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PROBLEMS FROM THE WORLD SURROUNDING PERFECT GRAPHS

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ABSTRACT

A family $\mathcal{G}$ of graphs is called χ-bound with binding function f if $\chi(G') \leqq f(\omega(G'))$ holds whenever G' is an induced subgraph of $G \in \mathcal{G}$. Here $\chi(G)$ and $\omega(G)$ denote the chromatic number and the clique number of G. The family of perfect graphs appear in this setting as the family of χ-bound graphs with binding function $f(x)=x$. The paper exposes open problems concerning χ-bound families of graphs.

CONTENT

0. INTRODUCTION

My aim is to introduce and propose a systematic study of χ-bound (and θ-bound) families of graphs and their binding functions. These families are natural extensions of the world of perfect graphs. Recall that the family $\mathcal{P}$ of perfect graphs contains the graphs G which satisfy $\chi(G')=\omega(G')$ for all induced subgraphs G' of G. Here $\chi(G)$ and $\omega(G)$ denote the chromatic number and the clique number of a graph G.

A family $\mathcal{G}$ of graphs is called *χ-bound* with *binding function* f if $\chi(G')\leq f(\omega(G'))$ holds whenever $G\in\mathcal{G}$ and G' is an induced subgraph of G. Without restricting generality, we may assume that a binding function is an $N\to N$ function where N denotes the set of positive integers, moreover $f(1)=1$ and $f(x)\geq x$ for all $x\in N$. Under these natural assumptions the smallest binding function is $f(x)=x$ and the family of graphs which is χ-bound with binding function $f(x)=x$ is the family of perfect graphs. The complementary notion of χ-bound families is the notion of *θ-bound* families. A family $\mathcal{G}$ of graphs is θ-bound with binding function f if $\bar{\mathcal{G}}$ is a χ-bound family with binding function f (here $\bar{\mathcal{G}}$ denotes the family containing the complements of the graphs of $\mathcal{G}$).

Section 1 introduces the notion of χ-bound and θ-bound families of graphs with several examples. The most frequently occuring problems concerning binding functions are formulated and illustrated there, namely: 1. Is there a binding function for a given family $\mathcal{G}$ of graphs? 2. What is the smallest binding function for $\mathcal{G}$? 3. Is there a linear binding function for $\mathcal{G}$? 4. Is there a polynomial binding function for $\mathcal{G}$?

The examples in 1.2 (e.g. circular arc graphs, multiple interval graphs, box graphs, polyomino graphs, overlap graphs) show that the behaviour (or at least the known properties) of these families are quite different concerning their binding functions. Although these families are usually χ-bound and θ-bound (the exception is the family of box graphs for more than two dimensions), in most cases the order of magnitude or linearity of their smallest binding function is not known.

The significance of binding functions from algorithmic point of view is discussed in 1.3. The idea is that families having "small" χ-binding functions (θ-binding functions) are natural candidates for approximation algorithms with a "good" performance ratio for the coloring problem (clique cover problem). The smaller is a binding function of a family, the better performance ratio is to be expected from an approximation algorithm operating on the graphs of the family.

Perfect families of graphs are often characterized by a set of forbidden induced subgraphs. The family of P_4-free graphs, Split graphs, Treshold graphs, Triangulated graphs, Meynel graphs are examples of such families. Analogous questions are discussed in sections 2,3 and 4 for χ-bound families of graphs: which forbidden induced subgraphs make a family χ-bound? Section 2 presents problems and results concerning the following conjecture: the family of graphs which does not contain a fixed forest as an induced subgraph is χ-bound. In section 3 we discuss problems when the set of forbidden induced subgraphs is infinite. The Strong Perfect Graph Conjecture fits into this problem area. It is surprising that a much weaker conjecture, namely that the family of graphs without odd holes and their complements is χ-bound, seems to be difficult. We should call this conjecture the Weakened Strong Perfect Graph Conjecture. In section 4 we consider the case when the set of forbidden subgraphs is closed under taking complementary graphs.

In section 5 we study the effect of taking union and intersection of graphs on binding functions. It is straightforward that the union of χ-bound families is again a χ-bound family. However, the intersection of two χ-bound families (even the intersection of two perfect families) is not necessarily χ-bound.

The situation of having the notion of χ-bound and θ-bound families resembles the time B.P.G.T. (Before Perfect Graph Theorem) when two types of perfectness had to be defined. It is easy to construct families which are χ-bound but not θ-bound although "natural" graph families are usually both χ-bound and θ-bound. In section 6 we try to find analogons of the Perfect Graph Theorem for certain χ-bound families of graphs. Let G_f denote the family of graphs θ-bound with θ-binding function f. If G_f is χ-bound then the smallest χ-binding function of G_f is called the *complementary binding function* of f. It turns out that the only self-complementary binding function is f(x)=x that is the Perfect Graph Theorem is stable in a certain sense. Only "small" binding functions may have complementary binding functions: if f has a complementary binding function then $\liminf f(x)/x=1$. However, it remains an open problem even to prove that f(x)=x+1 has a complemenetary binding function.

All results appearing here with proofs are unpublished elsewhere. They are expository in nature and mainly serve as background material and status information for the open problems. In fact, the main motivation of the author for writing this paper is his desire to see some of these 44 problems to be solved. I am indebted to my friend and collegue J.Lehel for several discussions which helped these ideas to take shape.

1. χ-BOUND AND θ-BOUND FAMILIES AND THEIR BINDING FUNCTIONS

1.1. Basic concepts. Let $\omega(G)$ and $\chi(G)$ denote the *clique number* and the *chromatic number* of a graph G, i.e. $\omega(G)$ is the maximum number of pairwise adjacent vertices of G and $\chi(G)$ is the minimum number k such that the vertices of G can be partitioned into k *stable* sets. A subset of vertices in a graph is called stable if it contains pairwise non-adjacent vertices.

A function f is a *χ-binding function* for a family $\mathcal{G}$ of graphs if

$$\chi(G') \leqq f(\omega(G'))$$

holds for all induced subgraphs G' of $G \in \mathcal{G}$.

Concerning the function f, we shall always assume that $f: N \to N$ where N denotes the set of positive integers, moreover $f(1)=1$, $f(x) \geqq x$ for all $x \in N$.

A family $\mathcal{G}$ of graphs is *χ-bound* if there exists a χ-binding function for $\mathcal{G}$.

The above definitions can be formulated for the complementary parameters of graphs. Let $\alpha(G)$ and $\theta(G)$ denote the *stability number* and the *clique-cover number* of a graph G, i.e. $\alpha(G)$ is the maximum number of vertices in a stable set of G and $\theta(G)$ is the minimum number k such that the vertices of G can be partitioned into k cliques.

A function f is a θ-*binding function* for a family $\mathcal{G}$ of graphs if

$$\theta(G') \leq f(\alpha(G'))$$

holds for all induced subgraphs G′ of $G \in \mathcal{G}$. A family $\mathcal{G}$ of graphs is θ-bound if there exists a θ-binding function for $\mathcal{G}$.

Since $\omega(G)=\alpha(\bar{G})$ and $\chi(G)=\theta(\bar{G})$ holds for any graph G by definition (where $\bar{G}$ denotes the complement of G), we observe:

f is a χ-binding function for $\mathcal{G}$ if and only if f is a θ-binding function for $\bar{\mathcal{G}}$;

$\mathcal{G}$ is χ-bound if and only if $\bar{\mathcal{G}}$ is θ-bound;

where $\bar{\mathcal{G}}$ denotes the family $\{\bar{G}:G\in\mathcal{G}\}$.

If a family $\mathcal{G}$ is χ-bound then it has obviously a *smallest χ-binding function* defined by

$$f^*(x) = \max\{\chi(G'):G'\ G\in\mathcal{G},\omega(G')=x\}.$$

Similarly, a θ-bound family has a smallest θ-binding function.

Due to the assumptions on binding functions, the smallest binding function a family may have is the identity function f(x)=x. The family of graphs with χ-binding function f(x)=x is the important family of perfect graphs. The family of perfect graphs is denoted by $\mathcal{P}$. The Perfect Graph Theorem of Lovász ([26]) states that $\mathcal{P}=\bar{\mathcal{P}}$ which implies that $\mathcal{P}$ can be equivalently defined as the family of graphs with θ-binding function f(x)=x.

The basic problems in our approach concerning a family G of graphs are:

Is G a χ-bound (or θ-bound) family?

What is the order of magnitude of the smallest χ-binding (or θ-binding) function for G?

Determine the smallest χ-binding (or θ-binding) function for G.

Before looking at some examples of χ-bound or θ-bound families, have a glance at the outside world. Let G_i be a graph such that $\omega(G_i)=2$ and $\chi(G_i)=i$, for each integer $i \geqq 2$. The existence of G_i is well-known, see for example [29]. Now the family $\{G_2, G_3, \ldots\}$ is obviously not χ-bound since it is impossible to define the value of a χ-binding function f(x) for x=2. A more surprising example of a family which is not χ-bound is provided by the intersection graphs of boxes in the three dimensional Euclidean space (see in 1.2.).

1.2. Some examples of χ-bound and θ-bound families. Now have a look at some well-known families of graphs and their binding functions. We start with three classical subfamilies of P which we need frequently later.

