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SU (1,1) SPIN COEFFICIENTS

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ABSTRACT

A comparative discussion of SU/2/ and SU/1,1/ spinor algebras is presented. Spin coefficients are introduced both in the SU/2/ and in the SU/1,1/ formalism. While in the present paper a particular flat space spin- and coordinate frame is used, the fundamental relations are given in a covariant notation such that the method can easily be adapted to spinor fields in curved three-spaces also. The SU/1,1/ spin coefficients are used to obtain the stationary axisymmetric gravitational equations in a form where all the field quantities appear as spin coefficients in a flat hyperbolic three-space.

РЕЗЮМЕ

В настоящей работе представляется сравнение спинорных алгебр SU(2) и SU(1,1). Как в формализм SU(2), так и в формализм SU(1,1) вводятся спиновые коэффициенты. Хотя в настоящей работе применяется специальная, ровная пространственная система спинов и координат, основные зависимости даются в ковариантной форме, поэтому этот метод может быть применен и в случае искривленных трехмерных пространств. Благодаря применению спиновых коэффициентов SU(1,1), стационарные, осесимметричные гравитационные уравнения приводятся к виду, в котором количества полей определяются спиновыми коэффициентами ровного трехмерного гиперболического пространства.

KIVONAT

Az SU(2) és SU(1,1) spinoralgebrák összehasonlitó tárgyalását adjuk meg. Spin koefficienseket vezetünk be mind az SU(2), mind az SU(1,1) formalizmusban. Noha e dolgozatban speciális, sima térbeli spin- és koordinátarendszert használunk, az alapvető összefüggéseket kovariáns alakban adjuk meg, ezért a módszer könnyen általánositható görbült háromdimenziós terekre. Az SU(1,1) spin koefficiensek felhasználásával a stacionárius, tengelyszimmetrikus gravitációs egyenleteket olyan alakra hozzuk, amelyben a térmennyiségeket sima, háromdimenziós hiperbolikus tér spin koefficiensei adják meg.

1. INTRODUCTION

The spin coefficient technique has been brought into being by the physicists' struggle with the essentially nonlinear character of the gravitational equations of Einstein. The rapid increase of the area of its applications is due to the extreme flexibility lent to the method essentially by the alternative uses of spinor and vector pictures in visualizing the geometric meaning of spin coefficients. Since the time when Newman and Penrose[1/] had developed the method, its uses have spread beyond the theory of general relativity. It is hard to give a comprehensive survey of all the papers involved in this field, yet I have tried to list some of the references[2/] containing the essential results.

Although spin coefficients were introduced originally in the SL/2,C/ spinor calculus, their use has come to be extended to SU/2/ as well. The way of formulating the SU/2/ calculus[3/] is easily adapted to SU/1,1/ spinors also. This will be done in the present paper. To facilitate comparison of the /already familiar/ SU/2/ spin coefficient formalism with the SU/1,1/ one, a parallel discussion will be given of the spinor algebras /Sec.2/, connecting quantities /Sec.3/, the dyad notation /Sec.4/ and field identities /Sec.5/. Although a particular flat-space coordinate- and- spin frame will be used throughout this paper, the covariant formulation of the basic relations opens the way for later applications to curved /Riemannian/ spaces. For the same purpose, the field identities given in Sec.5 do, in fact, contain curvature terms, although these are assumed to vanish, in all other parts of this paper.

In Sec.6 I use the SU/1,1/ spinor formalism to bring the field equations of the stationary axisymmetric vacuum to a form where all the field quantities are represented by spin coefficients. The research for the puzzling structure of the stationary axisymmetric gravitational equations has come into prominence since it was generally agreed[4/] that the external gravitational field of black holes must be restricted by the requirements of time-independence and axial symmetry. We can use here a flat-space spinor calculus since, as is well known, the field equations of the problem can be formulated on a flat "background" 3-space. As will be seen, however, the invariance of the field equations against changing the signature of the metric /thus turning the Euclidean flat space to a Minkowski-type hyperbolic space/ must be exploited in our construction. My final point will be the solution of the static subclass of the fields, using the method of SU/1,1/ spin coefficients.

2. A SIMULTANEOUS INTRODUCTION TO THE ALGEBRAS OF SU/2/ AND SU/1,1/ SPINORS

In this section I shall present a parallel discussion of the elements of both SU/2/ and SU/1,1/ spinor algebras. The notation adopted here is chosen so as to be the most convenient possible for the spin coefficient technique, and follows closely the conventions of reference 3. The parallel treatment of $SU\binom{2}{1,1}$ spinor algebras is achieved by a double-rowed notation, where necessary. The upper row refers always to SU/2/, while the lower one is for SU/1,1/.

