-vector and the parameters of Λ run through all the possible values./

To achieve the classification we must solve the equation:

$$\Lambda_v \left(\phi', \Theta, \xi, \alpha, \beta, \gamma\right) \overset{o}{q} = p$$

where $p = /p_o$, $p\sin\omega \ \cos\phi$, $p\sin\omega \ \sin\phi$, $p\cos\omega$ /is an optional four-vector. According to the value of p there are three cases:

$$p^2 > 0 \qquad p_o = \sqrt{p^2}\, ch\alpha' \qquad p = \sqrt{p^2}\, sh\alpha'$$

$$p^2 = 0 \qquad p_o = p = e^{\alpha}$$

$$p^2 < 0 \qquad p_o = \sqrt{-p^2}\, sh\alpha' \qquad p = \sqrt{-p^2}\, ch\alpha' \qquad /A26/$$

We choose $\overset{o}{q}$ as $q_o/1,0,0,V/$, Then the parameters α, β , γ are irrelevant. Working with the four-dimensional representation we get: $\phi = \phi''$,

$$\phi = \phi'' , \qquad ch\xi = \frac{p_o - v\, p \cos\omega}{q_o} + v^2$$

$$\frac{q_o}{p} \kappa v \, \{ch\xi(1-\cos\bar{\Theta}) + v^2 \cos\bar{\Theta} - 1\} = \sin\omega - \lambda ctg\bar{\Theta} \cos\omega \qquad /A27/$$

If $v < 1$ $/p^2 > o/$, there is no problem with eq. /A27/. For $v > 1/p^2 < o/$ we have to allow both $q_o > o$ and $q_o < o$ to cover the whole orbit. In a similar way we can find the Λ_v transformation which connects two fixed four - vectors $q_1 = m_1/ch\eta \ \ 0,0,sh\eta$ /, $q_2 = m_2/ch\,\eta \ \ 0,0,-sh\,\eta$ / with any p_1, p_2 pair satisfying $/q_1+q_2/^2 = /p_1+p_2/^2$. Our notation is now $v = (q_1+q_2)_z / (q_1+q_2)_o$.

That is, we are looking for transformation with the property:

$p_1 = \Lambda_v \ q_1$, $p_2 = \Lambda_v \, q_2$. We sketch the way of the calculation only: first from the equation $p_1+p_2 = \Lambda_v \ /q_1+q_2/$ we get ϕ', Θ, ξ as in eq. /A27/. Then

$$\Lambda_v^{-1}(\phi,\theta,\xi)(p_1-p_2) = S(\alpha,\beta,\gamma)(q_1-q_2) \qquad /A27a/$$

gives α and β ; γ remains unconstrained.

5. The function $\langle j_o \sigma k m;\ v | j_o\ \sigma k' m';v'\rangle$.

As is well-known, to label the irreducible representations of the Lorentz-group one needs four quantum numbers. Besides the eigenvalues of the Casimirians M^2-N^2, $\underline{MN}$ we can choose those of the Casimir-operator of a subgroup and M_3. Namely, we can choose as subgroup the IG, and the basis vectors can be labelled as $|j_o \sigma k m;v\rangle$ In the following we find the quantity

$$\langle j_o \sigma k m;v|\ j_o \sigma k' m';v'\rangle = \langle k_v | k'_{v'}\rangle$$

We apply the method described by Delbourgo et al. [20]. The basic equation is

$$e^{-i\xi N_3}\, e^{-i\chi_A J_A} = e^{-i\chi_B J_B}\, e^{-i\eta(N_1-M_2)}\, e^{-i\alpha N_3} \qquad /A28/$$