Interval graphs: the intersection graphs of closed intervals in a line.

Triangulated graphs: the graphs contining no C_k (a cycle of k vertices) for $k \geqq 4$ as an induced subgraph.

Comparability graphs: the graphs G whose edges can be oriented transitively (ab, bc$\in E(\vec{G})$ implies ac$\in E(\vec{G})$).

The proof of the perfectness of the above families can be found in [16]. We continue with some well-known non-perfect families of graphs defined as intersection graphs of geometrical figures. Proof techniques and results concerning their binding functions have been surveyed in [22].

Circular arc graphs (see in [16], p.188): the intersection graphs of closed arcs of a circle. The family of circular arc graphs is θ-bound, its smallest θ-binding function is $f(x)=x+1$. The family is χ-bound as well, the function $f(x)=2x$ is a suitable χ-binding function for $x \geq 2$. Both of these statements follow immediately from the perfectness of interval graphs. It is easy to construct circular arc graphs G_k for all k, satisfying $\omega(G_k)=k$, $\chi(G_k)=\lfloor 3k/2 \rfloor$. A.Tucker conjectures (see [36] that $\chi(G) \leq \lfloor 3\omega(G)/2 \rfloor$ holds for all circular are graphs G. In our terminology, Tucker's conjecture says:

Conjecture 1.1. The smallest χ-binding function for the family of circular arc graphs is $f(x)=\lfloor 3x/2 \rfloor$.

Multiple (or t-) interval graphs: intersection graphs of sets which are the union of t closed intervals of a line. In the special case when t=1, we get interval graphs. These graphs were introduced in [17] and in [24]. The results of [21] imply that the family of t-interval graphs is θ-bound for all fixed t. The order of magnitude of the smallest θ-binding function is not known even for t=2.

Problem 1.2. Determine the order of magnitude of the smallest θ-binding function for double interval graphs. In particular, is there a linear θ-binding function for double interval graphs?

It was proved in [20] that the family of t-interval graphs is χ-bound with a linear binding function $2t(x-1)$ for $x \geq 2$.

Box graphs (introduced in [33]): intersection graphs of sets of boxes in the d dimensional Euclidean space. A box is a parallelopiped with sides parallel to the coordinate axes. For $d=1$ we have the family of interval graphs.

It is easy to see that the family of d dimensional box graphs is θ-bound with θ-binding function x^d (see proposition 5.5. later). The order of magnitude of the smallest θ-binding function is not known even for $d=2$.

Problem 1.3. Determine the order of magnitude of the smallest θ-binding function for two dimensional box graphs.

Concerning χ-binding functions, it was proved by Asplund and Grunbaum ([1]) that two dimensional box graphs are χ-bound with an $O(x^2)$ χ-binding function. The order of magnitude of the smallest χ-binding function is not known, its value at $x=2$ is 6 as proved in [1]

Problem 1.4. Determine the order of magnitude of the smallest χ-binding function for two dimensional box graphs. In partucular, decide whether it is linear or not.

A surprising construction of Burling ([4]) shows that the family of three dimensional boxes is *not* χ-bound.

Polyomino graphs. This subfamily of two dimensional box graphs received some attention in the last few years. A polyomino is a finite set of cells in the infinite planar square grid. With a polyomino P we may associate a hypergraph H(P) whose vertices are the cells of P and whose edges are the set of cells in maximal boxes contained in P. The intersection graph G(P) of H(P) may be called a polyomino graph. Obviously, G(P) is a subfamily of two dimensional boxes thus it is both θ-bound and χ-bound. Answering a question of Berge et al. ([3]), J.B.Shearer proved ([35]) that G(P) is perfect if P is simply connected. It would be interesting to see whether the family of polyomino graphs has linear binding functions, these questions are attributed to P.Erdős.

Problem 1.5. Is there a linear θ-binding function for polyomino graphs?

Problem 1.6. Is there a linear χ-binding function for polyomino graphs?

Overlap graphs (alias Circle Graphs, Stack Sorting Graphs, see [16] p.242). These graphs are defined by closed intervals of a line as follows: the vertices are the intervals and two vertices are joined by an edge if the corresponding intervals overlap, i.e. they are intesecting but neither contains the other. An equivalent definition is obtained by considering the intersection graphs of chords of a circle. Golumbic calls these graphs "not so perfect" (see [16], p.235). A measure of "non-perfectness" can be the order of magnitude of the smallest binding functions. It is easy to give an $O(x^2)$ θ-binding function for the family of overlap graphs (see proposition 5.4 later). It is harder to prove that the family is χ-bound, the smallest known χ-binding function is exponential (see in [20]).

Problem 1.7. Is there a linear θ-binding function for the family of overlap graphs?

Problem 1.8. Is there a linear χ-binding function for the family of overlap graphs?

Intersection graphs of straight line segments in the plane. This family of graphs was introduced in [7]. The problem whether this family is χ-bound (θ-bound) arised during a conversation with P.Erdős. Denote this family by G_{SLS}, just to have a temporary name for reference.

Problem 1.9. Is G_{SLS} a χ-bound family?

Problem 1.10. Is G_{SLS} a θ-bound family?

1.3. Algorithmic aspects of binding functions. For various classes of perfect graphs there are fast polynomial algorithms to determine a largest stable set (of size $\alpha(G)$), a largest clique (of size $\omega(G)$), a good coloring of $V(G)$ with $\chi(G)=\omega(G)$ colors or a vertex-cover by $\theta(G)=\alpha(G)$ cliques. Many examples of such algorithms can be found in [16]. It turned out (see [18]) that all of these problems can be solved by polynomial algorithms for the family P of perfect graphs.

Families of χ-bound graphs are natural candidates for polynomial *approximation algorithms* for the vertex coloring problem. Similarly, polynomial approximation algorithms may work for the clique-cover problem in case of classes of θ-bound graphs. It is typical that the proof of a χ-binding function f for a family G of graphs provides a polynomial algorithm for a good coloring of the vertices of $G \in G$ with at most $f(\omega(G))$ colors. In this case we have a polynomial approximation algorithm with perfomrance ratio at most $f(\omega(G))/\omega(G)$ which may or may not be satisfactory in a particular situation. A very favorable case occures when a family G has a *linear* χ-binding function. Then the performance ratio of the algorithm is constant. The polynomial approximation algorithm can be useful if the coloring problem is known to be NP-complete for the family G which is again a typical case. Similar reasoning shows the role of θ-binding functions in approximation algorithms for the cligue cover problem. (The basic notions are used here as defined in [14]).

To see some examples, consider the coloring problem for circular arc graphs. This problem is NP-complete (see in [15]), on the other hand it is easy to give a polynomial approximation algorithm with performance ratio at most 2. The algorithm comes from the proof of the fact the 2x is a χ-binding function for the family of circular arc graphs. If $\lfloor 3x/2 \rfloor$ were known to be a χ-binding function (see conjecture 1.1.) then the proof would probably yield a polynomial approximation algorithm with performance ratio at most 3/2 .

The situation is similar if the coloring problem is considered for multiple interval graphs. The problem is NP-complete since the family of 2-interval graphs contains the family of circular arc graphs and the latter is NP-complete. The proof of the χ-binding function $2t(x-1)$ for the family of t-intervals $(x \geq 2)$ provides a very simple polynomial approximative algorithm with performance ratio less than 2t (see in [20]).

The above reasoning might convince the reader about the importance of the following vaguely formulated problem.

Problem 1.11. Find some applicable sufficient condition which implies that a family has a linear χ-binding function.

The existence of a linear binding function is an open problem for many χ-bound and/or θ-bound families. Problems 1.2.-1.8. provide examples and we shall see others later.

Concerning potential applications, we note that the coloring problem of circular arc graphs and multiple interval graphs occures in scheduling problems (see [38], [24], [16]), applications of the coloring problem of overlap graphs are discused in [16]. The clique cover problem of polyomino graphs is motivated by problem of picture processing as noted in [3].

2. BINDING FUNCTIONS ON FAMILIES WITH ONE FORBIDDEN SUBGRAPH

Let H be a fixed graph and consider the family $\mathcal{G}(H)$ of graphs which does not contain H as an induced subgraph:

$$\mathcal{G}(H) = \{G : H \not\subset G\}.$$

What choiches of H guarantee that $\mathcal{G}(H)$ is a χ-bound family? Assume that H contains a cycle, say length k. Let G_i be a graph of chromatic number i and of girth at least k+1. The existence of such graphs was proved by Erdős and Hajnal in [10]. Clearly $G_i \in \mathcal{G}(H)$ for $i=1,2,\dots$ showing that $\mathcal{G}(H)$ is not χ-bound. I conjectured that $\mathcal{G}(H)$ is χ-bound in all other cases, i.e. the following holds.

Conjecture 2.1. ([19]) $\mathcal{G}(F)$ is χ-bound for every fixed forest F.

Let S_n denote the star on n vertices and let $R(p,q)$ be the Ramsey function that is the smallest $m=m(p,q)$ such that all graphs of m vertices contain either a stable set of p vertices or a clique of q vertices. The following result shows that $\mathcal{G}(S_n)$ is χ-bound and its smallest χ-binding function is close to the Ramsey function.