A one-index covariant spinor ξ_A, is by definition, a quantity of two complex components /A = 0 , 1/ which transforms according to the rule

$$\hat{\xi}_A = U_A{}^B \, \xi_B \; . \qquad /2.1/$$

Here the transformation matrix has the forms[6]

$$\left[U_A{}^B\right] = \begin{bmatrix} \alpha & \beta \\ \pm\bar{\beta} & \bar{\alpha} \end{bmatrix} \qquad /2.2/$$

where the complex numbers α and β are restricted by the unimodularity condition

$$\alpha\bar{\alpha} \pm \beta\bar{\beta} = 1 \qquad /2.3/$$

The 2x2 matrices $\left[U_A{}^B\right]$ given by /2.2/ and /2.3/, inasmuch as the matrix multiplication is a group operation, constitute the group $SU\binom{2}{1,1}$ with the upper and lower signs in the definition, respectively.

The transformation rule of one-index contravariant spinors is given by

$$\xi^A = \xi^B (U^{-1})_B{}^A \quad , \qquad /2.4/$$

where $\underline{\underline{U}}^{-1}$ is the inverse of $\underline{\underline{U}}$:

$$U_A{}^B (U^{-1})_B{}^C = \delta_A{}^C \quad . \qquad /2.5/$$

Thus the contraction $\xi^A \zeta_A$ is an invariant,

$$\hat{\xi}^A \hat{\zeta}_A = \xi^R (U^{-1})_R{}^A U_A{}^S \zeta_S = \xi^R \zeta_R \quad . \qquad /2.6/$$

In accordance with the unimodularity of $\underline{\underline{U}}$, the rules for raising and lowering spinor indices are:

$$\xi^A = e^{AB} \xi_B \quad ; \quad \xi_A = \xi^B e_{BA} \qquad /2.7/$$

Here the "metric spinor"

$$\left[e_{AB}\right] = \left[e^{AB}\right] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad /2.8/$$

is left invariant by spin transformations. Definition /2.8/ implies

$$e^{AC} e_{BC} = \delta^A_B \quad . \qquad /2.9/$$

There exists an other invariant spinor. In order to show this, we take the complex conjugate of Eq.s /2.1/ and /2.4/:

$$\bar{\hat{\xi}}_{A'} = \bar{U}_{A'}{}^{B'} \xi_{B'} \; ; \; \bar{\hat{\xi}}^{A'} = \xi^{B'} \left(\overline{U^{-1}}\right)_{B'}{}^{A'} . \qquad /2.10/$$

The priming of spinor indices indicates that complex conjugates of spinors possess different transformation properties. We now introduce the Hermitian two-index spinor $a^{AB'}$ by

$$\left[a^{AB'}\right] = \begin{bmatrix} \pm 1 & 0 \\ 0 & 1 \end{bmatrix} . \qquad /2.11/$$

Using the transformation rules /2.4/ and /2.10/ for spinor indices, we can easily check that $a^{AB'}$ is invariant. By definition $a^{AB'}$ has the property

$$a_{AB'}\, a^{CB'} = \pm\, \delta_A{}^C \quad . \qquad /2.12/$$

We define the adjoint of a spinor ξ_A

$$\xi^{+A} = a^{AB'}\, \bar{\xi}_{B'} \quad . \qquad /2.13/$$

From /2.12/ it follows that by adjoining the spinor twice we again get ξ_A , however, in SU/2/, with opposite sign:

$$\xi^{++}{}_A = \mp\, \xi_A \quad . \qquad /2.14/$$

We also have the relations for the complex conjugates of contracted spinors:

$$\bar{\xi}_{A'}\, \bar{\eta}^{A'} = \pm \xi^{+}{}_A\, \eta^{+A} \; ; \quad \overline{\xi^{+}_{A'}}\, \bar{\eta}^{A'} = \eta^{+}_A\, \xi^A \quad . \qquad /2.15/$$

3. THE CONNECTING QUANTITIES

The parallel discussion of SU/2/ and SU/1,1/ spinors will extend to this section also. We now proceed to investigate the connection between SU $\binom{2}{1,1}$ spinors and geometric objects in a three-dimensional space. Throughout this section we will be contended with considering local relations in the space. Thus all the following relations hold in an arbitrary but fixed point P of the space; therefore we will not be concerned whether not this space is curved. We shall assume that, at least locally, an appropriate coordinate system exists in which the metric takes the form

$$\left[g_{ij}\right] = \begin{bmatrix} \pm 1 & & \\ & \pm 1 & \\ & & 1 \end{bmatrix} \; ; \quad \sqrt{g} \equiv \left(\det\left[g_{ij}\right]\right)^{1/2} = 1. \qquad /3.1/$$

So SU/1,1/ spinors will be related to objects in a Minkowski-type 3-space /with indefinite metric/.

