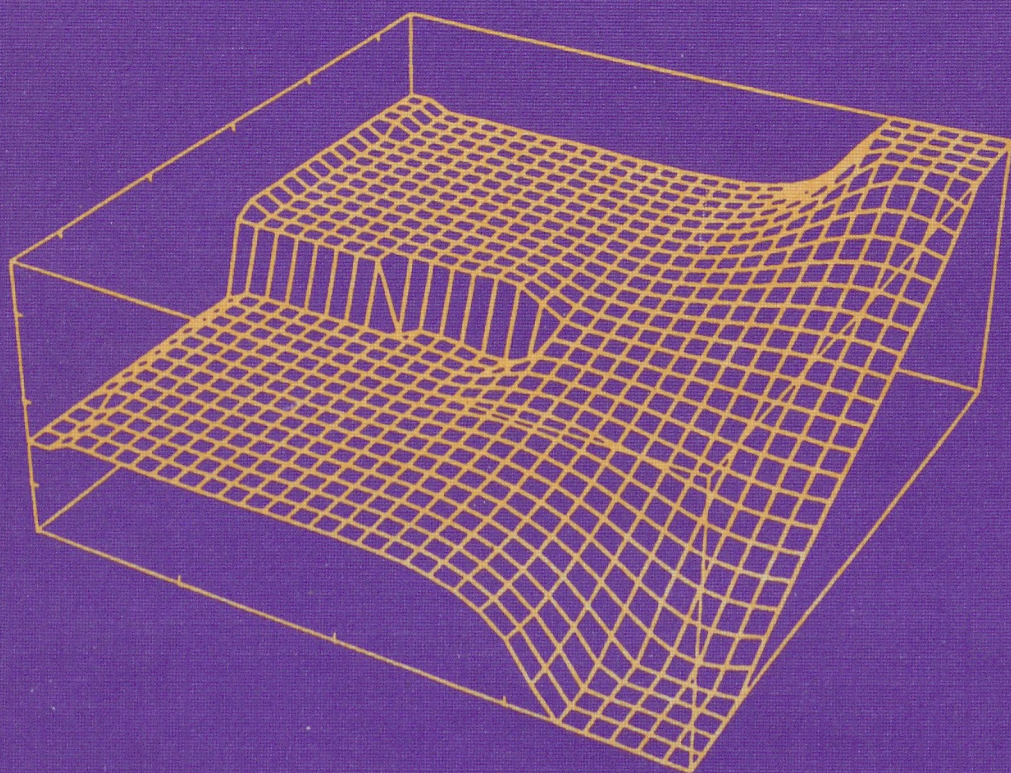


RELATIVITY TODAY

Proceedings of the
Fourth Hungarian Relativity
Workshop

Edited by R. P. KERR and Z. PERJÉS



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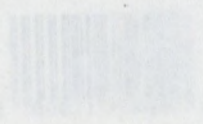
Proceedings of the
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Edited by

P. Kerr

1983

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Akadémiai Kiadó, Budapest

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ISBN 963 05 6766 0

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H-1519, Budapest, P.O. Box 245

Printed in Hungary

by Akadémiai Kiadó és Nyomda Vállalat, Budapest



M. TUD. AKADÉMIA KÖNYVTÁRA
Könyvtár 7620./19 34.... SZ.

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FOREWORD

This volume contains the written account of the lectures given at the *Fourth Hungarian Relativity Workshop*. The subjects cover some developing areas of the theory in a patchy fashion, corresponding to the representation of each subject by participants.

Past experience tells us that Hungarian Relativity Workshops are triennial events. Participants, both from Hungary and from abroad, had been lured by their common research interests. The subjects of common value drift only slightly as time goes by, as interest in them is motivated by the great unsolved mysteries. To name just a few of these, there is the symmetries of space-time (Part I) or the problem of matching a rotating body solution of the gravitational equations to an external field. This will suffice to explain why axisymmetric systems (Part II) appear on the agenda again. Another classical subject in relativity theory — canonical gravity — is discussed here from various standpoints.

Perhaps the most marked drift of interest in the subjects is towards the theory of gravitational radiation. This tendency is best understood in terms of the needs of the gravitational wave detectors under construction. The subject is discussed in both Part IV (Gravitational radiation) and Part V (Evolution in gravitational systems).

The organizers thank the Hungarian Academy of Sciences for generous financial and material support. Special praise is due to Ms. Zs. Bódi of the Institute staff for helping our workshop run smoothly.

December 1993

The Editors

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Mixed Invariants of the Riemann Tensor

C.B. DeWitt and D. DeWitt

Part I

Space-Time Symmetries

1. Introduction

It is well known that there are 20 independent algebraic invariants of the Riemann tensor in a four-dimensional Lorentzian space. In the general case, these invariants can be written in terms of four different types of quantities: 1) the real curvature scalar, two complex invariants, I and J formed from the Weyl tensor, 2) three real invariants L , P , and R formed from the symmetric Ricci tensor, 3) and three complex mixed invariants A , B , and C , formed from Φ and Ψ .

The literature on these and their role has been discussed by Wilson [9], Tolman [11], Grossmann [13], Synge [15] and many others. Penrose and Rindler [6] (p. 285) state that "the problem of classifying a complete set of space-time difficult (and, as far as we are aware, as yet unsolved). For it must be expected that a convenient set would be required, the members of which are not connected by a number of relations known as syzygies. Grossmann translated the mixed invariants in such a way which obscures the 'orientation' between the Weyl tensor and the Einstein curvature tensor. In a recent paper, Carmichael and McLerrigan [14] give some geometrical interpretations of the role played by the mixed invariants in Einstein-Maxwell and perfect fluid cases. They also give a set of 16 scalar invariants of the Riemann tensor which they claim to be a complete set, but in certain degenerate cases (Einstein-Maxwell and perfect fluid) and those that perfectly known and fail to be complete in the perfect fluid case. They also give other invariants, α , β , and γ formed from the tensors Φ and Ψ and Ψ , α is a new invariant.

In the general case, the invariants I and J essentially determine the components of the Weyl tensor in a canonical mixed tensor, likewise the invariants L , P , and R essentially determine the components of the trace-free Ricci tensor in a canonical mixed tensor.

Mixed Invariants of the Riemann Tensor

C.B. McIntosh and E. Zakhary

Abstract: There are at most 14 independent algebraic invariants of the Riemann tensor in a four-dimensional Lorentzian space. In the general case, these invariants can be written in terms of four different types of quantities: R , the real curvature scalar, two complex invariants I and J formed from the Weyl spinor, three real invariants I_6 , I_7 and I_8 formed from the trace-free Ricci spinor and three complex mixed invariants I_9 , I_{10} and I_{11} . Carminati and McLenaghan [1] give some geometrical interpretations of the role played by the mixed invariants in Einstein-Maxwell and perfect fluid cases. They show that 16 invariants are needed in certain degenerate cases (Einstein-Maxwell and perfect fluid) and show that previously known sets fail to be complete in the perfect fluid case.

In the general case, the invariants I and J essentially determine the components of the Weyl spinor in a canonical tetrad frame; likewise the invariants I_6 , I_7 and I_8 essentially determine the components of the trace-free Ricci spinor in a (in general different) canonical tetrad frame. The mixed invariants thus give the orientation between the frames of these two spinors. The six real pieces of information in I_9 , I_{10} and I_{11} is precisely the information needed to do this.

A table is given of invariants which give an independent set for each Petrov type of the Weyl spinor Ψ_{ABCD} and for each Segre type of the trace-free Ricci spinor $\Phi_{AB\dot{C}\dot{D}}$. This table involves 17 real invariants, including one real invariant and one complex invariant involving Ψ_{ABCD} , $\bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ and $\Phi_{AB\dot{C}\dot{D}}$ in some degenerate cases.

All invariants are written in the form $\text{Trace}(AB^T)$ where A and B are 3×3 complex matrices formed from the spinor components of the Riemann tensor; thus the invariants are easily calculated using a computer algebra program.

1. Introduction

It is well known that there are at most 14 independent algebraic invariants of the Riemann tensor in a four-dimensional Lorentzian space. In the general case, these invariants can be written in terms of four different types of quantities: R , the real curvature scalar, two complex invariants I and J formed from the Weyl spinor, Ψ , three real invariants I_6 , I_7 and I_8 formed from the trace-free Ricci spinor, Φ , and three complex mixed invariants I_9 , I_{10} and I_{11} formed from Φ and Ψ .

The fourteen invariants and their role have been discussed by Witten [9], Petrov [7], Greenberg [3], Sneddon [8] and many others. Penrose and Rindler [6] (p. 263) state that "the problem of obtaining a 'complete set' is much more difficult (and, as far as we are aware, as yet unsolved). For it must be expected that a redundant set would be required, the members of this set being connected by a number of relations known as *syzygies*. Greenberg introduces the mixed invariants as ones which describe the 'orientation' between the Weyl tensor and the Einstein curvature tensor. In a recent paper, Carminati and McLenaghan [1] give some geometrical interpretations of the role played by the mixed invariants in Einstein-Maxwell and perfect fluid cases. They also give a set of 16 scalar invariants of the Riemann tensor which they show to be a complete minimal set in certain degenerate cases (Einstein-Maxwell and perfect fluid) and show that previously known sets fail to be complete in the perfect fluid case. They use three mixed invariants m_3 , m_4 and m_5 formed from the spinors Φ and from both Ψ and $\bar{\Psi}$. m_5 is a new invariant.

In the general case, the invariants I and J essentially determine the components of the Weyl spinor in a canonical tetrad frame; likewise the invariants I_6 , I_7 and I_8 essentially determine the components of the trace-free Ricci spinor in a (in general different) canonical

tetrad frame. In neither case is the frame given unique, since there is always a six (real) parameter Lorentz group of motions of rotational freedom, $\Pi \otimes \bar{\Pi}$, which changes any frame but which leaves the invariants unchanged. For example, if the Weyl spinor is of Petrov type I , the invariants I and J determine the nonzero components $\Psi_0 = \Psi_4$, and Ψ_2 of the Weyl spinor Ψ_{ABCD} (in a canonical frame) or some tetrad rotation of these components (up to a re-labelling) in a canonical frame. If this canonical frame was taken for the Weyl spinor, the mixed invariants give the rotation between this frame and a canonical frame, given by I_9, I_{10} and I_{11} , of the components $\Phi_{AB\dot{C}\dot{D}}$ (again up to re-labelling). The six real pieces of information in I_9, I_{10} and I_{11} are precisely the information needed to determine this rotation. Examples will be given in a later paper.

All the invariants can be written in the form $\text{Tr}(AB^T)$, where A and B are 3×3 complex matrices formed from the spinor components of the Riemann tensor. Thus the invariants can be easily calculated using a computer algebra program.

A table is presented which gives the invariants which arise in various classes of the classification of the Riemann tensor. The Petrov type of the Weyl spinor is plotted against the Segre type of the trace-free Ricci spinor. The numbers of independent invariants, together with the invariants, are given in each class. It is shown that a new invariant M_2 , which plays a similar role to that played by the new invariant m_5 of Carminati and McLenaghan [1], is needed in the cases when the Weyl spinor is of Petrov type I, II or III and when the Petrov type of the Plebanski spinor formed from the nonzero Φ spinor is type O , i.e. when the Segre type is $[1(11,1)]$, $[(111),1]$ or $[(112)]$. These cases include the cases of perfect fluid (which is of Segre type $[(111),1]$ and null electromagnetic fields (which are of Segre type $[(112)]$) for any of the Weyl spinor of Petrov types I, II and III , which were the examples considered by Carminati and McLenaghan and shown by them to need their m_5 invariant.

2. Basic Equations

The metric g can be written as

$$g = g_{ab}\theta^a \otimes \theta^b, \quad (2.1)$$

where $\{\theta^a\}$ is a pseudo-orthonormal null tetrad of one-forms such that

$$\begin{aligned} \theta^1 \cdot \theta^2 &= +1, \\ \theta^3 \cdot \theta^4 &= -1 \end{aligned} \quad (2.2)$$

with other inner products zero. Then

$$g = \theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1 - \theta^3 \otimes \theta^4 - \theta^4 \otimes \theta^3. \quad (2.3)$$

In terms of a standard Newman-Penrose (NP) tetrad $\{l, n, m, \bar{m}\}$, the θ^a can be written as

$$\theta^1 \equiv n, \quad \theta^2 \equiv l, \quad \theta^3 \equiv -\bar{m}, \quad \theta^4 \equiv \bar{\theta}^3 \equiv -m. \quad (2.4)$$

l and n are each self-conjugate. The metric then has coordinate components

$$g_{\alpha\beta} = l_\alpha n_\beta + l_\beta n_\alpha - m_\alpha \bar{m}_\beta - m_\beta \bar{m}_\alpha. \quad (2.5)$$

The NP dual vectors $(D, \Delta, \delta, \bar{\delta})$ satisfy

$$\langle \theta^1, D \rangle = 1, \quad \langle \theta^2, \Delta \rangle = 1 \quad \langle \theta^3, \delta \rangle = 1 \quad \langle \theta^4, \bar{\delta} \rangle = 1 \quad (2.6)$$

with other contractions zero.

A natural bivector basis is

$$\begin{aligned} Z^1 &= \theta^1 \wedge \theta^3, & Z^2 &= \theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4, & Z^3 &= \theta^4 \wedge \theta^2, \\ \bar{Z}^1 &= \theta^1 \wedge \theta^4, & \bar{Z}^2 &= \theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4, & \bar{Z}^3 &= \theta^3 \wedge \theta^2. \end{aligned} \quad (2.7)$$

Note that, whenever the dual is given by

$$*\theta^a \wedge \theta^b \equiv \frac{1}{2} \eta^{ab}{}_{cd} \theta^c \wedge \theta^d, \quad (2.8)$$

the Z^i satisfy

$$\begin{aligned} *Z^j &= iZ^j, & *\bar{Z}^j &= -i\bar{Z}^j, \\ **Z^j &= -Z^j, & **\bar{Z}^j &= -\bar{Z}^j. \end{aligned} \quad (2.9)$$

The connection one-forms σ_i are given by Cartan's first equations of structure,

$$\begin{aligned} dZ^1 &= -2\sigma_2 \wedge Z^1 - \sigma_3 \wedge Z^2, \\ dZ^2 &= 2\sigma_1 \wedge Z^1 - 2\sigma_3 \wedge Z^3, \\ dZ^3 &= \sigma_1 \wedge Z^2 + 2\sigma_2 \wedge Z^3, \end{aligned} \quad (2.10)$$

and Cartan's second equations of structure are

$$\begin{aligned} \Sigma_1 &= d\sigma_1 + 2\sigma_1 \wedge \sigma_2, \\ \Sigma_2 &= d\sigma_2 + \sigma_1 \wedge \sigma_3, \\ \Sigma_3 &= d\sigma_3 + 2\sigma_2 \wedge \sigma_3, \end{aligned} \quad (2.11)$$

where the curvature two-forms Σ_i are defined by

$$\Sigma_i = (\Psi_{ij} + 2\Lambda H_{ij})Z^j + \Phi_{ij}\bar{Z}^j. \quad (2.12)$$

Here the tetrad components of the Weyl spinor Ψ_{ABCD} and of the trace-free Ricci spinor $\Phi_{AB\dot{C}\dot{D}}$ are the elements Ψ_{ij} and Φ_{ij} of the matrices Ψ and Φ respectively,

$$\Psi = \begin{bmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{bmatrix} \quad (2.13)$$

and

$$\Phi = \begin{bmatrix} \Phi_{00} & \Phi_{01} & \Phi_{02} \\ \Phi_{10} & \Phi_{11} & \Phi_{12} \\ \Phi_{20} & \Phi_{21} & \Phi_{22} \end{bmatrix}. \quad (2.14)$$

Also Λ is the curvature scalar and H is defined by

$$H = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1/2 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \quad (2.15)$$

3. Tetrad Freedom

A null spinor basis $\{o_A, \iota_A, \bar{o}_{\dot{A}}, \bar{\iota}_{\dot{A}}\}$ can be defined for the metric (2.3) where

$$o_A \iota^A = -\iota_A o^A = \bar{o}_{\dot{A}} \bar{\iota}^{\dot{A}} = -\bar{\iota}_{\dot{A}} \bar{o}^{\dot{A}} = 1. \quad (3.1)$$

and

$$l^a \leftrightarrow o^A \bar{o}^{\dot{A}}, \quad n^a \leftrightarrow \iota^A \bar{\iota}^{\dot{A}}, \quad m^a \leftrightarrow o^A \bar{\iota}^{\dot{A}}, \quad \bar{m}^a \leftrightarrow \iota^A \bar{o}^{\dot{A}} \quad (3.2)$$

The metric is then invariant under the $SL(2, C) \otimes S\bar{L}(2, C)$ Lorentz group of transformations on the spinor basis. This can be written as $\Pi \otimes \bar{\Pi}$ where (McIntosh and Hickman [4])

$$\begin{aligned} \Pi &= \left(\Pi_X^Y \right) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in SL(2, C), \\ \bar{\Pi} &= \left(\bar{\Pi}_{\dot{X}}^{\dot{Y}} \right) = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} \in S\bar{L}(2, C). \end{aligned} \quad (3.3)$$

Then

$$\begin{aligned} \begin{bmatrix} o^X \\ \iota^X \end{bmatrix}' &= \Pi \begin{bmatrix} o^X \\ \iota^X \end{bmatrix}, \\ \begin{bmatrix} o^{\dot{X}} \\ \iota^{\dot{X}} \end{bmatrix}' &= \bar{\Pi} \begin{bmatrix} o^{\dot{X}} \\ \iota^{\dot{X}} \end{bmatrix}. \end{aligned} \quad (3.4)$$

There are three obvious special cases of the form $\Pi \otimes \bar{\Pi}$, with

$$\Pi_K = \begin{bmatrix} 1 & 0 \\ K & 1 \end{bmatrix}, \quad \Pi_L = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}, \quad \Pi_M = \begin{bmatrix} M & 0 \\ 0 & M^{-1} \end{bmatrix}. \quad (3.5)$$

$\Pi_K \otimes \bar{\Pi}_K$ leaves the null direction l invariant, $\Pi_L \otimes \bar{\Pi}_L$ leaves the null direction n invariant and $\Pi_M \otimes \bar{\Pi}_M$ has the effect

$$D' = M \bar{M} D, \quad \Delta' = M^{-1} \bar{M}^{-1} \Delta, \quad \delta' = M \bar{M}^{-1} \delta, \quad \bar{\delta}' = M^{-1} \bar{M} \bar{\delta}. \quad (3.6)$$

If $M = A \exp(i\theta)$, this is a boost $l' = A^{-1}l$, $n' = An$ and a spatial rotation $m' = \exp(i\theta)m$.

A general transformation Π can be decomposed into basis transformations Π_K, Π_L and Π_M . For a general transformation,

$$\Psi' = \Xi \Psi \Xi^T, \quad \Phi' = \Xi \Phi \Xi^T, \quad Z' = (\Xi^T)^{-1} Z, \quad (3.7)$$

where

$$Z = [Z^1, Z^2, Z^3]^T \quad (3.8)$$

and

$$\Xi = \begin{bmatrix} A^2 & 2AB & B^2 \\ AC & AD + BC & BD \\ C^2 & 2CD & D^2 \end{bmatrix}. \quad (3.9)$$

Note that

$$(\Xi^T)^{-1} = \begin{bmatrix} D^2 & -CD & C^2 \\ -2BD & AD + BC & -2AC \\ B^2 & -AB & A^2 \end{bmatrix}. \quad (3.10)$$

Here, as in (3.3), $AD - BC = 1$.

4. Invariants

The matrix E is defined as the inverse of the matrix H of (2.15) by

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \quad (4.12)$$

Also P and Q are defined as the adjoint matrices of $E\Phi E$ and $E\Psi E$ respectively, that is, P satisfies

$$E\Psi EP^T / \det(E\Psi E) = I \quad (4.2)$$

and Q satisfies

$$E\Phi EQ^T / \det(E\Phi E) = I. \quad (4.3)$$

Explicit forms of P and Q in terms of NP-tetrad components of the Riemann tensor are given in the Appendix.

Quantities which are invariants of the Riemann tensor (and are thus invariant under $\Pi \otimes \bar{\Pi}$) can be grouped in five different classes as follows:

1. The Riemann scalar: $\Lambda = R/24$

$$\Lambda = \text{tr}(\Psi E)/6. \quad (4.4)$$

2. The Pure Ricci Invariants: I_6, I_7 and I_8

$$I_6 = \text{tr}(\Phi E \Phi^T E), \quad I_7 = \text{tr}(\Phi E Q^T E), \quad I_8 = \text{tr}(Q E Q^T E). \quad (4.5)$$

Note that $I_7 = 12\det(\Phi)$ and $I_7^2 = 9\det(Q)$.

3. The Pure Weyl Invariants: I and J

$$I = \text{tr}(\Psi E \Psi^T E), \quad J = \text{tr}(\Psi E P^T E). \quad (4.6)$$

Note that $J = 12\det(\Psi)$. The metric is algebraically degenerate if and only if $2I^3 - 3J^2 = 0$.

4. The Mixed Invariants: I_9, I_{10} and I_{11} – formed from Φ and Ψ

$$I_9 = \text{tr}(\Phi E (\Psi E \Phi)^T E), \quad I_{10} = \text{tr}(\Phi E (P E \Phi)^T E), \quad I_{11} = \text{tr}(Q E (\Psi E Q)^T E). \quad (4.7)$$

5. The Mixed Invariants: M_1 and M_2 – formed from Φ, Ψ and $\bar{\Psi}$.

$$M_1 = \text{tr}(\Phi E \bar{\Psi} E (\Psi E \Phi)^T E), \quad M_2 = \text{tr}(\Phi E \bar{\Psi} E (P E \Phi)^T E). \quad (4.8)$$

In [1] the second invariant M_2 is replaced by an invariant labelled m_5 which in matrix form is

$$m_5 = \text{tr}(\Psi E \Phi E \bar{\Psi} E (\Psi E \Phi)^T E). \quad (4.9)$$

These are related by

$$m_5 = (M_2 + I \cdot \bar{I}_9)/2. \quad (4.10)$$

In the general case, M_2 has less terms in it than does m_5 .

An inspection of the form of these invariants quickly shows a pattern; many other invariants could easily be written down. For example, another mixed invariant is $M_3 = \text{tr}(\Phi E \bar{\Psi} E (\Psi E Q)^T E)$.

In spinor language, the invariant m_5 of (4.9) is

$$m_5 = \Psi^{AB}{}_{CD} \Psi^{CD}{}_{EF} \Phi^{EF}{}_{\dot{E}\dot{F}} \bar{\Psi}^{\dot{E}\dot{F}}{}_{\dot{C}\dot{D}} \Phi^{AB}{}_{\dot{C}\dot{D}}. \quad (4.11)$$

Obviously the matrix E which appears in the matrix expressions of the invariants is equivalent to the process of raising subscript indices of the relevant spinors in order to obtain invariants which are contractions of spinor products.

A standard set of canonical forms of the Weyl spinor Ψ_{ABCD} is given in Table 1 for various Petrov types. In Table 2 canonical forms of the trace-free Ricci spinor $\Phi_{AB\dot{C}\dot{D}}$ for various Segre types are given. These are listed by Joly and MacCallum [2] who use a slightly modified version of the forms listed by McIntosh, Foyster and Lun [5]. Also given in Table 2 are the corresponding Segre types for each Petrov type of the Plebanski spinor χ_{ABCD} . (See also [5]; the NP components of χ_{ABCD} are given in (A.3).)

The nonzero invariants of from (4.5) to (4.8) for each Petrov type of the Weyl spinor and for each Segre class (and corresponding Petrov-Plebanski type) of the Ricci spinor are given in Table 3. The number of invariants of the Riemann tensor apart from the curvature scalar are listed for each possible case. It is shown that the new invariant M_2 is needed in the cases when the Weyl spinor is of Petrov type *I*, *II* or *III* and when the Petrov type of the Plebanski spinor formed from the nonzero Φ spinor is type *O*, i.e. when the Segre type is $[1(11,1)]$, $[(111), 1]$ or $[(112)]$. These cases include the cases of perfect fluid (which is of Segre type $[(111), 1]$) and null electromagnetic fields (which are of Segre type $[(112)]$) for any of the Weyl spinor of Petrov types *I*, *II* and *III*, which were the examples considered by Carminati and McLenaghan and shown by them to need their m_5 invariant.

In the general case, when the Weyl spinor is of Petrov type *I* and the trace-free Ricci spinor of Segre type $[111, 1]$ (or, equivalently, of type $[11Z\bar{Z}]$), the invariants I and J essentially determine the components of the Weyl spinor in a canonical tetrad frame; likewise the invariants I_6 , I_7 and I_8 essentially determine the components of the trace-free Ricci spinor in a (in general different) canonical tetrad frame. The mixed invariants then give the orientation between the frames of these two spinors. The six real pieces of information in I_9 , I_{10} and I_{11} are precisely the information needed to do this.

Carminati and McLenaghan [1] show that 16 invariants are needed in certain degenerate cases (Einstein-Maxwell and perfect fluid) and show that previously known sets fail to be complete in the perfect fluid case. Table 3 and the curvature scalar Λ involve 17 invariants. Thus 17 invariants can give a complete set in all possible cases. It is not known if this number can be reduced to 16. Carminati and McLenaghan [1] also give some geometrical interpretations of the role played by the mixed invariants in Einstein-Maxwell and perfect fluid cases.

Petrov type	Canonical form: non-zero Ψ
<i>I</i>	$\Psi_0 = \Psi_4, \Psi_2$
<i>II</i>	$\Psi_2, \Psi_4 = 1$
<i>III</i>	$\Psi_3 = 1$
<i>D</i>	Ψ_2
<i>N</i>	$\Psi_4 = 1$
<i>O</i>	none

Table 1. Canonical forms of the Weyl spinor Ψ_{ABCD} are listed for various Petrov types

PP type	Segre type	Canonical form: non-zero Ψ
<i>I</i>	$[111, 1]$ $[11Z\bar{Z}]$	$\Phi_{00} = \Phi_{22}, \Phi_{11}, \Phi_{02} = \Phi_{20} \neq \pm 2\Phi_{11} \pm \Phi_{00}$ $\Phi_{00} = -\Phi_{22}, \Phi_{11}, \Phi_{02} = \Phi_{20}$
<i>II</i>	$[112]$	$\Phi_{11}, \Phi_{22} = 1, \Phi_{02} = \Phi_{20}$
<i>III</i>	$[13]$	$-2\Phi_{11} = \Phi_{02} = \Phi_{20}, \Phi_{01} = \Phi_{10} = 1$
<i>D</i>	$[11(1, 1)]$ $[(11)1, 1]$ $[(11)Z\bar{Z}]$ $[(11)2]$ $[(11)(1, 1)]$	$\Phi_{11}, \Phi_{02} = \Phi_{20} \neq \pm 2\Phi_{11}$ $\Phi_{00} = \Phi_{22}, \Phi_{11} \neq \pm \Phi_{00}/2$ $\Phi_{00} = -\Phi_{22}, \Phi_{11}$ $\Phi_{11}, \Phi_{22} = 1$ Φ_{11}
<i>N</i>	$[1(12)]$ $[(13)]$	$2\Phi_{11} = \Phi_{02} = \Phi_{20}, \Phi_{22} = 1$ $\Phi_{01} = \Phi_{10} = 1$
<i>O</i>	$[1(11, 1)]$ $[(111), 1]$ $[(112)]$ $[(1111)]$	$2\Phi_{11} = \Phi_{02} = \Phi_{20}$ $2\Phi_{11} = \Phi_{00} = \Phi_{22}$ $\Phi_{22} = 1$ none

Table 2. Canonical forms of the Ricci spinor $\Phi_{AB\dot{C}\dot{D}}$ for various Segre types. The Petrov types of the Plebanski spinor χ_{ABCD} (PP-types) are also given for each Segre type

Petrov types of $\Psi_{ABCD} \Rightarrow$			I	II	III	D	N	O
$\Downarrow$ PP and $\Downarrow$ Segre types of $\Phi_{AB\dot{C}\dot{D}}$			4 $I\ J$	2 I		2 I		
I	$[111, 1]$ $[11Z\bar{Z}]$	3 $I_6\ I_7\ I_8$	6			4		
II	$[112]$	2 $I_6\ I_7$	$I_9\ I_{10}\ I_{11}$			$I_9\ I_{11}$		
III	$[13]$	1 I_6						
D	$[11(1, 1)]$ $[(11)1, 1]$ $[(11)Z\bar{Z}]$	2 $I_6\ I_7$	5			3		
	$[(11)2]$	1 I_6	$I_9\ I_{10}\ M_1$			$I_9\ M_1$		
	$[(11)(1, 1)]$	1 I_6	4 $I_9\ I_{10}$			2 I_9		
N	$[1(12)]$ $[(13)]$	1 I_6	5 $I_9\ I_{10}\ M_1$			3 $I_9\ M_1$		
O	$[1(11, 1)]$ $[(111), 1]$	1 I_6	3 $M_1\ M_2$			1 M_1		
	$[(112)]$							
	$[(1111)]$							

Table 3. The invariants and the number of invariants of the Riemann tensor apart from the curvature scalar Λ are listed for the various types of the Weyl spinor Ψ_{ABCD} and for various Segre classes (and corresponding Petrov-Plebanski types) of the Ricci spinor $\Phi_{AB\dot{C}\dot{D}}$. The invariants I_6, I_7, I_8 and M_1 are real; the invariants $I, J, I_9, I_{10}, I_{11}$ and M_2 are complex

5. Example: Petrov Type I and Segre Type $[111, 1]$

Consider a metric belonging to the most general class, i.e. one which has a Weyl spinor of Petrov type I and a trace-free Ricci spinor of Segre type $[111, 1]$. From Tables 1 and 2 the tetrad for this metric can be aligned so that the nonzero components of the Weyl spinor Ψ_{ABCD} can be

$$\Psi_0 = \Psi_4, \quad \Psi_2, \quad (5.1)$$

or it can be aligned so that the nonzero components of the Ricci spinor $\Phi_{AB\dot{C}\dot{D}}$ can be

$$\Phi_{00} = \Phi_{22}, \quad \Phi_{11}, \quad \Phi_{02} = \Phi_{20} \neq \pm 2\Phi_{11} \pm \Phi_{00}. \quad (5.2)$$

Λ will in general also be nonzero, but this is not influenced by the alignment of the tetrad. However, in general, the tetrad will not be such that the Ψ_{ABCD} and $\Phi_{AB\dot{C}\dot{D}}$ will both be in canonical forms of (5.1) and (5.2).

The invariants (4.5–4.8) in the aligned case, i.e. in the case when both the Weyl and Ricci spinors are in canonical form, are

$$I_6 = 2(\Phi_{00}^2 + \Phi_{02}^2 + 2\Phi_{11}^2), \quad I_7 = 12\Phi_{11}(\Phi_{00}^2 - \Phi_{02}^2),$$

$$I_8 = 4((\Phi_{00}^2 - \Phi_{02}^2)^2 + 8(\Phi_{00}^2 + \Phi_{02}^2)\Phi_{11}^2), \quad (5.3)$$

$$I = 2(\Psi_0^2 + 3\Psi_2^2), \quad J = 12\Psi_2(\Psi_0^2 - \Psi_2^2), \quad (5.4)$$

$$I_9 = 2(2\Phi_{00}\Phi_{02}\Psi_0 + (\Phi_{00}^2 + \Phi_{02}^2 - 4\Phi_{11}^2)\Psi_2),$$

$$I_{10} = 8(2\Phi_{00}\Phi_{02}\Psi_0\Psi_2 - \Phi_{11}^2\Psi_0^2 - (\Phi_{00}^2 + \Phi_{02}^2 - \Phi_{11}^2)\Psi_2^2),$$

$$I_{11} = -8(8\Phi_{00}\Phi_{02}\Phi_{11}^2\Psi_0 + ((\Phi_{00}^2 - \Phi_{02}^2)^2 - 4(\Phi_{00} - \Phi_{02})^2\Phi_{11}^2)\Psi_2) \quad (5.5)$$

and

$$M_1 = 2((\Phi_{00}^2 + \Phi_{02}^2)\Psi_0\bar{\Psi}_0 + (\Phi_{00}^2 + \Phi_{02}^2 + 8\Phi_{11}^2)\Psi_2\bar{\Psi}_2 + 2\Phi_{00}\Phi_{02}(\Psi_0\bar{\Psi}_2 + \bar{\Psi}_0\Psi_2)),$$

$$M_2 = 16(2(\Phi_{00}^2 + \Phi_{02}^2)\Phi_{11}^2\Psi_0\bar{\Psi}_0 - 4\Phi_{00}\Phi_{02}\Phi_{11}^2(\Psi_0\bar{\Psi}_2 + \bar{\Psi}_0\Psi_2) + ((\Phi_{00}^2 - \Phi_{02}^2)^2 + 2(\Phi_{00}^2 + \Phi_{02}^2)\Phi_{11}^2)\Psi_2\bar{\Psi}_2) \quad (5.6)$$

Acknowledgements

Colin McIntosh has great pleasure in thanking the Hungarian Academy of Sciences for its large contribution towards his expenses in attending the Fourth Hungarian Relativity Workshop in Gárdony, Hungary. He also very sincerely thanks Professor Roy Kerr, Dr. Zoltán Perjés and the Central Research Institute of Theoretical Physics in Budapest for their kind hospitality in July, 1992.

Appendix

Components of the P and Q matrices defined in (4.2) and (4.3) are

$$P = \begin{bmatrix} 4(\Psi_0\Psi_2 - \Psi_1^2) & 2(\Psi_0\Psi_3 - \Psi_1\Psi_2) & 4(\Psi_1\Psi_3 - \Psi_2^2) \\ 2(\Psi_0\Psi_3 - \Psi_1\Psi_2) & \Psi_0\Psi_4 - \Psi_2^2 & 2(\Psi_1\Psi_4 - \Psi_2\Psi_3) \\ 4(\Psi_1\Psi_3 - \Psi_2^2) & 2(\Psi_1\Psi_4 - \Psi_2\Psi_3) & 4(\Psi_2\Psi_4 - \Psi_3^2) \end{bmatrix}, \quad (\text{A.1})$$

$$Q = \begin{bmatrix} 4(\Phi_{00}\Phi_{11} - \Phi_{01}\Phi_{10}) & 2(\Phi_{00}\Phi_{12} - \Phi_{02}\Phi_{10}) & 4(\Phi_{01}\Phi_{12} - \Phi_{02}\Phi_{11}) \\ 2(\Phi_{00}\Phi_{21} - \Phi_{01}\Phi_{20}) & (\Phi_{00}\Phi_{22} - \Phi_{02}\Phi_{20}) & 2(\Phi_{01}\Phi_{22} - \Phi_{02}\Phi_{21}) \\ 4(\Phi_{01}\Phi_{21} - \Phi_{11}\Phi_{20}) & 2(\Phi_{01}\Phi_{22} - \Phi_{20}\Phi_{12}) & 4(\Phi_{11}\Phi_{22} - \Phi_{12}\Phi_{21}) \end{bmatrix}. \quad (\text{A.2})$$

NP components of the Plebanski spinor χ_{ABCD} are

$$\begin{aligned} \chi_0 &= \frac{1}{2}(\Phi_{00}\Phi_{02} - \Phi_{01}^2), \\ \chi_1 &= \frac{1}{4}(\Phi_{00}\Phi_{12} + \Phi_{10}\Phi_{02} - 2\Phi_{01}\Phi_{11}), \\ \chi_2 &= \frac{1}{12}(\Phi_{00}\Phi_{22} - 4\Phi_{11}^2 + \Phi_{02}\Phi_{20} + 4\Phi_{10}\Phi_{12} - 4\Phi_{01}\Phi_{21}), \\ \chi_3 &= \frac{1}{4}(\Phi_{10}\Phi_{22} + \Phi_{12}\Phi_{20} - 2\Phi_{11}\Phi_{21}), \\ \chi_4 &= \frac{1}{2}(\Phi_{20}\Phi_{22} - \Phi_{21}^2). \end{aligned} \quad (\text{A.3})$$

Note that these components could be used in various calculations. For example,

$$I_9 = \chi_0\Psi_4 - 4\chi_1\Psi_3 + 6\chi_2\Psi_2 - 4\chi_3\Psi_1 + \chi_4\Psi_0 = \text{Tr}(\Psi E \chi^T E) \quad (\text{A.4})$$

where χ is the matrix

$$\chi = \begin{bmatrix} \chi_0 & \chi_1 & \chi_2 \\ \chi_1 & \chi_2 & \chi_3 \\ \chi_2 & \chi_3 & \chi_4 \end{bmatrix}. \quad (\text{A.5})$$

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Symmetries of the Sectional Curvature Function in General Relativity

G.S. Hall and A.D. Hossack

Abstract: The problem of the existence on a space-time of vector fields whose associated local diffeomorphisms preserve the sectional curvature function of the space-time is solved.

1. Introduction

In the study of symmetries in differential geometry and in particular in general relativity attention is usually focussed on certain types of vector fields on manifolds. These vector fields are essentially equivalent to the existence everywhere of local groups of local diffeomorphisms of the manifold which reflect the desired symmetry. Thus a Killing vector field is really nothing more (or less) than a local group of local isometries in a neighbourhood of any point. In this paper the sectional curvature function of a space-time will be described and vector fields on space-times which give rise, in the manner just described, to local symmetries of the sectional curvature function will be discussed.

Throughout, M will denote a smooth space-time with smooth Lorentz metric g and the component form of the curvature and Ricci tensors will be denoted by R^a_{bcd} and R_{ab} ($\equiv R^c_{acb}$). As usual, a covariant derivative and a partial derivative will be denoted by a semi-colon and a comma respectively, and a Lie derivative by $\mathcal{L}$. Round and square brackets denote the usual symmetrisation and antisymmetrisation of indices. At each $p \in M$, $T_p M$ denotes the tangent space to M at p and G_p is the 4-dimensional Grassmann manifold of 2-dimensional subspaces (2-spaces) of $T_p M$. Of particular importance is the (4-dimensional open) submanifold $\bar{G}_p$ of G_p consisting of *non-null* members of G_p .

2. Local Symmetries

In this section some remarks will be made regarding the local symmetries that are generated by certain special vector fields on M . For reasons made clear in [1] one can, for the particular symmetries required here, regard such vector fields as smooth without any loss of generality. If X is a global smooth vector field on M then for any $p \in M$ there exists an open neighbourhood U of p and a positive real number ϵ such that for any $q \in U$ there exists an integral curve of X beginning at q and defined for parameter t values $-\epsilon < t < \epsilon$. Thus for each t in this range one has a local diffeomorphism φ_t which "moves" U a parameter distance t along the integral curves of X . More precisely, by taking the union of all subsets of the form $(-\epsilon, \epsilon) \times U$ constructed as above and denoting the resulting subset of $\mathbb{R} \times M$ by D , one can define a smooth map $\psi : D \rightarrow M$ by mapping $(t, p) \in D$ onto that point of M lying on the integral curve of X beginning at p and with parameter t . Then ψ and D have the following properties:

- (i) D is an open subset of $\mathbb{R} \times M$ such that for each $p \in M$, the subset $\{t \in \mathbb{R} : (t, p) \in D\}$ of $\mathbb{R}$ is an open interval containing 0.
- (ii) For each $p \in M$, $\psi(0, p) = p$.
- (iii) If $(r, p) \in D$, $(s, \psi(r, p)) \in D$ and $(r + s, p) \in D$ then $\psi(r + s, p) = \psi(s, \psi(r, p))$.