Theorem 2.2. The family $\mathcal{G}(S_n)$ is χ-bound and its smallest χ-binding function f^* satisfies

$$\frac{R(n-1,x+1)-1}{n-2} \leq f^*(x) \leq R(n-1,x)$$

for all fixed n, $n \geq 3$.

Proof. Let G be a graph on $R(n-1,x+1)-1$ vertices such that G contains neither a stable set of n-1 vertices nor a clique of x+1 vertices. Clearly $G \in \mathcal{G}(S_n)$ and $\chi(G) \geq |V(G)|/n-2$ which gives the lower bound for f^*.

To see the upper bound, let $G \in \mathcal{G}(S_n)$, $\omega(G)=x$. We claim that the degree of any vertex of G is less than $R(n-1,x)$. If some vertex $P \in V(G)$ has at least $R(n-1,x)$ neighbors then the neighborhood of P contains either a stable set of n-1 vertices or a clique of x vertices. The first possibility contradicts to $G \in \mathcal{G}(S_n)$ and the second contradicts to $\omega(G)=x$ and the claim follows. Therefore the chromatic number of G is at most $R(n-1,x)$. □

Note that for n=3 the lower and upper bounds are the same showing that $f^*(x)=x$, i.e. $\mathcal{G}(S_3)$ is a perfect family. It is easy to see that $\mathcal{G}(S_3)$ consists of graphs which can be written as the union of disjoint cliques.

Problem 2.3. Improve the estimates of theorem 2.2. for the smallest χ-binding function of $\mathcal{G}(S_4)$.

The next special case when conjecture 2.1. is solved occures if the underlying forest is a path.

Theorem 2.4. Let P_n denote a path on n vertices, $n \geq 2$. Then $\mathcal{G}(P_n)$ is χ-bound and $f_n(x)=(n-1)^{x-1}$ is a suitable χ-binding function.

Proof. Considering $n \geq 1$ fixed, we prove by induction on $\omega(G)$. To launch the induction, note that the theorem trivially holds for graphs G with $\omega(G)=1$. Suppose that $(n-1)^{x-1}$ is a binding function for all $G' \in \mathcal{G}(P_n)$ such that $\omega(G') \leq t$ for some $t \geq 1$.

Let $G \in \mathcal{G}(P_n)$ and $\omega(G)=t+1$. Assuming that $\chi(G)>(n-1)^t$, we shall reach a contradiction by constructing a path $(Q_1,Q_2,\dots,Q_n)$ induced in G. Technically we define nested vertex sets $V(G)\supset V(G_1)\supset \dots \supset V(G_i)$ and vertices $Q_1\in V(G_1)$, $Q_2\in V(G_2)$, $\dots,Q_i\in V(G_i)$ for all i satisfying $1\leq i\leq n$ with the following properties:

(i) G_i is a connencted subgraph of G

(ii) $\chi(G_i)>(n-i)(n-1)^{t-1}$

(iii) if $1\leq j<i$ and $Q\in V(G_i)$ then Q_jQ is an edge of G if and only if $j=i-1$ and $Q=Q_i$.

For i=1 we choose G_1 as a connected component of G with $\chi(G_1)>(n-1)^t$ because $\chi(G)<(n-1)^t$ was assumed. Let Q_1 be any vertex of G_1.

Assume that $G_1,G_2,\dots,G_i$ and $Q_1,Q_2,\dots,Q_i$ are already defined for some $i<n$, moreover (i)-(iii) are satisfied. Define G_{i+1} and Q_{i+1} as follows.

Let A denote the set of neihgbors of Q_i in G_i. Let $B=V(G_i)-(A\cup\{Q_i\})$. The graph G_A induced by A in G satisfies $\omega(G_A)\leq t$ because the presence of a (t+1)-clique in G_A would give a (t+2)-clique in the subgraph induced by $A\cup\{Q_i\}$. Now the inductive hypothesis implies $\chi(G_A)\leq(n-1)^{t-1}$.

Assume that $B\neq\emptyset$. Now $\chi(G_i)\leq\chi(G_A)+\chi(G_B)$ since a good coloring of G_A with $\chi(G_A)$ colors, a good coloring of G_B with $\chi(G_B)$ new colors and an assignment of any color used on $V(G_B)$ to Q_i defines a good coloring of G_i. Therefore

$$\chi(G_B)\geq\chi(G_i)-\chi(G_A)>(n-i)(n-1)^{t-1}-(n-1)^{t-1} =$$
$$(n-(i+1))(n-1)^{t-1}$$

which allows us to choose a connected component H of G_B satisfying $\chi(H)>(n-(i+1))(n-1)^{t-1}$. Since G_i is connected by (i), there exists a vertex $Q_{i+1}\in A$ such that $V(H)\cup\{Q_{i+1}\}$ induces a connected subgraph which we choose as G_{i+1}. It is easy to check that $G_1,G_2,\ldots,G_{i+1}$ and $Q_1,Q_2,\ldots,Q_{i+1}$ satisfy the requirements (i)-(iii).

Assume that $B=\emptyset$. Now $\chi(G_i)\leq\chi(G_A)+1$ which implies $(n-i)(n-1)^{t-1}<(n-1)^{t-1}+1$. That inequality implies $i=n-1$. Since $A\neq\emptyset$ by properties (i) and (ii) of G_i, Q_n can be defined as any vertex of A, $G_n=\{Q_n\}$. □

The proof of theorem 2.4. shows that for triangle free graphs a stronger statement holds.

Corollary 2.5. If G is a connected triangle free graph of chromatic number n then every vertex of G is an endpoint of an induced P_n in G.

Let $f_n^*(x)$ denote the smallest χ-binding function of $\mathcal{G}(P_n)$. Then

$$\frac{R(\lceil\frac{n}{2}\rceil,x+1)-1}{\lceil\frac{n}{2}\rceil-1} \leqq f_n^*(x) \leqq (n-1)^{x-1} \qquad (1)$$

where the upper bound comes from theorem 2.4. and the lower bound easily follows from the observation that an induced P_n in a graph G contains a stable set of size $\lceil\frac{n}{2}\rceil$. The truth is probably close to the lower bound. For example, for n=4 the lower bound is sharp, since the family $\mathcal{G}(P_4)$ is known to be perfect (see in [34]).

Problem 2.6. Improve the lower or the upper bound of (1) for the smallest χ-binding function $f_n^*(x)$ of $\mathcal{G}(P_n)$.

Problem 2.7. What is the order of magnitude of $f_5^*(x)$?

Problem 2.8. Determine $c=\lim_{n\to\infty} f_n^*(2)/n$. (It is easy to see that $1/2\leq c\leq 1$.)

Combining the ideas of the proofs of theorem 2.2. and theorem 2.4, it is possible to prove that $G(B)$ is X-bound where B denotes a broom. A broom is a tree defined by identifying an endvertex of a path with the center of a star. The broom is the maximal forest for which conjecture 2.1 is known to be true, in the following sense: if F is a forest which is not an induced subgraph of a broom then conjecture 2.1. is open. In particular, the following three special cases of conjecture 2.1 are open problems.

Problem 2.9. Prove that G is X-bound.

Problem 2.10. Prove that G  is X-bound.

Problem 2.11. Prove that G is X-bound.

It seems hard to attack the following special case of conjecture 2.1: a X-binding function f(x) for $G(F)$ can be defined at x=2 if F is a forest. To settle this problem, it is clearly enough to consider the case when F is a tree since every forest is an induced subgraph of some tree. Thus we have

Conjecture 2.12. Let T be a tree and let G be a triangle-free graph which does not contain T as an induced subgraph. Then $\chi(G)\leq c$ where c is a constant depending only on T.

Conjecture 2.12. was proved for trees of radius two in [23]. The smallest tree for which conjecture 2.12. is open looks like:

Problem 2.13. Prove conjecture 2.12. for the tree above.

In what follows, we consider problems concerning the smallest χ-binding functions of some special forests. The first example is mK_2, the union of m disjoint edges. Note that mK_2 is an induced subgraph of P_{3m-1} therefore $G(mK_2)$ is χ-bound by theorem 2.4. Theorem 2.4 gives an exponential χ-binding function for $G(mK_2)$. The methods used in [39] give better results.

Theorem 2.14. (Wagon [39]). The family $G(mK_2)$ has an $O(x^{2(m-1)})$ χ-binding function.

Theorem 2.15. (Wagon [39]). The function $\binom{x+1}{2}$ is a χ-binding function for $G(2K_2)$

Problem 2.16. What is the order of magnitude of the smallest χ-binding function for $G(2K_2)$?