In close analogy with the SL/2,C/ spinor calculus[7], we introduce the connecting quantities σ^{i}_{AB} , which are to be used later to relate spinors with tensors. The defining relation for the connecting quantities can be taken as:

$$\sigma_{iA}{}^{B}\,\sigma_{jB}{}^{C} + \sigma_{jA}{}^{B}\,\sigma_{iB}{}^{C} = g_{ij}\,\delta_{A}{}^{C} \qquad /3.2/$$

/Lower case Roman indices i,j,k,... denote tensor components with values 1,2 and 3./ In addition, the symmetry of σ^{i}_{AB} in its spinor indices will be required:

$$\sigma^{i}_{AB} = \sigma^{i}_{BA} \qquad /3.3/$$

An appropriate solution of Eq.s /3.2/ and /3.3/ is

$$\begin{aligned} \left[\sigma^{1}{}_{A}{}^{B}\right] &= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \pm 1 \\ 1 & 0 \end{bmatrix} \\ \left[\sigma^{2}{}_{A}{}^{B}\right] &= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \pm i \\ -i & 0 \end{bmatrix} \\ \left[\sigma^{3}{}_{A}{}^{B}\right] &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned} \qquad /3.4/$$

We see that for SU/2/ the connecting quantities are just the Pauli matrices divided by a common $\sqrt{2}$ factor. The explicit form /3.4/ of the SU/1,1/ connecting quantities will be used in Sec.6.

Using the expressions /3.4/, we can easily prove the covariant identity

$$\sigma_{iA}{}^{B}\,\sigma_{jB}{}^{C} - \sigma_{jA}{}^{B}\,\sigma_{iB}{}^{C} = \pm\sqrt{2}\, i\, e_{ijk}\,\sigma^{kC}_{A}\,\sqrt{g} \qquad /3.5/$$

Another useful relation is obtained from /3.2/ if we properly take into account the straightforward identity[8] $e_{A[B}\,e_{CD]} = 0$. Thus we have

$$\sigma_{iAB}\,\sigma^{i}_{CD} = -\frac{1}{2}\left(e_{AB}\,e_{BD} + e_{AD}\,e_{BC}\right) \qquad /3.6/$$

There are two different ways of defining the adjoint of σ^{iB}_{A} : we can take either $-a^{B}{}_{P'}\, a_{A}{}^{Q'}\, \bar{\sigma}_{iQ'}{}^{P'}$ or $-a_{AP'}\, a^{BQ'}\bar{\sigma}_{iQ'}{}^{P'}$ as the definition of the adjoint quantities. In order to retain the customary definition of matrix adjunction in the representation used here, we put

$$\sigma^{+}{}_{iA}{}^{B} = -a_{AP'}\, a^{BQ'}\, \bar{\sigma}_{iQ'}{}^{P'} \qquad /3.7/$$

Thus, by /3.4/, we are led to the adjunction properties of the connecting quantities:

$$\sigma^{+}{}_{iA}{}^{B} = \mp\, \sigma_{iA}{}^{B} \qquad /3.8/$$

4. SPINOR BASIS AND SPIN COEFFICIENTS

A normalized spinor basis and spin coefficients for SL/2,C/ spinor fields were first introduced by Newman and Penrose[1/]. The method was later extended to SU/2/ spinor fields by Perjés[3/]. Since spin coefficients refer to differential /nonlocal/ properties of the fields, it is a relevant question whether or not we are considering spinors in a curved space. Although in the following we shall confine ourselves to flat space, it is not hard to prove that all the following relations remain valid in curved spaces provided partial derivatives are properly replaced by covariant derivatives. This assumes the introduction of covariant spinor derivatives, which, along the lines of references 7 and 3, can be done without much difficulty[9/]. In this respect the comparative /double-rowed/ treatment of SU/2/ and SU/1,1/ spinor calculi is of especial use, therefore it will be maintained throughout the present section.

Let η_A be an arbitrary one-index spinor which is normalized by

$$\eta_A\, \eta^{+A} = 1 \qquad /4.1/$$

We choose a basic spinor dyad such that

$$\eta_{oA} \equiv \eta_A\,, \quad \eta_{1A} \equiv \overset{+}{\eta}_A \qquad /4.2/$$

where η_{aA} / $a = 0,1$/ are elements of the dyad. This basis in the spin space defines a complex vector basis in the 3-space according to the relations

$$\ell^i = \sqrt{2}\ \eta^A\ \overset{+}{\eta}_B\ \sigma_A^{i\ B}$$

$$m^i = \quad -\eta^A\ \eta_B\ \sigma_A^{i\ B} \qquad /4.3/$$

$$\bar{m}^i = \pm\ \eta^{+A}\ \overset{+}{\eta}_B\ \sigma_A^{i\ B}$$

An equivalent, but more compact, notation for the basic vector "triad" will also be used in the following;

$$z_{\underline{m}}^i = \left(\ell^i,\ m^i,\ \bar{m}^i\right) \qquad /4.4/$$

where $\underline{m}$ is a triad index ranging over the values 0,+ and –. From the adjunction properties /3.8/ of the connecting quantities, we obtain that ℓ^i is a real vector and $\bar{m}^i$ is indeed the complex conjugate of m^i. The orthogonality properties of the basis follow from /3.6/ and can be summarized as

$$z_{\underline{m}}^i\ z_{\underline{n}i} \equiv g_{\underline{mn}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \pm 1 \\ 0 & \pm 1 & 0 \end{bmatrix} \qquad /4.5/$$