Such a map ψ is called a *local $\mathbb{R}$ -transformation group acting on M* [2] (or just a local group of local diffeomorphisms of M — it being clear that each map $\varphi_t : U \rightarrow \varphi_t(U)$ mentioned earlier and defined by $\varphi_t(p) = \psi(t, p)$ is injective and smooth on U with a smooth inverse φ_{-t}). Conversely, the existence of local diffeomorphisms of M from which one can construct mathematical objects ψ and D as above satisfying (i), (ii) and (iii) leads to the existence of a global smooth vector field X on M which “generates” the transformations [2]. The vector field X is called the *infinitesimal generator* of ψ . The extra condition that $D = \mathbb{R} \times M$ (a *global $\mathbb{R}$ -transformation group acting on M*) is equivalent to the condition that X is a *complete* vector field on M , that is each integral curve of X is defined for the parameter t in the range $-\infty < t < \infty$ [2, 3]. Only local $\mathbb{R}$ -transformation groups will be required in this paper.

One can now ask whether these local maps φ_t preserve any particular aspect of the geometry of M . To do this one uses φ_t to map U into $\varphi_t(U)$ (or φ_{-t} to map $\varphi_t(U)$ into U), and to carry the geometry of the domain with it to its range in the obvious way (since φ_t (or φ_{-t}) are (local) diffeomorphisms) and then compare this carried over geometry with the geometry already present on the range. Preservation of the various aspects of the geometry then leads to the various symmetries studied in differential geometry. Thus preservation of the metric leads to $\varphi_t^* \circ g = g \circ \varphi_{-t}$ (usually written as $\varphi_t^* g = g$), which is equivalent to $\mathcal{L}_X g = 0$, that is to X being a *Killing* vector field on M . Similarly, preservation of the conformal structure leads to $\varphi_t^* \circ g = \alpha g \circ \varphi_{-t}$ for some positive function α on U (usually written as $\varphi_t^* g = \alpha g$) which is equivalent to $\mathcal{L}_X g = 2\phi g$ for some function ϕ , that is to X being a *conformal* vector field on M . If $\phi_{;ab} = 0$ in any coordinate chart, X is called a *special conformal* vector field on M and the associated φ_t maps preserve the Ricci tensor. *Homothetic* vector fields are conformal vector fields for which ϕ and hence α are constant functions on M and when X is homothetic but not Killing, it is referred to as *proper* homothetic. In the same way, conformal vector fields are said to be *proper* if they are not homothetic. In the next section the sectional curvature function will be introduced and the symmetries of this particular aspect of the geometry of M defined.

The above theory, suitably modified, applies also to a local vector field i.e. where attention is restricted to the domain of the vector field. This will always be understood in what is to follow.

3. The Sectional Curvature Function

The sectional curvature function at $p \in M$ is a real-valued function on the set of all non-null 2-spaces $\bar{G}_p$ at p . It is, in fact, the map $\sigma_p : \bar{G}_p \rightarrow \mathbb{R}$ given by

$$\sigma_p(B) = \frac{R_{abcd}B^{ab}B^{cd}}{2g_a[cg_d]_b B^{ab}B^{cd}}. \quad (3.1)$$

where $B \in \bar{G}_p$ and B_{ab} are the components of any non-null simple bivector whose blade is B . The definition of $\sigma_p(B)$ is easily shown to be independent of the choice of the bivector B_{ab} representing B . Further, the restriction of σ_p to $\bar{G}_p$ is necessary since the denominator in (3.1) vanishes if and only if B is null. When G_p (and hence $\bar{G}_p$) is given its standard manifold structure, σ_p is an analytic function. If it is agreed to say that M has *constant curvature* at p when σ_p is a constant function on $\bar{G}_p$ then no continuous extension of σ_p from $\bar{G}_p$ to G_p is possible unless M has constant curvature at p [4] (see also [5, 6]).

It is also clear that σ_p is uniquely determined once the space-time metric is given. It is also true that if a sectional curvature function σ_p , arising from some metric on M , is given at each $p \in M$ then provided that M does not have constant curvature at any $p \in M$ this metric is uniquely determined on M except, possibly, on some non-empty open subset of M in which the metric is locally isometric to a conformally flat generalised plane wave [7, 8]. It can be shown in a precise way that the situation when the metric is uniquely determined by the sectional curvature is generic [6, 9].

4. Symmetries of the Sectional Curvature Function

One can now define a symmetry of the sectional curvature function. Let X be a global vector field on M with associated local diffeomorphisms φ_t as described in section 2 above. If p is in the domain of φ_t and B is the 2-space at p spanned by u and v in $T_p M$, define the 2-space $\varphi_{t*} B$ at $q \equiv \varphi_t(p)$ to be the 2-space spanned by $\varphi_{t*} u$ and $\varphi_{t*} v$. Then X is called *sectional curvature preserving* if, for any φ_t associated with X and any 2-space B at any point p in the domain of φ_t , one always has $\sigma_q(\varphi_{t*} B) = \sigma_p(B)$ in the sense that either both sectional curvatures are undefined or both are defined and are equal [7]. Sometimes it is convenient, for local work in general relativity, not to insist that X is globally defined but to allow it to be a local vector field as described in section 2. The modifications to the above definition in this case are obvious.

This definition, together with the general remarks in section 2 and the final paragraph of section 3 can now be combined to give the following result. The maps φ_t arising from a sectional curvature preserving vector field carry the sectional curvature and corresponding metric from some open subset U of M to the open subset $\varphi_t(U)$. The sectional curvature preserving nature of X then means that the metric originally present on $\varphi_t(U)$ and the metric carried over to $\varphi_t(U)$ from U give the same sectional curvature function on $\varphi_t(U)$. Hence, *generically*, the metric originally present on $\varphi_t(U)$ and the metric on $\varphi_t(U)$ carried over from U agree on $\varphi_t(U)$ and it follows that X is a Killing vector field on M [7].

In the special cases not covered by this generic result some open subset of M is locally isometric to a conformally flat generalised plane wave. It is therefore of interest to consider the (simplified) case when M itself is a conformally flat generalised plane wave and to study sectional curvature preserving vector fields on M . Only local vector fields will be considered although in many cases they will be global. To be precise as to the nature of the space-time considered, M will, throughout this section, be a space-time admitting a global chart (u, v, x, y) in which the metric g takes the form

$$ds^2 = dx^2 + dy^2 + 2dudv + \frac{A(u)(x^2 + y^2)}{2} du^2 \quad (4.1)$$

for some smooth function A . Other more general definitions of conformally flat generalised plane waves are possible [10] but they all essentially lead locally to (4.1). In M the Weyl tensor is everywhere zero and the Ricci (and hence the energy-momentum) tensor takes the form $R_{ab} = -A(u)l_a l_b$ where $l_a = u_{,a}$ is covariantly constant and null ($l_{a;b} = 0, l^a l_a = 0$). M will be assumed non-flat in the sense that the curvature tensor does not vanish over a non-empty open subset of M and it follows from this that A cannot vanish over any non-empty open subset of M . It is well known that M admits, locally, six independent Killing vector fields all of which are orthogonal to the global null vector field l , i.e. which are tangent to the hypersurfaces of constant u . It may, but in general does not, admit a seventh independent

local Killing vector field but if it does this vector field is not tangent to the hypersurfaces of constant u over any non-empty open subset of M . M also admits a proper homothetic vector field tangent to these hypersurfaces and, being conformally flat, admits fifteen independent local conformal vector fields. Because M is a (non-flat) conformally flat generalised plane wave it follows that there are at most eight independent special conformal vector fields on M [11]. Clearly the Killing vectors are necessarily sectional curvature preserving and it is easy to show that the special conformal and proper homothetic vector fields are not.

Now it is known that with g as in (4.1) one can find local alternative metrics g' with the same sectional curvature function as g . Necessarily $g' = e^{2\sigma}g$ for some function σ and then g and g' have the same sectional curvature if and only if the Ricci tensor R'_{ab} derived from g'_{ab} satisfies $R'_{ab} = e^{2\sigma}g'_{ab}$ [7]. This latter condition is equivalent to the function σ being a function of u only, $\sigma = \sigma(u)$, and satisfying the differential equation $\ddot{\sigma} - \dot{\sigma}^2 = \frac{1}{2}A(u)(e^{2\sigma} - 1)$, where $\dot{\cdot} = \frac{d}{du}$, of which explicit solutions are known [8]. It now follows from the remarks in section 2 that if X is sectional curvature preserving then it preserves the conformal structure on M and is therefore a conformal vector field satisfying $\mathcal{L}_X g = 2\phi g$ with $\phi = \phi(u)$. Similar remarks apply to the relation between X and the Ricci tensor. These results can be collected together as the following theorem [7, 10].

Theorem 1: *Let (M, g) be a conformally flat generalised plane wave as described above. Then the following hold:*

- (i) *A smooth Lorentz metric g' on M (or on some open subset U of M) gives rise to the same sectional curvature function as g if and only if $g'_{ab} = e^{2\sigma}g_{ab}$ and $R'_{ab} = e^{2\sigma}R_{ab}$ for some smooth function σ on M (or U) or, equivalently, $g'_{ab} = e^{2\sigma}g_{ab}$ with $\sigma = \sigma(u)$ and satisfying the second-order differential equation given above.*
- (ii) *A global smooth vector field X on M (or U) is sectional curvature preserving if and only if $\mathcal{L}_X g_{ab} = 2\phi g_{ab}$ and $\mathcal{L}_X R_{ab} = 2\phi R_{ab}$ where ϕ is a smooth function on M (or U) or, equivalently, $\mathcal{L}_X g_{ab} = 2\phi g_{ab}$ with $\phi = \phi(u)$ and satisfying the second-order differential equation $\ddot{\phi} = A\phi$.*

The proof has mostly been described. It is merely remarked that the condition $\phi = \phi(u)$ of part (ii) follows from part (i), and the differential equation for ϕ follows from a substitution of the condition $\mathcal{L}_X R_{ab} = 2\phi R_{ab}$ into the differential relations which necessarily hold for conformal vector fields [10]. The following rephrased version of Theorem 1(ii) with some extensions can now be given [10].

Theorem 2: *Let (M, g) be a conformally flat generalised plane wave as described above. The set of global sectional curvature preserving vector fields on M is a finite-dimensional subalgebra of the (finite-dimensional) Lie algebra of conformal vector fields on M . It consists of precisely those conformal vector fields X satisfying $\mathcal{L}_X g = 2\phi(u)g$ with $\ddot{\phi} = A\phi$. It contains a six- (or seven-) dimensional Lie algebra of Killing vector fields as a (possibly but not necessarily) proper subalgebra and contains no proper homothetic or special conformal vector fields.*

Only the first part of the final sentence has not been justified and this will be done in the final theorem. Further, although Theorem 2 is stated for global vector fields it is easily modified to handle the local situation. This understanding will be maintained throughout the remainder of the paper.

To calculate the possibilities for the dimension of the Lie algebra of sectional curvature preserving vector fields (henceforth denoted by S), it is convenient to remove a special case first. It was remarked above that M may admit a seven-dimensional Lie algebra of Killing vector fields (the so-called *homogeneous* conformally flat generalised plane waves). If this is so then it has been shown [12] that the function $A(u)$ must take the form $A(u) = \pm(bu + c)^{-2}$ with $b, c \in \mathbb{R}$ not both zero and $u \neq -\frac{c}{b}$. Thus one can use the conclusions of Theorem 2 to show that any member of S is necessarily a Killing vector field and hence $\dim(S) = 7$ [10].

Henceforth M can be assumed to admit exactly six independent Killing vector fields and since S is a subalgebra of the Lie algebra of conformal vector fields it is convenient to calculate the general form of a conformal vector field X on M using the metric (4.1). This general form for the *covariant* components of X , using coordinates in the order u, v, x, y , can be shown to be [10,13]:

$$\begin{aligned}
 X_1(u, v, x, y) &= c_1 v^2 + vx\dot{f}_1(u) + vy\dot{f}_2(u) + c_3 v - \frac{1}{8}c_1 A(u)(x^2 + y^2)^2 \\
 &\quad + \frac{1}{2}A(u)(x^2 + y^2)P(u) + \frac{1}{4}A(u)(x^2 + y^2)(xf_1(u) + yf_2(u)) \\
 &\quad - \frac{1}{4}(x^2 + y^2)\ddot{P} - x\dot{f}_4(u) - y\dot{f}_3(u) + c_4, \\
 X_2(u, x, y) &= -\frac{1}{2}c_1(x^2 + y^2) + xf_1(u) + yf_2(u) + P(u), \\
 X_3(u, v, x, y) &= \frac{1}{2}(x^2 - y^2)\dot{f}_1(u) + xy\dot{f}_2(u) + c_1 vx + \frac{1}{2}x\dot{P}(u) + \frac{1}{2}xc_3 \\
 &\quad - vf_1(u) + c_2 y + f_4(u), \\
 X_4(u, v, x, y) &= -\frac{1}{2}(x^2 - y^2)\dot{f}_2(u) + xy\dot{f}_1(u) + c_1 vy + \frac{1}{2}y\dot{P}(u) + \frac{1}{2}yc_3 \\
 &\quad - vf_2(u) - c_2 x + f_3(u),
 \end{aligned} \tag{4.2}$$

while the associated function ϕ is

$$\phi(u, v, x, y) = c_1 v + x\dot{f}_1(u) + y\dot{f}_2(u) + \frac{1}{2}\dot{P}(u) + \frac{1}{2}c_3. \tag{4.3}$$

In this solution $c_1, \dots, c_4$ are arbitrary constants and $f_1, \dots, f_4$ and P are functions of u only, satisfying the differential equations

$$2\ddot{f}_i(u) = A(u)f_i(u) \quad (1 \leq i \leq 4) \quad \text{and} \quad \ddot{P} = 2A(u)\dot{P} + \dot{A}P(u). \tag{4.4}$$

One can see from the general structure of equations (4.2)–(4.4) and the fifteen arbitrary constants arising (four from the c_i , eight from the f_i and three from P) how a 15-dimensional local Lie algebra of conformal vector fields exists in some open neighbourhood of any $p \in M$. Further, one can identify within (4.2) and (4.3) a six-dimensional subalgebra of local Killing vector fields orthogonal to l by choosing

$$\begin{array}{llll}
 c_i = 0 & (i \neq 2) & f_i \equiv 0 & P = 0 \quad (\text{one vector field}), \\
 c_i = 0 & (i \neq 4) & f_i \equiv 0 & P = 0 \quad (\text{one vector field}), \\
 c_i = 0 & & f_i \equiv 0 \quad (i \neq 4) & P = 0 \quad (\text{two vector fields}), \\
 c_i = 0 & & f_i \equiv 0 \quad (i \neq 3) & P = 0 \quad (\text{two vector fields}).
 \end{array} \tag{4.5}$$

A proper homothetic vector field arises by choosing $c_i = 0$ ($i \neq 3$) and $f_i \equiv P \equiv 0$. A full basis for this conformal Lie algebra of M is given in the appendix.

The condition that a conformal vector field on M be in S can then be shown equivalent to $c_1 = 0$, $f_1 = f_2 \equiv 0$ and to P satisfying, as well as (4.4), the condition

$$\frac{d}{du}(AP) = Ac_3. \quad (4.6)$$

This leads to the final theorem (in which the Killing algebra of M is denoted by K).

Theorem 3: *If M is a conformally flat generalised plane wave then $\dim(S) \leq 8$. Thus there are only four possibilities:*

- (i) $\dim(K) = 6$, $\dim(S) = 6$,
- (ii) $\dim(K) = 7$, $\dim(S) = 7$,
- (iii) $\dim(K) = 6$, $\dim(S) = 7$,
- (iv) $\dim(K) = 6$, $\dim(S) = 8$,

and each possibility can occur.

The proof of the first part (which is suggested by the remarks preceding (4.6) — see (4.7) below) can be achieved by supposing that $\dim(S) \geq 9$ and using an “isotropy” argument to get a contradiction. In the second part note that (i) is the generic case and (ii) the homogeneous case. To establish (at least locally) that cases (iii) and (iv) are possible one returns to (4.2) and, for membership of S , sets $c_1 = 0$, $f_1 = f_2 \equiv 0$ and, without loss of generality, one can also use (4.5) to set $c_2 = c_4 = 0$, $f_3 = f_4 \equiv 0$. This leaves only the constant c_3 and the function $P(u)$ satisfying (4.4) and (4.6). For some open subset of M where A is nowhere zero, (4.6) integrates to give

$$P(u) = c_3 A^{-1}(u)F(u) + \alpha A^{-1}(u) \quad (4.7)$$

where F is an indefinite integral of A and α is an arbitrary constant. Equations (4.4) and (4.7) then give

$$\alpha \chi_0(A) + c_3 [\chi_0(A)F + \chi_1(A)] = 0 \quad (4.8)$$

where

$$\chi_0(A) = 6A\dot{A}\ddot{A} - 6\dot{A}^3 - A^2\ddot{\ddot{A}} + A^3\dot{A}, \quad (4.9)$$

$$\chi_1(A) = 3A^2\dot{A}^2 - 2A^3\ddot{A} - 2A^5. \quad (4.10)$$

The procedure then, in summary, is that one chooses c_3 , then solves (4.8) for A and finally finds P from (4.7). Then the metric (4.1) and the vector field (4.2) in S are determined. In the generic case A will not satisfy (4.8). However, if $h_1(u)$ and $h_2(u)$ are, respectively, solutions of the differential equations

$$2h_1\ddot{h}_1 = 1 + 2h_1^{-2} \quad \text{and} \quad 2h_2\ddot{h}_2 = 1 \quad (4.11)$$

and one sets $A_1(u) = h_1^{-2}(u)$ and $A_2(u) = h_2^{-2}(u)$ then $\chi_0(A_1) = 0$, $\chi_1(A_1) \neq 0$, $\chi_0(A_2) = \chi_1(A_2) = 0$. Thus (4.8) is satisfied by A_1 if one chooses $c_3 = 0$ (with α arbitrary) and by A_2 (with c_3 and α both arbitrary). Substitution of A_1 and A_2 for A in (4.1) leads to two metrics which (for possibly restricted values of u) satisfy, respectively, cases (iii) and (iv) of Theorem 3. It is mentioned in passing that the existence of a proper

special conformal vector field on M forces every sectional curvature preserving vector field to be Killing [13]. Consequently case (i) may be further subdivided into cases (ia) and (ib) according to whether or not M admits a proper special conformal vector field. It has been shown [13, 14] that M will possess an 8-dimensional special conformal Lie subalgebra if and only if $A(u) = \pm(au^2 + bu + c)^{-2}$ with $a, b, c \in \mathbb{R}$, not all zero and such that $au^2 + bu + c \neq 0$. If $a = 0$ the metric of the homogeneous case (iv) is recovered and in fact M admits a proper special conformal vector field (case (ia)) if and only if $a \neq 0$. Thus the generic subcase of case (i) is case (ib) for which there do not exist any proper special conformal or non-Killing sectional curvature preserving vector fields.

Finally it is remarked that whereas given (M, g) as above one can always find (for possibly restricted values of u) a function $\sigma(u)$ such that g and $e^{2\sigma}g$ have the same sectional curvature function, it will, in general, not admit a member of S apart from Killing vector fields. This highlights the essential difference between statements (i) and (ii) of Theorem 1. In other words, a conformal change $g \rightarrow e^{2\sigma}g$ which preserves the sectional curvature function cannot in general be simulated by a coordinate change represented essentially by the maps φ_t associated with a non-Killing member of S .

Acknowledgments

One of the authors (A.D.H.) wishes to thank Zoltán Perjés, Roy Kerr and the Hungarian Academy of Sciences for their hospitality and financial support during the 4th Hungarian Relativity Workshop at Gárdony where this paper was presented. He also gratefully acknowledges the award of an S.E.R.C. studentship. Both authors thank John Pulham in Aberdeen for many valuable discussions.

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Appendix

Let $\Phi(u)$ and $\Psi(u)$ be two independent solutions of the second-order linear differential equation in (4.4) (for the f_i). Then (Φ^2) , $(\Phi\Psi)$ and (Ψ^2) are easily shown to be three independent solutions of the third-order differential equation in (4.4) (for P). If each f_i is written as $f_i = a_i\Phi + b_i\Psi$ and if P is written as $P = \alpha\Phi^2 + \beta\Phi\Psi + \gamma\Psi^2$, then a substitution into equations (4.2) and (4.3) leads to the following basis $X_{(1)}, \dots, X_{(15)}$ for the fifteen-dimensional conformal Lie algebra on M corresponding, in order, to each of the fifteen constants $c_1, c_2, c_3, c_4, a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, \alpha, \beta$ and γ . In the list given below $h = x^2 + y^2$ and $k = x^2 - y^2$ for convenience:

Killing Vector Fields

$$\begin{aligned} X_{(2)} &= y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} & X_{(4)} &= \frac{\partial}{\partial v} \\ X_{(9)} &= -y\dot{\Phi} \frac{\partial}{\partial v} + \Phi \frac{\partial}{\partial y} & X_{(10)} &= -y\dot{\Psi} \frac{\partial}{\partial v} + \Psi \frac{\partial}{\partial y} \\ X_{(11)} &= -x\dot{\Phi} \frac{\partial}{\partial v} + \Phi \frac{\partial}{\partial x} & X_{(12)} &= -x\dot{\Psi} \frac{\partial}{\partial v} + \Psi \frac{\partial}{\partial x} \end{aligned}$$

Proper Conformal Vector Fields

$$\begin{aligned} X_{(1)} &= -\frac{h}{2} \frac{\partial}{\partial u} + \left(v^2 + \frac{Ah^2}{8} \right) \frac{\partial}{\partial v} + v \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \\ X_{(5)} &= x\Phi \frac{\partial}{\partial u} + x \left(v\dot{\Phi} - \frac{Ah\Phi}{4} \right) \frac{\partial}{\partial v} + \left(\frac{k}{2}\dot{\Phi} - v\Phi \right) \frac{\partial}{\partial x} + xy\dot{\Phi} \frac{\partial}{\partial y} \\ X_{(6)} &= x\Psi \frac{\partial}{\partial u} + x \left(v\dot{\Psi} - \frac{Ah\Psi}{4} \right) \frac{\partial}{\partial v} + \left(\frac{k}{2}\dot{\Psi} - v\Psi \right) \frac{\partial}{\partial x} + xy\dot{\Psi} \frac{\partial}{\partial y} \\ X_{(7)} &= y\Phi \frac{\partial}{\partial u} + y \left(v\dot{\Phi} - \frac{Ah\Phi}{4} \right) \frac{\partial}{\partial v} + xy\dot{\Phi} \frac{\partial}{\partial x} - \left(\frac{k}{2}\dot{\Phi} + v\Phi \right) \frac{\partial}{\partial y} \\ X_{(8)} &= y\Psi \frac{\partial}{\partial u} + y \left(v\dot{\Psi} - \frac{Ah\Psi}{4} \right) \frac{\partial}{\partial v} + xy\dot{\Psi} \frac{\partial}{\partial x} - \left(\frac{k}{2}\dot{\Psi} + v\Psi \right) \frac{\partial}{\partial y} \\ X_{(13)} &= \Phi^2 \frac{\partial}{\partial u} - \frac{h}{4} (2\dot{\Phi}^2 + A\Phi^2) \frac{\partial}{\partial v} + \Phi\dot{\Phi} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \\ X_{(14)} &= \Phi\Psi \frac{\partial}{\partial u} - \frac{h}{4} (2\dot{\Phi}\dot{\Psi} + A\Phi\Psi) \frac{\partial}{\partial v} + \frac{1}{2} (\dot{\Phi}\Psi + \Phi\dot{\Psi}) \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \\ X_{(15)} &= \Psi^2 \frac{\partial}{\partial u} - \frac{h}{4} (2\dot{\Psi}^2 + A\Psi^2) \frac{\partial}{\partial v} + \Psi\dot{\Psi} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \end{aligned}$$

Proper Homothetic Vector Field

$$X_{(3)} = v \frac{\partial}{\partial v} + \frac{x}{2} \frac{\partial}{\partial x} + \frac{y}{2} \frac{\partial}{\partial y}$$

Associated Scalar Functions

$$\phi_{(1)} = v$$

$$\phi_{(2)} = 0$$

$$\phi_{(3)} = 1/2$$

$$\phi_{(4)} = 0$$

$$\phi_{(5)} = x\dot{\Phi}$$

$$\phi_{(6)} = x\dot{\Psi}$$

$$\phi_{(7)} = y\dot{\Phi}$$

$$\phi_{(8)} = y\dot{\Psi}$$

$$\phi_{(9)} = 0$$

$$\phi_{(10)} = 0$$

$$\phi_{(11)} = 0$$

$$\phi_{(12)} = 0$$

$$\phi_{(13)} = \Phi\dot{\Phi}$$

$$\phi_{(14)} = (\dot{\Phi}\Psi + \Phi\dot{\Psi})/2$$

$$\phi_{(15)} = \Psi\dot{\Psi}$$

Let ρ be the constant $\rho = (\Phi\dot{\Psi} - \dot{\Phi}\Psi)(u)$ [10,13] and let $[i, j]$ and $[k]$ denote the vectors $[X_{(i)}, X_{(j)}]$ and $X_{(k)}$, respectively ($1 \leq i, j, k \leq 15$). Then the nonzero Lie brackets of the vectors of the basis $X_{(1)}, \dots, X_{(15)}$ of the conformal Lie algebra of M are:

$$[1, 4] = -2[3]$$

$$[1, 9] = [7]$$

$$[1, 10] = [8]$$

$$[1, 11] = [5]$$

$$[1, 12] = [6]$$

$$[2, 5] = [7]$$

$$[2, 6] = [8]$$

$$[2, 7] = -[5]$$

$$[2, 8] = -[6]$$

$$[2, 9] = -[11]$$

$$[2, 10] = -[12]$$

$$[2, 11] = [9]$$

$$[2, 12] = [10]$$

$$[3, 4] = -[4]$$

$$[3, 5] = [5]/2$$

$$[3, 6] = [6]/2$$

$$[3, 7] = [7]/2$$

$$[3, 8] = [8]/2$$

$$[3, 9] = -[9]/2$$

$$[3, 10] = -[10]/2$$

$$[3, 11] = -[11]/2$$

$$[3, 12] = -[12]/2$$

$$[4, 5] = -[11]$$

$$[4, 6] = -[12]$$

$$[4, 7] = -[9]$$

$$[4, 8] = -[10]$$

$$[5,6] = -\rho[1] \qquad [5,10] = -\rho[2] \qquad [5,11] = -[13]$$

$$[5,12] = \rho[3] - [14] \qquad [5,14] = \rho[5]/2 \qquad [5,15] = \rho[6]$$

$$[6,9] = \rho[2] \qquad [6,11] = -[14] - \rho[3] \qquad [6,12] = -[15]$$

$$[6,13] = -\rho[5] \qquad [6,14] = -\rho[6]/2$$

$$[7,8] = -\rho[1] \qquad [7,9] = -[13] \qquad [7,10] = \rho[3] - [14]$$

$$[7,12] = \rho[2] \qquad [7,14] = \rho[7]/2 \qquad [7,15] = \rho[8]$$

$$[8,9] = -[14] - \rho[3] \qquad [8,10] = -[15] \qquad [8,11] = -\rho[2]$$

$$[8,13] = -\rho[7] \qquad [8,14] = -\rho[8]/2$$

$$[9,10] = -\rho[4] \qquad [9,14] = \rho[9]/2 \qquad [9,15] = \rho[10]$$

$$[10,13] = -\rho[9] \qquad [10,14] = -\rho[10]/2$$

$$[11,12] = -\rho[4] \qquad [11,14] = \rho[11]/2 \qquad [11,15] = \rho[12]$$

$$[12,13] = -\rho[11] \qquad [12,14] = -\rho[12]/2$$

$$[13,14] = \rho[13] \qquad [13,15] = 2\rho[14]$$

$$[14,15] = \rho[15]$$

Holonomy in Classical Newton-Cartan Mechanics

G.S. Hall and B.M. Haddow

Abstract: Cartan's geometrisation of Newtonian mechanics is discussed and extended using the holonomy theory of connections. The idea of a degenerate metric connection is formalised and some of its properties are presented.

1. Introduction

Einstein's general theory of relativity is a geometrical theory. It is modelled on a four-dimensional manifold which admits a Lorentz metric (the space-time). The metric is required to satisfy Einstein's field equations together with, possibly, some appropriate boundary conditions. These field equations contain physics in the guise of the energy-momentum tensor and thus the metric acquires a dynamical content. In classical Newtonian mechanics, however, space is taken to be *a priori* Euclidean and, at least in this sense, the theory is not geometrised since the space geometry is absolute and given once and for all without recourse to experiment. Nevertheless, classical mechanics can still be described in an elegant geometrical fashion and was done so many years ago by Lagrange, Hamilton, Jacobi and others. An eminently readable account of this development can be found in the text of Lanczos [1]. Most of these discussions were involved with the geometrisation of the configuration space of Newtonian mechanics. More recently and in fact after Einstein's original papers on general relativity, Cartan [2] (see also [3, 4, 5]) published an interesting alternative geometrisation of space and time within the classical theory. In this latter theory the link between physics and geometry is provided not by a metric but by a connection on the manifold representing space and time. This connection is not a metric connection as in Einstein's theory and the techniques for handling it are a little different to those encountered in conventional general relativity. This paper will be concerned with Cartan's approach to Newtonian mechanics. It will describe the theory in the language of holonomy groups and then suggest how this approach leads to a generalisation of Cartan's theory with an interesting mathematical structure.

2. Holonomy Groups

Let M be a smooth, connected, Hausdorff, paracompact manifold carrying a smooth linear symmetric connection ∇ and for a fixed k ($1 \leq k \leq \infty$) and for any $p \in M$ let $C_k(p)$ denote the set of all piecewise C^k closed curves which start and end at p . Parallel transport using ∇ along any $c \in C_k(p)$ then gives rise to an isomorphism $f(c)$ of the tangent space $T_p M$ to M at p onto itself with the properties that $f(c^{-1}) = f(c)^{-1}$ and $f(c_1 \circ c_2) = f(c_1) \circ f(c_2)$, where the standard notation for inverting and combining curves has been used. The set of transformations arising in such a way from the members of $C_k(p)$ is therefore a subgroup of the group $GL(T_p M)$ of all isomorphisms of $T_p M$ and is called the k -holonomy group of M at p . The connectedness of M implies that it is path-connected and so [6] the k -holonomy groups at different points of M are isomorphic and one then speaks of the k -holonomy group of M . This group is also unaffected by the value of k [6] and thus one arrives at the holonomy group of M , denoted by Φ .

If the above operation is repeated using only those members of $C_k(p)$ which are homotopic to zero one arrives at the *restricted holonomy group* of M which is denoted by Φ^0 . Clearly if M is simply-connected we have $\Phi = \Phi^0$. It turns out that Φ is a Lie group and in fact a Lie subgroup of $GL(T_p M)$ and that Φ^0 is the (connected) component of the identity of Φ . It follows that Φ and Φ^0 have the same Lie algebra which is denoted by $\mathcal{A}$.

Now let V be a nontrivial proper subspace of $T_p M$ which is mapped onto itself by all members of the holonomy group of M at p . V is then called *holonomy invariant* and parallel transport of V from p to all other points of M gives rise to an m -dimensional distribution on M , where $m = \dim V$, which is smooth since ∇ is smooth. This distribution can also be shown to be *integrable* and its maximal integral submanifolds to be *totally geodesic* and (since ∇ is symmetric) *auto parallel*. If such a subspace V exists then the holonomy group of M is said to be *reducible*.

3. Classical Mechanics and the Cartan Connection

In this section Cartan's approach to the study of classical mechanics will be briefly reviewed and the essential details given. The manifold M representing space and time is taken to be $\mathbb{R}^4$ and the usual global coordinates labelled $x^0, \dots, x^3$. Here x^0 denotes Newtonian absolute time and each 3-dimensional submanifold of constant x^0 (a *space slice*) is assumed to be a Euclidean space with orthogonal coordinates x^α (Greek indices take the values 1, 2, 3 and Latin indices 0, ..., 3). Each space slice therefore has a global Euclidean metric with components $\text{diag}(1, 1, 1)$ in the coordinates x^α and this leads to a global smooth second-order symmetric tensor h on M with components h^{ab} in the above global chart given by

$$h^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.1)$$

The tensor h has rank 3 everywhere and so, being degenerate, is not a metric for M .

One now assumes that in this coordinate system Newton's equations for a particle acted on by a force described by a Newtonian potential ϕ (which may be time dependent) are

$$\frac{d^2 x^\alpha}{dx^{02}} = -\frac{\partial \phi}{\partial x^\alpha}. \quad (3.2)$$

Following Cartan one can regard the trajectories of such particles as geodesics in the manifold M if one endows M with the appropriate connection. To see this, note that $d^2 x^0 / dx^{02} = 0$ and that the geodesic equation on M given by

$$\frac{d^2 x^a}{d\lambda^2} + \Gamma_{bc}^a \frac{dx^b}{d\lambda} \frac{dx^c}{d\lambda} = 0 \quad (3.3)$$

where $\lambda = x^0$ is equivalent to (3.2) provided one chooses the functions Γ according to

$$\Gamma_{00}^\alpha = \frac{\partial \phi}{\partial x^\alpha} \quad (\text{all other } \Gamma_{bc}^a = 0). \quad (3.4)$$

This choice gives the connection coefficients Γ which define uniquely the required connection ∇ on M . It should be noted here that the above amounts to the assumption that the Newtonian absolute time x^0 is an affine parameter along the particle trajectories (geodesics) in M .

It can be seen that ϕ is not a well-defined function on M but requires a particular coordinate system for its existence. This is because the special coordinate systems in which (3.2) holds are not linked by the usual Galilean group but by a larger group which includes arbitrarily time dependent space translations [5]. Physically this is just the statement that no initial inertial frame can be specified and that time dependent space translations lead to extra (inertial) forces which are also derivable from a potential and which can then be added to the original function ϕ .

Cartan's approach thus consists of putting a connection ∇ on the manifold M representing space and time such that the particle trajectories (3.2), regarded as paths in M parametrised with absolute time x^0 , become affinely parametrised geodesics. It is then easily calculated from (3.4) that the space coordinate vector fields defined by $X = \partial/\partial x^1, Y = \partial/\partial x^2$ and $Z = \partial/\partial x^3$ are covariantly constant with respect to ∇ as are the tensor h in (3.1) above and the global one-form $t = dx^0$. Furthermore, the components of the curvature tensor arising from ∇ satisfy

$$R^\alpha{}_{0\beta 0} = -R^\alpha{}_{00\beta} = \frac{\partial^2 \phi}{\partial x^\alpha \partial x^\beta} \quad (3.5)$$

with all other $R^a{}_{bcd}$ zero.

It can then be seen that ∇ is not a flat connection on M unless ϕ is linear in the coordinates x^α . This leads to the important result that such a non-flat connection ∇ is *not* a metric connection. This follows because the covariant constancy of X, Y and Z and the Ricci identity give

$$R^a{}_{bcd}X^d = R^a{}_{bcd}Y^d = R^a{}_{bcd}Z^d = 0 \quad (3.6)$$

which if ∇ were a metric connection can be shown to imply that the curvature tensor would be identically zero on M . A connection ∇ defined above is called a *Cartan connection*.

4. Cartan's Theory and Holonomy

In this section Cartan's theory will be described in terms of holonomy groups. In the preceding section it was shown that the vector fields X, Y and Z and the 1-form t were covariantly constant for the Cartan connection on $\mathbb{R}^4$. It follows that for the basis $\partial/\partial x^a$ at any point in $\mathbb{R}^4$ the holonomy group of ∇ is easily seen to be a Lie subgroup (depending on the actual form of ϕ) of the 3-dimensional Lie group of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (a, b, c \in \mathbb{R}). \quad (4.1)$$

This Lie group has a natural structure as a 3-dimensional Lie subgroup of $GL(4, \mathbb{R})$ which is Lie-isomorphic to the Lie group $(\mathbb{R}^3, +)$ under the isomorphism which associates the matrix in (4.1) with $(a, b, c) \in \mathbb{R}^3$. This holonomy group is clearly reducible and its holonomy invariant distributions reveal the space slices of constant absolute time and, within each slice, the Euclidean nature of their geometry follows from the connection induced in them by ∇ .

To see the relevance of this holonomy group for Cartan's mechanics suppose one has the manifold $\mathbb{R}^4$ and some smooth connection ∇ on $\mathbb{R}^4$ whose holonomy group at some point

p with respect to some basis X, Y, Z, T of $T_p M$ is described by the linear transformations represented by a (not necessarily proper) subgroup of the matrices (4.1). Under parallel transport with ∇ the vectors X, Y and Z then yield three smooth covariantly constant vector fields on $\mathbb{R}^4$ (also called X, Y and Z) which span a holonomy invariant 3-dimensional integrable distribution on $\mathbb{R}^4$. Since any two of X, Y and Z have zero Lie bracket one may construct a local coordinate system about p in which X, Y and Z take the component form $X^a = \delta_1^a$, $Y^a = \delta_2^a$ and $Z^a = \delta_3^a$. In this coordinate system the 1-form t with constant components $(0, 0, 0, 1)$ is also covariantly constant under ∇ and in fact leads to a global covariantly constant 1-form on $\mathbb{R}^4$. One has thus, at least in this local coordinate system, reproduced Cartan's mechanics in the sense that the coordinate equations for the geodesics not lying in the holonomy invariant submanifolds generated by X, Y and Z are of the form (3.2) (but with the right-hand side not necessarily a gradient).

Each nontrivial $(a, b, c$ not all zero) matrix in the holonomy representation (4.1) has a Jordan form corresponding to the Segre type $\{(211)\}$ with all the eigenvalues equal to 1. It is thus not diagonalisable and has three independent eigenvectors $(1, 0, 0, 0)$, $(0, 1, 0, 0)$ and $(0, 0, 1, 0)$ each of which yields a holonomy invariant 1-dimensional subspace and which corresponds respectively to the covariantly constant vector fields X, Y and Z above. If the holonomy group is 3-dimensional $(a, b, c$ arbitrary in (4.1)) then the 3-dimensional subspace spanned by these eigenvectors together with any of its subspaces generates the only holonomy invariant distributions (but others are possible if the holonomy group is of lower dimension).

It is of interest to ask what happens if one again starts with the manifold $\mathbb{R}^4$ and a connection ∇ such that, with respect to some basis X, Y, Z, T of $T_p M$, the holonomy group at p is described by linear transformations represented by all matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & d \end{pmatrix} \quad (a, b, c, d \in \mathbb{R}, \quad d \neq 0). \quad (4.2)$$

Again X, Y and Z give rise to covariantly constant global vector fields on $\mathbb{R}^4$ and span a 3-dimensional holonomy invariant distribution on $\mathbb{R}^4$ and again a local coordinate system x^a about p exists in which $X^a = \delta_1^a$, $Y^a = \delta_2^a$ and $Z^a = \delta_3^a$. The (local) 1-form t with constant components $(0, 0, 0, 1)$ in this coordinate system is no longer covariantly constant (since we can have $d \neq 1$) but *properly recurrent*. By this it is meant that the covariant derivative of t satisfies $\nabla t = t \otimes \omega$ for some 1-form ω which is not an *exact* 1-form. The 1-form t is exact with $t = dx^0$ and the above holonomy invariant submanifolds are (locally) submanifolds of constant x^0 . An obvious converse of this result holds and the resulting affinely parametrised geodesics not lying in the submanifolds of constant x^0 can be written in the Newtonian form (3.2) (with the right-hand side not necessarily a gradient). In this case however, x^0 is not an affine parameter for these geodesics because one acquires an extra nonvanishing connection coefficient Γ_{00}^0 .