Problem 2.16. was posed in [39] and arose again in connection with a problem of Erdős and El-Zahar ([9]). Wagon notes in [39] that $3x/2$ is a lower bound for the smallest χ-binding function of $G(2K_2)$. A much better lower bound is

$$\frac{R(C_4,K_{x+1})}{3}$$

where $R(C_4,K_{x+1})$ denotes the smallest k such that every graph on k vertices contains either a clique of size x+1 or the complement of the graph contains C_4 (a cycle on four vertices). The above lower bound is non-linear because $R(C_4,K_t)$ is known to be at least $t^{1+\varepsilon}$ for some $\varepsilon>0$ as proved by Chung in [5]. Concerning particular values of the smallest χ-binding function f* for $G(2K_2)$, it is easy to see that $f^*(2)=3$. Erdős offered 20$ to decide whether $f^*(3)=4$. The prize went to Nagy and Szetmiklóssy who proved that $f^*(3)=4$. ([30])

Now we turn our attention to the smallest χ-binding function of $\mathcal{G}(F)$ where F is a forest of four vertices. The number of such forests is six and three of them (P_4, S_4 and $2K_2$) have been discussed before. The smallest χ-binding function of $\mathcal{G}(::)$ is asymptotically $\frac{1}{3}R(4,x+1)$ as the next proposition shows.

Proposition 2.17. Let $f^*(x)$ be the smallest χ-binding function for $\mathcal{G}(::)$. Then

$$\frac{R(4,x+1)-1}{3} \leqq f^*(x) \leqq \frac{R(4,x+1)+2R(3,x+1)}{3}-1.$$

Proof. The lower bound is obvious. Let p be the maximum number of disjoint three-vertex stable sets in $G\in\mathcal{G}(::)$. Let $|V(G)|=3p+q$, then $q \leqq R(3,x+1)-1$ and

$$\chi(G) \leqq p+q = \frac{|V(G)|+2q}{3} \leqq \frac{R(4,x+1)-1+2(R(3,x+1)-1)}{3} =$$

$$\frac{R(4,x+1)+2R(3,x+1)}{3}-1. \qquad \square$$

The smallest χ-binding function of $\mathcal{G}(< \cdot)$ is asymptotically $\frac{1}{2}R(3,x+1)$.

Theorem 2.18. Let $f^*(x)$ be the smallest χ-binding function of $\mathcal{G}(< \cdot)$. Then

$$\frac{R(3,x+1)-1}{2} \leqq f^*(x) \leqq \frac{R(3,x+1)+x-2}{2}$$

The lower bound is obvious. The proof of the upper bound is based on the following lemma.

Lemma 2.19. Assume that $G\in\mathcal{G}(<\ \cdot)$ and $\alpha(G)\geq 3$. Let S be a maximal stable set of G, i.e. $|S|=\alpha(G)$. Then $\omega(G-S)=\omega(G)-1$.

Proof.: Let $S=\{s_1,s_2,\ldots,s_\alpha\}$ and $v\in V(G)-S$. Since $G\in\mathcal{G}(<\ \cdot)$ v is adjacent with either exactly one vertex of S or with all vertices of S. Therefore $V(G)-S=V_1\cup V_2$ where $v\in V_1$ is adjacent with exactly one vertex of S and $v\in V_2$ is adjacent with all vertices of S. Let W be a clique of V(G)-S. Assume that $w_1,w_2\in W\cap V_1$, $w_1\neq w_2$ and $w_1s_i\in E(G)$, $w_2s_j\in E(G)$, $i\neq j$. Since $|S|\geq 3$, we can chose $s_k\in S$ such that $k\neq i$, $k\neq j$. Now $\{w_1,w_2,s_i,s_k\}$ (or $\{w_1,w_2,s_j,s_k\}$) induces $<\ \cdot$ in G, which contradicts to $G\in\mathcal{G}(<\ \cdot)$. We conclude that all vertices of $W\cap V_1$ are adjacent with the same vertex, say $s_i\in S$. Clarly s_i is adjacent with all vertices of $W\cap V_2$. Therefore any clique of V(G)-S can be completed to a larger clique by adding a suitable vertex of S. □

Proof of theorem 2.18. The theorem is trivial if $\alpha(G)=1$. Assume that $\alpha(G)=2$ and let $x_1y_1, x_2y_2,\ldots,x_py_p$ a maximum matching of $\bar{G}$. Let $q=|V(G)|-2p$, then $\chi(G)\leq p+q$ and $\omega(G)\geq q$. Thus

$$\chi(G)\leq p+q = \frac{|V(G)|+q}{2} \leq \frac{R(3,\omega(G)+1)-1+q}{2}$$

as stated in the theorem.

Now we can proceed by indcution on $\omega(G)$. The case $\omega(G)=1$ is trivial. The inductive step follows from lemma 2.17 and from the fact that the Ramsey function $R(x,3)$ is strictly increasing. Let $\alpha(G) \geqq 3$ and let S be a stable set of size $\alpha(G)$. The inductive hypothesis can be applied to $G'=G-S$ thus

$$\chi(G) \leqq \chi(G')+1 \leqq \frac{R(3,x)+x-2}{2}+1 \leqq \frac{R(3,x+1)+x-1}{2} . \quad \square$$

The sixth four vertex forest which was not discussed yet is .

Problem 2.19. What is the order of magnitude of the smallest χ-binding function for $G($$)$? The lower bound $\frac{R(3,x+1)-1}{2}$ is obvious and it is easy to prove that $\binom{x+1}{2}+x-1$ is an upper bound.

3. BINDING FUNCTIONS ON FAMILIES WITH AN INFINITE SET OF FORBIDDEN SUBGRAPHS

Let $\mathcal{H}$ be a set of graphs and let $\mathcal{G}(\mathcal{H})$ denote the family of graphs containing no graphs of $\mathcal{H}$ as induced subgraphs:

$$\mathcal{G}(\mathcal{H}) = \{G: H \not\subset G \text{ for all } H \in \mathcal{H}\}.$$

In section 2 we have dealt with χ-binding functions of $\mathcal{G}(\mathcal{H})$ for the case $|\mathcal{H}|=1$. Now we are concerned with the case $\mathcal{H}=\{H_1,H_2,\ldots,H_i,\ldots\}$.

If $H_i \in \mathcal{H}$ is acyclic for some i then conjecture 2.1. would imply that $\mathcal{G}(\mathcal{H})$ is a χ-bound family. Assume that for some fixed k, $g(H_i) \leq k$ for all i, where $g(H_i)$ denotes the girth (the length of the smallest cycle) of H_i. By the basic result of Erdős and Hajnal (see [10]), one can define G_i as a graph of chromatic number i and girth of at least k+1 for all i. Consequently, the family $\mathcal{G}=\{G_1,G_2,\ldots,G_i,\ldots\}$ is not χ-bound. Since $G_i \in \mathcal{G}(\mathcal{H})$ for all i, we observe:

Proposition 3.1. If $\mathcal{G}(\mathcal{H})$ is χ-bound then

$$\sup_{H \in \mathcal{H}} g(H) = \infty.$$

The most challenging open problem concerning perfect graphs is the Strong Perfect Graph Conjecture. Let us define $\mathcal{H}_o$ as $\{C_5,C_7,\ldots,C_{2i+1},\ldots\}$. The Strong Perfect Graph Conjecture states that $\mathcal{G}(\mathcal{H}_o \cup \bar{\mathcal{H}}_o)$ is the family of perfect graphs, i.e. $\mathcal{G}(\mathcal{H}_o \cup \bar{\mathcal{H}}_o)=\mathcal{P}$. Using our terminology, the Strong Perfect Graph Conjecture is equivalent with the statement that $G(H_o \cup \bar{H}_o)$ is a χ-bound family with χ-binding function $f(x)=x$. Surprisingly, it is not even known that $G(H_o \cup \bar{H}_o)$ is χ-bound.

Conjecture 3.2. (Weakened Strong Perfect Graph Conjecture.) The family $G(H_o \cup \bar{H}_o)$ is χ-bound.

The Strong Perfect Graph Conjecture gives a necessary and sufficient condition for perfectness in terms of forbidden subgraphs. To state similar conjectures for families having binding functions other than $f(x)=x$ seems difficult. Consider, for example, the family of graphs with θ-binding function $f(x)=x+1$. Graphs of that family do not contain the (disjoint) union of G_1 and G_2 as an induced subgraph where $G_1, G_2 \in H_o \cup \bar{H}_o$. The following proposition show that "critical" graphs can be much more complicated. Since its proof is based on case analysis, we state it without proof.

Proposition 3.3. Let G be the graph shown on figure 1. Then $\theta(G)=\alpha(G)+2$ and every induced proper subgraph $G' \subset G$ satisfies $\theta(G') \leqq \alpha(G')+1$.

A natural way to prove conjecture 3.2 is to prove the following stronger conjecture.

Conjecture 3.4. The family $G(H_o)$ is χ-bound.

Perhaps conjecture 3.4. can be strengthened further:

Conjecture 3.5. The family $G(H_o^m)$ is χ-bound for all $m \geqq 2$ where $H_o^m = \{C_{2m+1}, C_{2m+3}, \ldots\}$.