The physical components of an arbitrary tensor[10], say T_{ijk}, are given by $T_{\underline{mnp}} = T_{ijk}\ z_{\underline{m}}^i z_{\underline{n}}^j z_{\underline{p}}^k$, and conversely, as is easily proven, the relations $T_{ijk} = T_{\underline{mnp}}\ \ z_i^{\underline{m}} z_j^{\underline{n}} z_k^{\underline{p}}$ also hold. Here we remark that triad indices are raised and lowered by use of the triad metric $g^{\underline{mn}}$ and its inverse $g_{\underline{mn}}$ /as given by /4.5//, respectively. In a similar fashion, spinors can be given in terms of their dyad components. For example,

$$\phi_{ABC'} = -\phi_{abc'}\ \eta_A^a\ \eta_B^b\ \bar{\eta}_{C'}^{c'}$$

$$\phi_{abc'} = \phi_{ABC'}\ \eta_a^A\ \eta_b^B\ \bar{\eta}_{c'}^{C'} \qquad /4.6/$$

/dyad indices are chosen from the lower case Roman letters a,b,c,... and take the values 0 and 1/. The dyad components of a spinor, just like the physical components of tensors, are invariant scalar quantities. The algebraic properties of both spinors and tensors remain unaltered when transvecting with the basis. Care should be taken, however, of the order of dummy spinor indices /both ordinary and dyad/, since converting the position of a dummy index pair results in a change of sign, due to the skew symmetry of the spinor "metric" /cf. Eq./2.8//.

We now define the $SU\binom{2}{1,1}$ spin coefficients by the relations

$$\Gamma_{abcd} = \left(\partial_i\, \eta_{cA}\right)\eta^A_b\, \sigma^i_{CD}\, \eta^C_c\, \eta^D_d \qquad /4.7/$$

Here we have used the notation

$$\partial_i = \partial/\partial x^i \qquad /4.8/$$

As is easily inferred from the normalization /4.1/ of the spinor base, the spin coefficients exhibit symmetry both in their first and the second pair of indexes:

$$\Gamma_{abcd} = \Gamma_{bacd} = \Gamma_{abdc} \qquad /4.9/$$

Further relations among the spin coefficients can be deduced by considering their properties under adjunction. The rules for adjoining dyad components are needed at this point. Consider, for example, the dyad components of a one-index spinor, $\xi_a = \xi_A\, \eta^A_a$. From /2.15/ we obtain

$$\left(\overline{\xi_o}\right) = \pm\, \xi^+_A\, \eta^{+A} = \pm\, \xi^+_1$$
$$\left(\overline{\xi_1}\right) = -\xi^+_o \qquad /4.10/$$

The generalization of this rule for spinors with more then one index is a straightforward matter.

Taking into account all their symmetries, there are five independent $SU\binom{2}{1,1}$ spin coefficients altogether. We introduce an individual notation for these, according to the table

$\Gamma_{abcd} =$

cd \ ab	0 0	0 1, 1 0	11
0 0	$\frac{1}{\sqrt{2}}\sigma$	$\mp\frac{1}{2}\bar{\tau}$	$\pm\frac{1}{\sqrt{2}}\bar{\rho}$
0 1, 1 0	$-\frac{1}{2}\kappa$	$\mp\frac{\epsilon}{2\sqrt{2}}$	$\mp\frac{1}{2}\bar{\kappa}$
1 1	$\mp\frac{1}{\sqrt{2}}\rho$	$-\frac{1}{2}\tau$	$-\frac{1}{\sqrt{2}}\bar{\sigma}$

/4.11/

The $2^{-1/2}$ factors here are introduced for practical reasons and the simplicity thus attained in the expressions containing spin coefficients will become clear shortly.