It is remarked that the holonomy group represented by the matrices (4.2) is the *semi-direct* product $\mathbb{R}^3 \times_f \mathbb{R}$ of the groups $(\mathbb{R}^3, +)$ and $(\mathbb{R} - \{0\}, \times)$ under the homomorphism f from $\mathbb{R}$ to the automorphism group of $\mathbb{R}^3$ which associates $r \in \mathbb{R}$ with the automorphism $x \rightarrow rx$ of $\mathbb{R}^3$. From the algebraic viewpoint each nontrivial $(a, b, c, d$ not all zero) matrix of the form (4.2) is diagonalisable with independent eigenvectors $(1, 0, 0, 0)$, $(0, 1, 0, 0)$ and $(0, 0, 1, 0)$ corresponding to the eigenvalue 1 and an eigenvector $(a, b, c, d - 1)$ with eigenvalue d .

5. A Generalisation of Cartan's Theory

Cartan's theory extends Newton's Euclidean space arena for particle mechanics to a 4-dimensional space and time manifold carrying a non-metric connection. Prior to Cartan one of the most important geometrisations of classical mechanics had been a technique for dealing with holonomically constrained particle motion. Such a procedure is well known and involves a generalisation of the Euclidean space considered so far to an m -dimensional manifold (configuration space) M' carrying a positive-definite metric g . One obvious way to apply the Cartan approach here would be to construct the product manifold $M = M' \times \mathbb{R}$ with $\mathbb{R}$ playing the role of time and put a connection ∇ on M such that particle trajectories parametrised by time become affinely parametrised geodesics in M with respect to ∇ . This can be achieved but owing to the more general nature of the metric g one will not, except in special cases, have analogues of the covariantly constant vector fields X, Y and Z encountered earlier. One will however have an analogue of the covariantly constant tensor of equation (3.1) and this section will attempt to formalise and explain such structures on manifolds, It should be remarked however that the covariant constancy of the tensor in (3.1) on $\mathbb{R}^4$ is not by itself sufficient to derive exactly Cartan's version of Newtonian mechanics but it does lead to similar mathematical structures.

Let M be a connected paracompact n -dimensional smooth Hausdorff manifold carrying a global smooth symmetric tensor of type $(2,0)$ which has everywhere rank $n - 1$. In keeping with Cartan's ideas one would also assume that M admits a smooth connection ∇ for which h is covariantly constant (one also might add the assumption that h is *positive semi-definite*, i.e. $h(u, v) \geq 0 \quad \forall u, v \in T_p^*M$ and $\forall p \in M$ but this assumption is not needed in what is to follow). Unfortunately such a connection may not exist if no further assumptions are made regarding h . To see this note that the specified rank of h leads to a 1-dimensional subspace of the cotangent space T_p^*M to M at p (the *kernel* subspace of h at p) being uniquely determined at each $p \in M$. The kernel subspaces can be locally spanned by a smooth 1-form t satisfying $h^{ab}t_b = 0$. The covariant constancy of h with respect to ∇ then shows that the kernel subspaces of T_p^*M are holonomy invariant. There is also a uniquely determined $(n - 1)$ -dimensional subspace of T_pM at each $p \in M$ which is precisely the subspace of T_pM annihilated by each member of the kernel subspace of h at p . This $(n - 1)$ -dimensional distribution (called the *h-distribution*) is also holonomy invariant and is therefore integrable with its maximal integrable submanifolds being totally geodesic. It is not difficult however, to construct a manifold M and a global tensor h as above such that the h -distribution, though still well-defined, is not integrable and hence for which no connection with the above properties could exist. Now suppose that h is given on M and that the associated h -distribution is integrable. In fact a little more will be assumed which is in keeping with the particular application discussed in the first paragraph of this section, namely that the kernel subspaces of h can be spanned by a nowhere-zero global gradient. The existence of a global function $t : M \rightarrow \mathbb{R}$ is therefore assumed such that the kernel subspaces are everywhere spanned by the smooth 1-form dt . It now follows that a connection with the desired properties exists on M . To see this note that the paracompactness of M allows one to choose a smooth *positive definite* metric γ on M . One then defines a global positive smooth function $f : M \rightarrow \mathbb{R}$ in any coordinate chart by

$$f^2 = (t_{,a}t_{,b}\gamma^{ab})^{-1} \quad (dt = t_{,a}dx^a). \quad (5.1)$$

It follows that $\bar{\gamma} = f^2\gamma$ is a positive definite metric for M and that $\bar{\gamma}^{ab}t_{,a}t_{,b} = 1$ everywhere on M . One can then define a global smooth vector field T on M by the coordinate relation $T^a = \bar{\gamma}^{ab}t_{,b}$. It follows that $T^a t_a = 1$ everywhere on M . Now construct a smooth global metric on M by

$$g^{ab} = h^{ab} + T^a T^b \quad (5.2)$$

and let the components of the Levi-Civita connection associated with g be denoted by $\left\{ \begin{smallmatrix} a \\ bc \end{smallmatrix} \right\}$. Finally define a global connection Γ on M by the local connection coefficients

$$\Gamma_{bc}^a = \left\{ \begin{smallmatrix} a \\ bc \end{smallmatrix} \right\} + T^a t_{b|c} \quad (5.3)$$

where a stroke denotes a covariant derivative with respect to the Levi-Civita connection of g . If ∇ is the connection associated with the coefficients Γ and we use a semicolon for the ∇ covariant derivative one then finds $h^{ab};_c = 0$ and so h is covariantly constant with respect to ∇ . The following theorem has been established.

Theorem 1: *Let M be an n -dimensional smooth connected paracompact Hausdorff manifold and h a smooth symmetric tensor of type $(2, 0)$ and of rank $(n - 1)$ on M . If the kernel subspaces of h can be spanned by a global nowhere-zero gradient then there exists a connection on M with respect to which h is covariantly constant.*

If M is a manifold admitting a tensor h and a connection ∇ as above such that h is covariantly constant with respect to ∇ then ∇ is said to be a *degenerate metric connection* and is said to *fit* h . The holonomy group of ∇ can be described by choosing $p \in M$ and a basis for T_p^*M in which h takes the form

$$(h^{ab}) = \begin{pmatrix} & & 0 \\ & k & \vdots \\ & & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} = \alpha \quad (\text{say}), \quad (5.4)$$

where k is a nondegenerate symmetric $(n - 1) \times (n - 1)$ matrix in its Sylvester form. The signature of k is the same no matter what p is chosen for this construction. The holonomy group of ∇ is then a Lie subgroup of the Lie group of matrices L of $GL(n, \mathbb{R})$ for which $L\alpha L^T = \alpha$ where L^T is the transpose of L . One then has the following theorem.

Theorem 2: *Let M be an n -dimensional smooth connected Hausdorff manifold and h a smooth symmetric tensor of type $(2, 0)$ and rank $(n - 1)$ on M and let ∇ be a connection on M fitting h . With the above notation it then follows that the holonomy group of ∇ is then a Lie subgroup of the Lie group $\{L \in GL(n, \mathbb{R}) \mid L\alpha L^T = \alpha\}$ and with the basis as in (5.4) consists of matrices of the form*

$$\begin{pmatrix} & & a_1 \\ & Q & \vdots \\ & & a_{n-1} \\ 0 & \cdots & 0 & d \end{pmatrix} \quad (5.5)$$

which can be decomposed uniquely as

$$\begin{pmatrix} Q & \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} \\ 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} I & \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} \\ 0 & \dots & 0 & d \end{pmatrix} \begin{pmatrix} I & \begin{smallmatrix} p_1 \\ \vdots \\ p_{n-1} \end{smallmatrix} \\ 0 & \dots & 0 & 1 \end{pmatrix} \quad (5.6)$$

where Q is a member of the orthogonal group in $(n-1)$ dimensions appropriate to the signature of k in (5.4), $d \in \mathbb{R}$ with $d \neq 0$, $I = I_{n-1}$ and $Q\mathbf{p} = \mathbf{a}$. The entry d in (5.5) equals unity for all members of the holonomy group of ∇ if and only if the kernel subspaces of h can be spanned by a smooth 1-form on M which is covariantly constant with respect to ∇ .

6. Properties of the Generalised Connection

As in the previous section let M be a smooth n -dimensional connected Hausdorff paracompact manifold carrying a smooth $(2,0)$ symmetric tensor h of rank $n-1$ and a smooth symmetric linear connection fitting h . The holonomy invariance of the kernel subspaces of h show that there exists a local smooth 1-form t spanning these subspaces which is *recurrent* (i.e. $t_{a;b} = t_a \omega_b$) for some smooth local *recurrence 1-form* ω . The condition that t can be locally scaled so as to be covariantly constant is equivalent to ω being locally exact. The 1-form t is only defined locally and then only up to a smooth scaling function. The latter ambiguity in t has the effect of changing the recurrence form by the addition of a local exact 1-form. The smooth 2-form $F = d\omega$ with components $F_{ab} = \omega_{[a;b]}$ is therefore uniquely defined on M and is called the *recurrence curvature* associated with M, h and ∇ . The vanishing of F is equivalent to the statement that t can be locally scaled so as to be covariantly constant. The Ricci identities applied locally to the tensors t and h can be used to show that

$$R^a_{acd} = 2F_{cd} \quad (6.1)$$

where R^a_{bcd} are the curvature tensor components associated with ∇ . This not only confirms the global nature of the recurrence curvature F but associates the latter with the non-volume preserving property of ∇ (i.e. the condition $F = 0$ is equivalent to the 'volume preserving' condition $R^a_{acd} = 0$ for ∇ [7]).

The connection ∇ is a degenerate metric connection with respect to h . One can then ask when this connection is a metric connection, that is when is this connection compatible with some metric tensor on M . If the holonomy group of ∇ is a Lie subgroup of some orthogonal group in $(n-1)$ dimensions then it is a Lie subgroup of some orthogonal group in n dimensions and hence, by Schmidt's theorem, ∇ is a metric connection on M (which may be chosen of the signature compatible with the latter n -dimensional orthogonal group). Conversely if ∇ is compatible with a metric g on M and if the kernel subspaces of h are *non-null* with respect to g (this is of course always the case if g is positive definite) then a consideration of the $(n-1)$ -dimensional (necessarily non-null) holonomy invariant h -distribution shows that the holonomy group of ∇ is a subgroup of some orthogonal group in $(n-1)$ dimensions. The situation when the kernel subspaces of h are null with respect to g is not clear. Furthermore, if ∇ is compatible with a metric g on M then it is compatible with metrics other than just those of the form cg for some constant c . This follows because of the compatibility of ∇ with metrics on M of the form $g + \lambda\gamma$ where γ has components $\gamma_{ab} = h^{cd}g_{ac}g_{bd}$ and $\lambda \in \mathbb{R}$.

Finally one may ask how many connections ∇ are there on M fitting a specified tensor h as above. The *local* problem is easily solved. Starting with such a connection ∇ on M let U be a coordinate domain in which t_a are the components of a smooth 1-form on U which spans the kernel subspace of h at each point of U and let Γ_{bc}^a be the coefficients of ∇ on U . Any other such connection $\bar{\nabla}$ on U has coefficients $\bar{\Gamma}_{bc}^a$ which can be shown to be of the form

$$\bar{\Gamma}_{bc}^a = \Gamma_{bc}^a + h^{ad}\omega_{d(b}t_{c)} \quad (6.2)$$

where ω is any $(0,2)$ tensor on U with the property that $h^{ad}h^{cb}\omega_{db}$ is skew (so that for example one could choose ω_{ab} to be skew on U).

For the global problem if ∇ and $\bar{\nabla}$ are two connections which fit h then their connection coefficients certainly satisfy (6.2) in any coordinate chart on M . One can also show that if h is specified on M and ∇ is a global connection which fits h then ∇ is not unique. To see this one first uses the paracompactness of M to construct a global smooth (not identically zero) vector field k and a global smooth positive definite metric γ on M . Next one notes that there exists a coordinate neighbourhood U of any $p \in M$ on which the smooth 1-form t_a above satisfies $\gamma^{ab}t_at_b = 1$. At any point this 1-form is only defined up to a sign but the smooth $(0,2)$ tensor with local components t_at_b is globally defined on M . Finally one defines a global connection $\bar{\nabla}$ with coefficients $\bar{\Gamma}_{bc}^a$ given in any chart by

$$\bar{\Gamma}_{bc}^a = \Gamma_{bc}^a + k^at_bt_c \quad (6.3)$$

which is distinct from ∇ and which fits h .

Acknowledgements

One of the authors (B.M.H.) wishes to thank Zoltán Perjés, Roy Kerr and the Hungarian Academy of Sciences for their hospitality and financial support during the 4th Hungarian Relativity Workshop at Gárdony where this paper was presented. He also gratefully acknowledges the award of an S.E.R.C. studentship.

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Generalised Kerr-Schild Spaces

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Abstract: The original Kerr-Schild (KS) Ansatz is an example of a pencil of metrics

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \lambda h_{\alpha\beta}$$

that is a solution of the field equations, either Einstein or Einstein-Maxwell, for all values of the parameter λ . For these $\eta_{\alpha\beta}$ is flat and the perturbation is the square of a null vector, $h_{\alpha\beta} = k_{\alpha}k_{\beta}$. There is a natural generalization of this to quadratic perturbations of the type

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \lambda(k_{\alpha}n_{\beta} + n_{\alpha}k_{\beta}) + \lambda^2 A(n_{\sigma}n^{\sigma})k_{\alpha}k_{\beta},$$

where k_{α} is null and n_{α} is an orthogonal vector. It has been proved that either k_{α} is a geodesic principal null vector or A must take one of the special values 0 or 1.

1. Introduction

The original KS metrics [1, 2] were discovered in the following way. For any algebraically special Einstein space with diverging rays [3, 5], there is a coordinate system, $(\zeta, \bar{\zeta}, u, r)$, in which the metric is an explicit functional of two complex functions, m and Ω , both independent of the "radial" coordinate, r ,

$$\begin{aligned} ds^2 &= 2(r^2 + \Sigma^2)d\zeta d\bar{\zeta} - 2(dr + Wd\zeta + \bar{W}d\bar{\zeta} + Hk)k, \\ k &= du + \Omega d\zeta + \bar{\Omega} d\bar{\zeta}, \quad W = -(r + i\Sigma)\Omega + iD\Sigma, \\ H &= -\text{Re}\left(\bar{D}\partial_u\Omega + \frac{m}{r + i\Sigma}\right), \quad \Sigma = \text{Im}(\bar{D}\Omega), \end{aligned} \quad (1.1)$$

where

$$D = \partial_{\zeta} - \Omega\partial_u, \quad \bar{D} = \partial_{\bar{\zeta}} - \bar{\Omega}\partial_u. \quad (1.2)$$

The two functions, m and Ω , must satisfy the following differential equations,

$$\begin{aligned} \text{Im}(m - \bar{D}\bar{D}D\Omega) &= 0, \\ \partial_u(m - \bar{D}\bar{D}D\Omega) &= |\partial_u D\Omega|^2, \\ \bar{D}m &= 3\bar{\Omega}_u. \end{aligned} \quad (1.3)$$

When the Kerr metric was first discovered [3] it was transformed to the KS form,

$$g_{\alpha\beta} = \eta_{\alpha\beta} - \lambda k_{\alpha}k_{\beta}, \quad (1.4)$$

where the null vector k can be suitably normalised so that λ is an arbitrary real constant. Since both this and the general metric in (1.1) are linear in the complex mass m , or equivalently λ , it was natural to ask the following question.

Problem 1: Do other diverging algebraically special solutions exist where the metric is linear in some parameter?

It is easily shown that the field equations must then break into two disjoint sets,

$$\text{Im}(\bar{D}\bar{D}D\Omega) = 0, \quad \partial_u(\bar{D}\bar{D}D\Omega) = -|\partial_u D\Omega|^2 \quad (1.5)$$

and

$$\text{Im}(m) = 0, \quad m_u = 0, \quad m_\zeta = 3\Omega_u m. \quad (1.6)$$

From the integrability conditions for the second set,

$$m = \lambda P^3, \quad \Omega = -uP_\zeta/P + P\bar{\Phi}, \quad P_u = \Phi_u = 0, \quad (1.7)$$

where P and Φ are functions of ζ and $\bar{\zeta}$ and λ is a constant. Furthermore, the coordinates can be chosen so that $D\Omega = 0$. Using this and (1.7) gives

$$P_{,\zeta\zeta} = 0, \quad \Phi_{,\zeta} = 0. \quad (1.8)$$

Hence P is bilinear in ζ and $\bar{\zeta}$, and Φ is an analytic function of ζ . This gives the complete set of KS metrics. The coefficient of m , and therefore of λ , in the metric is just the square of the double principal null vector, k_α , so that all these metrics can be written as (1.4).

Perhaps the main reason why the field equations are comparatively easy to solve for the KS metrics is that the determinant and inverse of the metric tensor in (1.6) are themselves very simple, namely

$$|g_{\alpha\beta}| = |\eta_{\alpha\beta}|, \quad (1.9)$$

$$g^{\alpha\beta} = \eta^{\alpha\beta} + \lambda k^\alpha k^\beta, \quad (1.10)$$

provided, of course, that $k^\alpha k_\alpha = 0$.

2. Linear Pencils

The simplest generalization of the KS problem is the following:

Problem 2: To investigate all pencils of solutions of Einstein's equations,

$$g_{\alpha\beta} = \eta_{\alpha\beta} - \lambda h_{\alpha\beta}, \quad (2.11)$$

where λ is an arbitrary constant and the base metric $\eta_{\alpha\beta}$ must itself be an Einstein space, but need not be flat.

The cofactor matrix $G^{\alpha\beta}$ and determinant $|g|$ of such a pencil can each be expressed as polynomial tensor concomitants of $\eta_{\alpha\beta}$ and $h_{\alpha\beta}$. For instance, in 2-dimensions

$$G^{\rho\sigma} = |\eta|[\eta^{\rho\sigma} + \lambda(\eta^{\rho\sigma} h_\alpha^\alpha - h^{\rho\sigma})],$$

$$|g| = |\eta| \left[1 + \lambda h_\alpha^\alpha + \frac{1}{2} \lambda^2 \left((h_\alpha^\alpha)^2 - h_{\alpha\beta} h^{\alpha\beta} \right) \right].$$

The corresponding expressions in four dimensions are of degree three and four respectively in λ .

It is not always realized how complicated the Einstein equations are. The Riemann tensor density can be written as

$$|g|R_{\alpha\beta\gamma\delta} = \frac{1}{2}|g|(g_{\alpha\delta,\beta\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\alpha\gamma,\beta\delta} - g_{\beta\delta,\alpha\gamma}) + G^{\rho\sigma}([\alpha\delta,\rho][\beta\gamma,\sigma] - [\alpha\gamma,\rho][\beta\delta,\sigma]), \quad (2.12)$$

where

$$[\alpha\beta,\gamma] = \frac{1}{2}(g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma}). \quad (2.13)$$

This is already a polynomial of degree five in the covariant metric and its derivatives, but the Ricci tensor density components,

$$|g|^2 R_{\beta\gamma} = \frac{1}{2}|g|G^{\alpha\delta}(g_{\alpha\delta,\beta\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\alpha\gamma,\beta\delta} - g_{\beta\delta,\alpha\gamma}) + G^{\rho\sigma}G^{\alpha\delta}([\alpha\delta,\rho][\beta\gamma,\sigma] - [\alpha\gamma,\rho][\beta\delta,\sigma]), \quad (2.14)$$

are of degree eight.

We shall use matrix notation for the mixed index version[†],

$$g_{\beta}^{\alpha} = \delta_{\beta}^{\alpha} - \lambda h_{\beta}^{\alpha} \quad \Rightarrow \quad G = I - \lambda H. \quad (2.15)$$

Let us suppose that the characteristic equation for the matrix H is

$$|H - \mu I| = \Psi(\mu), \quad \Psi(H) = 0. \quad (2.16)$$

Replacing μ with λ^{-1} , we see that

$$|I - \lambda H| = f(\lambda) = \lambda^4 \Psi(1/\lambda). \quad (2.17)$$

This is independent of λ if $\Psi(\mu) = \mu^4$, $H^4 = 0$. The following theorem can now be readily proved, if we remember that the minimal equation for a matrix is always a factor of its characteristic equation.

Theorem 1: *A necessary and sufficient condition for the determinant of a pencil to be independent of λ , i.e.*

$$|\eta_{\alpha\beta} - \lambda h_{\alpha\beta}| = |\eta_{\alpha\beta}| \quad (2.18)$$

is that the minimal equation for the matrix $H = (h_{\alpha\sigma}\eta^{\sigma\beta})$ be

$$H^p = 0, \quad (2.19)$$

where $p \leq 4$. The matrix inverse of the pencil is then

$$(I - \lambda H)^{-1} = I + \lambda H + \dots + \lambda^{p-1} H^{p-1}. \quad (2.20)$$

If the minimal equation for H is $H^2 = 0$, then

$$h_{\alpha\sigma} h_{\beta}^{\sigma} = 0, \quad (2.21)$$

[†] The base metric $\eta_{\alpha\beta}$ and its inverse $\eta^{\alpha\beta}$ are used to raise and lower indices for the pencil.

so that the rows of the matrix are null and mutually orthogonal. Any two null vectors in a Lorentzian space must be parallel, and therefore the rows, and *perforce* the columns, of H must be proportional to some null vector k_α . After suitable normalization of this vector,

$$\begin{aligned} H^2 = 0 &\Rightarrow h_{\alpha\beta} = \pm k_\alpha k_\beta, \\ \mathbf{k} \cdot \mathbf{k} &= 0, \\ g_{\alpha\beta} &= \eta_{\alpha\beta} - \lambda k_\alpha k_\beta, \\ g^{\alpha\beta} &= \eta^{\alpha\beta} + \lambda k^\alpha k^\beta. \end{aligned} \tag{2.22}$$

This is just the classical KS Ansatz.

The situation where the minimal equation is $H^3 = 0$ is more complicated. The only possible Jordan canonical form for the matrix H is

$$H = (h_\alpha^\beta) = \begin{pmatrix} 0 & 1 & 0 & . \\ 0 & 0 & 1 & . \\ 0 & 0 & 0 & . \\ . & . & . & 0 \end{pmatrix}. \tag{2.23}$$

By lowering the second index with η , using the signature of the metric and making an appropriate change of base,

$$\begin{aligned} H^3 = 0 &\Rightarrow h_{\alpha\beta} = k_\alpha n_\beta + n_\alpha k_\beta, \\ \mathbf{n} \cdot \mathbf{n} &= 1, \quad \mathbf{k} \cdot \mathbf{k} = 0, \quad \mathbf{k} \cdot \mathbf{n} = 0, \\ g_{\alpha\beta} &= \eta_{\alpha\beta} - \lambda(k_\alpha n_\beta + n_\alpha k_\beta), \\ g^{\alpha\beta} &= \eta^{\alpha\beta} + \lambda(k^\alpha n^\beta + n^\alpha k^\beta) + \lambda^2 k^\alpha k^\beta. \end{aligned} \tag{2.24}$$

There is a problem with this. Although the covariant metric tensor is linear in the parameter λ , the contravariant one is quadratic. Furthermore, we know of no example of such a pencil. A typical example of a metric close to this type is the Goldberg-Kerr type III space [4, 5],

$$\begin{aligned} ds^2 &= 2d\zeta d\bar{\zeta} - 2du(dv + Wd\zeta + \bar{W}d\bar{\zeta} + Hdu), \\ W &= W(\bar{\zeta}, u), \quad H = H^0(\zeta, \bar{\zeta}, u) + \frac{1}{2}v(W_{,\bar{\zeta}} + \bar{W}_{,\zeta}), \end{aligned} \tag{2.25}$$

$$H_{,\zeta\bar{\zeta}}^0 = \text{Re}(W_{,\bar{\zeta}u} + W_{,\zeta}^2 + WW_{,\bar{\zeta}\bar{\zeta}}).$$

This can be thought of as a perturbation of flat space containing an arbitrary parameter λ , but only if we allow *quadratic perturbations*,

$$\begin{aligned} ds^2 &= 2d\zeta d\bar{\zeta} - 2dudv + 2\lambda kn + \lambda^2 H_2 k^2, \\ k &= du, \quad n = Wd\zeta + \bar{W}d\bar{\zeta} + H_1 du, \\ H_1 &= H_1^0 + v\text{Re}(W_{,\bar{\zeta}}), \quad W = W^0(\bar{\zeta}, u), \\ H_{1,\zeta\bar{\zeta}}^0 &= \text{Re}(W_{,\bar{\zeta}u}), \\ H_{2,\zeta\bar{\zeta}} &= \text{Re}(W_{,\zeta}^2 + WW_{,\bar{\zeta}\bar{\zeta}}). \end{aligned} \tag{2.26}$$

If we insist that this be *linear* in λ then we have to set H_2 to zero. This puts a totally unnecessary condition on W ,

$$W = \sqrt{ia(u)\bar{\zeta}^2 + \beta(u)\bar{\zeta} + \gamma(u)}, \quad (2.27)$$

where a is real. It is much more reasonable to assume that the metric is quadratic in λ , leading to quadratic field equations.

We think it unlikely that there are any significant examples of linear perturbations that are not Kerr-Schild.

3. Quadratic Perturbations

We shall generalize the KS ansatz in this paper by allowing the metric and its inverse to each be quadratic in the pencil parameter. The metric of (2.25) can then be naturally included in this class.

There are two simple *ansätze* where the perturbations are functions of two independent orthogonal vectors. For the first there exists an orthogonal pair p_α and q_α such that

$$\begin{aligned} g_{\alpha\beta}(\lambda) &= \eta_{\alpha\beta} + \lambda(p_\alpha q_\beta + p_\beta q_\alpha) + \lambda^2(q_\sigma q^\sigma)p_\alpha p_\beta, \\ g^{\alpha\beta}(\lambda) &= \eta^{\alpha\beta} - \lambda(p^\alpha q^\beta + p^\beta q^\alpha) + \lambda^2(p_\sigma p^\sigma)q^\alpha q^\beta. \end{aligned} \quad (3.28)$$

This has been studied by Bonanos [6], but there do not appear to be any nontrivial Einstein spaces of that type known as yet. For the second,

$$\begin{aligned} g_{\alpha\beta}(\lambda) &= \eta_{\alpha\beta} + \lambda(n_\alpha k_\beta + n_\beta k_\alpha) + \lambda^2 A(n_\sigma n^\sigma)k_\alpha k_\beta, \\ g^{\alpha\beta}(\lambda) &= \eta^{\alpha\beta} - \lambda(n^\alpha k^\alpha + n^\beta k^\beta) + \lambda^2(1 - A)(n_\sigma n^\sigma)k^\alpha k^\beta, \end{aligned} \quad (3.29)$$

where $k_\sigma k^\sigma = n_\sigma k^\sigma = 0$ and A is arbitrary.

When either p or q is null, the Bonanos ansatz, (3.28), reduces to that of (3.29) with $A = 0$ or $A = 1$. As an example of this, the metric (2.25) fits into the Bonanos ansatz when (2.27) is satisfied, but the general metric of this type does not.

We believe that any other quadratic pencil must be a superposition of two mutually orthogonal perturbations of these simple types. We have most of the proof of this statement, but there are still one or two possible degeneracies that need checking out.

We computed the curvature and Ricci tensors for quadratic pencils of the null type (3.29) using Reduce on a Sparcstation. By comparing various powers of the pencil parameter we established the following theorem:

Theorem 2: *Given any quadratic pencil of the type in (3.29), where the base space $\eta_{\alpha\beta}$ is flat, one of the following is true: (i) $A = 0$, (ii) $A = 1$, (iii) k is a geodesic principal null vector of the curvature for all values of the pencil parameter.*

We do not know whether cases (i) and (ii) allow k to be non-geodesic; we do not even know whether there are *any* interesting solutions of these types. Any metric in the third class must either be one of those considered by Newman and Tamburino [7], or algebraically special, in which case it must be of the Kundt class. It cannot have a diverging principal null vector.

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On Rational Representation of Stationary Axisymmetric Vacuum metrics I: Cubic and Quartic Coordinates

A.W.C. Lun and J.D. Steele

Part II

1. Statement of the Problem

There is an abundance of literature (for a recent review, see [1]) in the field of finding exact solutions of Einstein's field equations for stationary axisymmetric systems. The study of stationary axisymmetric spacetimes has been a central theme in general relativity. An overview of the subject can be obtained by considering the dual role played by the static vacuum and vacuum solutions (the so-called Weyl solutions) and, in particular, examining the relationship used to generate such solutions to the corresponding stationary ones. The aim of this paper is to investigate the links between various Weyl solutions as well as between the Weyl solutions and their stationary counterparts. We will build up a "picture" of these links in the Weyl case, and attempt to carry this structure across to the stationary case. Furthermore, we will point out some open problems in these attempts both for the Weyl metrics and their stationary counterparts.

Despite the great success and power of (i) the symmetry reduction introduced by Geroch [2], Kinnersley and Clement [3], and also Hawking and Ellis [4], (ii) the inverse scattering method (see e.g. [5]), and (iii) various bootstrap transformation techniques [6] in generating solutions of the stationary axisymmetric vacuum equations, it is in general difficult to have control over the solutions generated from a given seed and thereby capturing any one of these methods. We hope that our viewpoint can help elucidate some of the difficulties in these generating techniques.

In Weyl's canonical coordinates, the stationary axisymmetric metric is expressed in the form

$$ds^2 = e^{2U} \left[e^{2V} (dx^2 + dy^2) + e^{2W} dz^2 \right] - e^{2U} (dt + \lambda d\phi)^2, \quad (1.1)$$

where U , V and λ are functions of x and y only. The field equation is written as

$$(\Delta - U) (e^{2U} + e^{2V} e^{2W} e^{2U}) = 3(U^2 + V^2), \quad (1.2)$$

where the Ernst potential is

$$E = e^{2U} + e^{2V} e^{2W} e^{2U}. \quad (1.3)$$

The functions V and λ are determined by

$$\lambda_1 = \lambda_0 \frac{\sqrt{1-\lambda_0^2}}{(1+\lambda_0)}, \quad \lambda_2 = \lambda_0 \frac{\sqrt{1-\lambda_0^2}}{(1-\lambda_0)}, \quad (1.4)$$

where $\lambda_0 = \lambda_1 - \lambda_2$.

On Rational Representation of Stationary Axisymmetric Vacuum metrics I: Cubic and Quartic Coordinates

A.W.C. Lun and J.D. Steele

1. Statement of the Problem

There is an abundance of literature (for a recent review, see [1]) in the field of finding exact solutions of the Ernst equation (see (1.2) below), which is the cornerstone of the study of stationary axisymmetric vacuum metrics in General Relativity. However, an overview of the subject can be obtained by considering the vital role played by the static axisymmetric vacuum solutions (the so-called Weyl solutions) and, in particular, examining the techniques used to generalise such solutions to the corresponding stationary cases. The aim of this paper is to investigate the links between various Weyl solutions as well as between the Weyl solutions and their stationary counterparts. We will build up a picture of these links in the Weyl case, and attempt to carry this structure across to the stationary case. Furthermore, we will point out some open problems in these structures both for the Weyl metrics and their stationary counterparts.

Despite the great success and power of (i) the symmetry techniques introduced by Geroch [2], Kinnersley and Chitre [3], and also Hauser and Ernst [4]; (ii) the inverse scattering method (see e.g. [5]); and (iii) various Bäcklund transformation techniques [6] in generating solutions of the stationary axisymmetric vacuum equations, it is in general difficult to have control over the solutions generated from a given seed solution by applying any one of these methods. We hope that our viewpoint can help ameliorate some of the deficiencies in these generating techniques.

In Weyl's canonical coordinates, the stationary axisymmetric metric is expressed in the form

$$ds^2 = e^{-2U} \left[e^{2V} (d\rho^2 + dz^2) + \rho^2 d\phi^2 \right] - e^{2U} (dt + A d\phi)^2, \quad (1.1)$$

where U , V and A are functions of ρ and z only. The Ernst equation is written as

$$(\mathcal{E} + \bar{\mathcal{E}})(\mathcal{E}_{\rho\rho} + \rho^{-1}\mathcal{E}_{\rho} + \mathcal{E}_{zz}) = 2(\mathcal{E}_{\rho}^2 + \mathcal{E}_z^2), \quad (1.2)$$

where the Ernst potential is

$$\mathcal{E} = e^{2U} + i\omega. \quad (1.3a)$$

The functions V and A are determined by

$$V_{\zeta} = \sqrt{2}\rho \frac{\varepsilon_{\zeta} \bar{\varepsilon}_{\bar{\zeta}}}{(\varepsilon + \bar{\varepsilon})^2}, \quad A_{\zeta} = 2\rho \frac{(\varepsilon - \bar{\varepsilon})_{\zeta}}{(\varepsilon + \bar{\varepsilon})^2}, \quad (1.3b)$$

where $\sqrt{2}\partial = \partial_{\rho} - i\partial_z$.

The Ernst equation can be reformulated by introducing a new potential ξ via

$$\xi = (1 - \mathcal{E}) / (1 + \mathcal{E}). \quad (1.4)$$

Then ξ satisfies the differential equation

$$(\xi \bar{\xi} - 1)(\xi_{\rho\rho} + \rho^{-1}\xi_{\rho} + \xi_{zz}) = 2\bar{\xi}(\xi_{\rho}^2 + \xi_z^2). \quad (1.2')$$

In prolate spheroidal coordinates (see (2.1) below) the potential ξ has proved essential in the generation of the Tomimatsu-Sato-Cosgrove class of solutions [7].

It is straightforward to show that a stationary axisymmetric vacuum solution which is analytic at spatial infinity (i.e. when the reciprocal of the radial coordinate $R = \sqrt{\rho^2 + z^2}$ equals zero) is asymptotically flat if the Ernst potential satisfies

$$\operatorname{Re} \mathcal{E} = 1 - \frac{2M}{R} + O\left(\frac{1}{R^2}\right), \quad \operatorname{Im} \mathcal{E} = -\frac{2J}{R^2} \cos \theta + O\left(\frac{1}{R^3}\right) \quad (1.5)$$

where $\tan \theta = \rho/z$. The real constants M and J are the total mass and the total angular momentum of the asymptotically flat solution, respectively.

When the time-like Killing vector ∂_t in the metric (1.1) is hypersurface-orthogonal, the metric coefficient A can be put equal to zero, and as a consequence the axisymmetric Killing vector ∂_ϕ is also hypersurface-orthogonal and one has the Weyl metrics. In Weyl's canonical coordinates, the field equations become

$$\Delta U = \rho^{-1}(\rho U_\rho)_\rho + U_{zz} = 0, \quad (1.6a)$$

$$V_\rho = \rho(U_\rho^2 - U_z^2), \quad V_z = 2\rho U_\rho U_z. \quad (1.6b)$$

Note that the three-dimensional Laplace equation (1.6a) is the static form of the Ernst equation (1.2).

The "elementary" axisymmetric static metric has

$$U = \frac{1}{2}\delta \ln(R_c - (z - c)), \quad V = \frac{1}{2}\delta^2 \ln\left(\frac{R_c - (z - c)}{R_c}\right), \quad (1.7)$$

where $R_c^2 = \rho^2 + (z - c)^2$, with c and δ real constants. If one interprets U as a Newtonian potential (such an interpretation is purely for convenience in our classification and has no physical meaning in the context of General Relativity) then the "elementary" axisymmetric static metric corresponds to the Newtonian potential of a semi-infinite rod lying along the interval $[c, \infty)$ of the z -axis and having density δ . Following the literature, we will call δ a deformation parameter. Due to the linearity of the Laplace equation (1.6a), we can superimpose n oriented semi-infinite rods of arbitrary densities, and still have a solution of (1.6a). Consequently,

$$U = \sum_{i=1}^n \frac{\varepsilon_i \delta_i}{2} \ln(R_i - \varepsilon_i(z - c_i)), \quad (1.8a)$$

where $R_i^2 = \rho^2 + (z - c_i)^2$ and ε_i gives the orientation of the i th rod.

$$\varepsilon_i = \begin{cases} +1 & \text{when rod in } +ve \text{ } z \text{ direction,} \\ -1 & \text{when rod in } -ve \text{ } z \text{ direction.} \end{cases}$$

Without any loss of generality, we assume the n real parameters c_i satisfy $c_1 \leq c_2 \leq \dots \leq c_n$. The real constants δ_i are the deformation parameters. The metric function V is given by quadrature and can be written explicitly in the form

$$V = \sum_{i=1}^n \frac{\delta_i^2}{2} \ln \left(\frac{R_i - \varepsilon_i(z - c_i)}{R_i} \right) + \sum_{i=1}^{n-1} \sum_{j=i+1}^n \delta_i \delta_j \ln [\varepsilon_i R_i - \varepsilon_j R_j + (c_i - c_j)]. \quad (1.8b)$$

Both U and V are determined up to an additive constant.

One can construct asymptotically flat Weyl solutions from an even number ($2n$) of semi-infinite rods by orientating all of them in the same direction such that the densities of each pair are of the same magnitude but of the opposite signs (i.e. $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_{2n}$ and $\delta_{2i-1} = -\delta_{2i}$, $i = 1, 2, \dots, n$). Thus there are n finite segments of rods with various densities ($= \delta_{2i-1}$ for the i th rod). We shall call such asymptotically flat rods the n Voorhees-Zipoy solutions, whereas the $2n$ and $2n + 1$ semi-infinite rod solutions with arbitrary densities and orientations will be referred to as the generalised n Voorhees-Zipoy solutions and the generalised accelerated Voorhees-Zipoy solutions respectively. One can show, after some complicated algebra, that in the n Voorhees-Zipoy solutions equations (1.8) become

$$U = \sum_{i=1}^n \frac{\varepsilon_{2i-1} \delta_{2i-1}}{2} \ln \left[\frac{R_{2i-1} + R_{2i} + (c_{2i-1} - c_{2i})}{R_{2i-1} + R_{2i} - (c_{2i-1} - c_{2i})} \right], \quad (1.9a)$$

and

$$V = \sum_{i=1}^n \frac{\delta_{2i-1}^2}{2} \ln \left(\frac{(R_{2i-1} + R_{2i})^2 - (c_{2i-1} - c_{2i})^2}{4R_{2i-1}R_{2i}} \right) + \Gamma_0 \quad (1.9b)$$

where

$$\Gamma_0 = \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{\delta_{2i-1} \delta_{2j-1}}{2} \times \ln \left(\frac{[(R_{2i-1} + R_{2j})^2 - (c_{2i-1} - c_{2j})^2][(R_{2i} + R_{2j-1})^2 - (c_{2i} - c_{2j-1})^2]}{[(R_{2i-1} + R_{2j-1})^2 - (c_{2i-1} - c_{2j-1})^2][(R_{2i} + R_{2j})^2 - (c_{2i} - c_{2j})^2]} \right).$$

We have chosen the integration constants such that both U and V vanish when ρ approaches infinity. Consequently the asymptotic form of the Minkowski metric is given in cylindrical polar coordinates (t, ρ, z, φ) . We have to emphasize that there are many ways to construct distinct asymptotically flat Weyl solutions from semi-infinite rods ($n \geq 2$).