A weaker version of conjecture 3.5 seems also interesting:

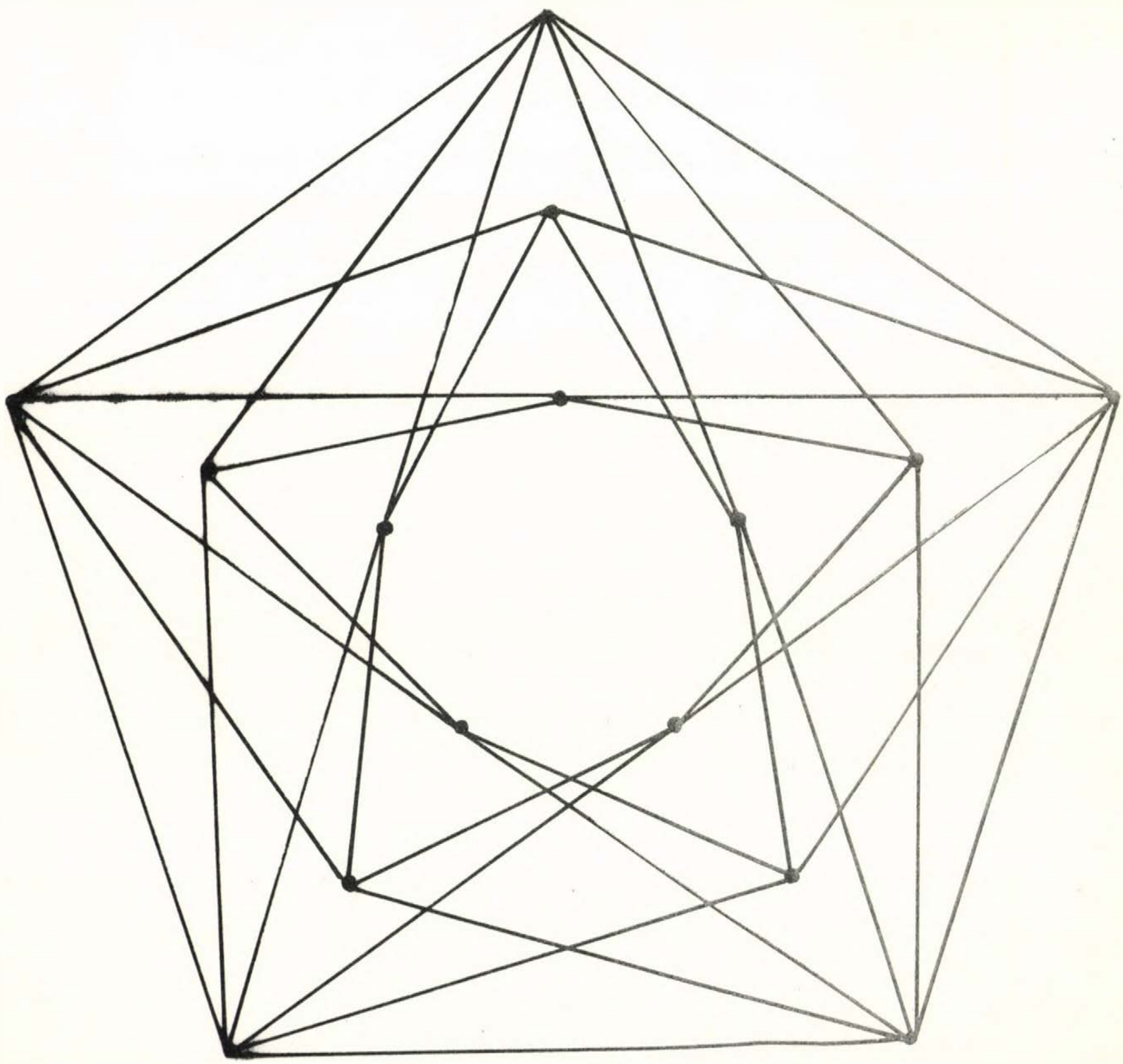


Figure 1

Conjecture 3.6. The family $G(\mathfrak{C}_l)$ is χ-bound for all $l \geq 4$, where $\mathfrak{C}_l = \{C_l, C_{l+1}, C_{l+2}, \ldots\}$.

Note that $G(\mathfrak{C}_4)$ is the family of triangulated graphs which is perfect. However, for $l \geq 5$, the conjecture is open.

Special cases of the Strong Perfect Graph Conjecture are known to be true. Some of these results say that $G(H)$ is perfect if $H = H_o \cup \bar{H}_o \cup \{H\}$ where H is a four-vertex graph. J.Lehel was curious about the four vertex graphs H for which the perfectness of $G(H_o \cup H_o \cup \{H\})$ is not known. The Perfect Graph Theorem reduces the eleven cases to six. The perfectness of $G(H_o \cup \bar{H}_o \cup \{H\})$ is known in the following cases:

H =  (A.Tucker [37])

H =  (K.R.Parthasarathy,G.Ravindra [32])

H =  (K.R.Parthasarathy,G.Ravindra [31])

H =  (Consequence of Meyniel's theorem [28] and a direct proof follows from lemma 2.19)

H =  (Seinsche proved that G () is perfect [34])

It remains to solve

Conjecture 3.7. (J.Lehel). The family $G(H_o \cup \bar{H}_o \cup \{C_4\}) = G(H_o \cup \{C_4\})$ is perfect.

4. BINDING FUNCTIONS ON FAMILIES HAVING A SELF-COMPLEMENTARY SET OF FORBIDDEN SUBGRAPHS

A family G of graphs is *self-complementary* if $G=\bar{G}$ i.e. $G \in G$ if and only if $\bar{G} \in G$. A self-complementary family G is χ-bound if and only if G is θ-bound. Moreover, if G is χ-bound then the smallest χ-binding function of G is the same as the smallest θ-binding function of G. Therefore we can speak about binding functions of G without referring to χ or to θ. We mention two well known families of perfect self-complementary graphs.

Permutation graphs (see in [16]): graphs G such that both G and $\bar{G}$ are comparability graphs.

Split graphs (see in [16]): graphs G such that both G and $\bar{G}$ are triangulated graphs. Equivalently, split graphs are graphs whose vertices can be partitioned into a clique and a stable set.

Let H be a family of graphs. Obviously $G(H)$ is self-complementary if and only if H is self-complementary. In what follows, we investigate binding functions of $G(H)$ for self-complementary H. To see some perfect families first, note that G is perfect ([34]), G is perfect and coincides with the family of split graphs as proved by Földes and Hammer ([13]). A slightly more general result is in [21] (theorem 3). The family G is a subfamily of both previous families, thus it is perfect. The family contains the so called treshold graphs (see in [16])

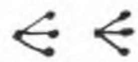
Concerning the existence of binding functions, the main open problem is a special case of conjecture 2.1.

Conjecture 4.1. The family $\mathcal{G}(F,\bar{F})$ has a binding function for every fixed forest F.

It seems useful to look at some special cases of conjecture 4.1. A straightforward attempt is to settle the following weaker versions of problems 2.9-2.11.

Problem 4.2. Prove conjecture 4.1 for F = 

Problem 4.3. Prove conjecture 4.1 for F = 

Problem 4.4. Prove conjecture 4.1 for F = 

Another problem is to determine or estimate the smallest binding function of $\mathcal{G}(F,\bar{F})$ when $\mathcal{G}(F)$ is known to be χ-bound. The rest of the section is devoted to problems and results of this kind.

Problem 4.5. Estimate the smallest binding function of $\mathcal{G}(S_n,\bar{S}_n)$. ($S_n$ is a star on n vertices.)

Concerning special cases of problem 4.5, note that the case n=3 is trivial since $\mathcal{G}(S_3,\bar{S}_3)$ contains only cliques and their complements.
The case n=4 is settled by the following theorem (cf. theorem 2.2).

Theorem 4.6. The smallest binding function of $\mathcal{G}(S_4,\bar{S}_4)$ (the claw and co-claw free graphs) is $f(x)=\left\lfloor \frac{3x}{2} \right\rfloor$.

Proof. Let G be a non-perfect member of $\mathcal{G}(S_4,\bar{S}_4)$. The result of Parthasarathy and Ravindra ([31]) implies that G contains an induced odd cycle or its complement. By symmetry we may assume that $C_{2k+1}=\{v_1,v_2,\dots,v_{2k+1}\}$ is an induced subgraph of G for some $k\geq 2$.

We claim that any vertex $x\in V(G)-V(C_{2k+1})$ is adjacent to all or to no vertices of C_{2k+1}.

To prove the claim, assume that x is adjacent to v_i. If x is not adjacent to v_{i-1} and x is not adjacent to v_{i+1} (indices are taken modulo 2k+1) then $\{v_{i-1}, v_i, v_{i+1}, x\}$ induces S_4 in G, a contradiction. We may assume that v_i and v_{i+1} are both adjacent to x. If there exists a vertex v_j in $C'=\{v_{i+3}, v_{i+4}, \dots, v_{i-3}, v_{i-2}\}$ such that v_j and x are not adjacent then $\{v_j, v_i, v_{i+1}, x\}$ induces $\bar{S}_4$ in G, a contradiction. Thus x is adjacent to all vertice of C'. Assume that x is not adjacent to v_{i-1} or to v_{i+2}, say x and v_{i-1} are not adjacent. If k=2 then x and v_{i+2} are adjacent otherwise $\{v_{i-1}, v_{i+1}, v_{i+2}, x\}$ would induce S_4 therefore $\{v_{i-1}, v_{i+1}, v_{i+2}, x\}$ induces $\bar{S}_4$. If $k \geq 3$ then $\{v_{i-1}, v_{i-3}, v_{i-4}, x\}$ induces $\bar{S}_4$. In all cases we got a contradiction. Therefore x is adjacent to all vertices of C_{2k+1} and the claim is proved.