Here J_A, J_B are the $S_2(v)$ generators with v_A, v_B, respectively. In some cases χ_A takes all its possible values but χ_B cover only a part of its domain of definition or vice versa. In these cases an additional factor $\exp -i\pi M_2$ is needed. In the 2x2 representation eq. /A28/ reads as

$$\begin{bmatrix} e^{\xi/2} & 0 \\ 0 & e^{-\xi/2} \end{bmatrix} \begin{bmatrix} \cos\bar{\chi}_A/2 & -\frac{1-v_A}{1+v_A}\sin\frac{\bar{\chi}_A}{2} \\ \frac{1+v_A}{1-v_A}\sin\frac{\bar{\chi}_A}{2} & \cos\frac{\bar{\chi}_A}{2} \end{bmatrix} = \begin{bmatrix} \cos\frac{\bar{\chi}_B}{2} & -\frac{1-v_B}{1+v_B}\sin\frac{\bar{\chi}_B}{2} \\ \frac{1+v_B}{1-v_B}\sin\frac{\bar{\chi}_B}{2} & \cos\frac{\bar{\chi}_B}{2} \end{bmatrix} \begin{bmatrix} e^{\alpha/2} & \eta e^{\alpha/2} \\ 0 & e^{-\alpha/2} \end{bmatrix}$$

hence

$$\frac{1+v_A}{1-v_A}\ e^{-\xi}\ \mathrm{tg}\ \frac{\bar{\chi}_A}{2} = \frac{1+v_B}{1-v_B}\ \mathrm{tg}\ \frac{\bar{\chi}_B}{2}\ ;\quad \frac{1+v_A}{1-v_B}\ \sin\bar{\chi}_A = e^{\alpha}\ \frac{1+v_B}{1-v_B}\ \sin\bar{\chi}_B$$

$$\frac{1-v_B}{1+v_B}\ e^{-\xi/2}\cos\frac{\bar{\chi}_A}{2}\ \sin\frac{\bar{\chi}_B}{2} - \frac{1-v_A}{1+v_A}\ e^{\xi/2}\ \sin\frac{\bar{\chi}_A}{2}\ \cos\frac{\bar{\chi}_B}{2} = e^{\alpha}\ \eta\ \frac{1}{1+v_B}\left(1+v_B\cos\bar{\chi}_B\right)$$

/A30/

As it can be seen, the aforementioned problem arises in eq. /A30/ if either v_A or v_B or both are bigger than >1 . Adding exp/ $-i\pi M_2$ / to the r.h.s. we get

$$-\frac{1+v_A}{1-v_A}\ e^{-\xi}\ \mathrm{ctg}\ \frac{\bar{\chi}_A}{2} = \frac{1+v_B}{1-v_B}\ \mathrm{tg}\ \bar{\chi}_B\ ;\ -\frac{1-v_A}{1+v_A}\ \sin\bar{\chi}_A = e^{\alpha}\ \frac{1+v_B}{1-v_B}\ \sin\bar{\chi}_B$$

/A31/

With the same trick as in ref. 20 we get:

$$d^{j_o\sigma}_{k_v m k'_v}(\xi) = N\int d\chi_B\ d^{k_v}_{mj_o}(\chi_B)\ d^{k'v'}_{mj_o}(\chi_A)\ \exp\ \alpha(\sigma-1) \qquad /A32/$$

We do not go intro more details, they can be extracted from ref. [20]. We note, that in the case v_A or/and v_B bigger than 1, $d^{j_o\sigma}$ splits into two irreducible representations according to eq. /A30/or eq. /A31/.

Appendix B. Generalized partial wave analysis.

In this section we compute the $\langle s,\ j_o\sigma km,\ W,\ \Sigma\Lambda|\ p_1\ p_2\ \lambda_1\ \lambda_2\rangle$ matrix element. We do it in several steps, defining successively the following functions /$P=p_1+p_2$, $Q=p_1-p_2$/:

$\langle P\ W\ W_3\ \lambda_1\ \lambda_2|\ P\ Q\ \lambda_1\ \lambda_2\rangle$, $\langle P\ W\ W_3\ \Sigma\Lambda|\ P\ W\ W_3\ \lambda_1\ \lambda_2\rangle$,