Many well-known Weyl solutions have the deformation parameter $\delta_i = 1$ or -1 . For instance, the one semi-infinite rod solution with $\delta = 1$ and $\varepsilon = \pm 1$ represents the Minkowski space-time; the Schwarzschild solution is a Voorhees-Zipoy [8] solution with $\delta_1 = 1$; the C-metric, which can be interpreted as a pair of uniformly accelerating Schwarzschild particles, is a three semi-infinite rod solution with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$ and $\delta_1 = \delta_3 = -\delta_2 = 1$; and the Bach-Weyl [9] two Schwarzschild particle metric has $\varepsilon_1 = \varepsilon_2$, $\varepsilon_3 = \varepsilon_4$ and

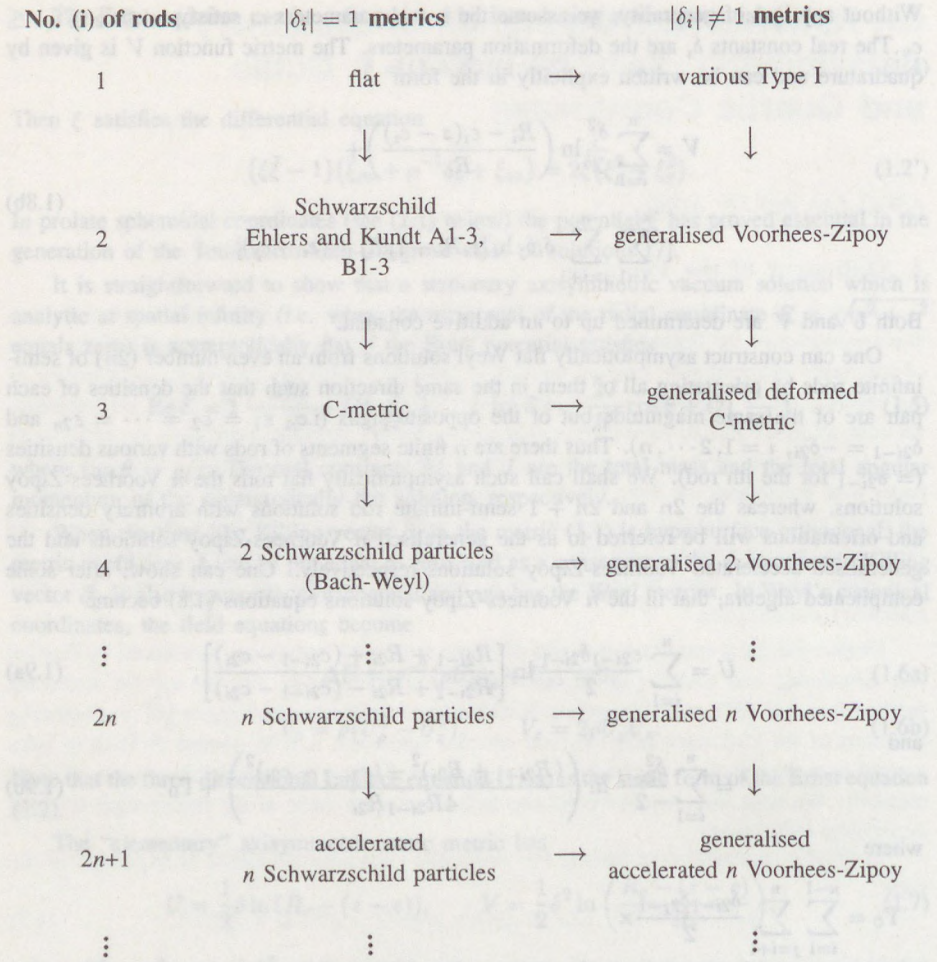


Figure 1 Hierarchy of Weyl solutions

$\delta_1 = \delta_3 = 1$, $\delta_2 = \delta_4 = -1$. Hence one can use a Minkowski one semi-infinite rod solution to build up a hierarchy of Weyl solution by (1) adding to it another Minkowski one rod solution, or (2) multiplying it by an arbitrary deformation parameter δ (compare with figure 1).

Most of the Weyl solutions have previously been given in the Weyl canonical coordinates, the exceptions being the C- and the deformed C-metrics. We will establish the connection between the cubic polynomial coordinates in the C-metric and the Weyl canonical coordinates in Theorem 1 (cf. Bonnor-Hoenselaers [10]). The deformed C-metrics are obvious in the Weyl canonical coordinates once this theorem is given, but the corresponding forms in cubic polynomial coordinates are more obscure and are given in section 2. It is straightforward to verify that our results coincide with those of Plebanski and Kerr quoted in [11].

No. (i) of rods	$ \delta_i = 1$ metrics	$ \delta_i \neq 1$ metrics
1	flat	→ not explicitly covered
	↓	↓
2	Kerr	→ Tomimatsu-Sato (δ integer) Cosgrove (δ arbitrary)
	↓	↓
3	Kinnersley (twisting C)	→ not explicitly covered
	↓	↓
4	double Kerr	→ Dietz [12]
⋮	⋮	⋮
$2n$	multiple Kerr (Yamazaki [13])	→ not explicitly covered
	↓	↓
$2n+1$	not explicitly covered	→ not explicitly covered
⋮	⋮	⋮

Figure 2 Hierarchy of stationary axisymmetric solutions

The generalisation of the Weyl solutions to their stationary axisymmetric vacuum counterparts is a formidable task, due to the non-linearity of the Ernst equation (1.2) in these cases. Figure 2 is the “stationary counterpart” to figure 1, most of these solutions are in principle known from application of HKX [6] transformations, for example; few seem to have been explicitly investigated.

The outstanding feature in Figure 2 is the two rod case, i.e. the Kerr and the Tomimatsu-Sato-Cosgrove class, which has only one (arbitrary) deformation parameter. This case has been thoroughly investigated because these solutions can be expressed easily in prolate spheroidal coordinates. The prolate spheroidal coordinates in our picture are the quadratic counterparts to the cubic coordinates in the C-metric. Our aim is to look for a similar Ansatz which can extend the twisting C-metric of Kinnersley to its deformed generalisations, and similarly for the double-Kerr. Partial results are given below.

2. Basic Results

We begin by looking at the relationship between the Weyl canonical coordinates and the prolate spheroidal coordinates in the two rod cases. The key part of the transformation is the (flat) 2-metric

$$\frac{d\rho^2 + dz^2}{\sigma^2 R_+ R_-} = \frac{dx^2}{x^2 - 1} - \frac{dy^2}{y^2 - 1}, \quad (2.1)$$

where x and y are the prolate spheroidal coordinates. The transformation is given by

$$\rho^2 = \sigma^2 (x^2 - 1)(1 - y^2)$$

$$z = \sigma xy$$

$$2\sigma x = R_+ + R_-$$

$$2\sigma y = R_+ - R_-$$

$$R_{\pm} = \rho^2 + (z \pm \sigma)^2.$$

The Schwarzschild solution has the Ernst potential

$$e^{2U} = \left(\frac{R_+ + R_- - 2\sigma}{R_+ + R_- + 2\sigma} \right) = \left(\frac{x - 1}{x + 1} \right).$$

In prolate spheroidal coordinates the potential ξ has the simple form $\xi = 1/x$. The Voorhees-Zipoy solutions have

$$e^{2U} = \left(\frac{R_+ + R_- - 2\sigma}{R_+ + R_- + 2\sigma} \right)^{\delta} = \left(\frac{x - 1}{x + 1} \right)^{\delta}.$$

and the potential ξ is then given by

$$\xi = \frac{(x + 1)^{\delta} - (x - 1)^{\delta}}{(x + 1)^{\delta} + (x - 1)^{\delta}}.$$

The extension of these metrics to include the rotation parameter leads to the Kerr (with $\delta = 1$), Tomimatsu-Sato (integral δ) and Cosgrove (arbitrary δ) metrics. For example, the Kerr metric in prolate spheroidal coordinates has a simple potential $\xi = (px - iqy)^{-1}$ where $p^2 + q^2 = 1$. Here q represents the rotation parameter: if $q = 0$ the metric is Schwarzschild (see [14]). In the Weyl canonical coordinates, the Ernst potential of the Kerr metric takes the form

$$\xi = \frac{e^{-i\theta}(R_+ - m) + e^{i\theta}(R_- - m)}{e^{-i\theta}(R_+ - m) - e^{i\theta}(R_- - m)},$$

where $e^{i\theta} = p + iq$ and $\sigma = mp$.

Now we consider the case of a 2-metric where the quadratic in equation (2.1) is replaced by a cubic, i.e.

$$\frac{d\rho^2 + dz^2}{R_1 R_2 R_3} = \frac{dx^2}{P_3(x)} - \frac{dy^2}{P_3(y)}, \quad (2.2)$$

where P_3 is some cubic and $R_i^2 = \rho^2 + (z - c_i)^2$.

The case where the cubic is replaced by a quartic is closely linked to this case because of the following result [15]:

Theorem 1: *There are coordinate systems (x, y) , (X, Y) , (ρ, z) , and (σ, Z) such that*

$$\frac{dx^2}{P_3(x)} - \frac{dy^2}{P_3(y)} = \frac{dX^2}{P_4(X)} - \frac{dY^2}{P_4(Y)} = \frac{d\rho^2 + dz^2}{R_1 R_2 R_3} = \frac{d\sigma^2 + dZ^2}{S_1 S_2 S_3 S_4},$$

where P_4 is a quartic and $S_i^2 = \Sigma^2 + (Z - C_i)^2$ for some constants C_i .

The result can be proved elegantly by the straightforward adaption of a result due to Lagrange and given in detail in Cayley [16]; essentially one uses a fractional linear transformation that maps one of the roots of the quartic to infinity.

In these coordinates the C-metric has the Ernst potential

$$e^{2U} = \frac{P_3(y)}{(x-y)^2} = \frac{(R_3 - (z - c_3))(R_1 - (z - c_1))}{(R_2 - (z - c_2))},$$

where $c_1 > c_2 > c_3$ are the roots of the cubic P_3 and correspond to the ends of the rods, as above. The function V can be found by applying the transformation equations to (1.8b).

The deformed generalisations of the C-metric in Weyl coordinates can be found from (1.8). In the cubic coordinates the Ernst potential takes the form

$$e^{2U} = \frac{2^{2\delta_3}}{(y-x)^{4\delta_3}} (c_1 - c_2 y)^{2\delta_1} (y^2 - c_2^2)^{2\delta_3} (c_1 - c_2 x)^{2(\delta_3 - \delta_1)},$$

where we have arranged that $c_3 = -c_2$ and have assumed that the density $\delta_2 = -\delta_3$. These deformation parameters are, of course, continuous.

The extension of this metric to include rotation parameters using, e.g. HKX transformations, give rise to the Kinnersley metric and its generalisations with various deformation parameters. Note that the Kinnersley metric is usually written using a quartic, but from Theorem 1 it can also be written in terms of a cubic — although this expression is not as pleasant.

The importance of the prolate spheroidal coordinates is that they allow the “deformed Kerr” (Tomimatsu-Sato-Cosgrove) metrics to be expressed in terms of rational functions of these coordinates, and hence in a way far better suited to computer (and human) manipulation. The application of the cubic and quartic coordinates to the generalised Kinnersley metrics is expected similarly to lead to more tractable expressions in these cases.

Acknowledgments

The authors would like to thank Cornelius Hoenselaers, Roy Kerr and Zoltan Perjés for many useful discussions.

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On the Equations for Twisting Type N and Perfect Fluid Solutions

C. Hoenselaers, R. Ferwagner and W. Schief

Abstract: We study a system of second-order partial differential equations which includes Einstein's equations for vacuum Petrov type N solutions with twisting rays. We also show that the equations governing rigidly rotating perfect fluid solutions can be cast into a similar form. From the analysis of the general system a transformation is derived which generates the Hauser solution from a trivial one.

1. Introduction

With the notable exception of stationary axisymmetric vacuum and electrovacuum solutions Einstein's equations have resisted any attempts to apply solution generating techniques. There is, of course, an abundance of *ad hoc* methods for solving the equations in special cases. The work presented here emerged from a study of rigidly rotating perfect fluid solutions and various solution generating methods [1, 2].

We had noticed that there is a superficial similarity between the equations for rigidly rotating perfect fluids and vacuum Petrov type N solutions with twisting rays. In order to exploit this similarity we have studied a system of linear second-order equations, the system (2.1) below, which exhibits features common to both cases.

It is known that the equations for twisting type N solutions with one Killing vector and a homothetic motion can be reduced to an ordinary differential equation of the third order [3, 4, 5]. The only solution known explicitly so far is the famous Hauser solution [6]. It will turn out that it can be generated by a new transformation of the general system from the most trivial seed solution.

Unfortunately, no such result was obtained for the Wahlquist solution [7, 8] which is the only known exact solution for rigidly rotating perfect fluids admitting a closed boundary and a regular axis. We are, however, able to derive many conditions on the Ernst potential which can be coded into two polynomials for a real function, a third-order polynomial with complex coefficients and a real quadratic one.

2. The General System

In this section we shall deal with the following system of second-order equations for a real function H ,

$$\partial^2 H = A\partial H + BH, \quad (2.1a)$$

$$\partial\partial^* H = CH, \quad (2.1b)$$

$$\partial^{*2} H = A^*\partial^* H + B^* H. \quad (2.1c)$$

∂ and ∂^* are commuting first-order differential operators and the functions A , B and C are to be determined from the integrability conditions. The first condition to be derived by

acting with ∂^* on (2.1a) and ∂ on (2.1b), equating the results and eliminating the second derivatives by (2.1) reads

$$(\partial^* A - C)\partial H + B\partial^* H = (\partial C - AC - \partial^* B)H$$

and its complex conjugate. Acting with ∂ and ∂^* on this equation we derive further equations for H and its derivatives. With the abbreviations

$$\begin{aligned}\alpha &= \partial^* A - C \\ \beta &= B\end{aligned}\tag{2.2}$$

$$\gamma = \partial C - AC - \partial^* B$$

they can be written as

$$\alpha\partial H + \beta\partial^* H = \gamma H,\tag{2.3a}$$

$$(\partial\alpha + \alpha A - \gamma)\partial H + \partial\beta\partial^* H = (\partial\gamma - \alpha\beta - \beta C)H,\tag{2.3b}$$

$$\partial^*\alpha\partial H + (\partial^*\beta + \beta A^* - \gamma)\partial^* H = (\partial^*\gamma - \alpha C - \beta\beta^*)H.\tag{2.3c}$$

These and their complex conjugate equations are six algebraic equations for H , ∂H and $\partial^* H$. To admit a nontrivial solution for H this system should at most be rank 2.

The simplest case is the one in which the system (2.3) is identically satisfied, i.e.

$$\alpha = \beta = \gamma = 0.\tag{2.4}$$

We get immediately from (2.2) for $C \neq 0$

$$C = e^\varphi, \quad A = \partial\varphi, \quad \partial\partial^*\varphi = e^\varphi.\tag{2.5}$$

The system (2.1) implies thus the Liouville equation in the rank 0 case.

Now we discuss the case in which the system (2.3) has rank 1.

Here we find from (2.3a) and its complex conjugate

$$\alpha\alpha^* = \beta\beta^*, \quad \beta\gamma^* = \gamma\alpha^*.\tag{2.6}$$

Furthermore, combining (2.3a) and (2.3b) yields

$$\partial\frac{\alpha}{\beta} + \frac{\alpha}{\beta}A - \frac{\gamma}{\beta} = 0,$$

$$\partial\frac{\gamma}{\beta} - \alpha - C = 0.$$

The last equation reads

$$\partial\frac{\gamma}{\beta} = \partial^* A,$$

which implies the existence of a complex potential,

$$A = \partial\varphi, \quad \frac{\gamma}{\beta} = \partial^*\varphi.$$

As

$$\frac{\alpha}{\beta} = \frac{\gamma \beta^*}{\beta \gamma^*} = \frac{\partial^* \varphi}{\partial \varphi^*}$$

we find

$$\partial(\ln \partial^* \varphi - \ln \partial \varphi^*) = \partial(\varphi^* - \varphi) \Rightarrow \partial^* e^\varphi = \partial e^{\varphi^*},$$

and thus there exists a real potential λ such that $\varphi = \ln \partial \lambda$ and consequently

$$A = \partial \ln \partial \lambda. \quad (2.7)$$

It can be checked by direct calculation that only one of the relations (2.3) remains, viz.

$$\partial^* \lambda \partial H + \partial \lambda \partial^* H = \partial \partial^* \lambda H,$$

and that the others are identically satisfied. The relations (2.2) become

$$B = \frac{\partial \lambda}{\partial^* \lambda} (\partial \partial^* \ln \partial \lambda - C), \quad (2.8)$$

$$\partial C - C \partial \ln \partial \lambda = B \partial^* \ln \partial \lambda + \partial^* B.$$

Upon elimination of B they yield

$$\begin{aligned} \partial^* \lambda \partial C + \partial \lambda \partial^* C + C \left(2 \partial \partial^* \lambda - \frac{\partial \lambda}{\partial^* \lambda} \partial^2 \lambda - \frac{\partial^* \lambda}{\partial \lambda} \partial^2 \lambda \right) = \\ \partial^2 \partial^2 \lambda - \frac{1}{\partial \lambda} \partial^2 \lambda \partial \partial^2 \lambda - \frac{1}{\partial^* \lambda} \partial^2 \lambda \partial^2 \partial^* \lambda + \frac{1}{\partial \lambda \partial^* \lambda} \partial^2 \lambda \partial^2 \lambda \partial \partial^* \lambda. \end{aligned} \quad (2.9)$$

To summarize: In the case in which equations (2.3) are of rank 1 the system to be considered reduces to the equations

$$\partial^* \lambda \partial H + \partial \lambda \partial^* H = H \partial \partial^* \lambda, \quad (2.10a)$$

$$\partial \partial^* H = CH. \quad (2.10b)$$

The coefficients A and B in (2.1) are defined in terms of λ and C by (2.7) and (2.8) while there is the one equation (2.9) relating λ and C .

Equation (2.10a) can be written as

$$\partial \left(\frac{\partial^* \lambda}{H^2} \right) + \partial^* \left(\frac{\partial \lambda}{H^2} \right) = 0,$$

which in turn implies

$$\begin{aligned} \partial \lambda = H^2 \partial \eta, \quad \partial^* \lambda = -H^2 \partial^* \eta, \\ \partial \eta \partial^* H + \partial^* \eta \partial H + H \partial \partial^* \eta = 0. \end{aligned} \quad (2.11)$$

It can be easily checked that $h = H\eta$ is another solution of (2.1), viz.

$$\partial^2 h = A \partial h + B h, \quad \partial \partial^* h = C h.$$

Consider now the function $\tilde{H} = HF(\eta)$. By virtue of (2.11) it satisfies

$$\partial\partial^*\tilde{H} = \left(\tilde{C} - \frac{\partial\lambda\partial^*\lambda F''}{H^4 F}\right)H,$$

while (2.10a) remains unchanged under substitution of $\tilde{H}$ for H . The transformation from one system (2.10), respectively (2.1) to another one, the twiddled one, can thus be written as

$$\begin{aligned}\tilde{H} &= HF(\eta), & \tilde{\lambda} &= \lambda, & \tilde{C} &= C - \frac{\partial\lambda\partial^*\lambda F''}{H^4 F}, \\ \tilde{A} &= A, & \tilde{B} &= B + \frac{\partial\lambda^2 F''}{H^4 F}.\end{aligned}\tag{2.12}$$

Indeed, it can be easily shown that C is again a solution of (2.9). The fact that $\tilde{\lambda} = \lambda$ implies that $\tilde{h}$ transforms as $\tilde{h} = HG(\eta)$ with $F'G - G'F = 1$.

There is another transformation for the system (2.10). It was originally discovered by Moutard [9] for (2.10b) only and can in the present context most conveniently be described by the interchange of λ and η . This implies transformations for the other functions, viz.

$$\begin{aligned}\tilde{H} &= \frac{1}{H}, & \tilde{\lambda} &= \eta, & \tilde{\eta} &= \lambda, & \tilde{C} &= -C + 2\frac{\partial H\partial^*H}{H^2}, \\ \tilde{h} &= \frac{\lambda}{H}, & \tilde{A} &= A - 2\partial\ln H, & \tilde{B} &= -B.\end{aligned}\tag{2.13}$$

Moutard transformations or their relatives, the Darboux transformations, have been useful in deriving Bäcklund transformations for a wide variety of nonlinear equations [10]. Combining a Moutard transformation (2.13) with (2.12) yields $\tilde{H} = H\tilde{F}(\lambda)$.

It should be noted that the rank 0 case, equations (2.4) and (2.5), can be cast into a form amenable to the transformations discussed above. To this end consider two independent solutions, H and h of Liouville's equation or, for $C = 0$, the wave equation and use them to define λ and η via (2.11). These functions can then be used in (2.12) to transform a rank 0 solution into a rank 1 system. Different seed solutions will in general lead to different rank 1 systems. This fact will be important in the next chapter.

Not much can be said in the general case in which the system (2.3) has rank 2. One can introduce a function μ such that

$$\begin{aligned}\mu &= \frac{\gamma\alpha^* - \gamma^*\beta}{\alpha\alpha^* - \beta\beta^*}, \\ \partial H &= \mu H.\end{aligned}\tag{2.14}$$

Equations (2.1) now become

$$\begin{aligned}\partial\mu + \mu^2 &= A\mu + B, \\ \partial^*\mu + \mu\mu^* &= C.\end{aligned}\tag{2.15}$$

Equations for A , B and C can be derived by inserting μ into the above expressions. The existence of H is guaranteed by C being real. As H is determined uniquely up to a constant factor, neither a Moutard transformation (2.13) nor a transformation (2.12) exist.

3. The Equations for Twisting Type N Solutions

Following Finley [11, 12] the Einstein equations for vacuum gravitational fields of Petrov type N can be written as the following triplet of equations for two functions Q and ξ :

$$\begin{aligned} \left(\partial_s \frac{1}{Q_s} \partial_s \frac{1}{Q_s} - \Delta(s) \right) \xi &= 0, \\ \left(R \partial_s \partial_z + \frac{1}{2} R_s \partial_z + \frac{1}{2} R_z \partial_s + \frac{1}{4} R_{sz} - \frac{3}{4} \frac{Q_{sz}^2}{Q_s Q_z} \right) \xi &= 0, \\ \left(\partial_z \frac{1}{Q_z} \partial_z \frac{1}{Q_z} - \delta(z) \right) \xi &= 0, \\ R &= \frac{Q_{sz}}{(Q_s Q_z)^3}. \end{aligned} \quad (3.1)$$

$\Delta(s)$ and $\delta(z)$ are arbitrary functions of their arguments which, however, should not be constant for nontrivial twisting and expanding solutions. Upon introduction of $H = R^{1/2} \xi$ the equations (3.1) take the form of the system (2.1) with

$$\begin{aligned} A &= \partial_s \ln (R Q_s^3), \\ B &= \frac{1}{2} \left(A_s - \frac{1}{2} A^2 - \partial_s^2 \ln Q_s + \frac{1}{2} (\partial_s \ln Q_s)^2 + 2 \Delta(s) Q_s^2 \right), \\ C &= \frac{1}{4} \left(\partial_{sz} \ln R + 3 \frac{Q_{sz}}{Q_s Q_z} \right), \end{aligned} \quad (3.2)$$

and the operation “*” being the interchange of s and z , e.g. $(\Delta(s))^* = \delta(z)$.

We are interested in the case in which the system has rank 1. The equation involving A can be integrated immediately to give two expressions for the potential λ , viz.

$$\lambda = \frac{-f_1}{2Q_s^2} + f_2 = \frac{-g_1}{2Q_z^2} + g_2, \quad (3.3)$$

where the f_i 's depend only on s and the g_i 's only on z . As a consequence λ has to satisfy the following first-order equation,

$$\frac{f_1^{1/2} \lambda_s}{(f_2 - \lambda)^{3/2}} = \frac{g_1^{1/2} \lambda_z}{(g_2 - \lambda)^{3/2}}. \quad (3.4)$$

Instead of replacing all the derivatives of Q by the appropriate expressions involving λ and working our way through the emerging long and rather unwieldy expressions, we try to generate a solution of the equations corresponding to (3.2) by the transformation (2.12) applied to a suitable seed solution and function $F(\eta)$. To this end we consider the most trivial starting system,

$$C = 0, \quad \partial_{sz} H = 0.$$

The solution is, of course, $H = f(s) + g(z)$. The most trivial system of independent solutions is $H = s$ and $h = z$, leading to $\lambda = sz$. Equation (3.4) can be solved immediately yielding

$$\begin{aligned} f_1 &= \frac{s}{(c_1 s + c_2)^3}, & g_1 &= \frac{z}{(c_1 z + c_3)^3}, \\ f_2 &= s \frac{c_4 - c_3 s}{c_1 s + c_2}, & g_2 &= z \frac{c_4 - c_2 z}{c_1 z + c_3}, \end{aligned}$$

with constants c_i . We can now calculate Q_s and Q_z from (3.3). The transformation (2.12) gives

$$\tilde{C} = -\frac{\partial \lambda \partial^* \lambda F''}{H^4 F} = -\frac{z F''}{s^3 F},$$

which is homogeneous of degree -2 in s and z . On the other hand, the expression calculated by inserting the derivatives of Q into (3.2) will not be homogeneous unless $c_1 = c_4 = 0$. Without loss of generality we can then choose $c_2 = c_3 = -1$. This gives

$$Q = \sqrt{2(s+z)}, \quad \tilde{C} = -\frac{3}{16} \frac{1}{(s+z)^2}. \quad (3.5)$$

As $\tilde{H} = sF(\eta)$ and $\eta = z/s$ this implies

$$16\eta(1+\eta)^2 F'' = 3F.$$

From the transformation (2.12) for $\tilde{B}$ we get, finally,

$$\tilde{B} = \frac{3}{16} \frac{z}{s} \frac{1}{(s+z)^2}, \quad \Delta = \frac{3}{8s}. \quad (3.6)$$

With $w = \frac{\eta-1}{\eta+1}$ and $F = \frac{2}{1-w} f(w)$ the above equation for F becomes

$$16(1-w^2) \ddot{f} = 3f, \quad (3.7)$$

which can be recognised as the hypergeometric equation appearing in the standard treatment of the Hauser solution.

We have derived the Hauser solution by the transformation (2.12) from the most trivial solution of the system (2.1). Unfortunately there are no other solutions to be generated as Finley [13] has shown that the Hauser solution is the only genuinely twisting type N solution for which the system (2.3) is of rank 1.

4. The Equations for Rigidly Rotating Perfect Fluid Solutions

Starting with the metric in the standard form, i.e.

$$ds^2 = \frac{1}{f} \left(e^{2\gamma} \left(\frac{dx^2}{P(x)} + \frac{dy^2}{Q(y)} \right) + H^2 d\varphi^2 \right) - f(dt - \omega d\varphi)^2,$$

the equations for rigidly rotating perfect fluid space-times take a form similar to (2.1), viz.

$$\partial^2 H = 2\partial\gamma\partial H - \frac{1}{2f^2}\partial\mathcal{E}\partial\mathcal{E}^*H, \quad (4.1)$$

$$\partial\partial^*H = 2e^{2\lambda}\frac{p}{f}H,$$

or

$$A = 2\partial\gamma, \quad B = -\frac{1}{2f^2}\partial\mathcal{E}\partial\mathcal{E}^*, \quad C = e^{2\gamma}\frac{p}{f}.$$

In this case the derivative operators are given by $\partial = \sqrt{P}\partial_x + i\sqrt{Q}\partial_y$ and its complex conjugate. Those equations, however, do not comprise all the Einstein equations. There are additional equations, the Ernst equation,

$$\partial\partial^*\mathcal{E} + \frac{1}{2H}(\partial H\partial^*\mathcal{E} + \partial^*H\partial\mathcal{E}) = \frac{1}{f}\partial\mathcal{E}\partial^*\mathcal{E} + e^{2\gamma}(\varepsilon + 3p), \quad (4.2)$$

and the equation relating the pressure p and the energy density ε to the real part of the Ernst potential, i.e.

$$\partial p = -\frac{1}{2f}(\varepsilon + p)\partial f. \quad (4.3)$$

From (4.1) we can derive integrability conditions. The analogue to (2.3a) becomes

$$2\partial\partial^*\gamma\partial H - \frac{1}{2f^2}\partial\mathcal{E}\partial\mathcal{E}^*\partial^*H = 2e^{2\gamma}\partial\frac{p}{f} + \frac{1}{2}\partial^*\left(\frac{1}{f^2}\partial\mathcal{E}\partial\mathcal{E}^*\right).$$

Using (4.2) and (4.3) the derivatives of H can be eliminated to give a second-order equation for γ ,

$$\partial\partial^*\gamma = e^{2\gamma}\frac{p}{f} - \frac{1}{8f^2}(\partial\mathcal{E}\partial^*\mathcal{E}^* + \partial\mathcal{E}^*\partial^*\mathcal{E}). \quad (4.4)$$

It can be shown that this equation together with the one above implies Ernst's equation. We therefore consider the Ernst equation (4.2) as a first-order equation for H analogous to (2.3a) and regard (4.3) and (4.4) as auxiliary equations.

Evidently there is no point in considering solutions for which (4.2) and its complex conjugate as a system for ∂H and ∂^*H is of rank 0.

Let us introduce the abbreviations

$$D = \partial\mathcal{E}\partial^*\mathcal{E}^* - \partial\mathcal{E}^*\partial^*\mathcal{E},$$

$$F = \frac{1}{D}(\varepsilon + 3p)(\partial\mathcal{E} - \partial^*\mathcal{E}),$$

$$G = \frac{1}{D}\left(\partial\mathcal{E}^*\partial\partial^*\mathcal{E} - \partial\mathcal{E}\partial\partial^*\mathcal{E}^* - \frac{1}{f}\partial\mathcal{E}\partial^*\mathcal{E}(\partial^*\mathcal{E} - \partial^*\mathcal{E}^*)\right).$$

The system being of rank 1 implies

$$D = e^{2\gamma}F + G = 0,$$

from which one derives

$$\partial\partial^*\text{Im } \mathcal{E} = 0,$$

$$e^{2\gamma}(\varepsilon + 3p) = \partial\partial^*\mathcal{E} - \frac{1}{f}\partial\mathcal{E}\partial^*\mathcal{E}^*. \quad (4.5)$$

Those solutions are thus not particularly interesting either. Hence we are stuck with the general rank 2 system (2.14) and (2.15). Inserting $\mu = e^{2\gamma}F + G$ into (2.15) we obtain

$$2G\partial\gamma = (e^{2\gamma}F + G) + e^{2\gamma}\partial F + \partial G - B, \quad (4.6a)$$

$$2Fe^{2\gamma}\partial^*\gamma = C - |e^{2\gamma}F + G|^2 - e^{2\gamma}\partial^*F - \partial^*G. \quad (4.6b)$$

Eliminating $\partial\gamma$ from the above equations yields

$$\begin{aligned} & e^{6\gamma}F^2F^* + e^{4\gamma}(3FF^*G + F^*\partial F) + \\ & e^{2\gamma}\left(2F^*G^2 + \partial(F^*G) - F^*B + FGG^* - 2G\frac{p}{f}\right) + \\ & G(\partial G^* + GG^*) = 0. \end{aligned} \quad (4.7)$$

There is yet another equation to be derived for $e^{2\gamma}$. Using the abbreviations

$$\begin{aligned} \Psi &= -\frac{1}{F}(\partial^*G + GG^*), \\ \Upsilon &= \frac{1}{F}\left(2\frac{p}{f} - \partial^*F - GF^* - FG^*\right), \end{aligned}$$

we get by eliminating $\partial\partial^*\gamma$ from (4.6b) and its complex conjugate

$$\text{Im}(\partial\Psi + \Upsilon\Psi^*) + e^{2\gamma}\text{Im}(\partial\Upsilon + 2F\Psi) + e^{4\gamma}\text{Im}(\partial^*F + F\Upsilon) = 0. \quad (4.8)$$

As equations to be satisfied by the real function $e^{2\gamma}$, (4.7) is a third-order polynomial with complex coefficients while (4.8) is a real quadratic polynomial. These two equations have to have at least one real root in common. This imposes severe restrictions on the coefficients. The resulting relations between F , G and their derivatives can, in principle, be written down in detail. Alas, due to their rather lengthy form we have not been able to draw any further conclusions and thus shall refrain from spelling them out.

Suffice to say that for vacuum solutions, i.e. $p = F = 0$, only the last term in (4.7) remains and it, or alternatively (4.6b), constitutes the integrability conditions for $\partial \ln H$ expressed in terms of the Ernst potential, cf. also [14]. The coefficients in (4.8) vanish identically. In this case there is no algebraic equation for $e^{2\gamma}$.

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Pauli-Briosch Type Identities and Pure Spinors

H. Urbantke

Abstract: A generalization to n for the well known identities of Pauli type of identities, first introduced in a systematic manner by Chen and Kostant, is given and applied to pure spinors.

Part III

1. Motivation and Some Prerequisites

Since the days of Killing, it has been known that in $2n$ space-time dimensions, and since the days of Pauli, it has been known that we have to know about spinors in $n > 4$ dimensions. For simplicity, I will restrict myself to even $n = 2m$ and consider a complex vector space V with (nondegenerate) metric tensor g . The identities I am going to present here, for special values of n , have been used in

- twistor theory, compactification ($n = 6$)
- supergravity ($n = 7, 8, \dots$)
- (classical) string theory ($n = 4, 6, 8, 10$)

For even $n = 2m$, the well known anticommutation relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} \cdot 1$$

admit, up to equivalence, only one irreducible representation by linear operators on a complex spinor space S ; we have $\dim S = 2^m$. Since we work in a complex V , we choose $\gamma_{\mu+1} = i\gamma_\mu$.

For each ordered subset $A \subset N := \{1, \dots, n\}$,

$$A = \{i_1, i_2, \dots, i_{|A|}\} \subseteq \{1, \dots, n\},$$

we write

$$\gamma_A := \gamma_{i_1} \gamma_{i_2} \dots \gamma_{i_{|A|}}, \quad \gamma_\emptyset := 1.$$

Then the operator algebra generated by the γ_i is linearly spanned by the γ_A and, after with $\text{End}(S)$, the set of all linear operators on S . More explicitly, every $F \in \text{End}(S)$ can be expanded,

$$F = \sum_A F_A \gamma_A.$$

The operator $\gamma_A = \gamma_{i_1} \dots \gamma_{i_{|A|}}$

$$\gamma_A \gamma_B = (-1)^{|A||B|} \gamma_B \gamma_A, \quad \gamma_A^2 = (-1)^{|A|} 1_S,$$

thus defining the usual (or half, Weyl, chiral) spinor projection

$$P_{\pm} := \frac{1}{2}(1_S \pm i^{|A|} \gamma_A).$$

We shall use small Latin indices for S , capital unbolded (bold) indices for S_+ (S_-).

Pauli-Briosch Type Identities and Pure Spinors

H. Urbantke

Abstract: A generalization to n ($= 2m$ even) dimensions of a type of identities, first considered in a systematic manner by Pauli and Kofink, is given, with notable special cases and an application to pure spinors.

1. Motivation and Some Prerequisites

Since the days of Kaluza-Klein we have to live with $n > 4$ space-time dimensions, and since the days of Pauli-Dirac we have to live with spinors. It follows that we have to know about spinors in $n > 4$ dimensions. For simplicity, I will restrict myself to even $n = 2m$ and consider a complex vector space V with (nondegenerate) metric tensor g . The identities I am going to present have, for special values of n , been used in

- twistor theory, compactification ($n = 6$)
- supergravity ($n = 7, 8, \dots$)
- (classical) string theory ($n = 4, 6, 8, 10$)

For even $n = 2m$, the well-known anticommutation relations

$$\gamma_\mu \gamma_\nu = g_{\mu\nu} \cdot 1$$

admit, up to equivalence, only one *irreducible* representation by linear operators on a complex spinor space S ; we have $\dim S = 2^m$. Since we work in a complex V , we assume $g_{\mu\nu} = \delta_{\mu\nu}$.

For each ordered subset $\Lambda \subset N := \{1, \dots, n\}$,

$$\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_{|\Lambda|} | 1 \leq \lambda_1 < \dots < \lambda_{|\Lambda|} \leq n\},$$

we write

$$\gamma_\Lambda := \gamma_{\lambda_1} \gamma_{\lambda_2} \dots \gamma_{\lambda_{|\Lambda|}}, \quad \gamma_\emptyset = id_S.$$

Then the operator algebra generated by the γ_Λ is linearly spanned by the γ_Λ and agrees with $\text{End}(S)$, the set of all linear operators on S . More explicitly, every $F \in \text{End}(S)$ can be expanded,

$$F = 2^{-m} \sum_{\Lambda} \text{tr} F \gamma_\Lambda^{-1} \gamma_\Lambda.$$

The operator $\gamma_N = \gamma_1 \dots \gamma_n$ has

$$\gamma_N \gamma_\Lambda = (-1)^{|\Lambda|} \gamma_\Lambda \gamma_N, \quad \gamma_N^2 = (-1)^m id_S,$$

thus defining the semi (or half, Weyl, chiral) spinor projections

$$S_\pm := \frac{1}{2} (id_S \pm i^m \gamma_N) S.$$

We shall use small Latin indices for S , capital undotted (dotted) indices for S_+ (S_-).

The operators $(-1)^m \gamma_\lambda^T$ (on $S^* = \text{dual of } S$) satisfy the same kind of commutation relations as the γ_λ themselves do: Hence there is an equivalence transformation B ($S \rightarrow S^*$) such that

$$B\gamma_\lambda B^{-1} = (-1)^m \gamma_\lambda^T.$$

B satisfies

$$(B\gamma_\Lambda)^T = (-1)^{(m-|\Lambda|)(m-|\Lambda|+1)/2} B\gamma_\Lambda$$

$$B\gamma_\Lambda S_\pm = S_{\pm(-1)^{m+|\Lambda|}}^*,$$

where $S_\pm^*$ are defined via the transposes of the above projections. Thus $B\gamma_\Lambda$ preserves (changes) chirality according to $m + |\Lambda|$ being even (odd). In particular, $B\gamma_\Lambda$ is symmetric and chirality preserving at the same time iff $|\Lambda| \equiv m \pmod{4}$. One further checks by induction, using the anticommutation rules,

$$\gamma_\lambda \gamma_\Lambda \gamma^\lambda \equiv (-1)^{|\Lambda|} (n - 2|\Lambda|) \gamma_\Lambda,$$

which in particular vanishes if $|\Lambda| = m$.

2. Fierz Rearrangements, Pauli-Kofink Identities

We now use the usual matrix notation, but note that no special kind of basis in S is being employed. Let M, L be any two linear operators on S , and consider, with spinors $\varphi, \psi, \omega, \chi$, the invariants $\varphi^T B M \psi$ and $\omega^T B L \chi$. A *Fierz rearrangement* of their product arises if in

$$\varphi^T B M \underbrace{\psi \omega^T B}_{F} L \chi$$

we replace the (rank one) operator $F := \psi \omega^T B$ by its expansion given above for any F

$$(\varphi^T B M \psi)(\omega^T B L \chi) \equiv 2^{-m} \sum_{\Lambda} (\omega^T B \gamma_\Lambda^{-1} \psi) (\varphi^T B M \gamma_\Lambda L \chi).$$

(Other versions are obtained by including M or L or both into F .)

Pauli-Kofink type identities are special cases where M, L are replaced by $\gamma_{[\mu_1 \dots \mu_p]}$, $\gamma_{[\nu_1 \dots \nu_q]}$ and some contractions are made between the μ_i and ν_j . (The case $p = q = 2$ with one contraction between them occurs in the Calabi-Yau compactification.)