There exists a close relationship between spin coefficients and the well known Ricci rotation coefficients given by

$$\gamma_{\underline{mnp}} = \left(\partial_j \, z^i_{\underline{m}}\right) z^j_{\underline{p}} \, z_{\underline{n}i} \quad . \qquad /4.12/$$

This is most easily seen by using the equivalent form of (4.7),

$$\Gamma_{abcd} = \frac{1}{2}\, \epsilon^{pq} \, \sigma^i_{aq} \, \sigma^j_{cd} \, \partial_j \, \sigma_{ibp} \quad . \qquad /4.13/$$

We have

$$\gamma_{\underline{nmp}} = \begin{array}{|c|c|c|c|} \hline \underline{p} \diagdown \underline{mn} & -\,0 & +\,0 & +\,- \\ \hline 0 & \bar{\kappa} & \kappa & \epsilon \\ \hline + & \bar{\rho} & \sigma & -\bar{\tau} \\ \hline - & \bar{\sigma} & \rho & \tau \\ \hline \end{array} \qquad /4.14/$$

Equation /4.14/ reveals the skewsymmetry of the rotation coefficients in their first and second indexes:

$$\gamma_{\underline{mnp}} = -\gamma_{\underline{nmp}} \qquad /4.15/$$

Hence we see that the quantity $\epsilon = \gamma_{+-0}$ is purely imaginary, $\epsilon = -\bar{\epsilon}$.

Like any tensor-type quantity, the vector operator of derivation can also be transvected with the basis to yield the scalar operators

$$\partial_{\underline{m}} \equiv z^i_{\underline{m}} \, \partial_i \qquad /4.16/$$

An alternative individual notation for the scalar derivative operators will prove useful in the following, namely,

$$D = z^i_0 \, \partial_i = \ell^i \, \partial_i$$

$$\delta = z^i_+ \, \partial_i = m^i \, \partial_i \qquad /4.17/$$

$$\bar{\delta} = z^i_- \, \partial_i = \bar{m}^i \, \partial_i \quad .$$

Using the expressions /4.3/ for the vector basis we have

$$D = -\sqrt{2}\,\sigma^{i}_{o1}\,\partial_i$$

$$\delta = \sigma^{i}_{oo}\,\partial_i \qquad /4.18/$$

$$\bar{\delta} = \mp\,\sigma^{i}_{11}\,\partial_i\,.$$

5. THE FIELD IDENTITIES

Though this paper is devoted to the flat-space spinor calculus, completeness requires the formulation of the field identities for the more general curved-space case. Therefore, in the present section we shall consider the 3-space to be a Riemannian one with the metric signature $(\pm,\pm,+)$, as the geometric arena of the SU $\binom{2}{1,1}$ spinor fields, respectively. The definition of the curvature tensor R_{ijkl} is given by the Ricci-identities

$$v_{i[;j;k]} = \frac{1}{2} R^{r}{}_{ijk}\, v_r\,, \qquad /5.1/$$

where v_i is an arbitrary vector field, and semicolon stands for the covariant derivation operation.

The equivalence of dyad and triad formalisms allows us to put down the field identities in the more simple triad notation. Applying the Ricci identities /5.1/ on the basic vectors $z^{i}_{\underline{m}}$ and taking the triad projections[3], we get

$$\gamma_{\underline{mnp};q} - \gamma_{\underline{mnq};p} - \gamma^{\ell}{}_{\underline{mp}}\,\gamma_{\underline{\ell nq}} + \gamma_{\underline{mn\ell}}\left(\gamma^{\ell}{}_{\underline{pq}} - \gamma^{\ell}{}_{\underline{qp}}\right) +$$
$$+\,\gamma^{\ell}{}_{\underline{mq}}\,\gamma_{\underline{\ell np}} = R_{\underline{mnpq}}\,. \qquad /5.2/$$

The curvature tensor of a three-space can be decomposed into the Ricci tensor $R_{ik} = R_{ij}{}^{j}{}_{k}$ and the curvature scalar $R = R_i{}^{i}$ since the conform tensor identically vanishes in this case[11]. In terms of triad components we have

$$R_{\underline{mnpq}} = -\,g_{\underline{mp}}\,R_{\underline{nq}} + g_{\underline{mq}}\,R_{\underline{np}} - g_{\underline{nq}}\,R_{\underline{mp}} + g_{\underline{np}}\,R_{\underline{mq}} -$$
$$-\,\frac{1}{2}\,R\left(g_{\underline{mq}}\,g_{\underline{np}} - g_{\underline{mp}}\,g_{\underline{nq}}\right)\,. \qquad /5.3/$$

Another important relation is the commutation rule of the scalar derivatives given by[1,3]

$$\varphi_{;\underline{m};\underline{n}} - \varphi_{;\underline{n};\underline{m}} = \left(\gamma_{\underline{m}}{}^{\ell}{}_{\underline{n}} - \gamma_{\underline{n}}{}^{\ell}{}_{\underline{m}}\right) \varphi_{;\underline{\ell}} \qquad /5.4/$$