$\langle s,\ j_o\sigma\ km,\ W,\ \Sigma\Lambda|\ P\ W\ W_3\ \Sigma\Lambda\rangle$.

a./ $\langle P\ W\ W_3\ \lambda_1\ \lambda_2|\ P\ Q\ \lambda_1\ \lambda_2\rangle$

This matrix element is nothing else what appears at the partial wave analysis in terms of the little group of P. W and W_3 are the eigenvalues of the little group Casimirian and a diagonalized generator, respectively. The PWA is elaborated in detail e.g. in [17] for SU/2/ and in [11] for all the other case, for E/2/ΛT in [19] also.

All the $|PQ\lambda_1\lambda_2\rangle$ vector can be obtained from a standard one $|P\overset{o}{Q}\lambda_1\lambda_2\rangle$ with the help of a little-group transformation R_w:

$$|PQ\lambda_1\lambda_2\rangle = R_w \; |P\overset{o}{Q}\lambda_1\lambda_2\rangle \qquad /B.1/$$

$\overset{o}{Q}$ can be chosen e.g. $\overset{o}{Q}_x = Q_y = o$. The representations of the little group form a full system:

$$\sum_{w\alpha\beta} D^{w}_{\alpha\beta}(\phi,\theta,\psi)\, D^{w}_{\alpha\beta}(\phi,\theta,\psi) \sim \delta(\phi-\phi')\,\delta(\psi-\psi')\,\delta(\theta-\theta') \qquad /B.2/$$

If the group is not compact $\sum_w$ means integration and summation as well over all the unitary representations. With eq./B.2./ the following operator-equality could be proved:

$$R(\phi,\theta,\psi) = \sum_{w\alpha\beta} \int dG(\phi',\theta',\psi')\, D^{w}_{\alpha\beta}(\phi,\theta,\psi)\, D^{w}_{\alpha\beta}(\phi',\theta',\psi')\, R(\phi',\theta',\psi') \qquad /B.3/$$

where $dG(\phi,\theta,\psi)$ is the group measure. Let's define

$$|P\; W\; \alpha\beta\lambda_1\lambda_2\rangle = \int dG\, D^{w}_{\alpha\beta}(R'_w)\, R'_w \; |P\overset{o}{Q}\lambda_1\lambda_2\rangle \frac{\sqrt{s}}{\Delta^{1/4}(s,m_1^2,m_2^2)} \qquad /B.4/$$

here $P^2=s$.

We see at once that the integration over ψ' can be performed and this gives relation between $\beta,\lambda_1\lambda_2$. So β is "superfluous" in the ket $|PW\alpha\beta\lambda_1\lambda_2\rangle$. This is the consequence of the fact that fo fix the direction of $\underline{Q}$ in $|PQ\lambda_1\lambda_2\rangle$ two angles are enough. This way, we omit β on the lhs. of eq.B.4. Inserting eq.B.4. and eq.B.3. into eq.B.1., we get:

$$|PQ\lambda_1\lambda_2\rangle = \sum_{ww_3} (2W+1)\, \frac{\sqrt{s}}{\Delta^{1/4}}\, \delta_{\beta,f(\lambda_1\lambda_2)}\, |PWw_3\lambda_1\lambda_2\rangle\, D^{w}_{w_3\beta}(\phi,\theta) \qquad /B.5/$$

$f(\lambda_1\lambda_2)$ means either $\lambda_1-\lambda_2$ or $\lambda_1+\lambda_2$ depending on the group structure.