The specific type of identity to be considered here arises if we take

$$M = \gamma_\mu, \quad L = \gamma^\mu, \quad \psi = \omega = \text{chiral}$$

and φ, χ of chirality $(-1)^{m+1}$ relative to ψ . Then the nonvanishing terms on the right will have $|\Lambda| \equiv m \pmod{4}$, the $|\Lambda| = m$ term dropping out by a remark above. Also, by chirality, it turns out that we can restrict the Λ -sum to $|\Lambda| \leq m - 4$ but taken twice. The final result is

$$(-2)^{m-4} (\varphi^T B \gamma_\mu \psi) (\psi^T B \gamma^\mu \chi) \equiv \sum_{\Lambda} \frac{m - |\Lambda|}{4} (\varphi^T B \gamma_\Lambda^{-1} \psi) (\varphi^T B \gamma_\Lambda \chi),$$

where the sum is over $\Lambda \subset N$ having $|\Lambda| = m - 4, m - 8, \dots (\geq 0)$.

Special Cases

We discuss our identity for some special values of $n = 2m$, using semispinor index notation. B will be used as a spinor metric to move spinor indices. If B preserves chirality, it defines separate metrics on S_+ and S_- ; if B changes chirality, it converts upper dotted (undotted) semispinor indices into lower undotted (dotted) ones.

[$m = 2$.] A well-known formula from 2-component spinor algebra arises.

[$m = 3$.] The quantities $\gamma^{\mu AB}$, m are antisymmetric, and the identity, being equivalent to

$$\gamma_{\mu A(B} \gamma^{\mu}_{C)D} \equiv 0, \quad \gamma^{\mu A(B} \gamma^{\mu}_{C)D} \equiv 0,$$

tells us that $\gamma_{\mu AB} \gamma^{\mu}_{CD}$ and $\gamma^{\mu AB} \gamma^{\mu}_{CD}$ are totally antisymmetric. This is the basic structure of twistor algebra (ε -twistors).

[$m = 4$.] This is the first (even-dimensional) case with a nonzero term on the right-hand side of the identity. The γ^{μ}_{ab} are symmetric, B is, again for the first time, symmetric and chirality preserving, defining separate metrics on $S_{\pm}$ which both are of dimension 8 just as V is. The identity becomes the pair

$$\begin{aligned} \gamma^{\mu \dot{A} B} \gamma^{\mu}_{\dot{D} C} + \gamma^{\mu \dot{A} C} \gamma^{\mu}_{\dot{D} B} &\equiv 2B_{BC} \delta^{\dot{A}}_{\dot{D}} \\ \gamma^{\mu \dot{A} \dot{B}} \gamma^{\mu}_{C \dot{D}} + \gamma^{\mu \dot{A} \dot{D}} \gamma^{\mu}_{C \dot{B}} &\equiv 2B_{\dot{B}\dot{D}} \delta^{\dot{A}}_C. \end{aligned}$$

When the basic anticommutation relations are also split into semispinor parts, one recognizes that the members of the triple (V, g) , $(S_+, B|S_+)$, $(S_-, B|S_-)$ are all on the same footing, e.g., V , S_+ are the semispinors for S_- , etc. ("Principle of Triality"). Putting $\varphi = \chi$, one obtains Brioschi's formula on using bases that diagonalize B .

[$m = 5$.] B is antisymmetric and mixing, $B\gamma_{\mu}$ is symmetric and splitting. Our identity is, semispinorially, equivalent to

$$\gamma_{\mu \dot{A}(\dot{B}} \gamma^{\mu}_{\dot{C} \dot{D})} = 0 = \gamma_{\mu A(B} \gamma^{\mu}_{C D)}.$$

These relations were termed "purity syzygies" by Hughston and Ward.

Pure Spinors

For a chiral spinor ψ , consider the subspaces of V given by

$$\begin{aligned} V_{\psi} &:= \{x = x^{\mu} e_{\mu} | x^{\mu} = \varphi^T B \gamma_{\mu} \psi, \varphi \in S\} \\ N_{\psi} &:= V_{\psi}^{\perp} = \{x | x^{\mu} \gamma_{\mu} \psi = 0\}, \end{aligned}$$

where $\{e_{\mu}\}$ is an orthonormal basis of V . From $0 = x^{\mu} \gamma_{\mu} x^{\nu} \gamma_{\nu} \psi = g_{\mu\nu} x^{\mu} x^{\nu} \psi$ one has that N_{ψ} is totally null, $N_{\psi} \subset N_{\psi}^{\perp} = V_{\psi}$, and V_{ψ} is null. ψ is called *pure* if N_{ψ} is maximal, i.e. $\dim N_{\psi} = m$; then $N_{\psi} = V_{\psi}$, and our identity implies:

ψ is pure iff $\psi^T B \gamma_{\Lambda} \psi = 0$ for $|\Lambda| = m - 4, m - 8, \dots$

Our identity also implies — on putting $\chi = \psi$ — that from $m = 5$ on there are certain identical relations satisfied by the $\psi^T B \gamma_{\Lambda} \psi$, to which further ones can be added by applying the Pauli-Kofink process to all products $(\varphi^T B \gamma_{\Lambda}^{-1} \psi)(\psi^T B \gamma_{\Lambda} \psi)$. These are the *purity syzygies* whose existence explains a discrepancy in the correspondence between the number of linearly independent purity conditions and the fact that the dimension of each of the two families of maximal totally null subspace of V equals $\frac{1}{2}m(m - 1)$.

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Two-Dimensional Sen Connections

L.B. Szabados

Abstract: The 2-dimensional version of the Sen connection is constructed for spacelike 2-surfaces. Two spinorial identities between the 2-dimensional Sen-Witten and the twistor operators and various quasi-local integrals of the curvature and the torsion of this Sen connection are derived.

1. Introduction

The (3-dimensional) Sen connection [1, 2] is known to play an important role in general relativity, e.g. in Ashtekar's canonical formulation [3] or in the energy-momentum and angular momentum problems [4–6] of general relativity. The main points of the construction of the Sen connection are the following. Let Σ be a spacelike hypersurface with unit future normal t^a . This defines a projection to Σ : $P_b^a := \delta_b^a - t^a t_b$.[†] By means of P_b^a one can define the induced (negative definite) metric h_{ab} and the Levi-Civita covariant differentiation operator D_a and define $\mathcal{D}_a := P_b^a \nabla_b$. The normal t^a , in its spinorial form, defines an isomorphism of the space $\bar{S}$ of primed spinors onto the space S of unprimed ones: $\bar{S} \rightarrow S : \lambda^{A'} \mapsto -\sqrt{2} t^A{}_{A'} \lambda^{A'}$. One can therefore convert all the primed indices into unprimed, and, instead of complex conjugation, one has the adjoint operation: $^+ : S \rightarrow S : \lambda^A \mapsto -\sqrt{2} t^A{}_{A'} \bar{\lambda}^{A'}$. If $\chi_{ABCD} := 2 t^{A'}{}_{B'} t^{C'}{}_{D'} D \chi_{AA'CC'}$ is the unprimed spinor form of the extrinsic curvature of Σ then

$$\mathcal{D}_{AB} \lambda_C = D_{AB} \lambda_C + \frac{1}{\sqrt{2}} \chi_{ABC}{}^D \lambda_D.$$

With these notations the Sen identity [2, 4] is

$$\begin{aligned} (\mathcal{D}^A{}_B \lambda^B)^+ (\mathcal{D}_{AC} \lambda^C) &= D_{AB} \left(\lambda^{+B} \mathcal{D}^A{}_C \lambda^C - \frac{1}{2} \lambda_C^+ \mathcal{D}^{AB} \lambda^C \right) + \\ &+ \frac{1}{2} (\mathcal{D}^{AB} \lambda^C)^+ (\mathcal{D}_{AB} \lambda_C) - \frac{1}{2\sqrt{2}} G_{aBB'} t^a \lambda^B \bar{\lambda}^{B'}, \end{aligned}$$

where G_{ab} is the Einstein tensor. If Σ extends to the spacelike or null infinity in an asymptotically flat space-time and λ^A is an asymptotically constant solution of the Sen-Witten equation $\mathcal{D}_{AB} \lambda^B = 0$ then, provided the dominant energy condition holds, the positivity of the ADM or Bondi - Sachs masses follows from this identity [4, 5]. Thus $\mathcal{D}_{AB}$, the so-called Sen operator, seems to be the “natural” differential operator in these problems. It is important that both the intrinsic and extrinsic geometrical properties of Σ are coded in $\mathcal{D}_{AB}$.

If we would like to define the gravitational energy, momentum and angular momentum quasi-locally; i.e. to associate these quantities to closed spacelike 2-submanifolds $\mathcal{S}$ in M , then an appropriate formalism for the calculations on $\mathcal{S}$ is needed. Such a formalism may be the well known Geroch-Held-Penrose formalism [7], which is not covariant in its form. But the covariance of a formalism may help and suggest why and what to calculate. In the present contribution we try to develop a covariant formalism which might be considered as the 2-dimensional version of the Sen connection.

In the next section we discuss the algebra of surface spinors. The 2-dimensional version of the Sen operator, its “curvature” and “torsion” and the induced 2-dimensional spin connection will be discussed in section 3. Two spinorial identities will also be derived.

[†] The space-time signature is assumed to be $+- - -$.

2. The Algebra of 2-surface Spinors

Let (M, g) be a 4-dimensional Lorentzian geometry with signature -2 and $\S$ be a 2-dimensional spacelike submanifold. Let t^a and v^a be orthogonal timelike and spacelike unit normals to $\S$: $t^a t_a = 1$, $v^a v_a = -1$ and $t^a v_a = 0$.[†] The vectors t^a and v^a are, of course, not unique; there is the gauge freedom,

$$\begin{aligned} t'^a &= t^a \cosh u + v^a \sinh u \\ v'^a &= t^a \sinh u + v^a \cosh u \end{aligned} \quad (2.1)$$

which we call a boost gauge transformation. If $t^{AA'}$ and $v^{AA'}$ are the spinor forms of the normals to $\S$ then

$$2t^{AR'} t_{BR'} = \delta_B^A, \quad 2v^{AR'} v_{BR'} = -\delta_B^A. \quad (2.2)$$

Let us define

$$\gamma^A_B := 2t^{AR'} v_{BR'}. \quad (2.3)$$

It is easy to show that γ^A_B is boost gauge independent, and that

$$\gamma^R_R = 0, \quad \gamma^A_R \gamma^R_B = \delta_B^A. \quad (2.4)$$

Thus γ^A_B is non-degenerate and $\gamma : S \rightarrow S : \lambda^A \mapsto \gamma^A_R \lambda^R$ is an isomorphism of the spinor space S onto itself. Its eigenvalues are ± 1 , and hence γ^A_B plays a role similar to the γ_5 matrix. Its eigenspinors may be called left handed and right handed with respect to γ^A_B . γ_{AB} can also be considered as a complex metric of signature $(+ -)$ on S . The null spinors of the metric γ_{AB} are just the eigenspinors of γ^A_B , and hence $(\lambda^A, \gamma^A_R \lambda^R)$ is an independent system iff λ^A is not a null spinor. The group leaving invariant the complex metric γ_{AB} is isomorphic to $Z_2 \times C$. As a consequence of the existence of the *extra* structure γ_{AB} on S the decomposition $\phi_{AB} = \frac{1}{2} \varepsilon_{AB} \phi^R_R + \phi_{(AB)}$ is no longer irreducible. Its symmetric part can be decomposed further as

$$\phi_{(AB)} = -\frac{1}{2} \gamma_{AB} \gamma^{RS} \phi_{RS} + \left(\phi_{(AB)} + \frac{1}{2} \gamma_{AB} \gamma^{RS} \phi_{RS} \right),$$

the sum of the γ -trace and the trace-free symmetric part of ϕ_{AB} . The spinor form of the induced volume form on $\S$ is given by

$$\begin{aligned} t^a v^b \varepsilon_{abcd} &= t^{AA'} v^{BB'} i \left(\varepsilon_{AC} \varepsilon_{BD} \varepsilon_{A'D'} \varepsilon_{B'D'} - \varepsilon_{AD} \varepsilon_{BC} \varepsilon_{A'C'} \varepsilon_{B'D'} \right) = \\ &= \frac{i}{2} \left(\gamma_{CD} \varepsilon_{C'D'} - \bar{\gamma}_{C'D'} \varepsilon_{CD} \right). \end{aligned} \quad (2.5)$$

The elements of the spinor space S will be called spacelike 2-dimensional spinors if a spinor γ^A_B satisfying (2.4) is given on S .

Let (o_A, ι_A) be a normalized spinor dyad such that

$$t^a = \frac{1}{\sqrt{2}} (o^A \bar{o}^{A'} + \iota^A \bar{\iota}^{A'}), \quad v^a = \frac{1}{\sqrt{2}} (o^A \bar{o}^{A'} - \iota^A \bar{\iota}^{A'}). \quad (2.6)$$

[†] If $\S$ is orientable and an open neighbourhood of $\S$ in M is space and time orientable then t^a and v^a are globally well defined.

Then $\gamma^A_B = o^A \iota_B + \iota^A o_B$ and hence

$$\gamma^A_B o^B = -o^A, \quad \gamma^A_B \iota^B = \iota^A. \quad (2.7)$$

Thus o^A and ι^A are null spinors, o^A being left handed and ι^A right handed. The linear isomorphism $\gamma : \mathbf{S} \rightarrow \mathbf{S}$ can obviously be extended to the whole tensor algebra over $\mathbf{S}$. Its action on the vectors of the complex null tetrad is

$$\begin{aligned} \gamma(l^a) &= \gamma(o^A) \bar{\gamma}(\bar{o}^{A'}) = l^a, \\ \gamma(n^a) &= \gamma(\iota^A) \bar{\gamma}(\bar{\iota}^{A'}) = n^a, \\ \gamma(m^a) &= \gamma(o^A) \bar{\gamma}(\bar{\iota}^{A'}) = -m^a. \end{aligned} \quad (2.8)$$

Therefore a vector X^a is tangent to $\mathcal{S}$ iff $\gamma(X^a) = -X^a$ and X^a is orthogonal to $\mathcal{S}$ iff $\gamma(X^a) = X^a$. The spinor form of the projection $\Pi_b^a := \delta_b^a - t^a t_b + v^a v_b$ is therefore

$$\Pi_b^a = \frac{1}{2} (\delta_B^A \delta_{B'}^{A'} - \gamma^A_B \bar{\gamma}^{A'}_{B'}), \quad (2.9)$$

and hence the induced metric on $\mathcal{S}$ is $q_{AA'B'B'} = \frac{1}{2} (\varepsilon_{AB} \varepsilon_{A'B'} - \gamma_{AB} \bar{\gamma}_{A'B'})$.

The spinor γ^A_B can be defined intrinsically too. If $x^{AA'}$ and $y^{AA'}$ are q_{ab} -orthonormal vectors tangent to $\mathcal{S}$ then $J^A_B := 2x^{AR'} y_{BR'} = i\gamma^A_B$. J^A_B , and hence γ^A_B also, are rotation gauge invariant. The fact that γ^A_B is globally well defined is a consequence of its definition (2.3) and the globality of t^a and v^a . If we define γ^A_B by the intrinsic properties of $\mathcal{S}$ then, by definition, it will not be globally well defined. However, using its rotation gauge invariance, it is easy to show that it is, in fact, globally well defined for orientable $\mathcal{S}$.

Finally, for any spinor λ^A let us define

$$\begin{aligned} T^a &:= \frac{1}{2} (\lambda^A \bar{\lambda}^{A'} + \gamma^A_R \lambda^R \bar{\gamma}^{A'}_{R'} \bar{\lambda}^{R'}), \\ V^a &:= \frac{1}{2} (\lambda^A \bar{\gamma}^{A'}_{R'} \bar{\lambda}^{R'} + \gamma^A_R \lambda^R \bar{\lambda}^{A'}), \\ X^a &:= \frac{1}{2} (\lambda^A \bar{\lambda}^{A'} - \gamma^A_R \lambda^R \bar{\gamma}^{A'}_{R'} \bar{\lambda}^{R'}), \\ Y^a &:= \frac{i}{2} (\gamma^A_R \lambda^R \bar{\lambda}^{R'} - \lambda^A \bar{\gamma}^{A'}_{R'} \bar{\lambda}^{R'}). \end{aligned} \quad (2.10)$$

They are mutually orthogonal real vectors, satisfying

$$T^a T_a = -X^a X_a = -Y^a Y_a = -V^a V_a = \frac{1}{2} |\gamma_{AB} \lambda^A \lambda^B|^2 \quad (2.11)$$

$$\varepsilon_{abcd} T^a X^b Y^c V^d = \frac{1}{4} |\gamma_{AB} \lambda^A \lambda^B|^4. \quad (2.12)$$

Thus $\{T^a, X^a, Y^a, V^a\}$ is a future directed, right-handed orthogonal vector basis unless λ^A is a null spinor. X^a and Y^a are tangent to $\mathcal{S}$ while T^a and V^a are orthogonal to it. If λ^A becomes null then X^a, Y^a will be zero and T^a and V^a parallel null vectors. Furthermore,

$$M^a := X^a - iY^a = \frac{1}{2} (\delta^A_R + \gamma^A_R) (\delta^{A'}_{R'} - \bar{\gamma}^{A'}_{R'}) \lambda^R \bar{\lambda}^{R'} \quad (2.13)$$

is a complex, null vector tangent to $\mathcal{S}$, and $M^a \bar{M}_a = -|\gamma_{AB} \lambda^A \lambda^B|^2$.

3. The 2-dimensional Sen and Spin Connections

Let us define $\Delta_a := \Pi_a^b \nabla_b$, which may be called the 2-dimensional Sen operator. If we define

$$Q^e_{cd} := -\Pi_r^e \Delta_c \Pi_d^r, \quad (3.1)$$

then the action of the commutator of the Δ_a 's on functions and on spinor fields is

$$(\Delta_{CC'} \Delta_{DD'} - \Delta_{DD'} \Delta_{CC'}) \phi = -2Q^e_{[cd]} \Delta_e \phi, \quad (3.2)$$

$$(\Delta_{CC'} \Delta_{DD'} - \Delta_{DD'} \Delta_{CC'}) \xi^A = -R^A_{Bef} \Pi_{CC'}^e \Pi_{DD'}^f \xi^B - 2Q^e_{[cd]} \Delta_e \xi^A. \quad (3.3)$$

Thus the "torsion" of the operator Δ_a is not zero, but $2Q^e_{[cd]}$, while the "curvature" is the pull-back to $\mathcal{S}$ of the anti selfdual part of the space-time curvature. In terms of the Weyl and Ricci spinors and the Λ scalar it is given by

$$\begin{aligned} F_{ABCC'DD'} &:= R_{ABef} \Pi_{CC'}^e \Pi_{DD'}^f = \\ &= -\frac{1}{4} \varepsilon_{C'D'} (\psi_{ABCD} - \psi_{ABEF} \gamma^E_C \gamma^F_D + \gamma_{CD} \phi_{ABE'F'} \bar{\gamma}^{E'}_{F'} + \\ &\quad + \Lambda (\varepsilon_{AC} \varepsilon_{BD} + \varepsilon_{AD} \varepsilon_{BC} - \gamma_{AC} \gamma_{BD} - \gamma_{AD} \gamma_{BC})) - \\ &\quad - \frac{1}{4} \varepsilon_{CD} (\bar{\gamma}_{C'D'} \psi_{ABEF} \gamma^{EF} + \phi_{ABC'D'} - \phi_{ABE'F'} \bar{\gamma}^{E'}_{C'} \bar{\gamma}^{F'}_{D'} + \\ &\quad + 2\Lambda \gamma_{AB} \bar{\gamma}_{C'D'}). \end{aligned} \quad (3.4)$$

Its contraction with the induced volume form is

$$F_{ABcd} t^e v^f \varepsilon_{ef}^{cd} = -i (\psi_{ABCD} \gamma^{CD} - \phi_{ABC'D'} \bar{\gamma}^{C'D'} + 2\Lambda \gamma_{AB}). \quad (3.5)$$

To find the spinor form of the "torsion" and the expression of the "curvature" in terms of extrinsic and intrinsic geometrical quantities, let us define

$$Q^E_{aF} := \frac{1}{2} \Delta_a \gamma^E_R \gamma^R_F. \quad (3.6)$$

Obviously $Q^R_{aR} = 0$, and, by the definition (3.1) of Q^e_{ab} and the spinor form of the projection,

$$Q^e_{ab} = \frac{1}{2} (\delta_{B'}^E Q^E_{aB} + \delta_B^E \bar{Q}^{E'}_{aB'} + Q^E_{aR} \gamma^R_B \bar{\gamma}^{E'}_{B'} + \bar{Q}^{E'}_{aR'} \bar{\gamma}^{R'}_{B'} \gamma^E_B). \quad (3.7)$$

Since $Q_{eab} = Q_{aeb}$ by the hypersurface orthogonality of t^a and v^a , Q^E_{aF} has a "hidden" symmetry,

$$Q_{EAA'F} = \frac{1}{2} (\delta_{A'}^{R'} \delta_F^R - \bar{\gamma}^{R'}_{A'} \gamma^R_F) Q_{AER'R} + \frac{1}{2} (\varepsilon_{AF} \varepsilon^{E'F'} - \gamma_{AF} \bar{\gamma}^{E'F'}) \bar{Q}_{A'EE'F'}. \quad (3.8)$$

The spinor form of the "torsion" is therefore

$$T_{EE'AA'BB'} = -(\varepsilon_{A'B'} Q_{AEE'B} + \varepsilon_{AB} \bar{Q}_{A'EE'B'}), \quad (3.9)$$

and the expression for the “curvature” is

$$\begin{aligned}
 F^A_{BCC'DD'} = & \frac{1}{2} {}^8R^{AA'}{}_{BA'cd} - \frac{1}{2} \gamma^A{}_B (\delta_c A_d - \delta_d A_c) + \Delta_{DD'} Q^A{}_{CC'B} - \\
 & - \Delta_{CC'} Q^A{}_{DD'B} + Q^A{}_{CC'R} Q^R{}_{DD'B} - Q^A{}_{DD'R} Q^R{}_{CC'B} + \\
 & + Q^A{}_{RR'B} \left(\delta_{C'}^R Q^R{}_{DD'C} + \delta_C^R \bar{Q}^{R'}{}_{DD'C'} - \delta_{D'}^R Q^R{}_{CC'D} - \right. \\
 & \left. - \delta_D^R \bar{Q}^{R'}{}_{CC'D'} \right), \tag{3.10}
 \end{aligned}$$

where δ_c is the induced Levi-Civita covariant derivative, ${}^8R_{AA'BB'cd}$ is the spinor form of its curvature tensor, and

$$A_c := \Pi_c^r \nabla_r t_s v^s. \tag{3.11}$$

A_c seems to be an $SO(1,1)$ gauge field on $\mathbb{S}$ and, under the boost gauge transformation (2.1), transforms as $A'_c = A_c - \delta_c u$.

To motivate how to define the action of δ_a on spinors, let us consider the δ_a derivative of the complex, null surface vector M^a , defined by (2.13),

$$\begin{aligned}
 \delta_a M^b &= \Delta_a M^b - \Delta_a \Pi_f^b M^f = \\
 &= \frac{1}{2} \left(\Delta_a (\lambda^B + \gamma^B{}_R \lambda^R) - \frac{1}{2} \Delta_a \gamma^B{}_R (\lambda^R + \gamma^R{}_S \lambda^S) \right) (\bar{\lambda}^{B'} - \bar{\gamma}^{B'}{}_{R'} \bar{\lambda}^{R'}) + \\
 &+ \frac{1}{2} (\lambda^B + \gamma^B{}_R \lambda^R) \left(\Delta_a (\bar{\lambda}^{B'} - \bar{\gamma}^{B'}{}_{R'} \bar{\lambda}^{R'}) + \frac{1}{2} \Delta_a \bar{\gamma}^{B'}{}_{R'} (\bar{\lambda}^{R'} - \bar{\gamma}^{R'}{}_{S'} \bar{\lambda}^{S'}) \right). \tag{3.12}
 \end{aligned}$$

It seems natural to define the action of δ_a on spinors by

$$\delta_a (\lambda^B \pm \gamma^B{}_R \lambda^R) := \Delta_a (\lambda^B \pm \gamma^B{}_R \lambda^R) \mp \frac{1}{2} \Delta_a \gamma^B{}_R (\lambda^R \pm \gamma^R{}_S \lambda^S), \tag{3.13}$$

that is by

$$\delta_a \lambda^B := \Delta_a \lambda^B - Q^B{}_{aR} \lambda^R. \tag{3.14}$$

δ_a annihilates both the symplectic and complex metrics,

$$\delta_a \varepsilon_{RS} = 0, \quad \delta_a \gamma_{RS} = 0. \tag{3.15}$$

Using (3.14), the commutator (3.3) and (3.3-4) one has the following identity for any two spinor fields λ^A and μ^A ,

$$\begin{aligned}
 \frac{1}{2} \bar{\gamma}^{A'B'} (\Delta_{A'A} \lambda^A) (\Delta_{B'B} \mu^B) &= \delta_{AA'} \left(\bar{\gamma}^{A'B'} \lambda_B \Delta_{B'}^A \mu^B \right) - \\
 &- \bar{\gamma}^{A'B'} (\Delta_{A'(A} \lambda_{B)}) (\Delta_{B'}^A \mu^B) - \frac{i}{2} \lambda^A \mu^B R_{ABcd} t^e v^f \varepsilon_{ef}{}^{cd}. \tag{3.16}
 \end{aligned}$$

This is the 2-dimensional version of the Sen identity [1, 2, 4, 5]. Apart from the total divergence, it is a relation between the first derivatives of the spinor fields and the curvature, which is actually the integrand of Penrose’s quasi-local charge integrals.

We have a second identity too. Using (3.14), (3.2-3.5) and (3.9),

$$\begin{aligned}
 \frac{3}{2} \varepsilon^{A'B'} (\Delta_{A'A} \lambda^A) (\Delta_{B'B} \mu^B) &= \delta_{RR'} \left(\lambda^R \Delta_S^{R'} \mu^S + \lambda^S \Delta_S^{R'} \mu^R \right) + \\
 &+ \varepsilon^{A'B'} (\Delta_{A'(A} \lambda_{B)}) (\Delta_{B'}^{(A} \mu^{B)}) - \frac{i}{2} \lambda^A \mu^B \gamma^K{}_A R_{K B c d} t^e{}_v f^c{}_e f^d{}_v - \\
 &- \frac{1}{2} \left(2 \varepsilon^{R A} \lambda^B \Delta_S^{R'} \mu^S - \lambda^R \Delta^{R'(A} \mu^{B)} - \lambda^B \Delta^{R'(A} \mu^{R)} - \right. \\
 &\quad \left. - \varepsilon^{R A} \lambda_S \Delta^{R'(S} \mu^{B)} \right) \varepsilon^{A'B'} T_{RR' AA' BB'}.
 \end{aligned} \tag{3.17}$$

The second derivatives of the spinor fields again appear only in the form of a total divergence. However, in contrast to the identity (3.16), (3.17) also contains the "torsion" T_{rab} of the operator Δ_a . The original Sen identity contains a single spinor field, and moreover the 3-dimensional counterpart of our second identity does not seem to be known. That seems to be related to the angular momentum of the gravitating system, and will be discussed in a separate paper. $\Delta_{A'A} \lambda^A = 0$ may be called the 2-dimensional Sen-Witten equation, and one can show that the vanishing of the trace-free part of $\Delta_{A'(A} \lambda_{B)}$ is just the 2-surface twistor equation. Thus, apart from total divergences, (3.16) and (3.17) are identities between the 2-dimensional Sen-Witten and twistor operators and various quasi-local integrals of the "curvature" and the "torsion" of Δ_a , for any pair of spinor fields. A more detailed discussion and the application of these identities will be published elsewhere.

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Rarita-Schwinger Fields from Supercovariantly Constant Spinors

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Abstract: Non-gauge solutions to the Rarita-Schwinger equation on space-times that admit supercovariantly constant spinors are constructed. As an example we derive spin-3/2 perturbations on the extreme Reissner-Nordström metric. We also give a relation between supercharge and first-order perturbations in mass and charge of the background.

1. The Method

Consider a triplet (M, e^a, Φ) , where M is an n -dimensional, pseudo-Riemannian manifold possessing a spinor structure, e^a is an orthonormal frame field on M and Φ denotes collectively some bosonic fields. We wish to consider (M, e^a, Φ) as a background space-time and examine the spin-3/2 excitations which are governed by the Rarita-Schwinger equations on this background. The Rarita-Schwinger equations may be written as

$$\gamma \wedge \hat{D}\psi = 0, \quad (1.1)$$

where the spin-3/2 fields are represented by the spinor-valued one-forms $\psi = \psi_a e^a$ and their possible internal symmetry indices are omitted for notational convenience. The symbol $\hat{D}$ denotes the supercovariant exterior differential operator on M which involves the torsion-free connection one-form $\omega^{ab}(e^c)$ and the matter fields Φ . Although the precise form of $\hat{D}$ is dictated by the particular supersymmetric coupling of the fields, it is always defined in such a way that the spin-3/2 fields having the form

$$\psi = \hat{D}\alpha, \quad (1.2)$$

where α is a spinor-valued zero-form, automatically satisfy (1.1) by the virtue of the background field equations. With this property of $\hat{D}$ one ensures the supersymmetry invariance of the theory at the linearized level: The action of the supersymmetry transformation on ψ is to map the spin-3/2 fields, which are initially zero on the background, precisely into the form (1.2). Conversely, a solution (1.2) to the Rarita-Schwinger equations can always be reduced to zero by a change of the supergauge. Therefore, the solutions of (1.1) having the form (1.2) are called gauge-generated.

In supergravity theories the backgrounds (M, e^a, Φ) which admit nonzero supercovariantly constant spinors α_{sc} , satisfying

$$\hat{D}\alpha_{sc} = 0, \quad (1.3)$$

are of particular interest. In four space-time dimensions, the backgrounds where nontrivial solutions to (1.3) exist saturate a generalized Bogomolny bound [1]. Let us therefore consider a one-parameter family of backgrounds $(M_\lambda, e^a(\lambda), \Phi(\lambda))$ which has a limit $(M_0, e^a(0), \Phi(0))$ as $\lambda \rightarrow 0$ and which admits supercovariantly constant spinors in this limit. Choose λ to be real, positive and in such a manner that the expansion of $\hat{D}_\lambda$ for small λ yields

$$\hat{D}_\lambda = \hat{D}_0 + \lambda W + O(\lambda^2), \quad (1.4)$$

where $\hat{D}_0$ is the supercovariant derivative with respect to $(M_0, e^a(0), \Phi(0))$ and W is a certain matrix-valued one-form which is independent of λ . Consider next any spinor $\alpha(\lambda)$ defined with respect to $e^a(\lambda)$ which has, for small λ , the form

$$\alpha(\lambda) = \lambda^{-1} \alpha_{sc} + \beta + O(\lambda), \quad (1.5)$$

where α_{sc} is a solution to $\hat{D}_0 \alpha_{sc} = 0$ and β is a certain λ -independent spinor. Using such an $\alpha(\lambda)$ construct now a family of gauge-generated solutions to (1) on $(M_\lambda, e^a(\lambda), \Phi(\lambda))$. According to (1.2), (1.4) and (1.5) these gauge solutions have the form

$$\psi(\lambda) = \lambda^{-1} \hat{D}_0 \alpha_{sc} + W \alpha_{sc} + \hat{D}_0 \beta + O(\lambda) \quad (1.6)$$

for small λ , and by construction, the first term on the right-hand side is zero. Therefore, in the limit $\lambda \rightarrow 0$ this family goes into

$$\psi = \lim_{\lambda \rightarrow 0} \psi(\lambda) = W \alpha_{sc} + \hat{D}_0 \beta, \quad (1.7)$$

and consists of two pieces. The term $\hat{D}_0 \beta$ is manifestly a pure gauge solution on $(M_0, e^a(0), \Phi(0))$ and can be transformed away by a change of the supergauge. However, the first term $W \alpha_{sc}$ is in general *not* gauge-generated but nevertheless satisfies the Rarita-Schwinger equation at $\lambda = 0$. Note that the gauge spin or (1.5) does not exist at $\lambda = 0$. Moreover, the necessary conditions for $W \alpha_{sc}$ to be gauge-generated on $(M_0, e^a(0), \Phi(0))$ require

$$(\hat{D}_0 W) \alpha_{sc} = \hat{\Omega}_0 \varepsilon, \quad (1.8)$$

where ε is a certain gauge spinor and $\hat{\Omega}_0$ denotes the matrix of supercurvature two-forms of M_0 . In general, (1.8) cannot be satisfied for the bosonic background considered. Hence, starting from gauge-generated $\psi(\lambda)$ which are defined on $(M_\lambda, e^a(\lambda), \Phi(\lambda))$ it is possible to construct non-gauge spin-3/2 fields on a background which admits supercovariantly constant spinors.

2. Motivation

The study of spin-3/2 fields on curved space-times is motivated by the fundamental role that these fields play in supergravity theories. The invariance of a theory under local supersymmetry transformations gives rise to conserved spinor charges, which can be expressed as surface integrals at spatial infinity. It is therefore of some interest to look for black hole solutions which carry supercharge. However, the fact that black holes are highly selective with respect to their allowed charges has led to the so-called no-hair theorems.

Within the Einstein-Maxwell theory a generic stationary black hole is completely characterized by its mass, angular momentum, electric and magnetic charge.

This theory may be embedded into the $O(2)$ extended ($N = 2$) supergravity whose field content is the gravitational frame field e^a , the electromagnetic potential one-form $A = A_a e^a$ and two Majorana spinor-valued one-forms Ψ^j ($j = 1, 2$) combined into a complex (Dirac) field,

$$\Psi = \Psi^1 + i \Psi^2 = \Psi_a e^a. \quad (2.1)$$

Setting $\Psi = 0$ gives back Einstein-Maxwell theory. Thus any solution to Einstein-Maxwell equations is trivially a solution to the $N = 2$ theory.

A possible way to study the question whether black holes can carry supercharge is to consider perturbations of the spin-3/2 field on the given background. We wish to find solutions of (1.1) where

$$\hat{D} = d + \frac{1}{2}\omega^{ab}\sigma_{ab} - \frac{\kappa}{2}F^{ab}\sigma_{ab} \quad (2.2)$$

where d is the exterior derivative, and where the ω^{ab} are torsion-free connection one-forms,

$$\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b], \quad \gamma = \gamma_a e^a.$$

As backgrounds we take the family of Kerr-Newman black holes and demand that the Rarita-Schwinger field be

- (i) stationary
- (ii) flat at spatial infinity
- (iii) regular (on and outside the horizon)
- (iv) not of the gauge form, (1.2).

A detailed analysis (see [2]) of the spin-3/2 modes on the Kerr-Newman black holes has led to the result that there are no solutions to (1.1) that satisfy these conditions unless the background parameters are such that

$$m^2 = e^2 + q^2, \quad (2.3)$$

where m , e and q are the mass, electric and magnetic charge of the hole respectively, i.e. the charge of the hole equals its mass in units where $4\pi G = 1$. This singles out the extreme Reissner-Nordström (ERN) metric since the angular momentum has to vanish in order that a horizon exists.

Later it was shown [3, 4] that the extreme Reissner-Nordström black holes admit perturbations satisfying (i-iv). These perturbations are characterized by a nonzero supercharge. By a rather tedious calculation the linear solution was generalized to an exact one, taking into account the full back-reaction of the Rarita-Schwinger field onto the Einstein-Maxwell fields [5].

3. Application of the Method

The uniqueness of the ERN metric in supporting a spin-3/2 supercharge has a geometrical explanation: Among the Kerr-Newman black holes the ERN metric is singled out by the existence of supercovariantly constant spinors α_{sc} satisfying (1.3). These spinors give rise to additional gauge invariant RS modes [6].

In what follows we wish to apply the limiting procedure sketched at the beginning, to derive the mentioned RS solution.

Consider a family of Reissner-Nordström black holes with mass m and electric charge e . Only for $e^2 = m^2$ the background admits a supercovariantly constant spinor. Let us therefore choose

$$\lambda = m^2 - e^2 \geq 0 \quad (3.1)$$

and denote by $(e_{(\lambda)}^a, A^{(\lambda)})$ the family of Reissner-Nordström fields, which in isotropic coordinates (t, x^i) reads

$$\begin{aligned} e_{(\lambda)}^0 &= \left(1 - \frac{\lambda}{4r^2}\right) / \left(1 + \frac{m}{r} + \frac{\lambda}{4r^2}\right) dt, \\ e_{(\lambda)}^i &= \left(1 + \frac{m}{r} + \frac{\lambda}{4r^2}\right) dx^i, \\ A^{(\lambda)} &= \sqrt{m^2 - \lambda} \left(r + m + \frac{\lambda}{4r}\right), \end{aligned} \quad (3.2)$$

with $r^2 = (x^i)^2$.

For $\lambda = 0$ this becomes simply

$$e^0 = V^{-1} dt, \quad e^i = V dx^i, \quad A = -V^{-1} dt, \quad V = 1 + \frac{m}{r}. \quad (3.3)$$

On this background the supercovariantly constant spinor is given by

$$\alpha_{\text{sc}} = \frac{1}{\sqrt{V}} \begin{pmatrix} c \\ c \end{pmatrix}, \quad (3.4)$$

where c is any two-component complex constant spinor.

Consider the RS field obtained by the limiting procedure

$$\Psi = \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \hat{D}_{(\lambda)} \alpha_{\text{sc}} \quad (3.5)$$

(we use Weyl representation for the γ -matrices). If we write Ψ as

$$\Psi = \begin{pmatrix} w \\ v \end{pmatrix}, \quad (3.6)$$

where w, v are two-component spinor-valued one-forms, a straightforward calculation then leads to the terms that remain in the limit

$$w = \frac{1}{mr^2} \left(\frac{x^i \sigma_i}{V} dt + \frac{V}{2} x^i dx^i + i \varepsilon_{k\ell j} x^k \sigma_j dx^\ell \right) \frac{c}{\sqrt{V}}, \quad (3.7)$$

while v is obtained from w by replacing dt by $-dt$. From this one may read off Ψ (omitting the second term on the right hand side of (3.7), which is pure gauge)

$$\Psi = \frac{1}{Mr^3} (x^i \gamma_i e^0 - \varepsilon_{ijk} x^j \gamma_5 \gamma_k V^{-1} e^i) \alpha_{\text{sc}}. \quad (3.8)$$

The so constructed RS spinor satisfies (1.1). This solution was first given in [3] and rederived by the above method in [7] although in different coordinates and different gauge.

The nonzero supercharge associated by the solution (3.8) is

$$S = -\frac{1}{4\pi} \oint_{S_\infty^2} \gamma_5 \gamma \wedge \Psi = -\frac{2}{M} \alpha_{\text{sc}}. \quad (3.9)$$

4. An Integral Relation

Supercharge is only gauge invariant if the gauge spinor vanishes at spatial infinity. Suppose that $\alpha \rightarrow \alpha_0 \neq 0$ for $r \rightarrow \infty$ then this induces a pure gauge field with supercharge which can be expressed as in [8],

$$S = -\frac{1}{4\pi} \int_{S_\infty^2} \gamma_5 \gamma \wedge \hat{D}\alpha = (\gamma^a P_a^{\text{ADM}} - e + \gamma_5 q) \alpha_0, \quad (4.1)$$

where P^{ADM} , e and q are the ADM four momentum, electric and magnetic charge respectively of the background configuration. (A similar expression was used by Gibbons and Hull to show the Bogomolny inequality $m^2 \geq e^2 + q^2$, [1].) From (3.9) one defines the scalar supercharge,

$$S = \text{Re} \int \bar{\alpha} \gamma_5 \gamma \wedge \hat{D}\alpha. \quad (4.2)$$

If, on the ERN background we take $\alpha = \alpha_{\text{sc}}$ then evidently S is zero and using the right hand side of (4.1) tells us that $m^2 = e^2$ with $q = 0$.