As a simple example of manipulating with triad labels in the above identities, we put down here the detailed form of the curvature scalar:

$$R = g^{\underline{\underline{mn}}} R_{\underline{\underline{mn}}} = R_{oo} \pm 2R_{+-} \qquad /5.5/$$

Combining /5.2/, /5.3/ and /5.5/, and using the detailed notation for rotation coefficients and scalar derivatives /Eq.s /4.14/ and /4.17//, we obtain

$$D\sigma - \delta\kappa \mp e\sigma \mp \bar{\tau}\kappa - \kappa^2 \mp \sigma(e+\bar{\rho}) \mp \rho\sigma = -R_{++} \qquad /5.6a/$$

$$D\rho - \bar{\delta}\kappa \mp \tau\kappa - \kappa\bar{\kappa} \mp \sigma\bar{\sigma} \mp \rho^2 = -\frac{1}{2}R_{oo} - (1\mp 1)R_{+-} \qquad /5.6b/$$

$$D\tau - \bar{\delta}e + \kappa\bar{\sigma} - \rho\bar{\kappa} \mp \tau e - e\bar{\kappa} \pm \bar{\tau}\bar{\sigma} \mp \tau\rho = -R_{o-} \qquad /5.6c/$$

$$\delta\tau + \bar{\delta}\bar{\tau} + \sigma\bar{\sigma} - \rho\bar{\rho} \mp 2\tau\bar{\tau} + e(-\bar{\rho}+\rho) = \frac{1}{2}R_{oo} - (2\mp 1)R_{+-} \qquad /5.6d/$$

$$\bar{\delta}\sigma - \delta\rho \mp \tau\sigma - \kappa(\rho-\bar{\rho}) \mp \sigma\tau = R_{o+} \quad . \qquad /5.6e/$$

Similarly, the relations /5.5/ can be written as

$$(D\delta - \delta D)\varphi = \pm\sigma\bar{\delta}\varphi \pm (\bar{\rho}+e)\delta\varphi + \kappa D\varphi \qquad /5.7a/$$

$$(\delta\bar{\delta} - \bar{\delta}\delta)\varphi = \pm\bar{\tau}\delta\varphi \mp \tau\delta\varphi + (\bar{\rho}-\rho)D\varphi \; . \qquad /5.7b/$$

An application of the SU/2/ spin coefficient technique relying mainly on identities /5.6/ and /5.7/ has been discussed at length in Ref. 3. In the following section another example will be given demonstrating the use of SU/1,1/ spin coefficients. So we now dispense with the double-rowed notation and from the next section on we confine our attention to the SU/1,1/ spinor calculus in a flat hyperbolic space.

6. APPLICATION OF SU/1,1/ SPIN COEFFICIENTS: THE STATIONARY AXISYMMETRIC VACUUM

We are considering here the vacuum as one being described in the framework of the general relativity theory. Under the assumption of stationarity and axial symmetry, the vacuum field equations of Einstein considerably simplify. As Kramer and Neugebauer[12/] have pointed out, the corresponding Lagrangian can be written in the form

$$L = \nabla A \nabla A + \nabla B \nabla B - \nabla C \nabla C \qquad /6.1/$$

where the field quantities A, B and C are invariant scalars in a fictitious Euclidean 3-space, each of them being independent of the azimuthal angle. The usual notation for gradient and Laplacian operators / ∇ and Δ , correspondingly/ will be adopted in the following.

Taking account of the constraint equation

$$A^2 + B^2 - C^2 = -1 \qquad /6.2/$$

the field equations can be derived from the Lagrangian /6.1/ and are of the form

$$\begin{aligned} \Delta A &= \lambda A \\ \Delta B &= \lambda B \\ \Delta C &= \lambda C \end{aligned} \qquad /6.3/$$

where $\lambda = L$ is the Lagrange-multiplier.

From our point of view, an essential remark is that the field equations /6.3/ are unaffected when changing the signature of the metric from (+ +,+) to (-,-,+). This can easily be seen if we introduce, for example, cylindrical coordinates ρ, z, ϕ,

$$X = \rho \sin\phi$$
$$Y = \rho \cos\phi \qquad /6.4/$$
$$Z = z$$

where X,Y, and Z are the usual Cartesian coordinates. In view of the axial symmetry of the field variables, the g_{33} component of the metric

$$\left[g_{ij}\right] = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \rho^2 \end{bmatrix} \qquad /6.5/$$

does not enter the field equations /6.3/. Thus, in place of /6.5/, it is permissible to take the metric

$$\left[\hat{g}_{ij}\right] = \begin{bmatrix} -1 & & \\ & -1 & \\ & & \rho^2 \end{bmatrix}. \qquad /6.6/$$

Next the coordinate transformation

$$x = \rho\ \mathrm{sh}\phi$$
$$y = \rho\ \mathrm{ch}\phi \qquad /6.7/$$
$$z = z$$

leads to the metric form

$$\left[g_{ij}\right] = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \qquad /6.8/$$

which has been used in previous sections when developing the flat-space version of SU/1,1/ spin coefficient technique.