b./ $\langle P\,W\,W_3\,\lambda_1\,\lambda_2 | P\,W\,W_3\,\Sigma\Lambda\rangle$

This discussed in sect. 2., here we only repeat the result:

$$\langle\lambda_1\lambda_2|\Sigma\Lambda\rangle = \sum_{S'} A_{\Sigma S'}\, C^{s_1\lambda_1 s_2-\lambda_2}_{S'\lambda} \qquad /B.6/$$

c./

The method is similar to that one used in a./. All the $|P\Omega\lambda_1\lambda_2\rangle$ states can be obtained from a fixed one $|\overset{o}{P}\overset{o}{\Omega}\lambda_1\lambda_2\rangle$ by homogeneous Lorentz transformation, where $\overset{o}{P}=q_o/1,0,0,v/$ and $\overset{o}{Q}_x=\overset{o}{Q}_y=o$, as we have discussed it in sect.3.1.:

$$\Lambda|\overset{o}{P}\overset{o}{Q}\lambda_1\lambda_2\rangle = \sum_{\lambda'_1\lambda'_2} D^{s_2}_{\lambda'_1\lambda_1} D^{s_2}_{\lambda'_2\lambda_2} |\Lambda\overset{o}{P},\, \Lambda\overset{o}{Q},\, \lambda'_1\, \lambda'_2\rangle \qquad /B.7/$$

The Λ Lorentz transformation is Euler-parametrized in terms of the $S(v)$group, the little group of $\overset{o}{P}$ vector; the k quantum number in $|j_o\sigma km\rangle$ is the eigenvalue of the Casimirian of the little group, c.f. Appendix A. The analogue of eq.B.3. here is

$$\Lambda' = \sum_{j_o\sigma kmk'm'} \int d\Lambda\, D^{j_o\sigma}_{kmk'm'}(\Lambda)\, D^{j_o\sigma}_{kmk'm'}(\Lambda')\,\Lambda \qquad /B.8/$$

Here again $\sum$ could mean integration, e.g. over σ :
Applying the results of a./. we can write

$$|\overset{o}{P}\overset{o}{Q}\lambda_1\lambda_2\rangle = \sum_{W} 2W+1\, \frac{\sqrt{s}}{\Delta^{1/4}}\, |\overset{o}{P}\, W\,\lambda,\, \lambda_1\,\lambda_2\rangle \qquad /B.9/$$

As the Casimirians of the Lorentz group do not commute with W_o, /the eigenvalue of that is λ_i /, in eq.B.9. it is necessary to perform the eq.B.6. type diagonalization. Since

$$\int d\Lambda\, D^{j_o\sigma}_{k\mu k'\mu'}(\Lambda)\,\Lambda\,|\,\overset{o}{P}\,W\,\lambda\,\Sigma\Lambda\rangle = \delta_{k'W}\,\delta_{\mu'\Lambda}\,|s,\, j_o\sigma k\mu,\, W,\, \Sigma\Lambda\rangle \qquad /B.10/$$

we can write at last:

$$|P\Omega\Sigma\Lambda> = \sum_{\substack{j_o\sigma jm \\ \lambda_i\lambda_i'}} D^{j_o\sigma}_{jmW\Lambda}(\Lambda)\, d^{s_1}_{\lambda_1'\lambda_1}\, d^{s_2}_{\lambda_2'\lambda_2}\, |S, j_o\sigma km, W, \Sigma\Lambda> \; .$$

$$<\Sigma\Lambda\,|\lambda_1\lambda_2> <\lambda_1'\,\lambda_2'\,|\Sigma\Lambda> \left(j_o^2 - \sigma^2\right) \frac{\sqrt{s}}{\Delta^{1/4}} \qquad /B.11/$$

Appendix C. Evaluation of the overlap function

$$<s,\, j_o\sigma jm,\, W^+\, \Sigma^+\Lambda^+|\; \tau,\, l_o\rho l\mu,\, W^-,\, \Sigma^-\Lambda^->$$

We shall use the shorthand notation $<+|->$ for the matrix element in question.