New information is obtained by applying the limiting procedure to (4.1) and (4.2) (see [9]). Consider the scalar supercharge given by

$$S = \text{Re} \int \bar{\alpha}_{\text{sc}} \gamma_5 \gamma \wedge \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \hat{D}_{(\lambda)} \alpha_{\text{sc}} = \lim_{\lambda \rightarrow 0} \frac{2c\bar{c}}{\lambda} (m(\lambda) - e(\lambda)), \quad (4.3)$$

where the specific form of α_{sc} given in (3.4) has been used. $m(\lambda)$ and $e(\lambda)$ are mass and charge of the non-extreme RN background. If we expand these quantities around their extreme values,

$$\begin{aligned} m(\lambda) &= m(0) + \lambda \delta m + O(\lambda^2), \\ e(\lambda) &= e(0) + \lambda \delta e + O(\lambda^2), \end{aligned} \quad (4.4)$$

with

$$m(0) = |e(0)|$$

then the right-hand side of (4.3) becomes

$$S = 2c\bar{c}(\delta m - \delta e). \quad (4.5)$$

One sees that there is a supercharge associated with the first-order perturbation in mass and charge of the ERN background.

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On Quasinormal Modes

B.G. Schmidt

I. Introduction — Normal Modes

Before discussing quasinormal modes, it is useful to remember what normal modes are! Compact linear oscillating systems (e.g. strings, membranes, or cavities) have preferred time harmonic states of motion.

Gravitational Radiation

If dissipation is neglected, there is generally an infinite collection of such periodic solutions, and the "general solution" can be expressed as a superposition

$$\Phi(t, x) = \sum_{\omega} a_{\omega} e^{-i\omega t} \phi_{\omega}(x) \quad (1.1)$$

of such normal modes. The simplest example is a string of length L which is fixed at the ends.

All such systems can be described by an equation of the type (Φ may be a vector)

$$\frac{\partial \Phi}{\partial t} = A \Phi \quad (1.2)$$

where A is a linear differential operator acting only on the spatial variables. A Hilbert space is chosen such that the differential operator becomes symmetric. Due to the boundary conditions dictated by the physical problem, A becomes a self-adjoint operator on the appropriate Hilbert space and has a pure point spectrum. The eigenfunctions and eigenvalues determine the periodic solutions (1.1).

The definition of self-adjointness is rather subtle from a physicist's point of view because fairly complicated domain issues play an essential role here [1] where a mathematical exposition for physicists is given). We have to decide which class of solutions we want to consider. The wave equation modeling the finite string had solutions of various degrees of differentiability. To describe all "realistic situations", clearly C^∞ functions would be sufficient. Sometimes it may, however, also be convenient to consider more general solutions.

From the mathematical point of view, the collection of all smooth functions is not a natural setting to study the wave equation because sequences of solutions exist which converge to non-smooth functions. To establish such powerful statements like (1.1) one has to study the equation on certain subsets of the Hilbert space of square integrable functions. For "nice" equations it usually happens that the eigenfunctions are, in fact, analytic. They can then be used to generate, for example, all smooth solutions by a pointwise converging sequence (1.1).

The key point is that we need some mathematical regularity to obtain the "completeness property" of the eigenfunctions.

All this changes when we consider "open systems" which can lose energy to infinity. The simplest case is waves on an infinite string. The general solution of this problem is

$$\Phi(x, t) = A(x - t) + B(x + t) \quad (1.3)$$

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$$\Phi_n(t, x) = e^{i\omega_n t} \phi_n(x), \quad n = 1, 2, 3 \dots \quad (1.1)$$

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$$\Phi(t, x) = \sum_{n=1}^{\infty} a_n e^{i\omega_n t} \phi_n(x) \quad (1.2)$$

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$$\Phi(t, x) = A(t - x) + B(t + x) \quad (1.4)$$

with "arbitrary" functions A and B . Which solutions should we study? Since we have all solutions, this is not a serious question. In more general cases in which the general solution is not known, we have to select a certain class of solutions which we consider as relevant for the physical problem.

Let us, as an example, consider the scattering of waves (scalar or electromagnetic) at a fixed obstacle in space. This is the subject of the "Lax Phillips scattering theory" [2]. In this framework one studies finite energy solutions to the wave equation outside a fixed domain, together with boundary conditions at the obstacle, such that the total energy is conserved (an example is the complete reflection of the waves).

Again there are many solutions. We can consider solutions for which the fields are spatially compact or — the other extreme — solutions with infinite total energy for which the fields grow at spatial infinity in a quite arbitrary way!

From the point of view of physics smooth solutions with spatially compact support should be the relevant class — who cares what happens near infinity! Again it turns out that mathematically it is more convenient to study all solutions of finite total energy. Then the relevant operator is again self-adjoint, but now its spectrum is purely "continuous". There are no eigenfunctions which are square integrable. Only "improper eigenfunctions" like plane waves exist. In more physical terms the purely continuous spectrum means that there is no periodic solution of finite energy! The representation of the general solution — the analogue of (1.2) — is an integral over "improper eigenfunctions". A representation by a discrete sum is not possible.

Where are the quasinormal modes? We just saw that they cannot possibly be eigenvalues and eigenfunctions of some self-adjoint operator. The next section deals with the issue of defining quasinormal modes for linearized fields propagating on the static Schwarzschild space-time. In Section 3 the meaning and physical relevance of quasinormal modes are explained. Section 4 contains a brief discussion of the quasinormal modes of relativistic stellar models. In an appendix it is sketched that quasinormal modes show up as resonances in quantum mechanics.

2. Quasinormal Modes of the Schwarzschild Space-time

"Quasinormal modes of the Schwarzschild space-time" were first considered by Vishveshwara [3] in 1970. I do not want to discuss the following development in detail. I just list some of the very early and some of the recent papers I am aware of and apologize for the incompleteness [4–16].

What is the problem addressed in the investigations of quasinormal modes of the Schwarzschild space-time? One studies the propagation of linear test fields Φ on the static part of the Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \quad (2.1)$$

to understand the scattering, reflection and absorption of the linear field by the black hole. The cases usually considered are scalar fields, electromagnetic fields, and linearized gravitation. All three cases can, by introducing potentials, be reduced to a wave equation for one function [17,18]. Using the radial coordinate

$$x = r + \ln\left(\frac{r}{2M} - 1\right) \quad (2.2)$$

such that the horizon is at $x = -\infty$, and decomposing Φ into the appropriate spherical harmonics,

$$\Phi = \frac{1}{r} \phi(t, r) Y_{lm}(\theta, \phi), \quad (2.3)$$

the function Φ has to satisfy

$$\frac{\partial^2}{\partial t^2} \phi + \left(-\frac{\partial^2}{\partial x^2} + V(x) \right) \phi = 0. \quad (2.4)$$

The potential,

$$V(x) = \left(1 - \frac{2M}{r} \right) \left(\frac{l(l+1)}{r^2} + \frac{M\sigma}{r^3} \right), \quad (2.5)$$

depends on the spin of the linear field: $\sigma = 1, 0, -3$ for the scalar field, the Maxwell field and the linearized gravitational, respectively. In (2.5) we have to consider r as the function of x defined by (2.2). The potential $V(x)$ decays exponentially near the horizon, $x \rightarrow -\infty$, and as x^{-2} for $x \rightarrow +\infty$. The aim is to study solutions of this equation.

The usual approach is as follows: Since the equation is linear and the coefficients are independent of time, the ansatz

$$\phi = e^{i\omega t} \hat{\phi}(\omega, x) \quad (2.6)$$

implies that $\hat{\phi}$ satisfies the ordinary differential equation

$$-\omega^2 \hat{\phi} - \hat{\phi}'' + V \hat{\phi} = 0. \quad (2.7)$$

Which solutions of this equation are interesting, or relevant, or the ones we want to study? The next step is to impose boundary conditions, either without any explanation or by an appeal to quantum mechanics, or by just mentioning "outgoing waves". As a consequence of some boundary conditions, there are only solutions for certain complex values of ω , called "normal modes of the black hole". In analogy with properties of normal modes discussed in the introduction, the general picture seems to be that the general solution can be expressed as a superposition of these "modes", and the question of completeness is considered as an important issue.

All this is quite misleading! The spatial differential operator defined by (2.4) has a continuous real spectrum [19] and the complex modes thus determined define quasi-eigenfunctions diverging for large x , hence they cannot be used to represent general solutions of finite energy.

To clarify the situation, let us go back to the time-dependent equation and pose a problem (I repeat here the argument given in [14]). We want to study solutions determined by initial data of compact support. As outlined in the introduction one can study more general situations, but this case is certainly of particular physical importance (in quantum mechanics it is clear that one has to consider square integrable solutions).

Kay and Wald [20] have established that all solutions of (2.4) determined by data with compact support are bounded in space-time by some constant C which depends on the data, i.e.

$$|\phi(t, x)| < C. \quad (2.8)$$

Therefore the solution has a Laplace transform

$$\hat{\phi}(s, x) = \int_0^\infty e^{-st} \phi(t, x) dt, \quad (2.9)$$

where $s > 0$ is real. The Laplace transform $\hat{\phi}$ satisfies

$$s^2 \hat{\phi} - \hat{\phi}'' + V \hat{\phi} = +s \phi(0, x) + \partial_t \phi(0, x). \quad (2.10)$$

For $\omega = is$ the left-hand side is like (2.7). Note, however, that in contrast to the Fourier transform the data appear as an inhomogeneous term in the time-independent equation.

Which solution $\hat{\phi}$ of this inhomogeneous equation gives the unique solution in space-time determined by the data? There is no arbitrariness; only one of the Green functions for the inhomogeneous equation is correct!

All Green functions can be constructed by the following well-known method. Choose any two linearly independent solutions of the homogeneous equation $f_-(s, x)$ and $f_+(s, x)$, and define

$$G(s, x, x') = \frac{1}{W(s)} \begin{cases} f_-(s, x') f_+(s, x) & (x' < x), \\ f_-(s, x) f_+(s, x') & (x' > x), \end{cases} \quad (2.11)$$

where $W(s)$ is the Wronskian of f_- and f_+ . If we denote the inhomogeneity of (2.10) by j , a solution of (2.10) is

$$f(s, x) = \int_{-\infty}^{\infty} G(s, x, x') j(s, x') dx'. \quad (2.12)$$

We still have to select a unique pair f_-, f_+ . Here the information that the solution in space-time is bounded can be used. The definition of the Laplace transform implies that $\hat{\phi}$ is bounded as a function of x . Pretend for a moment that the potential V vanishes for $|x| > a$. Then the solutions of the homogeneous equation (2.10) for $|x| > a$ are

$$\hat{\phi} = e^{\pm sx}, \quad (2.13)$$

and only the pair

$$f_+ = e^{-sx}, \quad f_- = e^{+sx}, \quad (2.14)$$

gives a Green's function which defines a bounded solution for j of compact support.

Note that the usual boundary conditions appear, but now we understand why we have to impose them. Only the time-dependent problem can determine the boundary condition for a time-independent formulation (Laplace transform or Fourier transform). This is also true for scattering theory in quantum mechanics.

For the potential given by the Schwarzschild space-time, investigations of the asymptotic properties of the solutions of the homogenous part of (2.12) show that there is exactly one function, f_+ , bounded for large x and one, f_- , bounded for small x [21]. Hence these determine the correct Green function.

Up to now the Laplace variable s was real by definition. It is, however, a general property of the Laplace transforms of bounded functions that they have a unique analytic extension

onto the complex half plane $\text{Re}(s) > 0$. Therefore G must also have an analytic extension, and the Wronskian must be nonvanishing.

If we now try to perform further analytic extension in s of $f_{\pm}$ to the half plane $\text{Re}(s) < 0$, we may find values for s_n such that W vanishes. For such a value the functions $f_{\pm}$ are linearly dependent; s_n has necessarily a negative real part such that the corresponding "quasi-eigenfunction" grows exponentially at infinity as we see from (2.14). This is immediately clear for the case of the truncated potential in which (2.14) holds.

Now we have obtained the usual quasinormal modes s_n and the corresponding solutions of the homogeneous equation (2.14) by different ways! Note that we started from real functions $f_{\pm}$ defined for real s and did an analytic extension. The boundary behaviour for the quasi-eigenfunctions can only be used as a definition in the truncated case, where it holds exactly. It cannot be used as an asymptotic condition, as was pointed out by Nollert [22], because one has to find the correct exponentially growing solution.

The definition we proposed [14] is: quasinormal modes s_n are the zeros of the analytically extended Wronskian formed from the solution of the homogeneous equations defining the unique Green function of the problem.

3. The Meaning of Quasinormal Modes

The definition of quasinormal modes we developed in the last section does not shed any light on their mathematical or physical meaning. This is completely different in the case of normal modes which simply describe periodic solutions. A quasinormal mode corresponds to a damped periodic solution. The most irritating property, however, is the exponential growth of the solution for large x .

The resolution of this puzzling fact is given by the "Lax Phillips scattering theory" [2]. A simple application of it in which everything can be calculated explicitly, is the scattering of scalar waves at a sphere. We consider only solutions with $l = 1$ spherical harmonics angular behaviour. The general $l = 1$ solution of the scalar wave equation has as its (t, r) -dependent part

$$\Phi(t, r) = \frac{1}{r} [A'(t - r) + B'(t + r)] + \frac{1}{r^2} [A(t - r) - B(t + r)] . \quad (3.1)$$

The condition of reflection $\Phi(t, 1) = 0$ relates A and B through

$$0 = A'(t - 1) + B'(t + 1) + A(t - 1) + B(t + 1) . \quad (3.2)$$

Only one function can be chosen freely. If we prescribe B , then A is determined as a solution of

$$A'(t) + A(t) = -B'(t + 2) + B(t + 2) =: f(t) , \quad (3.3)$$

an ordinary differential equation with the general solution

$$A(t) = e^{-t} \left(\int_0^t e^{t'} f(t') dt' + A(0) \right) . \quad (3.4)$$

The function $B(t)$ describes a wave coming in from "past null infinity" which is modified by the reflection at the sphere and travels outward as a wave described by $A(t)$ to "future null infinity".

We may choose $B(t)$ such that $f(t)$ vanishes outside an interval, say, $2 < t < C$, where C is an arbitrary constant larger than 2. This represents an incoming pulse of finite duration. We then have to set $A(0) = 0$ to obtain a solution of finite energy — otherwise, the solution would grow exponentially in the past. The result is a finite pulse coming in and a pulse of infinite (!) duration going out.

Inserting (3.4) into (3.1), we observe that for large t (i.e. $t > r + 2 - C$), the solution is

$$\phi(t, r) = \alpha \left(-\frac{1}{r} e^{-t+r} + \frac{1}{r^2} e^{-t+r} \right) = \alpha e^{-t} e^r \left(-\frac{1}{r} + \frac{1}{r^2} \right), \quad (3.5)$$

where

$$\alpha = \int_0^\infty e^{t'} f(t') dt'. \quad (3.6)$$

Note that the solution (3.5) is independent of the particular structure of the incoming wave, which is characterized by the function $B(t)$. Only the constant α depends on $B(t)$.

We will see that the spatial part of this solution represents a "quasinormal mode" with frequency $s = -1$. In this particular example there exists only one quasinormal mode. Furthermore the frequency s is real, therefore the solution is not oscillatory but just damped. In the case $l = 2$ there would be $-2/3 \pm i\sqrt{3}$ in the exponent; for higher l there are l quasinormal modes. In other cases (more complicated obstacles), there may be an infinite number of quasinormal modes.

Let us go quickly through the argument of the last section to find the quasinormal modes for this example. The analogue of (2.10) is the equation ($\phi = \frac{1}{r}\chi$)

$$s^2 \hat{\chi} - \hat{\chi}'' + \frac{2}{r^2} \hat{\chi} = +s \chi(0, x) + \partial_t \chi(0, x). \quad (3.7)$$

Two linearly independent solutions of the homogeneous equation are

$$f_1 = e^{sr} \left(1 - \frac{1}{sr} \right), \quad f_2 = e^{-sr} \left(1 + \frac{1}{sr} \right).$$

The solution satisfying the boundary condition at $r = 1$, $\hat{\phi} = 0$ is

$$f_0 = -e^{-2s} \left(\frac{s+1}{s-1} \right) e^{sr} \left(1 - \frac{1}{sr} \right) + e^{-sr} \left(1 + \frac{1}{sr} \right). \quad (3.8)$$

The Green function has to be constructed from f_0 and f_2 by analogy with (2.14). The Wronskian of these solutions is

$$W(s) = \frac{2s(s+1)}{(s-1)}. \quad (3.9)$$

It vanishes for $s = 0$, the static solution $\chi = 1/r$, and for $s = -1$, the quasinormal mode. The quasi-eigenfunction is

$$\phi = \chi \frac{1}{r} = e^r \left(\frac{1}{r} - \frac{1}{r^2} \right) \quad (3.10)$$

which leads together with the $e^{st} = e^{-t}$ time behaviour to the solution (3.5).

This example shows quite explicitly that the quasinormal mode contains information about the late time behaviour at a fixed point in space. This is a general phenomenon. A

general theorem is proved by Lax and Phillips [2] which states that for "nice obstacles" (for example a convex surface) the quasinormal modes describe the exponential decay of the solutions in time at a fixed point in space. The quasinormal mode with the biggest (negative) real part determines the decay of generic solutions with data of compact support.

Recently an analogous theorem was proved by Bachelot and Motet-Bachelot [23, 24] which states for the case we are interested in, i.e. fields on the Schwarzschild space-time, the following: consider equation (2.4) with a cutoff of the potential (2.5). Then there exists an infinite number of quasinormal modes $s_n = \alpha_n + i\beta_n$, $n = 1 \dots \infty$, $\dots \alpha_{n+1} < \alpha_n \dots < \alpha_1 < 0$, poles of the Green function. For solutions with compactly supported data and every $\epsilon > 0$, there are constants a_n depending on the data and constants $C(n, \epsilon)$ depending on the cutoff such that for any j

$$\left| \phi(t, x) - \sum_{n=1}^{n=j} a_n e^{(\alpha_n + i\beta_n)t} \phi_n(s_n, x) \right| \leq C(j, \epsilon) (1 + |x|)^{\frac{1}{2}} e^{(-|\alpha_{n+1}| + \epsilon)t} \quad (3.11)$$

holds. The functions $\phi_n(s_n, x)$ are the quasi-eigenfunctions.

The quasinormal modes determine the time decay! The quasi-eigenfunctions give the wave form of the late time solution in a compact part of space. Therefore it is irrelevant that these functions grow at infinity.

The origin of this theorem is the inverse Laplace transformation. The pole structure of the Laplace transform is related to large t behaviour of the original function. (The analyticity properties of the Green function determine which type of convergence holds for the asymptotic expansion in time.)

What is the situation for the actual potential? The quasinormal modes can be defined as we did. Naturally we expect that they should appear as the limit of the quasinormal modes of the truncated potentials if the cutoff is moved to infinity. This is shown in [23, 24] if besides the truncation of the potential a damping factor $e^{-\alpha|x|^2}$ is introduced, which approaches 1, ($\alpha \rightarrow 0$), in the limit in which the cutoff is removed. Here we have also the first proof that the number of quasinormal modes of the Schwarzschild space-time is infinite.

The actual nature of the decay in time for solutions of wave equations on the Schwarzschild space-time is not known. Certain calculations indicate [5, 7] that, due to the backscatter at the curvature of the non-flat metric up to infinity, the exponential decay changes to a polynomial one. Nevertheless numerical calculations show that a formula like (3.11) seems to give a very good approximation at some intermediate times. It is very impressive that Bachelot and Motet-Bachelot can fit a time-dependent solution, calculated numerically, to an expansion of the type appearing in (3.13). Such a fit is used to calculate values for the quasinormal modes. These agree up to six decimal places with the values calculated in [10, 25], using completely different methods. Hence the observation that quasinormal modes are related to the decay in time does not only increase our understanding, but also offers a new method of calculating the quasinormal modes.

4. Quasinormal Modes of Stars

The question of normal modes versus quasinormal modes becomes particularly interesting when considering stellar oscillations. Adiabatic oscillations of stars treated by Newton's theory of gravitation are superpositions of periodic motions, a classical case of "compact oscillating systems". There exists a complete set of normal modes.

This changes if we treat the problem using Einstein's theory. If the oscillations are non-radial the star will emit gravitational radiation, and "energy" will be lost to infinity. The relevant linear operator has no longer a pure point spectrum, but a continuous one. We can repeat the line of argument of section 2 and define quasinormal modes. Numerical calculations were performed by various authors.

Recently Kokkotas and Schutz [15, 16] found an "unexpected collection of quasinormal modes"; first for some model problem, then for neutron stars. Let us try to understand those new "w-modes".

Using the formalism of the Laplace transform the eigenvalues of the Newtonian oscillations are imaginary if there are no growing modes (these would correspond to real positive s_0 ; the Laplace transform of such a solution would only be analytic for $s > s_0 > 0$). From the point of view of the Newtonian limit we expect to reach the Newtonian normal modes as limits of relativistic quasinormal modes. Numerical calculations show that indeed the damping part is by about four orders of magnitude smaller than the part which determines the oscillation. This is of particular importance because certain methods of calculation of quasinormal modes depend on the property that they lie very near the imaginary axis (Breit-Wigner formula).

For the new modes found by Kokkotas and Schutz this is not the case. A recent time-dependent formulation of relativistic oscillations [26] suggests a very simple and physical explanation of this phenomenon.

In Regge-Wheeler gauge the time-dependent polar oscillations, with Y_{lm} angular behaviour, are described by solutions of the following system of linear equations:

$$-\frac{1}{a^2}\ddot{f} + f'' = F(f, f', g, g', H, H'), \quad (4.1)$$

$$-\frac{1}{c^2}\ddot{g} + g'' = G(f, f', g, g', H, H'), \quad (4.2)$$

$$-\frac{1}{w^2}\ddot{H} + H'' = M(f, f', g, g', H, H'). \quad (4.3)$$

The functions F, G, M are linear in their arguments. f and g are perturbed metric coefficients, and H is proportional to the linearized density. The characteristic speeds of gravitational and hydrodynamic perturbations are a and b , respectively.

Suppose we consider just the equation for H and put f, g together with its derivatives identical to zero in M . Then we have what we could call the "relativistic Cowling approximation". The gravitational field generated by the perturbed density is ignored. We do just linearized hydrodynamics on a fixed background stellar model. The boundary conditions lead to a Sturm-Liouville problem as in the Newtonian Cowling approximation. We expect a discrete spectrum and (imaginary) normal modes. Coupling to gravity should distort the normal modes to quasinormal ones with small negative damping part. These are the quasinormal modes accessible by the techniques used in [12,13].

There is, however, another extreme case we might consider. Take the equations for f and g ; if we put $H = \dot{H} = 0$ in F and G we describe linearized gravitational waves

propagating on and outside the background star. Now we are in a situation analogous to linearized gravitational waves on the Schwarzschild space-time and we expect quasinormal modes to appear. These modes have no corresponding Newtonian normal modes.

In the coupled system both types of modes will reappear with changed values of the frequency. There is no natural parameter in the equations which could be used to decouple the equations as is the case in certain model systems [15,16]. Therefore, it is not clear how to turn these remarks into mathematical limit statements. One could insert a "decoupling parameter" by hand into the equations and try to show that the modes behave in the expected way as this was done for a model problem in [27].

The interpretation of the quasinormal modes is as in section 3. They encode information about the generic asymptotic decay in time. Asking for completeness is again irrelevant. One should be mainly interested in the quasinormal modes with the slowest damping because these determine the decay of generic data.

In principle normal and quasinormal modes are quite different. "In practice", however, they may be very near! The linear functional analysis point of view is always a global one. The more local aspect of differential equations, which enters in particular into the construction of the Green function, should result in a "smooth transition" of the poles of the Green function from the normal mode to the quasinormal mode case. Work is in progress to find out how the Newtonian limit of relativistic oscillations can be performed and in which sense the Newtonian modes can be reached as a limit of the relativistic ones.

Acknowledgments

I want to thank J. Ehlers and H. Beyer for their careful reading of this article and valuable discussions.

Appendix

The purpose of this appendix is a brief comparison of solutions of the Schrödinger equation,

$$i\dot{\Psi} = \left(-\frac{d^2}{dx^2} + V(x)\right)\Psi, \quad (A1)$$

and the wave equation,

$$-\ddot{\Phi} = \left(-\frac{d^2}{dx^2} + V(x)\right)\Phi, \quad (A2)$$

for the same potential $V(x)$.

The ansatz of "stationary states",

$$\Psi = e^{-iEt}\psi(x), \quad (A3)$$

leads to the time independent Schrödinger equation,

$$-\psi'' + V\psi = E\psi. \quad (A3)$$

The ansatz (corresponding to Laplace transformation)

$$\Phi = e^{st}\phi(x) \quad (A4)$$

gives for the wave equation

$$s^2 \phi - \phi'' + V \phi = 0. \quad (\text{A5})$$

The equations (A3) and (A5) are the same if we set $E = -s^2$. Hence time-independent scattering theory — the theory of the operator $-\frac{d^2}{dx^2} + V(x)$ — does not know whether we deal with the wave equation or the Schrödinger equation.

Consider first a positive potential of compact support. From quantum mechanics we know that there are no bound states. Hence the operator $-\frac{d^2}{dx^2} + V(x)$ has no eigenfunctions and there are only scattering states. Using the method of section 2 we can construct the Green function and define the quasinormal modes. I do not know whether there is a relation between the poles of the Green function (which are complex) and the maxima of the scattering cross section as a function of real E .

For a negative potential of compact support and sufficient depth, there will be eigenfunctions with negative eigenvalues describing bound states of the quantum system. What are the properties of the corresponding solutions of the wave equation? Let $\psi_n(x)$ be an eigenfunction of $-\frac{d^2}{dx^2} + V(x)$ with eigenvalue $E_n < 0$. Then ($E = -s^2$)

$$\Phi = e^{\sqrt{-E_n} t} \psi_n(x) \quad (\text{A6})$$

is a solution of the wave equation. For large positive x we have

$$\psi_n = e^{-\sqrt{-E_n} x} \quad (\text{A7})$$

as the only square integrable solution of the Schrödinger equation with vanishing potential. Thus the solution of the wave equation for large x is

$$\Phi = e^{\sqrt{-E_n} t} e^{-\sqrt{-E_n} x} = e^{\sqrt{-E_n} (t-x)}. \quad (\text{A8})$$

This solution grows exponentially in time and falls off exponentially in space for $x \rightarrow \infty$. This apparently strange behaviour is possible because the energy density of the conserved energy of the wave equation,

$$\frac{1}{2} \left((\dot{\Phi})^2 + (\Phi')^2 + V \Phi^2 \right), \quad (\text{A9})$$

is not positive definite if the potential is somewhere negative. In this situation the solution can grow in time and nevertheless have conserved finite energy.

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Trapped Surfaces for Purely Radiative Initial Data

R. Beig

1. Introduction

Trapped surfaces, when they were first introduced by Penrose [1], played mainly the role of an ingredient, an object assumed to exist in (some of) the singularity theorems.

In recent years, motivated in part by the problem of cosmic censorship, the focus has shifted somewhat towards studying the correctness of the underlying picture, namely that, when gravitational collapse has proceeded beyond a certain critical stage, trapped surfaces (TS's) are likely to be formed. In particular, people have begun to study condition for the actual presence of TS's in certain classes of initial-data sets.

The present lecture is concerned with TS's (rather: outer trapped surfaces) for asymptotically flat, pure-vacuum, globally regular initial data. Their existence was conjectured by Bartnik [2]. An affirmative answer in the time-symmetric case was first given by Beig and Ó Murchadha [3], and the maximal case is work, again jointly with Ó Murchadha, in progress. This paper summarizes our findings in the time-symmetric and maximal case.

I should point out that our trapped surfaces will generally be trapped both towards the future and the past, so that the associated space-time has to be geodesically incomplete both in the future and the past. Thus, presumably, these data cannot arise from ones having no TS's, and they can certainly not arise in a space-time with a regular past null and timelike infinity. If cosmic censorship is assumed, space-time will start out as a white hole and settle down to a black hole.

Consider the space of all (vacuum, maximal, asymptotically flat) initial data, given in terms of "free data". These are

- (a) a 3-manifold M
- (b) a conformal, positive definite, metric g_{ab} on M
- (c) a TT-tensor (w.r.t. g_{ab}) ψ_{ab} on M .

These "free data" are not completely unrestricted. Namely, g has to be such that the Hamiltonian constraint can be solved. This, together with the global conditions we shall impose on M , puts a restriction on (M, g) in the form of a global inequality, called $\lambda(g) > 0$. (It also restricts the topology of M , e.g. S^3 is allowed, but not T^3 .) Then we look at "critical sequences" $({}^n g, {}^n \psi)$ on M , for which $\lambda(g) > 0$ but $\lambda({}^n g) \rightarrow 0$ as $n \rightarrow \infty$. We prove there is a number N , such that all $(M, {}^n g, {}^n \psi)$ have TS's for $n > N$.

Abrahams et al. [4] have numerically constructed time-symmetric sequences $({}^n g, {}^n \psi_{ab} = 0)$ which become singular as $n \rightarrow \infty$ and for which there are no TS's even for arbitrarily large n . The following interpretation suggests itself: The phase space of G.R. has some sort of boundary which consists, firstly, of metrics with $\lambda(g) = 0$: all points near that part of the boundary correspond to initial data with TS's. Secondly, there are parts of the boundary corresponding to singular data: these can have neighbourhoods with no TS's. These latter regions in phase space, according to Abrahams et al., could be related to possible violations of cosmic censorship.

2. Solving the Hamiltonian Constraint

Recall the definition of the conformal Laplacian,

$$L_g = -\Delta_g + \frac{1}{8}\mathcal{R}[g] \quad (2.1)$$

with $\Delta_g = g^{ab}D_a D_b$ and $\mathcal{R}[g]$ the scalar curvature. Let

$$\bar{g}_{ab} = \omega^4 g_{ab}, \quad \bar{\phi} = \omega^{-1}\phi, \quad \omega > 0. \quad (2.2)$$

Then

$$L_{\bar{g}}\bar{\phi} = \omega^{-5}L_g\phi. \quad (2.3)$$

When in particular $\phi > 0$ and we set $\omega = \phi$, we have that $\bar{\phi} \equiv 1$, so that there results the well-known identity

$$\frac{1}{8}\mathcal{R}[\bar{g}] = \phi^{-5}L_g\phi. \quad (2.4)$$

Thus $\bar{g}$ is scalar-flat iff $\phi > 0$ satisfies the Lichnerowicz equation $L_g\phi = 0$. More generally, let ψ_{ab} be a TT-tensor,

$$g^{ab}\psi_{ab} = 0, \quad D^a\psi_{ab} = 0 \quad (2.5)$$

and

$$L_g\phi = \frac{1}{8}\phi^{-7}\psi_{ab}\psi^{ab}, \quad \phi > 0. \quad (2.6)$$

Then $\bar{g}_{ab} = \phi^4 g_{ab}$, $\bar{p}_{ab} = \phi^{-2}\psi_{ab}$ satisfy

$$\bar{D}^a\bar{p}_{ab} = 0, \quad \mathcal{R}[\bar{g}] = \bar{p}_{ab}\bar{p}^{ab}, \quad (2.7)$$

i.e. the initial-value constraints.

We now turn to global issues. Let M be compact and Λ a point $\in M$. Our aim is to solve the equation (2.6) on $M \setminus \{\Lambda\} = \bar{M}$ and prescribe conditions near Λ such that $(\bar{M}, \bar{g}_{ab} = \phi^4 g_{ab}, \bar{p}_{ab} = \phi^{-2}\psi_{ab})$ is an asymptotically-flat (near Λ) initial-data set. These conditions turn out to be as follows: Let $\Omega^{1/2}$ be an ‘‘asymptotic distance function’’ (ADF). This means that Ω is a smooth function on M , $\Omega > 0$ outside Λ and $(\Omega_a := D_a\Omega, \Omega_{ab} := D_a D_b\Omega, \dots)$

$$\Omega|_{\Lambda} = 0, \quad \Omega_a|_{\Lambda} = 0, \quad (\Omega_{ab} - 2g_{ab})|_{\Lambda} = 0, \quad \Omega_{abc}|_{\Lambda} = 0. \quad (2.8)$$

For example, Ω could be taken to be the square of the geodesic distance from Λ in a neighbourhood Λ and smoothly extended to a positive function on all of M . Then we require

$$\phi = \Omega^{-1/2} + (\text{bounded function}), \quad \psi_{ab} = O(\Omega^{-2}). \quad (2.9)$$

The singularity in ϕ secures that $\bar{g}_{ab}$ is flat to leading order near Λ in the following sense: Let x^a be coordinates centered at Λ with $g_{ab}|_{\Lambda} = \delta_{ab}$. Then the metric $\bar{g}_{ab} = \phi^4 g_{ab}$ satisfies

$$\bar{g}_{ab} - \delta_{ab} = O(\Omega^{1/2}) \quad (2.10)$$

in inverted coordinates $\bar{x}^a = x^a/\Omega$. The singular behaviour for ψ_{ab} has to be allowed if we want a nonvanishing contribution to $\bar{p}_{ab}$ in order $O(\Omega)$, which is the asymptotic order

at which total linear ADM-momentum appears. In fact, the ADM-momentum will be zero in our case: the regularity of g_{ab} near Λ forces the bounded function in (2.9) for ϕ to be continuous (in fact: Lipschitz-continuous) near Λ . In particular, $\phi - \Omega^{-1/2}$ is "spherically symmetric" near Λ . This implies that $\bar{g}_{ab}$ is asymptotically Schwarzschildian. Hence we are in the centre-of-mass frame whence the total momentum is zero.

The above conditions imply that ϕ is a (positive) Green function for the Lichnerowicz operator, i.e.

$$(L_g f, \phi) = 4\pi f|_{\Lambda} + \frac{1}{8}(f, \phi^{-7}\psi_{ab}\psi^{ab}) \quad (2.11)$$

for all test functions f and where $(\ , \)$ is the standard L^2 -inner product of g_{ab} . In particular

$$L_g \phi = \frac{1}{8}\phi^{-7}\psi_{ab}\psi^{ab} \quad \text{outside } \Lambda, \quad (2.12)$$

as it should be.

Next, note the following crucial facts. L_g is a self-adjoint operator on $(M, (\ , \))$ with a pure point spectrum. Its lowest eigenvalue, $\lambda(g)$, is non-degenerate and the associated eigenfunction is nowhere zero. It easily follows from (2.11) that $\lambda(g) > 0$, when $\phi > 0$ exists. In the time-symmetric case, i.e. $\psi_{ab} = 0$, one knows that the converse is true as well. In this case ϕ is simply the Green function of L_g with source at Λ . We now solve the nonlinear equation.

Let ϕ_0 be the time-symmetric solution and set $\phi = \phi_0 + h$. Thus

$$L_g h = \frac{1}{8}(1 + \phi_0^{-1}h)^{-1}\phi_0^{-7}\psi_{ab}\psi^{ab} =: \rho \quad (2.13)$$

outside Λ with h bounded at Λ . Note that $\phi_0^{-7}\psi_{ab}\psi^{ab}$ diverges only like $\Omega^{-1/2}$ at Λ . We write (2.13) symbolically as

$$h = F[h], \quad (2.14)$$

where

$$F[h] = L_g^{-1} \circ \rho. \quad (2.15)$$

The operator L_g^{-1} is given by convolution with the Green function of L_g , with source taken at an arbitrary point. Since the Green function is positive, $F[h]$ is a positive operator in the sense of mapping positive functions into positive functions. Next define two functions h_- and h_+ by

$$h_- \equiv 0, \quad h_+ = F[h_-] = \frac{1}{8}L_g^{-1}[\phi_0^{-7}\psi_{ab}\psi^{ab}]. \quad (2.16)$$

Clearly $h_- \leq h_+$ whence $h_- \leq F[h_-]$. But F is also a decreasing map. Thus $F[h_+] \leq F(h_-) = h_+$. So h_- , h_+ are respectively sub- and supersolutions for (2.18). This can now be solved by a standard trick. Define

$$0 \leq c := \frac{7}{8} \sup_M \phi_0^{-8}\psi_{ab}\psi^{ab}.$$

(Note that the last expression is in fact bounded.) We replace (2.13) by the equation

$$(L_g + c)h = \rho(h) + ch. \quad (2.17)$$

The constant c has been chosen so that $\rho + ch$ is *increasing* in h , as opposed to ρ . Equivalently to (2.17) we have

$$h = G[h], \quad (2.18)$$

where $G = (L_g + c)^{-1} \circ [\rho(h) + c \cdot h]$. We now take sequences $h_n = G[h_{n-1}]$, $h_0 = h_-$ and $h_n^- = G[h_{n-1}^2]$, $h_0' = h_+$. The sequence h_n is increasing, the sequence h_n' is decreasing. Both converge to the (unique) fixed point of G . (We omit the analytical details.) This completes our discussion of the Hamiltonian constraint.

3. Critical Sequences

We consider the time-symmetric case first. Here a critical sequence (CS) is a sequence of smooth metrics ${}^n g$ with $\lambda({}^n g) > 0$, converging uniformly up to sufficiently many derivatives to a metric ${}^\infty g$ with $\lambda({}^\infty g) = 0$. We also assume that ${}^n g$ and the connections $\partial {}^n g$ all coincide at the point Λ so that an ADF $\Omega^{1/2}$ can be chosen independently of n . We now have (see [3])

Theorem: *Along any CS, the ADM-mass m_n goes to infinity: $\lim_{n \rightarrow \infty} m_n = \infty$. In particular this implies a positive-mass proof.*

The connection between $\lambda(g)$ and the mass arises as follows: The Green function ϕ_0 has the expansion

$$\phi_0 = \Omega^{-1/2} + \frac{m}{2} + O(\Omega^{1/2}). \quad (3.1)$$

One shows that, as $\lambda(g) \rightarrow 0$, ϕ_0 blows up in such a way that m blows up. Moreover

$$\begin{aligned} a^n \phi_0 &= \Omega^{-1/2} + \frac{m_n}{2} + m_n O(\Omega^{1/2}), \\ \partial {}^n \phi_0 &= \partial \Omega^{-1/2} + m_n O(1), \end{aligned} \quad (3.2)$$

where the constants involved in the O -symbols are meant to be n -independent. Next consider level-surfaces of $\Omega : \Omega = \Omega_0$ for small $\Omega_0 > 0$. These are regular, topologically spherical surfaces in $\bar{M} = M \setminus \{\Lambda\}$. The surface $\Omega = \Omega_0$ is trapped, iff H , the trace of its extrinsic curvature w.r.t. the metric $\bar{g}_{ab} = \phi_0^4 g_{ab}$, with the normal pointing towards Λ , is negative. We compute

$${}^n H = -\frac{{}^n \phi_0^{-3} \Omega^{-1/2}}{(\Omega_c \Omega^c)^{1/2}} \left[\Omega^{1/2} \phi_0 \left({}^n g^{ab} - \frac{\Omega^a \Omega^b}{\Omega_a \Omega_b} \right) \Omega_{ab} + 4 \Omega^{1/2} {}^n \phi_0^a \Omega_a \right]. \quad (3.3)$$

Thus, using (3.2) and (2.8),

$${}^n H = \frac{4 {}^n \phi_0^{-3} \Omega^{-1/2}}{(\Omega_c \Omega^c)^{1/2}} \left[1 - \frac{m_n \Omega^{1/2}}{2} + m_n O(\Omega) \right]. \quad (3.4)$$

It is now easy to see that for any sufficiently small $\Omega_0 > 0$ there is a number N , such that the surfaces $\Omega = \Omega_0$ are TS's for $n > N$.