Let $V(G)-V(C_{2k+1})=A \cup NA$ where A(NA) denotes the set of vertices adjacent (non-adjacent) to C_{2k+1}. We claim that either A or NA is empty.

Assume that $a \in A$, $b \in NA$ and $ab \in E(G)$. Let $v_i v_j \notin E(G)$, now $\{a, b, v_i, v_j\}$ induces S_4. Similarly, if $ab \notin E(G)$ then we choose i and j such that $v_i v_j \in E(G)$ and now $\{a, b, v_i, v_j\}$ induces $\bar{S}_4$. Thus the claim is true.

The theorem follows by induction on the number of vertices of $G \in \mathcal{G}(S_4, \bar{S}_4)$. The inductive step goes as follows.

Let $G \in \mathcal{G}(S_4, \bar{S}_4)$. If G is perfect then $\chi(G)=\omega(G) \leq \left\lfloor \frac{3\omega(G)}{2} \right\rfloor$. Otherwise $G=C_{2k+1} \cup A$ or $G=C_{2k+1} \cup NA$ as was proved above. In the first case we use the inductive hypothesis for A:

$$\chi(G) = \chi(A)+3 \leqq \left\lfloor \frac{3\omega(A)}{2} \right\rfloor +3 = \left\lfloor \frac{3(\omega(G)-2)}{2} \right\rfloor +3 = \left\lfloor \frac{3\omega(G)}{2} \right\rfloor$$

In the second case we use the inductive hypothesis for NA $\chi(G) = \chi(NA)$ and $\omega(G)=\omega(NA)$ thus

$$\chi(G) = \chi(NA) \leqq \left\lfloor \frac{3\omega(NA)}{2} \right\rfloor = \left\lfloor \frac{3\omega(G)}{2} \right\rfloor .$$

We proved that $f(x) = \left\lfloor \frac{3x}{2} \right\rfloor$ is a binding function for $\mathcal{G}(S_4,\bar{S}_4)$. To see that it is the smallest one, let G_m be defined as follows. Consider K_m and remove the edges of $\left\lfloor \frac{m}{5} \right\rfloor$ vertex disjoint C_5. Now it is easy to see that $G_m \in \mathcal{G}(S_4,\bar{S}_4)$ for all m, moreover $\omega(G_{5k})=2k$, $\chi(G_{5k})=3k$, $\omega(G_{5k+1})=2k+1$, $\chi(G_{5k+1})=3k+1$. □

Problem 4.7. Estimate the smallest binding function of $\mathcal{G}(P_n,\bar{P}_n)$ (cf. theorem 2.4 and problem 2.6).

Problem 4.8. What is the order of magnitude of the smallest binding function for $\mathcal{G}(P_5,\bar{P}_5)$? (Cf. problem 2.7.)

Problem 4.9. What is the order of magnitude of the smallest binding function for $\mathcal{G}(mK_2,\overline{mK_2})$? (Cf. theorem 2.14).

The case m=2 in problem 4.9 is settled by the following theorem.

Theorem 4.10. The smallest binding function $\mathcal{G}(2K_2, \overline{2K_2})$ is $f(x)=x+1$ (Cf. problem 2.16).

Proof.: Let $G \in \mathcal{G}(2K_2,\overline{2K_2})$ and let S be a stable set of G such that $|S|=\alpha(G)$. Assume that $x,y \in V(G)-S$, $xy \notin E(G)$. The definition of S and $2K_2 \not\subset G$ imply that $\Gamma(x) \cap S$ and $\Gamma(y) \cap S$ are non-empty sets and one contains the other, say $\Gamma(x)\cap S \subseteq \Gamma(y) \cap S$. ($\Gamma(p)$ denotes the set of neighbors of $p \in V(G)$.) Now $\overline{2K_2} \not\subset G$ implies $|\Gamma(x)\cap S|=1$.

Let $K_1=\{x: x\in V(G)-S, |\Gamma(x)\cap S|>1\}$, then K_1 is clique in G by the argument above. We proceed to show that $V(G)-(S\cup K_1)$ is again a clique of G. Assume that $p,q\in V(G)-(S\ K_1)$ and $pq\notin E(G)$. By definition, $|\Gamma(p)\cap S|=|\Gamma(q)\cap S|= 1$. However, $\Gamma(p)\cap S=\Gamma(q)\cap S$ contradicts the maximality of S and $\Gamma(p)\cap S \neq \neq \Gamma(q)\cap S$ contradicts the assumption $2K_2 \not\subset G$.

We have shown that the deletion of a stable set S of G results in a perfect graph (the complement of a bipartite graph). Thus $\chi(G)\leq\chi(G-S)+1=\omega(G-S)+1 \leq \omega(G)+1$, showing that $f(x)=x+1$ is a binding function for $\mathcal{G}(2K_2,\overline{2K_2})$. To see that $f(x)=x+1$ is the smallest binding function, it is enongh to consider complete graphs from which the edges of a C_5 are deleted. □

The proof of theorem 4.10 gives

Corollary 4.11. If $G\in\mathcal{G}(2K_2,\overline{2K_2})$ then V(G) can be partitioned into two cliques and a stable set. By symmetry, V(G) can be also partitioned into two stable sets and a clique.

Using lemma 2.19, it is easy to prove

Theorem 4.12. Let F denote the forest  . Then $\mathcal{G}(F,\bar{F})$ contains complete multipartite graphs and their complements, moreover the graph C_5.

Using the result of Parthasrathy and Ravindra ([32]) which proves the Strong Perfect Graph Conjecture for $\mathcal{G}($$)$ (or equivalenly for $\mathcal{G}($$)$) it is easy to derive

Theorem 4.13. Let F denote the forest . Then the non-perfect members of $\mathcal{G}(F,\bar{F})$ are

1. The graph of figure 2 and its non-perfect subgraphs.

2. A clique K whose vetices are adjacent with two consecutive vertices of a C_5.
3. The complements of the graphs defined in 1. and 2.

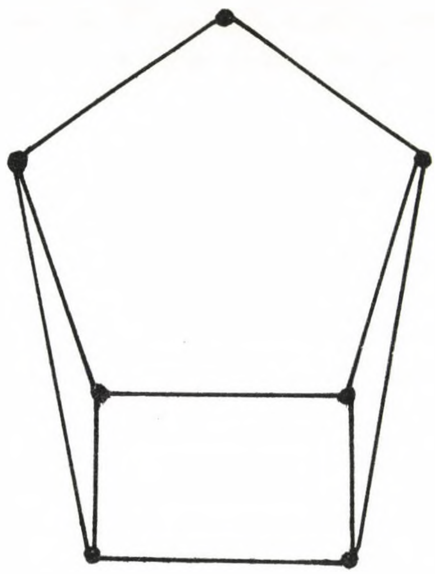


Figure 2

Putting together the previous two theorems, we have the following corollary.

Corollary 4.14. Let F denote either  or
Then the smallest binding function of $G(F,\bar{F})$ is

$$f(x) = \begin{cases} 3 & \text{if } x=2 \\ x & \text{if } x>2. \end{cases}$$

Before finishing this section, note that the smallest binding function of $G(F,\bar{F})$ was found for four-vertex forests F with one exception. The exceptional case occures when $F=\bar{K}_4$, i.e. F is a stable set of four vertices. The family $G(\bar{K}_4,K_4)$ is very excentric since it is finite (like $G(\bar{K}_m,K_m)$ in general for fixed m). Its smallest binding function f*(x) is determined by the values f*(2) and f*(3). It is easy to deduce f*(2)=3 from the facts that R(3,4)=9 and that a graph G with $\omega(G)=2$, $\chi(G)\geq 4$ satisfies $|V(G)|\geq 9$ (In fact, $|V(G)|\geq 11$ is true as proved by Chvatal in [6]). It is possible to determine f*(3) without brute force?

5. BINDING FUNCTIONS ON UNION AND INTERSECTION OF GRAPHS

For graphs $G_1, G_2, \ldots, G_k$, the graphs $\bigcup_{i=1}^{k} G_i$ and $\bigcap_{i=1}^{k} G_i$ are defined usually as follows.

$$V(\cup G_i) = \cup V(G_i) \qquad E(\cup G_i) = \cup E(G_i)$$

$$V(\cap G_i) = \cap V(G_i) \qquad E(\cap G_i) = \cap E(G_i).$$

If $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k$ are families of graphs then their union is the family $\{\cup G_i : G_i \in \mathcal{G}_i\}$ and their intersection is the family $\{\cap G_i : G_i \in \mathcal{G}_i\}$. By definition, $\cap \mathcal{G}_i$ is a χ-bound family if and only if $\cup \bar{\mathcal{G}}_i$ is a θ-bound family. This fact combined with $\chi(G_1 \cup G_2) \leq \chi(G_1)\chi(G_2)$ gives the following obvious observation.

Proposition 5.1.

a) If $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k$ are χ-bound families with binding functions $f_1, f_2, \ldots, f_k$ then $\cup \mathcal{G}_i$ is a χ-bound family and $\prod_{i=1}^{k} f_i$ is a suitable χ-binding function.

b) If $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k$ are θ-bound families with binding functions $f_1, f_2, \ldots, f_k$ then $\cap \mathcal{G}_i$ is a θ-bound family and $\prod f_i$ is a suitable θ-binding function. □

Proposition 5.1, trivial as it is, sometimes can be conveniently applied to prove the existence of binding functions.