In the coordinate system /6.7/ we define a basic spinor η_A with the components

$$\eta_A = \begin{pmatrix} A + i\ B \\ C \end{pmatrix}. \qquad /6.9/$$

The adjoint spinor η^{+A} is given by

$$\eta^{+A} = \bar{\eta}_{B'}\ a^{AB'} = (A - iB,\ C)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = (-A + iB, C)\ . \qquad /6.10/$$

So $\eta_{cA} = \left(\eta_A,\ \eta^+_A\right)$ is a properly normalized SU/1,1/ spinor base,

$$\eta_A\ \eta^{+A} \equiv \eta_o\ \eta^{+0} + \eta_1\ \eta^{+1} = C^2 - A^2 - B^2 = 1\ . \qquad /6.11/$$

In terms of the basic spinor η_A, the field equations take the spinor form

$$\Delta\eta_A = \lambda\eta_A \qquad /6.12/$$

with

$$\lambda = -\nabla\eta_A \nabla\eta^{+A} = \partial_{BC}\,\eta_A\,\partial^{BC}\,\eta^{+A}\ ; \qquad \partial_{BC} \equiv \sigma^{i}_{BC}\,\partial_i\ . \qquad /6.13/$$

A more convenient form of the field equations is obtained if we observe that

$$\epsilon_{BC}\,\epsilon_{DE}\,\Delta\eta_A = \left(-\partial_{BD}\,\partial_{CE} + \partial_{CD}\,\partial_{BE} - \partial_{CE}\,\partial_{ED} + \partial_{BE}\,\partial_{CD}\right)\eta_A =$$

$$= 2\left(\partial_{CD}\,\partial_{BE} - \partial_{BD}\,\partial_{CE}\right)\eta_A \qquad /6.14/$$

Multiplying both sides of Eq. /6.12/ by $\epsilon_{BC}\,\epsilon_{DE}$, we get

$$\left(\partial_{CD}\,\partial_{BE} - \partial_{BD}\,\partial_{CE}\right)\eta_A = \frac{1}{2}\,\lambda\eta_A\,\epsilon_{BC}\,\epsilon_{DE} \qquad /6.15/$$

We now proceed to obtain the spin coefficient version of the field equations. Insertion of the Kronecker symbols

$$\delta_A{}^B = \eta_{aA}\,\eta^{aB} \qquad /6.16/$$

in the expression /6.13/ for λ gives

$$\lambda = \Gamma_{Oabc}\,\Gamma_1{}^{abc} \qquad /6.17/$$

Next we contract the field equations /6.15/ with $\eta^A_a\ \eta^B_b\ \eta^C_c\,\eta^D_d\ \eta^E_e$. After rearranging properly the derivative operators and making use of /6.17/, we arrive at the set of spin coefficient equations

$$\partial_{cd}\,\Gamma_{Oabe} - \partial_{bd}\,\Gamma_{Oace} = \Gamma_{Oabr}\,\Gamma_e{}^r{}_{cd} - \Gamma_{Oacr}\,\Gamma_e{}^r{}_{bd} +$$

$$+ \Gamma_{Oaer}\left(\Gamma_b{}^r{}_{cd} - \Gamma_c{}^r{}_{bd}\right) + \Gamma_{Orbe}\,\Gamma_a{}^r{}_{cd} - \Gamma_{Orce}\,\Gamma_a{}^r{}_{bd} +$$

$$+ \frac{1}{2}\,\epsilon_{Oa}\,\epsilon_{bc}\,\epsilon_{dc}\,\Gamma_{Opqr}\,\Gamma_1{}^{pqr} \qquad /6.18/$$

In the detailed notation we can write the above form of the field equations as

$$D\kappa - 2\delta\rho = \sigma(\bar{\kappa}-\tau) + \rho(\kappa+\bar{\tau}) - 2\kappa\bar{\rho} \qquad /6.19a/$$

$$De - 2\delta\tau = -\rho\bar{\rho} + \sigma\bar{\sigma} + 2\tau\bar{\tau} - \bar{\tau}\bar{\kappa} + \tau\kappa - 2e\bar{\rho} \quad . \qquad /6.19b/$$