$$<+|-> = \int \frac{d^3p_1\, d^3p_2}{p_{1o}\, p_{2o}} \sum_{\lambda_1,\lambda_2} <+|p_1\lambda_1\, p_2\lambda_2> <p_1\lambda_1\, p_2\lambda_2|-> \delta\left(p^{+2}-s\right)\delta\left(p^{-2}-s\right) =$$

$$= \int \frac{d^3p_1\, d^3p_2}{p_{1o}\, p_{2o}} \sum_{\lambda_1\lambda_2} <j_o\sigma jm|j_o\sigma j_v m> <s, j_o\sigma j_v m, W^+, \Sigma\Lambda|p_1\lambda_1\, p_2\lambda_2> \,.$$

$$<p_1\lambda_1\, p_2\lambda_2|\tau,\, l_o\rho l_v\mu,\, W^-,\, \Sigma^-\Lambda^-> <l_o\rho l_v\mu|l_o\rho l\mu>\, \delta\, \ldots\, \delta\, \ldots \qquad /C.1/$$

where $P^+=p_1+p_2$, $P^-_o=p_{1o}-p_{2o}$, $\underline{P}^- = \underline{p}_1+\underline{p}_2$, j_v and l_v are the eigenvalue of the Casimirian of the little group of P^+ and P^-, respectively. The quantity $<j|j_v>$, $<l_v|l>$ is known from eq. A.32. Inserting eq, B.11. into eq.C.1. and performing the substitution we can write:

$$<+|-> = \frac{1}{2} \frac{\sqrt{s\tau}}{\Delta^{1/4}(s)\, \Delta^{1/4}(\tau)} \quad d\Lambda^- \sum_{j_v l_v} \left(j_o^2 - \sigma^2\right)\left(l_o^2 - \rho^2\right) <j|j_v> \,.$$

$$D^{j_o\sigma}_{j_v mW^+\Lambda^+}\left(\Lambda^+\right) D^{l_o\rho}_{l_v\mu W^-\Lambda^-}\left(\Lambda^-\right) <l_v|l>\, \delta\left(p^2-s\right)\delta\left(p^{-2}-\tau\right)$$

$$\sum_{\lambda_i'\lambda_i''} <\Sigma^+\Lambda^+|\lambda_1'\lambda_2'>\, d^{s_1}_{\lambda_1'\lambda_1''}\left(\Theta_1\right)\, d^{s_2}_{\lambda_2'\lambda_2''}\left(\Theta_2\right) <\lambda_1''\lambda_2''|\Sigma^-\Lambda^-> \qquad /C.2/$$

Here the parameters of Λ^+ and Θ_1 , Θ_2 depend on the parameters of Λ^-.

We need eq.C.2. only at $\tau = /m_1+m_2/^2$. The $\Delta(\tau)$ factor in the nominator seems to give a zero here, but a similar factor in the scattering amplitude /that appears in $< m_1, m_3, p_1=0, p_3=0 | \ldots 1_0 \rho 1 \mu \ldots >$ c.f. eq. 3.1.14./ just cancels it out, so we need not bother ourselves for that. At $\tau = /m_1+ m_2/^2$ $v'=1$, hence $<1_v|1> = 1$ what makes eq.C.2. simpler.

To find out the explicit dependence between the parameters of Λ^+ and Λ^- we use eqs. A 27, A.27/a with

$$p_1 = \Lambda^- \left(m_1, \underline{0},\right) \quad p_2 = \Lambda^- \left(-m_2, \underline{0}\right).$$

the η parameter of the fixed $q_{1,2}$ vectors is $s=m_1^2+m_2^2-2m_1 m_2 ch2\eta$
To get Θ_1 and Θ_2, we have to write

$$\Lambda^+ = e^{-i\alpha S_3} e^{-i\beta S_2} e^{-i\gamma S_3} e^{-i\xi N_1} e^{-i\Theta S_2} e^{-i\phi S_3}$$

in the form $e^{-i\alpha_1 M_3} e^{-i\alpha_2 M_2} e^{-i\alpha_3 M_3} e^{-i\alpha_4 N_3} e^{-i\alpha_5 M_2} e^{-i\alpha_6 M_3}$

We do not go into its detail here.

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