Corollary 5.2. Let $\mathcal{P}$ denote the family of all perfect graphs. The union (intersection) of k copies of $\mathcal{P}$ is χ-bound (θ-bound) with binding function x^k. □

Problem 5.3. What is the smallest χ-binding function for $P \cup \overline{P}$?

Proposition 5.4. The family of overlap graphs is θ-bound with θ-binding function x^2.

Proof. Let G_1 denote the family of co-interval graphs, let G_2 denote the family of interval inclusion graphs. Since G_1 and G_2 are perfect families, x^2 is a χ-binding function for $G_1 \cup G_2$ by corollary 5.2. The family of overlap graphs is a subfamily of $\overline{G_1 \cup G_2}$. □

Proposition 5.5. The family of d-dimensional box graphs is θ-bound with θ-binding function x^d.

Proof. The family in question is the intersection of d families of interval graphs and we can apply corollary 5.2. □

It is tempting to think that $\bigcap_{i=1}^{k} G_i$ is χ-bound provided that G_i is χ-bound for i=1,2,...,k. However, this is not the case. It may happen that $G_1 \cap G_2$ is not χ-bound although G_1 and G_2 are perfect families. A suprising contruction of Burling ([4]) gives three dimensional box graphs B_n for all positive integers n such that $\omega(B_n)=2$, $\chi(B_n)=n$. The result shows that $I \cap I \cap I$ is not χ-bound, where I denotes the family of interval graphs. The analysis of Burling's construction shows however that $I \cap J$ is not χ-bound, where J is the family of "crossing graphs" of boxes in the plane. The vertices of crossing graphs are boxes in the plane and two vertices are adjacent if and only if the corresponding boxes cross each other. It is immediate to check that J is a subfamily of the family of comparability graphs. Note that $I \cap I$ is χ-bound with an $O(x^2)$ χ-binding function as proved by Asplund and Grünbaum ([1]).

Therefore the results in [4] and in [1] imply

Theorem 5.6. Let $I, \mathcal{C}$ denote the family of interval graphs and comparability graphs, respectively. Then

a) $I \cap I$ is χ-bound

b) $I \cap I \cap I$ is not χ-bound

c) $I \cap \mathcal{C}$ is not χ-bound.

Perhaps part a) holds in a stronger form.

Problem 5.7. Let τ denote the family of triangulated graphs. Is $\tau \cap \tau$ χ-bound? In particular, is $\tau \cap I$ χ-bound?

Since the graphs of τ can be represented as subtrees of a tree (see in [16]), problem 5.7 can be wieved as a geometrical problem.

The following result shows a pleasant property of comparability graphs.

Proposition 5.8. Let $\mathcal{C}$ denote the family of comparability graphs. The intersection of k copies of $\mathcal{C}$ is χ-bound and x^{2k-1} is a suitable χ-binding function.

Proof.: Let $G_1, G_2, \ldots, G_k \in \mathcal{C}$ and assign a transitive orientation to the edges of G_i for all i ($1 \leq i \leq k$). Assume that $xy \in E(\bigcap_{i=1}^{k} G_i)$. The edge xy is oriented according to its orientation in G_k, moreover we assign a type to it as follows. The type of xy is a 0-1 sequence of length $k-1$. For all j, $1 \leq j \leq k-1$ the j-th element of the sequence is 0 if xy is oriented in G_j from x to y and it is 1 otherwise. It is immediate to check that the edges of a fixed type of $\bigcap_{i=1}^{k} G_i$ define a transitively oriented graph. The number of possible types is at most 2^{k-1} which implies that $\bigcap_{i=1}^{k} G_i$

can be written as the union of at most 2^{k-1} comparability graphs. Now the proposition follows from corollary 5.2. □

Problem 5.9. Estimate the smallest χ-binding function of $\mathcal{C} \cap \mathcal{C}$.

A subfamily of perfect graphs, the permutation graphs occure in many applications. Permutation graphs can be defined as graphs G such that both G and $\bar{G}$ are comparability graphs. Corollary 5.2 and proposition 5.8 gives

Proposition 5.10. Let k be fixed and consider the family $\mathcal{G}$ of graphs obtained by at most k applications of intersections and unions from permutation graphs. Then $\mathcal{G}$ is χ-bound and θ-bound. □

Now we want to determine the smallest θ-binding function of families obtained as the union of k bipartite graphs. Observe that this family contains exactly the graphs of chromatic number at most 2^k. Therefore we are interested in finding the smallest θ-binding function for the family $\mathcal{G}_m$ of at most m-chromatic graphs.

Proposition 5.11. Let $f_m^*(x)$ denote the smallest θ-binding function for $\mathcal{G}_m$. Then

a) $f_m^*(x) \leq \left\lfloor \frac{m+1}{2} \right\rfloor x$

b) $f_m^*(x) \geq \frac{m}{2} x$ for $x > x_o = x_o(m)$.

Proof. It is trivial to cover the vertex set of an at most m-chromatic graph G by the vertices of at most $\left\lfloor \frac{m+1}{2} \right\rfloor = s$ bipartite graphs, $B_1, B_2, \ldots, B_s$. Now

$$\theta(G) \leq \sum_{i=1}^{s} \theta(B_i) = \sum_{i=1}^{s} \alpha(B_i) \leq s \cdot \alpha(G)$$

and a) follows.

The lower bound is pointed out by Erdős, remarking that for $n \geq n_0$ and for arbitrary m, there is a graph $G=G(n,m)$ on kn vertices satisfying $\alpha(G)=n$, $\omega(G)=2$, $\chi(G)=m$ (see in [8]). □

Proposition 5.12. The smallest binding function $f_3^*(x)$ of $\mathcal{G}_3$ satisfies:

a) $f_3^*(x) \geq \frac{5}{3}x$

b) $f_3^*(x) \geq \frac{8}{5}x$ if x is divisible by 5.

Proof. First we prove a. We may assume that $G \in \mathcal{G}_3$ is 3-chromatic. Let A_1, A_2, A_3 be the color classes of G in a good 3 coloring of V(G). Let G_{12}, G_{13}, G_{23}, be the subgraphs of G induced by $A_1 \cup A_2$, $A_1 \cup A_3$, $A_2 \cup A_3$, respectively. Since G_{ij} is a bipartite graph, $\theta(G_{ij})=\alpha(G_{ij})$ which shows that $V(G_{ij})$ can be covered by at most $\alpha(G)$ cliques (vertices or edges) of G_{ij} for $1 \leq i < j \leq 3$.

We may assume that the clique cover of $V(G_{ij})$ covers all vertices of $V(G_{ij})$ exactly once. The cliques in the covers of $V(G_{12})$, $V(G_{13})$, $V(G_{23})$ form a clique cover of G with at most $3\alpha(G)$ elements and all vertices of G are covered exactly twice by these cliques. This cover can be partitioned into components where the cliques (edges and vertices) of each component are either the edges and the two endvices of a path (allowing two identical vertices as a degenerate case) or the edges of a cycle of length divisible by 3. It is easy to check that the vertices of a component of m cliques can be covered by at most 5m/9 cliques. These cliques are edges and vertices except for a component which forms a triangle, in this case the triangle is used instead of three edges. Therefore we get a clique cover of V(G) with at most $3\alpha(G)\cdot\frac{5}{9} = \frac{5\alpha(G)}{3}$ cliques.

The lower bound b) was quessed by Erdős who devised to find a graph G with $|V(G)|=15$, $\alpha(G)=5$, $\chi(G)=3$, $\omega(G)=2$. Really, such G exists as a subgraph of a 17-vertex graph H containing neither triangles nor six independent vertices (see H in [25]). The graphs containing disjoint copies of G form a family with θ-binding function $\frac{8x}{5}$ for the cases when x is divisible by 5. □

Problem 5.14. Let $f_3^*(x)$ be the smallest binding function of G_3. Determine $\lim_{x\to\infty} f_3^*(x)/x$. (It is at least $\frac{8}{5}$ and at most $\frac{5}{3}$ by proposition 5.12.)

6. COMPLEMENTARY BINDING FUNCTIONS. THE STABILITY OF THE PERFECT GRAPH THEOREM

We say that a binding function f *has a complementary binding function* if the family G_f of graphs with θ-binding function f is χ-bound. The smallest χ-binding function of G_f is called the *complementary binding function of f*. Note that θ and χ can change roles in the definitions. We are interested in the following general problem.

Problem 6.1. Which binding functions have complementary binding functions and what are their complementary binding functions?

Using the notion of complementary binding function, the Perfect Graph Theorem says that the function $f(x)=x$ is a self-complementary binding function. (The converse statement is also true, see theorem 6.6 later.)

One feels that only "small" functions may have complementary binding functions. This is really the case as the next theorem shows.

Theorem 6.2. If $f(x)$ has a complementary binding function then $\liminf f(x)/x=1$.