We next turn to the maximal case. Here we have additionally a sequence ${}^n \psi_{ab}$ of TT-tensors with

$${}^n \psi_{ab} = O(\Omega^{-1}), \quad (3.5)$$

where the O -symbol is understood as before. The ${}^n g_{ab}$ are also as before. Such sequences of ${}^n \psi_{ab}$'s exist. There is no obstruction to the solvability of the momentum constraint, namely $D^a \psi_{ab} = 0$, analogous to $\lambda(g) > 0$ for the Hamiltonian constraint, except that known existence theorems apply only to the case $M \cong S^3$. We now also have to estimate the ${}^n h$'s. One finds that

$${}^n h = m_n^{-1} O(1), \quad \partial {}^n h = O(\lg \Omega), \quad (3.6)$$

where the m_n 's are the masses of the time-symmetric case.

The TS-condition now acquires a term coming from $\bar{p}_{ab} = \phi^{-2} \psi_{ab}$ given by

$$\frac{{}^n \phi^{-3} \Omega^{-1/2}}{(\Omega_c \Omega^c)^{1/2}} \left[\frac{{}^n \psi_{ab} \Omega^a \Omega^b}{(\Omega_d \Omega^d)^{1/2}} {}^n \phi^{-3} \Omega^{1/2} \right]. \quad (3.7)$$

The square bracket in (3.7) is $O(\Omega^{1/2})$. The time-symmetric contribution to ${}^n H$ becomes

$$\frac{4 {}^n \phi^{-3} \Omega^{-1/2}}{(\Omega_c \Omega^c)^{1/2}} \left[1 - \frac{m_n}{2} (1 + O(m_n^{-1})) \Omega^{-1/2} + m_n O(\Omega) + O(\Omega \lg \Omega) \right]. \quad (3.8)$$

Thus the term (3.8) dominates (3.7), and we reach the same conclusion as before.

Finally, let us point out that there are many methods of obtaining CS's. It is known, for example, that one can always deform a metric with $\lambda(g) > 0$ into one with $\lambda(g) < 0$ by changing it in a small subset of M .

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Chaotic Trajectories Around a Black Hole

L. Bombelli

Abstract: The Melnikov method for identifying chaos in near-integrable systems is briefly summarized. I then describe an application of the method to relativistic particles moving around a black hole. In addition to the chaotic trajectories present in Schwarzschild gravity, there are in the general relativity case trajectories around an unperturbed Schwarzschild black hole. These give rise to other orbits, which approach the unstable ones for $t \rightarrow \pm\infty$ and wind outward up to some maximum radius in between. The latter are an example of the so-called homoclinic trajectories. As an example of the method, I consider the case of a particle moving around a black hole.

Part V

Evolution in Gravitating Systems

1. Introduction

Relativistic chaos has been known for a long time. In both types of systems one may look at: the dynamics of the space-time metric itself, and the motion of matter in a curved space-time. In the first case, one usually reduces the system to a finite-dimensional one by imposing some symmetries, and then studies the dynamics by using some approximations and/or numerical methods to check for chaos. This has produced an extensive literature on chaos in cosmology (see for example [1-9]), including comparisons on the results obtained, partly concerning the effects of the various simplifying steps used and the differences between various criteria for chaos. Regarding the second type of system, chaos in geodesic motion in a curved manifold has been known mathematically for an even longer time, and a considerable amount of work has been done on applications to particles in curved space-time (see for example [10-12]).

These studies point thus to confirm the natural expectation that, given the non-linear nature of the equations involved, chaos should be a generic feature of relativistic systems. It is therefore interesting, on the one hand, to look for interesting examples of relativistic chaotic behaviour, in particular in cases where it may be possible to understand its physical significance, and on the other hand to explore the possibility of using other methods to identify the occurrence of chaos, perhaps suited in different situations.

One method for detecting chaotic behaviour that is well known and widely used in other contexts, but is almost unknown in the relativity community, is the method for homoclinic chaos (described in several texts, such as [13, 14]). This method nicely complements other ones, in that it is analytical, but it applies to linearized perturbations of integrable Hamiltonian systems. When such systems exhibit wild behaviour in phase space that are potential sources of chaos under a perturbation, the Melnikov method detects them but, but one can detect the onset of chaos. If this is found, then one may wish to use other methods to determine the nature of the chaotic region or other specific or integral.

More specifically, consider a one-dimensional Hamiltonian system (the method can be applied in higher dimensions as well [15]) with an unstable equilibrium point $x = x_{un}$ in the potential $U(x)$. There are then trajectories which approach x_{un} as $t \rightarrow \pm\infty$, and, if there are other points at which the potential has a higher value, there is also at least one trajectory γ asymptotically approaching x_{un} for both $t \rightarrow -\infty$ and $t \rightarrow +\infty$; this is called a homoclinic

Chaotic Trajectories Around a Black Hole[†]

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1. Introduction

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[†] Work done in collaboration with E. Calzetta.

and it is a separatrix or boundary between orbits in phase space which are confined to one side of the hyperbolic point and those which are not. Homoclinic orbits are likely seeds of chaos when a time-periodic perturbation is added to the Hamiltonian. If

$$\begin{aligned} H(p, q, t) &= H_0(p, q) + \epsilon G(p, q, t - t_0) \\ G(p, q, t - t_0) &= \operatorname{Re} G(p, q) e^{i\omega(t-t_0)}, \end{aligned} \quad (1.1)$$

where t_0 is the initial phase of the perturbation, the criterion for establishing whether a stochastic region will form near ℓ is to evaluate the integral,

$$I(t_0) = \int_{-\infty}^{\infty} dt \{ H_0(p(t), q(t)), G(p(t), q(t), t - t_0) \}, \quad (1.2)$$

where the integral is taken along the unperturbed homoclinic loop ℓ , and $\{ \cdot, \cdot \}$ stands for the Poisson bracket; if $I(t_0)$ has transverse zeroes at discrete values of t_0 , then chaos does develop.

These considerations can be generalised to the case where there are *heteroclinic* s , joining two different unstable equilibrium points, but we will not need to do it here. In fact, in a way homoclinic/heteroclinic chaos is quite generic, since first-order perturbations of integrable systems generically lead to the replacement of some of the tori in phase space by patterns with stable and unstable fixed points, with the corresponding separatrices, even if these were not initially present, so one may view this "perturbative chaos" as the first step of what will eventually become the stochastic layer studied in KAM theory [15,16].

The work described here concerns the chaotic nature of relativistic particle motion around a Schwarzschild black hole to which a time-dependent perturbation of the metric has been added. Thus, physically, the results obtained are a new indication of the pervasiveness of chaotic phenomena in general relativity, and a first step in the study of an effect which may have interesting astrophysical consequences. But another underlying motivation for the work has been to become familiar with the Melnikov method, which can be applied not just to the refinement of the present model, but also to other problems in relativity and cosmology, where having one more technique available to study chaos is particularly interesting.

Section 2 contains the description of the application to motion around a perturbed black hole and the results. Additional remarks are contained in the concluding section 3. For more details both on the method and the application, as well as on the literature, I refer to [17].

2. Motion around a Perturbed Black Hole

We begin by studying geodesic motion around the unperturbed Schwarzschild black hole, described by the metric

$$g_{ab} dx^a dx^b = -f dt^2 + f^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.1)$$

where $f(r) := 1 - 2M/r$. (This part of the problem is of course a standard one—see, e.g., [18]—but we will need to study one particular set of orbits in more detail than is usually done.)

For a particle of mass μ on a curved background, a possible form for the action is

$$S = \int \mathcal{L} ds = \frac{\mu}{2} \int g_{ab} \dot{x}^a \dot{x}^b ds, \quad (2.2)$$

where s is the proper time along the world-line $x^\mu(s)$, with the constraint

$$g^{ab} p_a p_b = -\mu^2, \quad (2.3)$$

where as usual we have defined $p_a := \partial \mathcal{L} / \partial \dot{x}^a = \mu g_{ab} \dot{x}^b$. In the case of the Schwarzschild metric, the momenta $p_t = -H_0$ and p_ϕ are constants of the motion, and we set them to the fixed values $-E$ and L , respectively. These two first integrals of the motion, together with the Hamiltonian corresponding to the action (2.2), generating s -evolution, or equivalently the constraint (2.3), take the form

$$\begin{aligned} \mu f \frac{dt}{ds} &= E, \\ \mu r^2 \sin^2 \theta \frac{d\phi}{ds} &= L, \\ \mu^2 \left(\frac{dr}{ds} \right)^2 + f(r) \left[\mu^2 + \mu^2 r^2 \left(\frac{d\theta}{ds} \right)^2 + \frac{L^2}{r^2 \sin^2 \theta} \right] &= E^2. \end{aligned} \quad (2.4)$$

Restricting ourselves to orbits on the equatorial plane $\theta = \pi/2$, we obtain for the radial motion the one-dimensional problem of a particle moving in the potential

$$U(r) = -\frac{2M\mu^2}{r} + \frac{L^2}{r^2} - \frac{2ML^2}{r^3}. \quad (2.5)$$

In order for a homoclinic orbit to exist, $U(r)$ must have an unstable equilibrium point at which its value is less than zero, the asymptotic value at $r = \infty$. The existence of such points is better discussed in terms of the parameter

$$\beta := \sqrt{1 - 12M^2\mu^2/L^2}. \quad (2.6)$$

The potential is plotted in Figure 1 for different parameter values; it has a local maximum (and a minimum) for β real and positive, and its value there is negative if $\beta < \frac{1}{2}$; in other words, we must have

$$2\sqrt{3}M < \frac{L}{\mu} < 4M. \quad (2.7)$$

Notice that if the general relativistic r^{-3} term had been absent in (2.5) there would always have been stable circular orbits for $L \neq 0$, as in the Newtonian theory, but no unstable ones, and thus no homoclinic orbit.

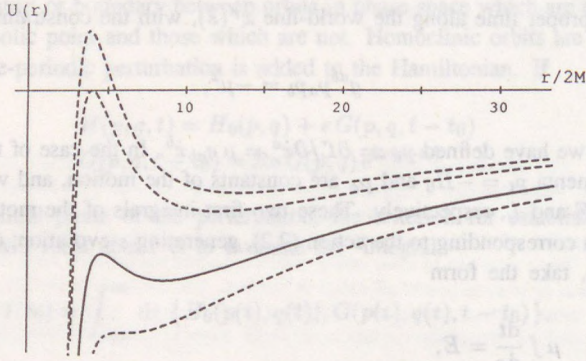


Figure 1. The potential U plotted as a function of $r/2M$ for $\beta = 0.55, 0.5, 0.4$, and 0.0 , from top to bottom

The explicit form for the homoclinic orbit can be found by solving the equations (2.4), in which ℓ is picked out by setting its energy to be that of the unstable circular orbit, with $dr/ds = 0$ at the unstable point. To solve the radial equation, it is convenient to use an inverse radial coordinate; we shall reexpress the solution in terms of r , but parametrized by

$$z := \exp(\sqrt{\beta}\phi), \quad (2.8)$$

rather than s or t , which gives

$$r(z) = \frac{6M}{(1 - 2\beta) + 3\beta[(z - 1)/(z + 1)]^2}. \quad (2.9)$$

A homoclinic orbit in physical (configuration) space is shown in figure 2. To find its shape in phase space, necessary for later calculations, we also need $p_r = \mu f^{-1} dr/ds$, which gives

$$p_r(z) = \pm \frac{1}{f(r(z))} \left[E^2 - f(r(z)) \left(\mu^2 + \frac{L^2}{r^2(z)} \right) \right]^{1/2}, \quad (2.10)$$

where the two signs correspond to the outgoing and ingoing parts of the orbit, respectively. Finally, knowing $r(z)$, the first two equations in (2.4) can be combined and integrated to give

$$t(z) = \frac{6\sqrt{6}(2 - \beta)M}{\sqrt{\beta}(1 + \beta)^{3/2}} \left[\frac{\ln z}{2 - \beta} - \frac{2\beta}{1 - 2\beta} \frac{(1 - 5\beta)z + (1 + \beta)}{(1 + \beta)z^2 + 2(1 - 5\beta)z + (1 + \beta)} + \right. \\ \left. - \frac{\sqrt{\beta}(1 + \beta)^{3/2}}{3\sqrt{6}(2 - \beta)} \ln \frac{(2 - \beta)z + (2 + 5\beta) - 2\sqrt{6\beta}(1 + \beta)}{(2 - \beta)z + (2 + 5\beta) + 2\sqrt{6\beta}(1 + \beta)} + \right. \\ \left. + \frac{\sqrt{\beta}(11 - 11\beta - 4\beta^2)}{3\sqrt{3}(1 - 2\beta)^{3/2}} \arctan \frac{(1 + \beta)z + (1 - 5\beta)}{2\sqrt{3\beta}(1 - 2\beta)} \right], \quad (2.11)$$

up to an additive constant, which will not be important for us because it is a real number, as we shall see below.

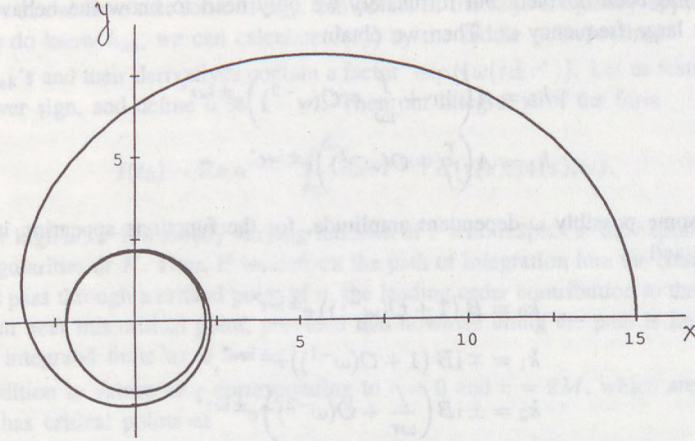


Figure 2. The homoclinic orbit around the unperturbed black hole for $\beta = 0.4$; the radial coordinate is the Schwarzschild coordinate r

We now want to add a time-periodic perturbation to the Schwarzschild metric. For convenience, we will use only perturbations which are ϕ -independent and even under reflection with respect to the $\theta = \pi/2$ plane; then L will still be conserved, the perturbed trajectory will stay in the equatorial plane, and we can still analyse the motion using the (r, p_r) phase space. Components of the metric perturbation can be decomposed into tensor spherical harmonics labelled by (l, m) on the $r = \text{constant}$ spheres [17]. Then the requirements above translate into $m = 0$, and restrictions on the values of l . Specifically, if we call even (odd) a tensor spherical harmonic whose sign changes by $(-1)^l$ ($(-1)^{l+1}$) under spatial inversion, the most general such perturbation in the Regge-Wheeler gauge [17] can be written as (the real part of)

$$h_{ab}(t, r, \theta) = \begin{bmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0(r) & h_1(r) & 0 & 0 \end{bmatrix} e^{i\omega(t-t_0)} \sin \theta \frac{d}{d\theta} P_l(\cos \theta), \quad (2.12)$$

where P_l is the Legendre polynomial of order l , and we are interested in odd l (and real ω), while the most general even perturbation can be written in the form

$$h_{ab}(t, r, \theta) = \begin{bmatrix} f k_0(r) & k_1(r) & 0 & 0 \\ k_1(r) & f^{-1} k_0(r) & 0 & 0 \\ 0 & 0 & r^2 k_2(r) & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta k_2(r) \end{bmatrix} e^{i\omega(t-t_0)} P_l(\cos \theta), \quad (2.13)$$

where we are interested in even l .

The most reasonable and simplest perturbation of the form above we can consider is a solution of the linearized vacuum Einstein equation. Solving this equation even for a simple background such as Schwarzschild is a difficult task to which a considerable amount

of literature has been devoted, but fortunately we only need to know the behavior of the solutions for large frequency ω . Then we obtain

$$\begin{aligned} h_0 &= A \left(\pm r + \frac{f}{i\omega} + \mathcal{O}(\omega^{-2}) \right) e^{\pm i\omega r^*}, \\ h_1 &= A \left(\frac{r}{f} + \mathcal{O}(\omega^{-2}) \right) e^{\pm i\omega r^*}, \end{aligned} \quad (2.14)$$

where A is some possibly ω -dependent amplitude, for the functions appearing in the odd perturbation, and

$$\begin{aligned} k_0 &= B (1 + \mathcal{O}(\omega^{-1})) e^{\pm i\omega r^*}, \\ k_1 &= \mp iB (1 + \mathcal{O}(\omega^{-1})) e^{\pm i\omega r^*}, \\ k_2 &= \pm iB \left(\frac{f}{\omega r} + \mathcal{O}(\omega^{-2}) \right) e^{\pm i\omega r^*}, \end{aligned} \quad (2.15)$$

where B is some possibly ω -dependent amplitude, for the even perturbations. Here we have set, as usual,

$$r^* := r + 2M \ln \left(\frac{r}{2M} - 1 \right), \quad (2.16)$$

for the modified radial coordinate going to $-\infty$ at the horizon.

To calculate the function (1.2), we need to know the Hamiltonian perturbation, $G(p, q, t - t_0)$. This is obtained by solving the mass shell constraint (2.3) with the perturbed metric,

$$f^{-1}(1 + \epsilon f h^{tt}) p_t^2 + 2\epsilon h^{it} p_i p_t - (g^{ij} - \epsilon h^{ij}) p_i p_j = \mu^2, \quad (2.17)$$

for $-p_t = H_0 + \epsilon G$. Having set $p_\theta = 0$, $p_\phi = L$, we obtain

$$\begin{aligned} H_0 &= \sqrt{f(g^{ij} p_i p_j + \mu^2)} = \sqrt{f(f p_r^2 + \mu^2 + L^2/r^2)}, \\ G &= f h^{tt} p_t - \frac{f}{2H_0} (h^{tt} H_0^2 + h^{ij} p_i p_j) \\ &= -f p_r h_{tr} - L r^{-2} h_{t\phi} - \frac{1}{2} H_0 f^{-1} h_{tt} + \\ &\quad - \frac{f}{2H_0} (f^2 p_r^2 h_{rr} + 2L f p_r r^{-2} h_{r\phi} + L^2 r^{-4} h_{\phi\phi}). \end{aligned} \quad (2.18)$$

The calculation of the Poisson brackets then gives, changing the variable of integration,

$$\begin{aligned} I(t_0) &= \int_0^\infty \left\{ \left[\frac{2f p_r}{r} h_{t\phi} - f p_r h_{t\phi,r} + \frac{1}{Er^2} \left(ME^2 - \frac{L^2 f^2}{r} - 7M f^2 p_r^2 + 2f^2 p_r^2 r \right) h_{r\phi} \right] \right. \\ &\quad + \left[\frac{EM p_r}{Lf} h_{tt} - \frac{p_r r^2}{2L} h_{tt,r} + \frac{1}{Lf} \left(ME^2 - \frac{L^2 f^2}{r} - M f^2 p_r^2 \right) h_{tr} \right. \\ &\quad - \frac{f^2 p_r^2 r^2}{L} h_{tr,r} + \frac{f p_r}{EL} \left(ME^2 - \frac{L^2 f^2}{r} - 2M f^2 p_r^2 \right) h_{rr} \\ &\quad \left. \left. - \frac{f^4 p_r^3 r^2}{2EL} h_{rr,r} + \frac{L f p_r}{Er^3} \left(2f - \frac{M}{r} \right) h_{\phi\phi} - \frac{L f^2 p_r}{2Er^2} h_{\phi\phi,r} \right] \right\} \frac{dz}{\sqrt{\beta z}}, \end{aligned} \quad (2.19)$$

where the orbit equations $r(z)$ and $p_r(z)$ are given by (2.9) and (2.10) respectively, and the perturbation h_{ab} by (2.12–2.15). This integral most likely could not be evaluated exactly

even if we knew the exact form of h_{ab} . However, in the high frequency approximation, in which we do know h_{ab} , we can calculate $I(t_0)$ by the saddle point method.

All h_{ab} 's and their derivatives contain a factor $\exp\{i\omega(t \pm r^*)\}$. Let us restrict ourselves to the lower sign, and define $u := t - r^*$. Then our integral is of the form

$$I(t_0) \sim \text{Re} \, e^{-i\omega t_0} \int_0^\infty dz \, e^{i\omega u(z)} F(r(z), p_r(z), \omega), \quad (2.20)$$

where, for high ω , F is a slowly varying function of z with respect to the exponential, except at the singularities of F . Thus, if we deform the path of integration into the complex z -plane, making it pass through a critical point of u , the leading order contribution to the integral will come from near this critical point, provided that nowhere along the path is $\text{Im} \, u(z) < 0$, to keep the integrand finite as $\omega \rightarrow \infty$.

In addition to values of z corresponding to $r = 0$ and $r = 2M$, which are singularities of F , u has critical points at

$$z_\pm(\beta) = \frac{-2(1 - 2\beta) \pm 3i\sqrt{2\beta(1 - \beta)}}{2(1 + \beta) + \sqrt{6\beta(1 + \beta)}}, \quad (2.21)$$

and it has poles only at $r = 0$ and $r = 2M$, corresponding to negative real values of z . We find that $\text{Im} \, u(z)$ is positive at $z_+(\beta)$ and negative at $z_-(\beta)$, for all β . In fact, from plots of $\text{Im} \, u(z)$ in the complex z -plane (an example of which is shown in figure 3) we see that for any β the positive real axis can be deformed into a path through $z_+(\beta)$, which is always in the half plane $\text{Im} \, z > 0$, without crossing a pole of F , and such that $\text{Im} \, u(z) > 0$ all along the path.

The above shows that the saddle point method can be used to calculate the leading order contribution to $I(t_0)$. Since, in addition, the function F (and its second derivative) can be evaluated at $z_\pm(\beta)$ and shown not to vanish there, the integral in (2.20) is not zero, and $I(t_0)$ vanishes only at the isolated, transverse zeroes of $\exp(-i\omega t_0)$, at least for high enough ω . It seems reasonable to conclude then that $I(t_0)$ in fact vanishes identically at most for a discrete set of values of ω , if we assume it is an analytic function of ω . This is hardly surprising. Given the form of the perturbation, the only alternative would have been for $I(t_0)$ to be always identically zero; but this would have meant that all terms in the expansion of the integral in inverse powers of ω had to vanish, which they are unlikely to do unless there was some simple reason for it, e.g., as a consequence of the linearised Einstein equation. We conclude that for a generic vacuum perturbation h_{ab} of the type considered here, a region of phase space near the homoclinic orbit becomes chaotic.

3. Concluding Remarks

We have shown that, under a large class of perturbations of the Schwarzschild metric, there exists a set of orbits near the homoclinic orbit, corresponding to the unperturbed situation, that is chaotic. In order to better understand the consequences of this simple result, our model can be refined in a number of ways. Staying within the one particle situation, more general periodic perturbations h_{ab} (with arbitrary l and m) can be considered, and for a given perturbation with finite frequency ω one should be able to obtain Poincaré sections for the perturbed trajectories, explicitly showing the chaotic behaviour. For the latter purpose, it would be useful to have an exact solution of the linearized Einstein equation, which may be

available for a small set of parameter values [18]. Similarly, all calculations can be repeated for a rotating, Kerr black hole, or for null geodesics. In the latter case, the goal would be to investigate possible consequences of these chaotic effects on the way black holes appear to us, rather than on the behaviour of matter around a black hole.

A more realistic model, however, would be obtained by treating combinations of the pure frequency perturbations that are everywhere regular (in particular at the horizon) and possibly non-periodic, and doing the calculations with a gas of relativistic particles around the black hole, where each one can be seen as moving in a fluid composed of the other ones. This may help us relate the present considerations to our understanding of the behaviour of matter around a black hole in astrophysics. It would also be interesting to compare the properties of the gravitational radiation emitted by particles in regular and stochastic orbits around black holes (if such orbits survive for radiating particles), and to extend the analysis to perturbations generated by other types of physically relevant mechanisms, such as due to another black hole passing by. In the case of a fluid surrounding the black hole, an axisymmetric fluid configuration would not radiate, so one could study perturbations of such a configuration including radiation reaction.

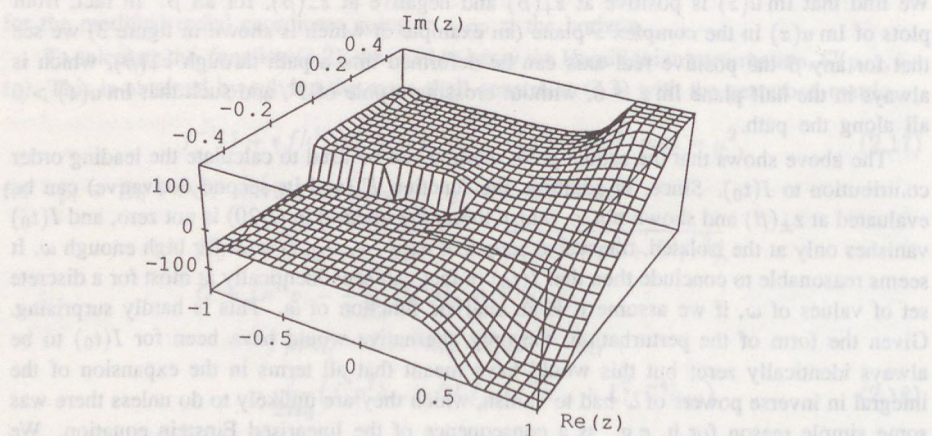


Figure 3. The imaginary part of $u(z)$ in the region $-1 < \text{Re } z < 1$, $-0.5 < \text{Im } z < 0.5$, for $\beta = 0.4$. A branch cut from $z = 0$ in the negative real direction is visible

Acknowledgments

I would like to thank the organizers for the opportunity to come to Gárdony and participate in a very fruitful and enjoyable conference, and the participants for stimulating discussions. This work is supported by the Directorate-General for Science, Research and Development of the Commission of the European Communities under contract no. CII*-0540-M(TT).

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Numerical Evolution of Spherically Symmetric Deviations from an Isotropic Universe

J. Frauendiener

Abstract: We describe the evolution of spherically symmetric deviations from a $k = 0$, radiation filled isotropic universe. We compare the solution to the full Einstein equations with those of the linearized equations. The evolutions are indistinguishable for initial data that is small enough. However, for large data, i.e. for initial density contrasts in the 1 percent range, trapped surfaces appear and singularities form.

1. Introduction

We live in a lumpy universe. This fact has to be taken into account when one calculates the anisotropy of the cosmic microwave background radiation. Usually this is done by treating the “lumps” as linear perturbations on an isotropic universe. One of the earliest calculations along these lines was performed by Sachs and Wolfe [1], who determined the general solution of the linearized Einstein equations on a dust or radiation filled universe ($k = 0$). This raises the question as to how justified is the use of perturbation theory. That is one would like to compare the linear and nonlinear evolution of deviations from an isotropic universe. This problem will be treated here numerically for spherically symmetric deviations [2].

It is clear that there are two parameters that are important when one tries such a comparison. First, the time interval over which one follows the evolution: qualitatively, the longer the interval the bigger the difference will be. Second, the initial deviation from the background solution: again qualitatively, the bigger the initial difference the bigger the final difference. In fact, for most evolving systems there are theorems to the effect that one can make the final difference between linear and nonlinear evolution arbitrarily small by choosing the initial difference or the time interval small enough. Therefore, we will choose here for both parameters values that come from physical considerations: the time span is taken to be the interval between the decoupling and the present; and we consider initial deviations whose amplitudes are in the range usually taken for fluctuations at the time of decoupling.

The numerical treatment of this problem is simplified enormously by the assumption of spherical symmetry. This reduces the problem to a $1 + 1$ hyperbolic system for which there is available an elaborate theory for existence and uniqueness of solutions as well as many methods for its numerical treatment.

We shall confine ourselves here to an essentially qualitative discussion without going into too much detail (see [2]). In section 2 we present a way to write the field equations for spherical symmetry developed by S. Kind [4]. The system is in quasilinear symmetric hyperbolic form. The singularity due to the polar coordinates is not present in the principal part. We describe the linearization of this system and its solutions in section 3. Finally, in section 4 we present our results.

2. The Field Equations in Spherical Symmetry

Our model is a time-dependent spherically symmetric space-time containing radiation that is described as an ideal isentropic fluid with energy density ρ and pressure $p = \frac{1}{3}\rho$. The general spherically symmetric metric can be written as

$$g = -e^{2\Phi} dT^2 + e^{2\Lambda} dR^2 + r^2 d\Omega, \quad (2.1)$$

with Φ , Λ , and r functions of the coordinates (T, R) only. $d\Omega$ is the metric of the standard unit 2-sphere. The ‘‘area radius’’ r is related to the area A of the spheres of transitivity labeled by (T, R) by $A = 4\pi r^2$. We adopt a Lagrangian point of view, i.e., we assume that R is a comoving radial coordinate such that the 4-velocity of the fluid is $e^{-\Phi}\partial_T$. We also require the existence of a regular center, i.e., a fluid line on which r vanishes and we label it by $R = 0$. Then the coordinates T and R are fixed up to the choice of T along the central fluid line and a reparametrization $R \mapsto g(R)$ where $g(0) = 0$. The essential field equations in this coordinate system are then (a dot means ∂_T and a prime is ∂_R)

$$\begin{aligned} \dot{\rho} + (\rho + p)(\dot{\Lambda} + 2\frac{\dot{r}}{r}) &= 0, \\ p' + (\rho + p)\Phi' &= 0, \\ (e^{-\Phi}\dot{r})' - e^{-\Phi}r'\dot{\Lambda} &= 0, \\ 8\pi p &= -\frac{1}{r^2}[1 - r'^2e^{-2\Lambda} + \dot{r}^2e^{-2\Phi}] + \frac{2}{r}r'\Phi'e^{-2\Lambda} - \frac{2}{r}e^{-\Phi}(\dot{r}e^{-\Phi})', \\ 8\pi\rho &= \frac{1}{r^2}[1 - r'^2e^{-2\Lambda} + \dot{r}^2e^{-2\Phi}] + \frac{2}{r}r'\dot{\Lambda}e^{-2\Phi} - \frac{2}{r}e^{-\Lambda}(r'e^{-\Lambda})'. \end{aligned} \quad (2.2)$$

Here, p is to be replaced by $p = \frac{1}{3}\rho$. Note, that this system is quasilinear, of second-order and singular at places where r , one of the dependent variables, vanishes.

Recently, Kind and Ehlers [5] have shown how to cast this system into a symmetric hyperbolic system. The key idea is to introduce a ‘‘logarithmic density’’ $\mathcal{B}$,

$$\mathcal{B}(\rho) = \int_{\rho_0}^{\rho} \frac{s(\bar{\rho})}{\bar{\rho} + p(\bar{\rho})} d\bar{\rho},$$

where s is the speed of sound in the fluid, $s^2 = \frac{dp}{d\rho}$. In our case we get $\mathcal{B} = \frac{\sqrt{3}}{4} \log \frac{\rho}{\rho_0}$ or, inverting, $\rho = \rho_0 \exp(\frac{4}{\sqrt{3}}\mathcal{B})$. Thus, we may regard ρ and p as functions of $\mathcal{B}$. The second field equation can be integrated immediately to yield

$$\Phi(\rho) = - \int_{\rho_0}^{\rho} \frac{s^2(\bar{\rho})}{\bar{\rho} + p(\bar{\rho})} d\bar{\rho},$$

thus defining Φ as a function of $\mathcal{B}$: $\Phi = -(1/\sqrt{3})\mathcal{B}$. The other variables are

$$\begin{aligned} Q &= \log \frac{R}{r}, & \omega &= r', \\ W &= \frac{v}{r}, & \text{where } v &= e^{-\Phi}\dot{r}, \\ \mu &= \frac{3}{r^3} \int_0^R \rho r'^2 dR, \\ P &= e^{-\Lambda}\mathcal{B}', & K &= \frac{v'}{\omega} + 2\frac{v}{r}, \end{aligned} \quad (2.3)$$

The meaning of some of the variables is the following: v is a coordinate velocity and as such it is not restricted by the requirement $v < c$. μ is a mean density within a ball of coordinate radius R . P is proportional to the density gradient and K is the trace of the extrinsic curvature of the constant T surfaces. The variables obey the following sets of equations:

$$\begin{aligned}
 \dot{P} + se^{\Phi-\Lambda}K' &= e^{\Phi} \left[PK \left(s^2 - \frac{ds}{dB} - 1 \right) + 2WP \right], \\
 \dot{K} + se^{\Phi-\Lambda}P' &= e^{\Phi} \left[P^2 \left(s^2 - \frac{ds}{dB} \right) - 2s\omega e^{Q-\Lambda} \frac{P}{R} \right. \\
 &\quad \left. - (K - 2W)^2 - 2W^2 - 4\pi(\rho + 3p) \right], \\
 \dot{B} &= -e^{\Phi} sK, \\
 \dot{W} &= -e^{\Phi} \left[s\omega e^{Q-\Lambda} \frac{P}{R} + W^2 + \frac{4\pi}{3} (\mu + 3p) \right], \\
 \dot{\Lambda} &= e^{\Phi} (K - 2W), \\
 \dot{\omega} &= e^{\Phi} \left[\omega (K - 2W) - se^{\Lambda-Q} RPW \right], \\
 \dot{Q} &= -e^{\Phi} W, \\
 \dot{\mu} &= -3e^{\Phi} W (\mu + p);
 \end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
 Z_1 &:= RQ' + \omega e^Q - 1 = 0, \\
 Z_2 &:= B' - e^{\Lambda}P = 0, \\
 Z_3 &:= Re^{-Q}W' - \omega(K - 3W) = 0, \\
 Z_4 &:= \frac{4\pi}{3} Re^{-Q}\mu' + 3\omega(\mu - \rho) = 0, \\
 Z_5 &:= e^{-2\Lambda}(\omega' - \omega\Lambda') + Re^{-Q} \left[\frac{4\pi}{3} (3\rho - \mu) - W(K - 2W) \right] = 0.
 \end{aligned} \tag{2.5}$$

This system has the following properties:

1. There are eight evolution equations (2.4) for the eight variables. It is easy to see that one can put the system into symmetric hyperbolic form by taking the sum and difference of the first two equations. Thus it can be written in the form $\partial_T u + A \partial_R u = b$ where u is column vector containing the eight dependent variables, $A = A(u, T, R)$ is a symmetric matrix and $b = b(u, T, R)$ is a column vector. In fact, here A is diagonal with only the first two entries nonvanishing. The first two equations describe the incoming and outgoing sound waves traveling within the fluid. Due to the spherical symmetry there are no gravity waves present.
2. There are five equations (2.5) for five of the variables containing only spatial derivatives. These "constraints" can be shown to be propagated by the evolution equations: taking time derivatives of the zero-quantities Z_i , commuting derivatives and using (1.6) one derives a quasilinear symmetric hyperbolic system for the Z 's which is homogeneous. Therefore, if $Z_i = 0$ at $T = T_0$ then $Z_i = 0$ always due to uniqueness of the solution of symmetric hyperbolic systems. The remaining three functions whose spatial derivatives do not appear in (2.5) are free functions. These are P , K and a combination of ω and Λ other than $e^{-\Lambda}\omega$.

3. Due to the use of polar coordinates the system is singular at $R = 0$. However, the singularity does not appear in the principal part of the system, it shows up only at two places on the right-hand side in the combination (P/R) .

The background Friedmann solution with $k = 0$ is described in our coordinate system by

$$\Phi = 0, \quad e^{2\Lambda} = 2H_0T, \quad r = e^\Lambda R, \quad \rho = 3/(4T^2).$$

H_0 is the present value of the Hubble parameter which we set equal to one by appropriately choosing the physical units. From these expressions one easily derives the values of the Kind-Ehlers variables for the Friedmann solution.

In the actual computation we use yet another set of variables, $u_0 \dots u_7$, which are the deviations of the values of the variables from their Friedmann values. Thus the Friedmann solution corresponds to $u_i = 0$. These variables are less singular near $T = 0$ which can be seen from the linearized equations.

The evolution of initial data now proceeds as follows: first, we need to determine the initial data for all the variables. To this end we specify the free functions on the initial timeslice $T = T_0$ as outlined below and then solve the constraint equations to obtain the values for the other variables. These are used subsequently to determine the variables at later times by using the evolution equations. Solving the constraint equations is also very much simplified by the assumption of spherical symmetry. Here we need to solve ordinary differential equations whereas usually one has to solve an elliptic boundary value problem.

The free functions are P , K and ω , say. $\omega = r'$ can be considered to be "pure gauge" because prescribing ω is exactly exploiting the freedom in the choice of the comoving radial coordinate. We choose $\omega = 1$, i.e., $r = R$. P is proportional to the density gradient and K , given ω , is determined by the velocity and its gradient. Thus the free data are exactly what one expects on physical grounds, namely the pressure and velocity distribution of the fluid.

3. The Linearized System and its Solution

It is quite straightforward to linearize the nonlinear system of equations around the Friedmann background. The resulting linear system has exactly the same properties as the nonlinear one: eight evolution equations together with five constraints that are propagated by the evolution equations and containing three free functions, the linearized versions of P , K and ω . We omit the linear system here since it is not very illuminating (cf. [2]).

The general solution of the linearized Einstein equations (no symmetry assumed) on a dust or radiation filled isotropic universe has been derived by Sachs and Wolfe [1]. They write the perturbed metric in the form

$$g = g_F + h = t^2 \{-dt^2 + dR^2 + R^2 d\Omega + h_{ab} dx^a dx^b\}, \quad (3.1)$$

where $x^a = (t, R, \theta, \phi)$ and $t = \sqrt{2T}$. The general solution is characterized by three different "modes": density perturbations, that travel through the fluid with sound speed; the perturbations of the gravitational field, i.e., gravity waves moving on light characteristics and velocity perturbations that induce vorticity into the fluid flow. Under the assumption of spherical symmetry only the density perturbations survive. These can be obtained from a single "potential", a function $E(t, R)$ that satisfies the flat spherically symmetric wave equation,

$$3\partial_t^2 E - \partial_R^2 E - \frac{2}{R} \partial_R E = 0. \quad (3.2)$$

To obtain an explicit expression of the perturbed metric in our coordinate system from the expressions given in [1] we need to perform a gauge transformation on the metric (3.1) by adding a term $\partial_{(a}\xi_{b)}$ for an appropriate 1-form ξ_a . This enables us to write the linearized metric in [1] as a linearization of our metric (2.1). We obtain

$$\begin{aligned}\delta\Phi &= -\frac{1}{4t}\left(\ddot{t}E - \ddot{E}\right), \\ \delta\Lambda &= -\frac{1}{4t}\left(t\partial_{RR}\dot{E} - 2\partial_{RR}E + \ddot{E}\right), \\ \delta r &= -\frac{1}{4t}\left(t\partial_R\dot{E} - 2\partial_RE + R\ddot{E}\right), \\ \delta\rho &= -\frac{1}{4t^5}\left(\ddot{t}E - \ddot{E}\right).\end{aligned}$$

Given these perturbations of the metric functions it is easy to derive the perturbed variables (2.3) from an arbitrary solution of (3.2). This may be written generally as

$$E(t, R) = \frac{(g_1(t - \sqrt{3}R) - g_2(t + \sqrt{3}R))}{R},$$

with free functions g_1 and g_2 of one real variable. The requirement of regularity at $R = 0$ reduces this freedom to one function g of one real variable,

$$E(t, R) = \frac{(g(t - \sqrt{3}R) - g(t + \sqrt{3}R))}{R}.$$

We use the explicit solution of the linearized equations to check the accuracy of the code.