In addition, we have the spin coefficient equations arising from the field identities /5.6/ with the lower signs /corresponding to SU/1,1/ on imposing the flat-space condition $R_{\underline{mp}} = 0$:

$$D\sigma - \delta\kappa = -(\rho+\bar{\rho})\sigma - 2e\sigma - \kappa\bar{\tau} + \kappa^2 \qquad /6.20a/$$

$$D\rho - \bar{\delta}\kappa = -\rho^2 - \sigma\bar{\sigma} - \kappa\tau + \kappa\bar{\kappa} \qquad /6.20b/$$

$$D\tau - \bar{\delta}e = -\rho\tau + \bar{\sigma}\bar{\tau} - \kappa\bar{\sigma} + \rho\bar{\kappa} + (\bar{\kappa}-\tau)e \qquad /6.20c/$$

$$\delta\tau + \bar{\delta}\bar{\tau} = \rho\bar{\rho} - \sigma\bar{\sigma} - 2\tau\bar{\tau} + e(\bar{\rho}-\rho) \qquad /6.20d/$$

$$\delta\rho - \bar{\delta}\sigma = 2\sigma\tau - \kappa(\rho-\bar{\rho}) \ . \qquad /6.20e/$$

The set consisting of Eq.s /6.19/ and /6.20/ is completely equivalent to the stationary axisymmetric vacuum equations /6.3/. The most interesting feature of the present formulation of the problem is that <u>the gravitational field quantities are represented here merely by spin coefficients</u> and we do not have any additional terms in the field equations. This situation is to be compared with the SU/2/ spinor form of the <u>stationary</u> gravitational equations[3/], in which the excess of a complex vector field appears in the spin coefficient version of the field equations.

Although a more detailed study of the structure of field equations /6.19/ and /6.20/ lies beyond the scope of the present paper, it is perhaps useful to depict here the way of manipulating with SU/1,1/ spin coefficient equations on a very simple example. Let us take the class of solutions for which

$$e = \tau = 0 \ , \quad \kappa = \bar{\kappa}, \quad \rho = \bar{\sigma} \qquad /6.21/$$

holds. Using the representation /3.4/ for the connecting quantities and taking the spinor basis as in /6.9/, we can prove that conditions /6.21/ are characteristic to the static axisymmetric fields with[12] B = 0. Without, however, relying on the detailed structure of the spin coefficients, we can find the solution of the field equations /6.19/ and /6.20/ by imposition of /6.21/. We find that Eq.s /6.19b/, /6.20c/ and /6.20d/ are identically satisfied. The remaining field equations are

$$D\kappa - 2\delta\bar{\sigma} = -\sigma\kappa + \bar{\sigma}\kappa \qquad /6.22a/$$

$$D\sigma - \delta\kappa = -(\sigma+\bar{\sigma})\sigma + \kappa^2 \qquad /6.22b/$$

$$\delta\bar{\sigma} - \bar{\delta}\sigma = \kappa(\sigma-\bar{\sigma}). \qquad /6.22c/$$

Comparison of Eq.s /6.22b/ and /6.22c/ with commutators /5.7/ shows that the former are just the integrability conditions of a real scalar field ϕ such that

$$\kappa = D\phi, \qquad \sigma = \delta\phi . \qquad /6.23/$$

Further, taking the sum of Eq.s /6.22a/ and /6.22c/, and inserting /6.23/, we obtain

$$\left(DD - \delta\bar{\delta} - \bar{\delta}\delta\right)\phi = 0 , \qquad /6.24/$$

which is the Laplace equation written down in the spin coefficient notation[13]. The solution of our problem is thus reduced to finding the solutions of the equation

$$\Delta\phi = 0 \qquad /6.25/$$

where, owing to the axial symmetry required for any solution ϕ , no matter what signature is chosen for the metric.

In our representation, as is easily seen, the scalar ϕ should be identified with $-2\ell n(A+C)$, hence it is not hard to prove that the procedure given in the above example is the spin coefficient version of obtaining Weyl's static axisymmetric solutions[14].

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[5] Spinor indices for which the values 0 and 1 can be assigned and the Einstein summation convention holds are denoted by capital Roman letters A, B, C,...

[6] According to the convention of Bade and Jehle[7], whenever a matrix notation is used for a two-component quantity, the first /second/ index refers to the row /column/, independently of the position of the indices. No confusion of matrix rows with our twofold notation for SU/$^{2}_{1,1}$/ can occur, since the former are always enclosed in matrix brackets.

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[13] The complete spin coefficient form of Laplace's equation is given by
$$[DD - \delta\bar{\delta} - \bar{\delta}\delta + (\rho+\bar{\rho})D - (\kappa+\bar{\tau})\bar{\delta} - (\bar{\kappa}+\tau)\delta]\phi = 0 .$$
In obtaining /6.24/, conditions /6.21/ and /6.23/ have been used.

61.913

[14] For the conventional solution procedure, see, for example, J.L. Anderson: Principles of Relativity Physics /Academic Press, 1967/, p. 393. Two erroneous sentences, however, should be ignored here; namely those preceding Eq. /11-5.8/, since this condition, being trivially satisfied by the field equations, makes no restriction on the field quantities.

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