Proof. To prove the theorem, it is enough to show that $f_\varepsilon(x)=(1+\varepsilon)x$ has no complementary binding function if ε is a real number satisfying $0<\varepsilon\leq 1$. The proof is based on graphs defined by Erdős and Hajnal in [11]: for every $\varepsilon\in(0,1]$ and for every natural number k there exists a graph G_k^ε with the following properties:

$$\chi(G_k^\varepsilon) = k \qquad (1)$$

$$\frac{|V(G)|}{\alpha(G)} < 2+\varepsilon \text{ for all induced subgraphs } G \subseteq G_k^{\varepsilon} \qquad (2)$$

Note that (2) implies that G_k^{ε} is a triangle-free graph. Therefore (1) implies that the family $\mathcal{G}_{\varepsilon}=\{G_1^{\varepsilon},G_2^{\varepsilon},\ldots\}$ is not χ-bound. We are going to prove that $\mathcal{G}_{\varepsilon}$ is a θ-bound family with θ-binding function $f_{\varepsilon}(x)$.

Let G be an induced subgraph of G_k^{ε}. We have to prove that $\theta(G)\leq(1+\varepsilon)\alpha(G)$. Since G is triangle-free, $\theta(G)=|V(G)|-\nu(G)$ where $\nu(G)$ is the cardinality of a maximum matching in G. We can express $\nu(G)$ by the Tutte-Berge formula (see in [38] and in [2]) as follows:

$$\nu(G) = \min_{A\subseteq V(G)} \frac{|V(G)|+|A|-\sigma(H)}{2} \qquad (3)$$

where H denotes the subgraph induced by V(G)-A in G and $\sigma(H)$ denotes the number of odd components of H. Using (3) and $\theta(G)=|V(G)|-\nu(G)$, we can rewrite $\theta(G)\leq(1+\varepsilon)\alpha(G)$ equivalently as

$$\alpha(G) \geq \frac{|V(H)|+\sigma(H)}{2(1+\varepsilon)} \quad \text{for all } H\subseteq G. \qquad (4)$$

In order to prove (4), let H be an induced subgraph of G with connected components $H_1,H_2,\ldots,H_m$. Consider the partition of $\{1,2,\ldots,m\}$ into I_1,I_2,I_3 defined as follows:

$$\begin{aligned} &i\in I_1 \text{ if } H_i \text{ is bipartite and } |V(H_i)| \text{ is even;} \\ &i\in I_2 \text{ if } H_i \text{ is bipartite and } |V(H_i)| \text{ is odd;} \qquad (5) \\ &i\in I_3 \text{ if } H_i \text{ is not bipartite.} \end{aligned}$$

We claim that

$$\alpha(H_i) \geqq \frac{|V(H_i)|}{2} \quad \text{if} \quad i \in I_1,$$

$$\alpha(H_i) \geqq \frac{|V(H_i)|+1}{2} \quad \text{if} \quad i \in I_2, \qquad (6)$$

$$\alpha(H_i) > \frac{|V(H_i)|+1}{2(1+\varepsilon)} \quad \text{if} \quad i \in I_3 .$$

The first two inequalities are obvious. To prove the third one, let C_{2t+1}, be a minimal odd cycle of H_i for some $i \in I_3$. Using (2) for C_{2t+1}, we get $t=\alpha(C_{2t+1})>\frac{2t+1}{2+\varepsilon}$ i.e. $t>\frac{1}{\varepsilon}$ which implies

$$|V(H_i)| \geqq 2t+1 > \frac{2}{\varepsilon}+1. \qquad (7)$$

Observing that (7) is equivalent with

$$\frac{|V(H_i)|}{2+\varepsilon} > \frac{|V(H_i)|+1}{2(1+\varepsilon)}$$

and $\alpha(H_i) > \frac{|V(H_i)|}{2+\varepsilon}$ by (2), we get the third inequality of (6).

Now we use (6) to estimate $\alpha(G)$. Clearly

$$\alpha(G) \geqq \sum_{i=1}^{m} \alpha(H_i) = \sum_{i \in I_1} \alpha(H_i) + \sum_{i \in I_2} \alpha(H_i) + \sum_{i \in I_3} \alpha(H_i) \geqq$$

$$\geqq \frac{|V(H)|+|I_2 \cup I_3|}{2(1+\varepsilon)} \geqq \frac{V(H)+\sigma(H)}{2(1+\varepsilon)}$$ since $|V(H_i)|$ is even for $i \in I_1$ by (5). Thus we proved (4) and the theorem follows. □

Theorem 6.2 gives a necessary condition for the existence of complementary binding functions. Concerning sufficient conditions, the main open problem is the following.

Conjecture 6.3. The function $f(x)=x+c$ has complementary binding function for any fixed positive integer c.

Conjecture 6.3 is open even in the case c=1. Probably this case already contains all the difficulties. An evidence supporting conjecture 6.3 is the following result.

Proposition 6.4. If $\mathcal{G}$ denotes the family of graphs with θ-binding function $f(x)=x+c$ then, for all $G\in\mathcal{G}$, $\omega(G)=2$ implies $\chi(G)\leq 6c+2$.

Proof. Assume that $G\in\mathcal{G}$, $\omega(G)=2$. Clearly, $\frac{|V(G)|}{2}\leq\theta(G)\leq\alpha(G)+c$ which implies

$$\alpha(G)\geq\frac{|V(G)|-2c}{2}\ . \qquad (8)$$

Let C_1 be an odd cycle of minimum length in G, let C_2 be an odd cycle of minimum length in the subgraph induced by $V(G)-V(C_1)$ in G, etc. We continue to define $C_1,C_2,\ldots,C_m$ until the subgraph induced by $V(G)-\bigcup_{i=1}^{m}V(C_i)$ in G does not contain odd cycles. Applying (8) for the subgraph C induced by $\bigcup_{i=1}^{m}V(C_i)$ in G, we get

$$\frac{|V(C)|-2c}{2}\leq\alpha(C)\leq\sum_{i=1}^{m}\alpha(C_i)=\frac{|V(C)|-m}{2}$$

from which $m\leq 2c$ follows. A good coloring of V(G) can be defined by coloring V(C) with 3m colors and using two additional colors for the bipartite graph induced by $V(G)-V(C)$. Therefore $\chi(G)\leq 3m+2\leq 6c+2$. □

By a deep result of Folkman ([12]) which answers a conjecture of Erdős and Hajnal, condition (8) implies $\chi(G) \leq 2c+2$. Therefore proposition 6.4 holds with 2c+2 instead of 6c+2.

The existence of complementary binding functions is known only for "very small" functions. We mention a modest result of this type.

Proposition 6.4. Let t be a fixed positive integer. If f(x) is a binding function such that f(x)=x for all $x \geq t$ then f(x) has a complementary binding function. □

It does not seem to be a trivial problem to determine the complementary binding functions of *any* function different from f(x)=x. Perhaps the simplest problem of this type is

Problem 6.5. Let f be the binding function defined as

$$f(x) = \begin{cases} x \text{ if } x \neq 2 \\ 3 \text{ if } x = 2 \end{cases}.$$

What is the complementary binding function of f? Perhaps $\left\lfloor \frac{3x}{2} \right\rfloor$ is the truth.

The following result shows that the Perfect Graph Theorem is stable in a certain sense.

Theorem 6.6. If f(x) is a self-complementary binding function then f(x)=x for all positive integers.

Proof. Assume that f is self-complementary.

Case 1. Assume that f(2)=2. If $f(x) \neq x$ for some $x \in N$ then we can choose $k \in N$ such that $k \geq 3$, $f(k) > k$ and $f(x)=x$ for $x<k$. Clearly f is a θ-binding function for $\{C_{2k+1}\}$ but fails to

be a χ-binding function for $\{C_{2k+1}\}$, i.e. f is not self-complementary. The contradiction shows $f(x)=x$ for all $x\in N$.

Case 2. Assume that $f(2)>2$ and $f(k)<\left\lceil\frac{3k-1}{2}\right\rceil$ for some k. Consider the graph G_k whose complement is $\left\lfloor\frac{k}{2}\right\rfloor$ disjoint C_5 and, for odd k, an additional isolated vertex. Now f is a θ-binding function for $\{G_k\}$ $(\alpha(G_k)=2,\ \Theta(G_k)=3)$ but fails to be a χ-binding function for $\{G_k\}$ $(\omega(G_k)=k,$ $\chi(G_k)=\left\lceil\frac{3k-1}{2}\right\rceil)$.

Case 3. $f(k)\geq\left\lceil\frac{3k-1}{2}\right\rceil$ for all $k\in N$. In this case theorem 6.2 implies that f(x) has no complementary binding function, again a contradiction. □

A generalization of the Perfect Graph Theorem (proved also by Lovász in [27]) says that a graph G is perfect if

$$\alpha(G')\cdot\omega(G')\geq |V(G')|$$

holds for all induced subgraph G′ of G. The first step in searching analogous properties would be to settle

Problem 6.7. Let $\mathcal{G}$ be the family of graphs G satisfying

$$\alpha(G')\cdot\omega(G')\geq |V(G')|-1$$

for all induced subgraphs G′ of G. Is it true that $\mathcal{G}$ is a χ-bound (or, equivalently, θ-bound) family? If yes, what is the smallest binding function for $\mathcal{G}$?

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