4. Results

In our calculations we use $t = \sqrt{2T}$ as our time coordinate. In the unperturbed background solution t is the conformal time. The time interval from decoupling to the present is described by $0.03 \leq t \leq 1$ in this time coordinate. We specify the initial data at $t = 0.03$ in the following way. First, as mentioned above, we choose the area radius as the comoving radial coordinate initially, i.e., $\omega = 1$. Then we prescribe the density contrast as a function of R . Finally we set $v = v_F$, the Friedmann velocity at $t = 0.03$, i.e., initially the density distribution is at rest with respect to the expanding universe.

We want to compare linear and nonlinear evolution of the same initial configuration. By “same” we mean that the density contrast in both cases is described by the same function in terms of the radial coordinate R . We considered two initial configurations: a shell of matter with a higher density than the homogeneous Friedmann density and a lump of denser matter at the center. The density contrast was given by the function

$$\frac{\delta\rho}{\rho_F} = A \exp\left(\frac{(R - R_0)^2}{(R - R_0)^2 - h^2}\right),$$

If $R_0 = 0$ we have the central lump and $h \leq R_0$ describes a shell with thickness $2h$. A is the maximal value of the density contrast, reached at $R = R_0$. It turns out that for amplitudes in the range $10^{-4} \leq A \leq 10^{-2}$, which we consider here, the two cases do not differ in an essential way, so we concentrate on the shell only.

The linear evolution of the shell is as follows: the shell starts to contract under the pull of its own gravitational field, reaches a state of maximal density at $R = 0$, bounces off and disperses. After the bounce, a region around the center appears, where the density contrast is virtually constant and different from zero. Further outward it decreases to zero. As time moves on this region of homogeneity expands and, at the same time, the maximal value of the density contrast decreases. It seems likely, that for $t \rightarrow \infty$ a $k = 0$ Friedmann universe will remain, the perturbation dispersing without any trace, cf. Figure 1.

As to the nonlinear evolution, we can confirm the expectation based on the theory of symmetric hyperbolic equations, that for small values of A the evolution very much resembles the linear one. In Figure 2 the difference in the final density contrast for the linear and the nonlinear evolution at $A = 10^{-4}$ is shown, the relative difference being approximately 2%.

As A grows this difference increases rapidly. The nonlinear evolution is always “faster” than the linear one. In Figure 3, we plot the logarithm of the maximal value of the final density contrast due to the nonlinear evolution against the one due to the linear evolution. The dotted line indicates the diagonal. For $A = 10^{-4}$ there is not much difference. But for $A = 8 \times 10^{-3}$ the difference is about 2.5 orders of magnitude.

For $A = 0.01$ the nonlinear evolution is as follows: at $t \approx 0.15$ the maximal density contrast has increased by approximately one order of magnitude. At $t \approx 0.4$ the density contrast is at 1.5 and the dense region has reached the center, the density profile becomes quite steep and the gravitational attraction has grown such that the interior region ($R < 0.6$) has a considerably smaller expansion rate than the rest of the universe. At $t \approx 0.63$ the expansion in the interior has actually turned negative, indicating that this region is “cut off” from the evolution of the outside universe and bound to collapse under its own gravitational

Figure 1: The linear evolution of a shell with $A = 10^{-4}$ for different values of t .

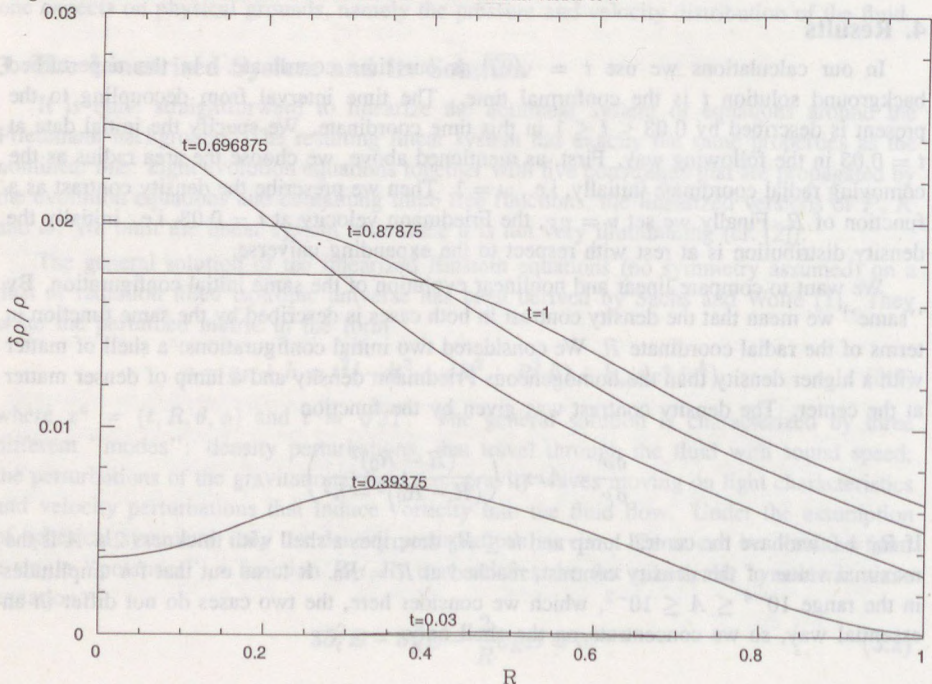


Figure 2: Final density contrast for $A = 10^{-4}$ for the nonlinear (solid line) and the linear (dotted line) evolution.

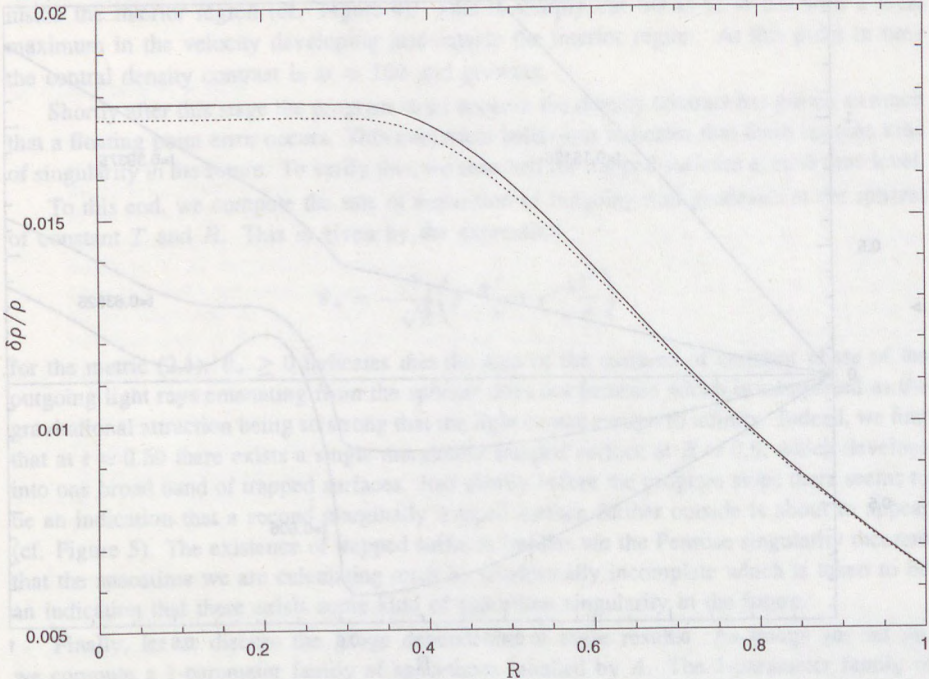


Figure 3: Logarithm of the maximal final density contrast for different initial density contrasts.

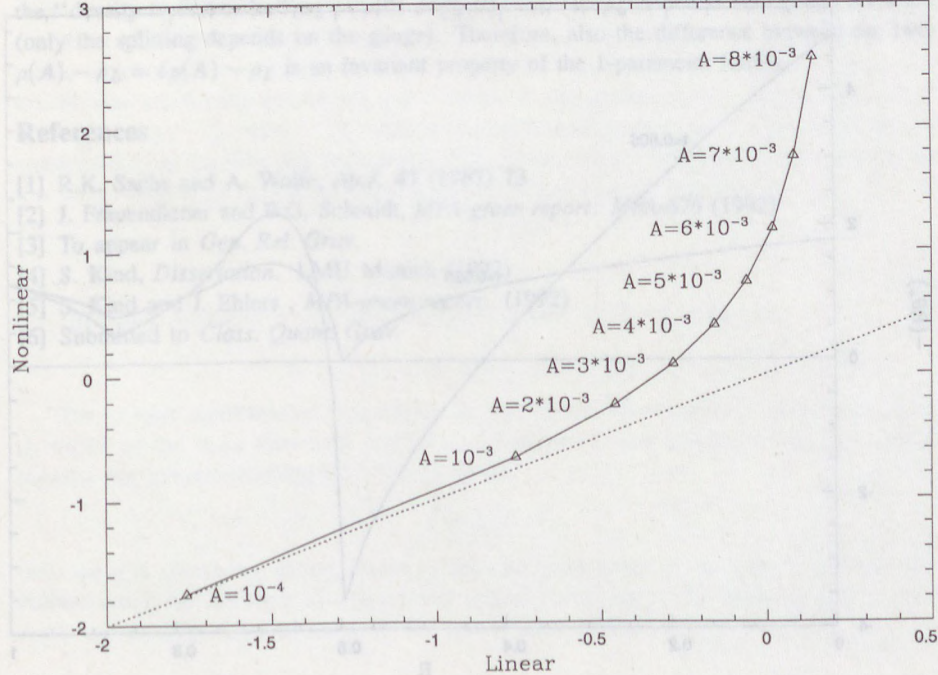
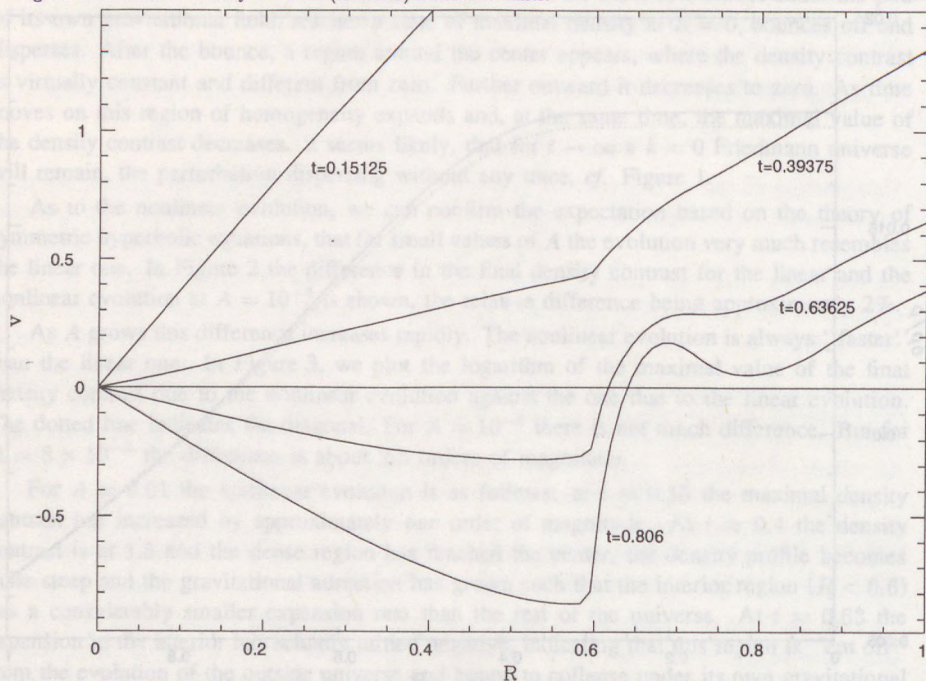
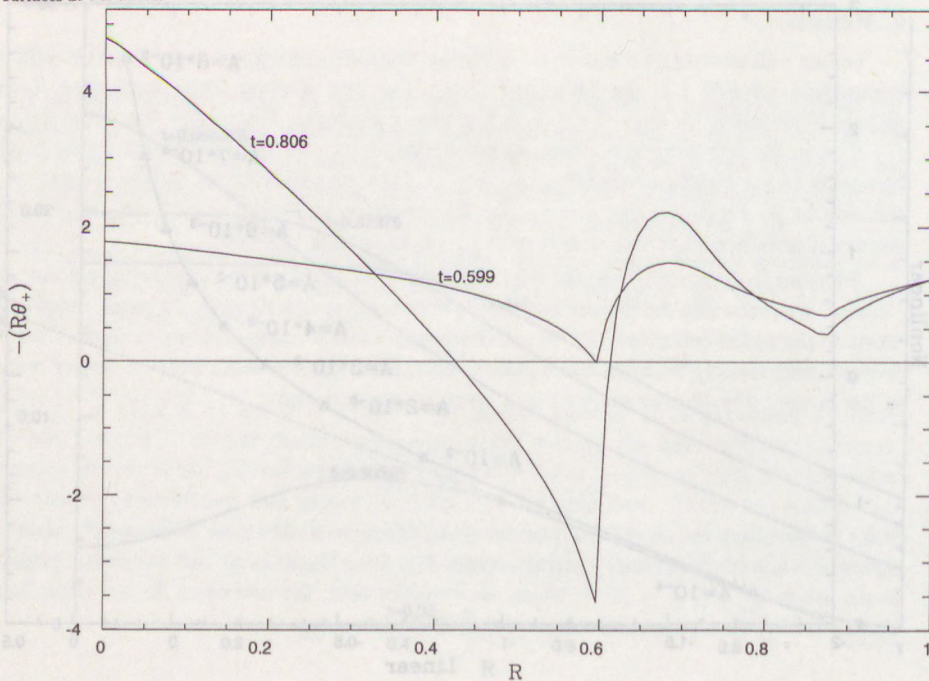


Figure 4: Coordinate velocity $v = e^{-\Phi} \dot{r}$ ($A = 0.01$) at different times.Figure 5: Expansion of the outgoing light rays showing the marginally trapped surface at $t = 0.599$ and the band of trapped surfaces at $t = 0.806$.

attraction. This continues until $t \approx 0.8$ where there is a big matter flow towards the center inside the interior region (cf. Figure 4). This is sharply cut off at $R = 0.6$ with a local maximum in the velocity developing just outside the interior region. At this point in time the central density contrast is at ≈ 150 and growing.

Shortly after this stage the program stops because the density contrast has grown so much that a floating point error occurs. This numerical behaviour indicates that there is some kind of singularity in the future. To verify this, we searched for trapped surfaces at each time level.

To this end, we compute the rate of expansion of outgoing null geodesics at the spheres of constant T and R . This is given by the expression

$$\theta_+ = -\frac{1}{\sqrt{2}} \left(e^{-\Phi} \frac{\dot{r}}{r} + e^{-\Lambda} \frac{r'}{r} \right)$$

for the metric (2.1). $\theta_+ \geq 0$ indicates that the area of the surfaces of constant phase of the outgoing light rays emanating from the spheres does not increase which is interpreted as the gravitational attraction being so strong that the light cannot escape to infinity. Indeed, we find that at $t \approx 0.59$ there exists a single marginally trapped surface at $R \approx 0.6$, which develops into one broad band of trapped surfaces. Just shortly before the program stops there seems to be an indication that a second marginally trapped surface further outside is about to appear (cf. Figure 5). The existence of trapped surfaces implies *via* the Penrose singularity theorem that the spacetime we are calculating must be geodesically incomplete which is taken to be an indication that there exists some kind of spacetime singularity in the future.

Finally, let us discuss the gauge dependence of these results. As things are set up, we compute a 1-parameter family of spacetimes labelled by A . The 1-parameter family of densities $\rho(A) = \rho_F(1 + \frac{\delta\rho}{\rho_F}(A))$ consists of scalars for each value of A and determines the “density to first-order” $\rho_L = \rho(0) + \frac{d\rho}{dA}(0) = \rho_F + \delta\rho_L$ which is also gauge invariant (only the splitting depends on the gauge). Therefore, also the difference between the two, $\rho(A) - \rho_L = \delta\rho(A) - \rho_L$ is an invariant property of the 1-parameter family.

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The Canonical Structure of Parametric Manifolds

G. Fodor and Z. Perjés

Abstract: Canonical structures based on a 3+1 decomposition of the Einstein-Hilbert Lagrangian with respect to a generic congruence of timelike curves are investigated. The non-Riemannian geometry of the curves, considered to be points of a 3 - D differentiable manifold, incorporates time as a parameter in the differential structure of the manifold.

The Lagrangian of the system contains as canonical variables the spatial 3-metric g_{ik} , its conformal multiplier f and the connection one-form ω . One of the fields, the redshift factor f , emerges as a secondary entity that interacts with a system consisting of the 3-metric g and the connection form ω . We are employing the (g, ω) system for developing a canonical approach. We find an extension of the phase-space of the system which is computationally viable. In fact, we obtain the Hamiltonian density $\mathcal{H} = \sqrt{g} \left(R^{(3)} - \omega_k D_i H^{ik} - \lambda_{ik} H^{ik} \right)$ where D_i is the Riemannian derivative, H^{ik} and λ_{ik} are canonical coordinates in the extended phase space. The primary constraints and most of the equations for the multiplying functions are reasonably simple in the extended phase space.

The present approach to canonical gravity reduces to the ADM formulation when the congruence is chosen hypersurface-orthogonal. It can be adopted when space-times with no global spacelike foliation need to be considered. New gauge fixings are made possible here, though our discussion is fully gauge-independent.

1. Introduction

The notion of a parametric manifold emerges when considering the differential geometry of congruences of curves in a space-time. There are two mathematical features which suggest that these congruences represent a special class of manifolds. The first (which has already been implied in computations by Zelmanov [1]) is that the projected coordinate derivatives, non-commutative in nature, entirely appropriate the role of the (commutative) coordinate derivatives commonly encountered in General Relativity. All differentiable and metric structures in these congruences are subsequently built up as in Riemannian geometry, except that coordinate derivatives are substituted throughout by the "starry derivative" $\partial_i^* = \frac{\partial}{\partial x^i} - \omega_i(x^j) \frac{\partial}{\partial t}$ where t is an adapted coordinate along the curves. The congruence is considered to be timelike but at least non-null. There exists, then, a preferred vector field $a = \partial/\partial t$, tangent to the curves. The allowed coordinate transformations are

$$\begin{aligned} t' &= t + F(x^i), \\ x^{i'} &= x^i(x^k). \end{aligned} \quad (1.1)$$

The starry derivative is invariant provided the field $\omega_i(x)$ transforms as

$$\omega'_i = \omega_i - \nabla_i F.$$

The second mathematical observation is that in the decomposition of the curvature quantities of the space-time with respect to a congruence with tangent a , one can clearly separate the terms containing the squared length,

$$f = a_\mu a^\mu,$$

from the rest. There is a unique choice of the conformal gauge of the 3-space in which the Hilbert action has the form of a dynamical system consisting of the fields g_{ik} and ω_i plus terms that describe the interaction of this system with the field f .

Definition: A *parametric manifold* is a pair $(\mathcal{M}, \omega)$, where $\mathcal{M}$ is a differentiable manifold and

$$\omega : R \rightarrow \{s(T^*(\mathcal{M}))\},$$

is a smooth one-parameter family of 1-form fields with $\{s(T^*(\mathcal{M}))\}$ the module of sections of the cotangent bundle, such that a reparametrization

$$t' = t + F(x) \quad t, t' \in R \quad (1.2)$$

is escorted by the transformation

$$\omega'(t') = \omega(t) - dF. \quad (1.3)$$

Definition: A (p, q) -*tensor* in a parametric manifold is a map

$$T_q^p : R \rightarrow \{s(T_q^p(\mathcal{M}))\} \quad (1.4)$$

into the module of sections of the tensor bundle $T_q^p(\mathcal{M})$ which transforms under reparametrization (1.2): $T_q^p(t') = T_q^p(t)$. A $(0, 0)$ -tensor is a parametrized family $\varphi(t)$ of functions on $\mathcal{M}$.

2. Generalized Connections

The mathematical theory of generalized connections has been worked out by Otsuki [2]. Projected derivatives have been introduced by Lottermoser [3].

Definition: A *generalised connection* [2] ∇ is a map $\nabla : T_q^p \rightarrow T_{q+1}^p$ with the properties

(a) On a scalar φ , $\nabla \varphi = d_* \varphi$ where the “starry differential” is defined

$$d_* \varphi \stackrel{\text{def}}{=} d\varphi - \omega \dot{\varphi} \quad (2.1)$$

and a dot denotes $\frac{\partial}{\partial t}$;

(b) ∇ is linear and commutes with contractions;

(c) The Leibnitz rule holds for the tensor product $\nabla(S \otimes T) = \nabla S \otimes T + S \otimes \nabla T$;

(d) ∇ has zero torsion, but nonvanishing deficiency, $\mathcal{D} = d_* \omega$;

(e) ∇ is a metric connection on a metric $\mathcal{M}$.

Definition: The *deficiency* $\mathcal{D}$ of a generalized connection is

$$\nabla_{[i} \nabla_{j]} \varphi = \mathcal{D}_{ij} \dot{\varphi}. \quad (2.2)$$

In local coordinates, the covariant derivative of a vector v is

$$\nabla_i v^k = v_{*i}^k + \Gamma_{ij}^k v^j \quad (2.3)$$

where $i = 1, 2, \dots, D$. From property (e), the connection components can be expressed in terms of the metric components and starry derivatives: $\Gamma_{ik}^l = \frac{1}{2} g^{lr} (g_{ir*k} + g_{kr*i} - g_{ik*r})$.

The *parameter derivative* ∇_o of a (p, q) tensor,

$$\nabla_o T_{kl\dots}^{ij\dots} = \dot{T}_{kl\dots}^{ij\dots} \quad (2.4)$$

is again a (p, q) tensor.

The commutators on a scalar φ are

$$\begin{aligned}(\nabla_i \nabla_k - \nabla_k \nabla_i) \varphi &= (\omega_{i*k} - \omega_{k*i}) \dot{\varphi}, \\ (\nabla_i \nabla_o - \nabla_o \nabla_i) \varphi &= \dot{\omega}_i \dot{\varphi}.\end{aligned}\quad (2.5)$$

The *Zelmanov curvature*, Z_{rijk} , is defined by

$$[\nabla_k \nabla_j - \nabla_j \nabla_k + (\omega_{j*k} - \omega_{k*j}) \nabla_o] v_i = Z_{ijk}^r v_r, \quad (2.6)$$

and has the algebraic properties

$$Z_{ij(kl)} = 0, \quad Z_{i[jkl]} = 0. \quad (2.7)$$

The action of the commutator on the metric yields the decomposition

$$Z_{ijkl} = K_{ijkl} + \dot{g}_{ij} \omega_{[k*l]} \quad (2.8)$$

with

$$K_{ijkl} = K_{[ij][kl]}. \quad (2.9)$$

The Riemannian part R_{ijkl} of the curvature is

$$R_{ijkl} = R_{[ij][kl]}, \quad R_{i[jkl]} = 0. \quad (2.10)$$

We have

$$Z_{ijkl} = R_{ijkl} + \dot{g}_{ij} \omega_{[k*l]} + \frac{1}{2} (\dot{g}_{ik} \omega_{[j*l]} + \dot{g}_{il} \omega_{[k*j]} + \dot{g}_{jk} \omega_{[l*i]} + \dot{g}_{jl} \omega_{[i*k]}). \quad (2.11)$$

The *parametric curvature*, W_{ikl} , is defined by the commutator

$$[\nabla_i \nabla_o - \nabla_o \nabla_i - \dot{\omega}_i \nabla_o] v_k = [W_{ikl} + \frac{1}{2} (\nabla_i - \dot{\omega}_i) \dot{g}_{kl}] v^l. \quad (2.12)$$

This is given by

$$W_{ijk} = \partial_o g_{i[k*j]} - \dot{g}_r [{}^r \Gamma_j]_i, \quad (2.13)$$

and has the symmetry properties

$$W_{ijk} = W_{i[jk]}, \quad W_{i[jk]} = 0. \quad (2.14)$$

Finally, the Bianchi identity generalizes to

$$\nabla_{[i} K_{j]k[lm]} = 2 \omega_{[i*l} W_{m]kj}. \quad (2.15)$$

3. Space-time as a Parametric Manifold

The decomposition of space-time tensors will be exemplified by the decomposition of the 4-vector,

$$\hat{u} = \{u^i, u_o\}. \quad (3.1)$$

The vertical and horizontal parts of $\hat{u}$ are, respectively,

$$u_o \stackrel{\text{def}}{=} \hat{u}^\mu a_\mu, \quad (u^i, 0) \stackrel{\text{def}}{=} (\delta_\nu^\mu - f^{-1}a^\mu a_\nu)\hat{u}^\nu. \quad (3.2)$$

The 4-metric can then be decomposed,

$$(\hat{g}_{\mu\nu}) = \begin{pmatrix} -f^{-1}g_{ij} + f\omega_i\omega_j & f\omega_i \\ f\omega_j & f \end{pmatrix} \quad (3.3)$$

with inverse

$$(\hat{g}^{\mu\nu}) = \begin{pmatrix} -fg^{ij} & f\omega^i \\ f\omega^j & f^{-1} - f\omega^2 \end{pmatrix}, \quad (3.4)$$

where g is the metric of the parametric 3-manifold of trajectories.

It follows that the components of the vector $\hat{u}$ are

$$\hat{u}^\mu = (u^i, f^{-1}u_o - u^r\omega_r), \quad \hat{u}_\mu = (-f^{-1}u_i + \omega_i, u_o). \quad (3.5)$$

The scalar product of vectors $\hat{u}$ and $\hat{v}$ has the parametric decomposition

$$\hat{u}^\mu \hat{v}_\mu = f^{-1}(u_o v_o - u^i v^k g_{ik}). \quad (3.6)$$

The soldering forms σ satisfy the defining relation,

$$\sigma_{\mu AC} \sigma_{\nu B}^{C'} + \sigma_{\nu AC} \sigma_{\mu B}^{C'} = \hat{g}_{\mu\nu} \epsilon_{AB}. \quad (3.7)$$

Contraction with $a^\mu a^\nu$ yields the product rule for the $a_{AB'}$ spinors:

$$a_{AC'} a_B^{C'} = \frac{1}{2} f \epsilon_{AB}, \quad (3.8)$$

where $a_{AB'} = \sigma_{oAB'}$. From the mixed components with $(\mu, \nu) = (i, 0)$ we get

$$\sigma_{AC'}^i a_B^{C'} + a_{AC'}^i \sigma_B^{C'} = 0. \quad (3.9)$$

We now introduce the soldering forms in the 3-space by

$$\sigma_{AB}^i = -\frac{\sqrt{2}}{f} \sigma_{AC'}^i a_B^{C'}. \quad (3.10)$$

(3.9) then expresses the symmetry of the 3-dimensional soldering forms in their spinor indices.

Expanding the covariant derivative ∇_i in terms of the Riemannian three-dimensional covariant derivative D_i , the Hilbert action of General Relativity is

$$\mathcal{L} = \sqrt{g} \left[-^{(3)}R \right. \\ \left. + \frac{1}{4}\omega^2 \dot{g}^{ij} \dot{g}_{ij} + \frac{1}{4}\omega^2 (g^{ij} \dot{g}_{ij})^2 - \frac{1}{2}\omega_i \dot{g}^{ij} \dot{g}_{jk} \omega^k \right. \\ \left. - \frac{1}{2}\omega^i \omega^j \dot{g}_{ij} g^{kl} \dot{g}_{kl} + \omega_i \dot{\omega}_j \dot{g}^{ij} + \omega^k \dot{\omega}_k \dot{g}_{ij} g^{ij} \right. \\ \left. - \dot{g}^{ij} D_i \omega_j - g^{ij} \dot{g}_{ij} D^k \omega_k \right. \\ \left. - \frac{3}{2f^2} \dot{f}^2 - \frac{1}{2f^2} \dot{f}^2 \omega^2 + \frac{1}{2} g^{ij} \dot{\omega}_i \dot{\omega}_j f^2 \omega^2 - \frac{1}{2} f^2 (\omega^i \dot{\omega}_i)^2 - \frac{1}{4f^2} g^{ij} \dot{g}_{ij} \right. \\ \left. - \frac{1}{4f^2} (g^{ij} \dot{g}_{ij})^2 - 2 \frac{\dot{f}}{f} \omega^i \dot{\omega}_i + \frac{\dot{f}}{f^3} g^{ij} \dot{g}_{ij} + \frac{\dot{f}}{f^2} \omega^i D_i f + \frac{2}{f} \dot{\omega}_i D^i f \right. \\ \left. + f^2 \omega^i \dot{\omega}_j D^j \omega_i - f^2 \omega^i \dot{\omega}_j g^{jk} D_k \omega_i - \frac{1}{2f^2} (D^i f)(D_i f) \right. \\ \left. + \frac{1}{2} f^2 g^{ij} (D_k \omega_i)(D^k \omega_j) - \frac{1}{2} f^2 (D^j \omega_i)(D^i \omega_j) \right]. \quad (3.11)$$

Here there are three kinds of terms: $R^{(3)}$, the true Ricci scalar of the 3-space, does not contain either ω or f ; the terms in the box do not contain any f factor, but the remaining terms are quite general.

4. A Free Parametric Manifold

Keeping the terms in (3.11) which do not contain the scalar f , the action of a free parametric manifold is

$$\mathcal{L}^{(\omega)} = \sqrt{g} \left[-R^{(3)} - g^{ij} D_i \omega_j - g^{ij} \dot{g}_{ij} D^k \omega_k + \omega^k \dot{\omega}_k g^{ij} \dot{g}_{ij} + \omega_i \dot{\omega}_j \dot{g}^{ij} \right. \\ \left. - \frac{1}{2} \omega^i \omega^j \dot{g}_{ij} g^{kl} \dot{g}_{kl} - \frac{1}{2} \omega_i \dot{g}^{ij} \dot{g}_{jk} \omega^k + \frac{1}{4} \omega^2 (g^{ij} \dot{g}_{ij})^2 + \frac{1}{4} \omega^2 \dot{g}^{ij} \dot{g}_{ij} \right]. \quad (4.1)$$

Varying $\mathcal{L}^{(\omega)}$ with respect to ω_i and g_{ij} yields the Euler-Lagrange equations, respectively

$$\nabla_j H^{ij} = 0, \quad (4.2)$$

where

$$H^{ij} = \dot{g}^{ij} + (g^{kl} \dot{g}_{kl}) g^{ij}, \quad (4.3)$$

and

$$R_{ij} + \frac{1}{2} g^{kl} (\dot{g}_{ki} \nabla_l [\omega_j] + \dot{g}_{kj} \nabla_l [\omega_i]) - g_{ij} g^{kl} (\nabla_k - \dot{\omega}_k) \dot{\omega}_l + \dot{\omega}_i \dot{\omega}_j - \nabla_i (\dot{\omega}_j) = 0. \quad (4.4)$$

Taking algebraic combinations yields the constraint

$$C_1 \equiv \omega^2 D^i \dot{\omega}_i - \omega^i \omega^j D_i \dot{\omega}_j + (\omega^i \dot{\omega}_i)^2 - \omega^2 g^{ij} \dot{\omega}_i \dot{\omega}_j - \frac{1}{2} \omega^2 \dot{g}^{ij} \omega_i \dot{\omega}_j - \frac{1}{2} \omega^2 \omega^k \dot{\omega}_k g^{ij} \dot{g}_{ij} \\ - \frac{1}{8} \omega^4 \dot{g}^{ij} \dot{g}_{ij} + \frac{1}{4} \omega^2 \omega_i \dot{g}^{ij} \dot{g}_{jk} \omega^k + \frac{1}{4} \omega^2 \omega^i \omega^j \dot{g}_{ij} g^{kl} \dot{g}_{kl} - \frac{1}{8} \omega^4 (g^{ij} \dot{g}_{ij})^2 \\ + \frac{1}{2} \omega^2 \dot{g}^{ij} D_i \omega_j + \omega^i \omega^j \dot{g}_{jk} D^k \omega_i - \frac{1}{2} g^{kl} \dot{g}_{kl} \omega^i \omega^j D_i \omega_j - \frac{1}{2} \omega^i \omega^j \dot{g}_{ij} D^k \omega_k \\ + \frac{1}{2} \omega^2 \dot{g}^{ij} \dot{g}_{ij} D^k \omega_k + \omega^i \omega^j {}^{(3)}R_{ij} - \frac{1}{2} \omega^2 R^{(3)} = 0. \quad (4.5)$$

5. The Hamiltonian

The canonical momenta of the (g, ω) system with Lagrangian (4.1) are obtained by variation with respect to $\dot{\omega}_i$ and $\dot{g}_{ik}$:

$$p^i = \sqrt{g}(\dot{g}^{ij}\omega_j + \omega^i g^{jk}\dot{g}_{jk}) \quad , \quad (5.1)$$

$$\pi^{ij} = \sqrt{g}\left(D^{(i}\omega^{j)} - g^{ij}D^k\omega_k + g^{ij}\omega^k\dot{\omega}_k - \omega^{(i}g^{j)k}\dot{\omega}_k - \frac{1}{2}\omega^i\omega^j g^{kl}\dot{g}_{kl} - \frac{1}{2}g^{ij}\omega^k\omega^l\dot{g}_{kl} - \omega^{(i}\dot{g}^{j)k}\omega_k + \frac{1}{2}\omega^2 g^{ij}g^{kl}\dot{g}_{kl} + \frac{1}{2}\omega^2\dot{g}^{ij}\right) \quad . \quad (5.2)$$

We have one primary constraint,

$$\omega_i\omega_j\pi^{ij} = \sqrt{g}(\omega^i\omega^j D_i\omega_j - \omega^2 D^i\omega_i) \quad . \quad (5.3)$$

The Hamiltonian density is complicated.

An alternative approach to the Hamiltonian structure of the (g, ω) system arises by enlarging the phase space by the canonical coordinates H^{ik} and by the corresponding momenta. The quantities H^{ik} have been first introduced in the field equation (4.2) for no other reason than economy of writing.

We can eliminate all derivatives of the field ω_i by subtracting divergence terms,

$$\begin{aligned} \mathcal{L}^{(\omega)} = \sqrt{g} \left\{ -R^{(3)} + \omega_k D_i H^{ik} - \frac{1}{2} \dot{H}^{ik} \omega_i \omega_k \right. \\ \left. - H^{ik} \left(\frac{1}{4} \omega_i \omega_k g^{ab} \dot{g}_{ab} + \frac{1}{2} \dot{g}_{il} \omega_k \omega^l - \frac{1}{4} \omega^2 \dot{g}_{ik} \right) \right\} \quad (5.4) \end{aligned}$$

This form of the action suggests that ω_i is the potential of an external field.

We can achieve a first-order form of the Lagrangian by considering independent variables and adding the definition (4.3) of H^{ik} to the Lagrangian with a Lagrange multiplier function λ_{ik} ,

$$\begin{aligned} \mathcal{L}^{(H)} = \sqrt{g} \left\{ -R^{(3)} + \omega_k D_i H^{ik} - \frac{1}{2} \dot{H}^{ik} \omega_i \omega_k - H^{ik} \right. \\ \left. \left(\frac{1}{4} \omega_i \omega_k g^{ab} \dot{g}_{ab} + \frac{1}{2} \dot{g}_{il} \omega_k \omega^l - \frac{1}{4} \omega^2 \dot{g}_{ik} \right) + \lambda_{ik} (H^{ik} - \dot{g}^{ik} - g^{ik} g^{rs} \dot{g}_{rs}) \right\} \quad (5.5) \end{aligned}$$

As a result, the fields g^{ik} , H^{ik} , λ_{ik} and ω_i , are allowed to be treated as independent coordinates.

The momenta are obtained by varying the Lagrangian with respect to the time derivatives of the canonical coordinates.

The Hamiltonian is $H = \int \mathcal{H} d^3x$, where the Hamiltonian density is

$$\mathcal{H} = \sqrt{g} \left(R^{(3)} - \omega_k D_i H^{ik} - \lambda_{ik} H^{ik} \right) \quad (5.6)$$

and does not contain any momentum variable.

None of the momenta $p_n = (\pi_{ik}, \Pi_{ik}, \mathcal{P}^{ik}, P^i)$ corresponding to the coordinates $q^n = (g^{ik}, H^{ik}, \lambda_{ik}, \omega_i)$ contain any time derivative. Hence the momenta yield the primary Hamiltonian constraints:

$$C_{ik}^g = \pi_{ik} + \sqrt{g} \left(\frac{1}{2} \omega_a \mu_{ik}^a + \lambda_{ik} - \lambda_a^a g_{ik} \right), \quad (5.7)$$

$$C_{ik}^H = \Pi_{ik} + \frac{1}{2} \sqrt{g} \omega_i \omega_k, \quad C_\lambda^{ik} = P^{ik}, \quad C_\omega^i = P^i,$$

where we introduce the tensor

$$\mu_{abi} = \frac{1}{2} (\omega_i H_{ab} - H_{ij} \omega^j g_{ab} - \omega^j H_{j(a} g_{b)i} - H_{i(a} \omega_{b)}) \quad (5.8)$$

with the symmetry property $\mu_{abi} = \mu_{(ab)i}$. These constraints are all second class, since the Poisson brackets $\{C^g, C_\lambda\}$ and $\{C^H, C_\omega\}$ do not weakly vanish.

Following Dirac [3], we introduce the total Hamiltonian $H_T = \int \mathcal{H}_T d^3x$ of the constrained system,

$$\mathcal{H}_T = \mathcal{H} + u_g^{ik} C_{ik}^g + u_H^{ik} C_{ik}^H + u_\lambda^i C_\lambda^{ik} + u_\omega^i C_\omega^i, \quad (5.9)$$

where the unknown coefficients u_m are functions of the coordinates x^i and parameter t .

The condition that the primary constraints are preserved can be expressed that the Poisson brackets of these constraints with the total Hamiltonian vanish. Since none of the primary constraints contain any derivative term, we can employ a distributional notation where the multiplier functions of the constraints are not displayed:

$$\begin{aligned} \{C_{ab}^g, H_T\} = & \sqrt{g} \left[-^{(3)}R_{ab} + \frac{1}{2} R^{(3)} g_{ab} + \frac{1}{2} g_{ab} H^{ik} D_i \omega_k + D^j (H_{j(a} \omega_{b)}) \right. \\ & - \frac{1}{2} D^j (H_{ab} \omega_j) - \frac{1}{2} g_{ab} \lambda_{ik} H^{ik} + u_H^{ik} \left(\frac{1}{4} \omega^2 g_{ai} g_{bk} - \frac{1}{2} \omega_{(a} g_{b)i} \omega_k \right) \\ & + u_g^{ik} \left(\frac{1}{4} g_{ab} \mu_{ik}^j \omega_j + \frac{1}{4} \omega_i \omega_k H_{ab} + \frac{1}{2} g_{i(a} \omega_{b)} H_k^j \omega_j - \frac{1}{2} g_{ab} \lambda_{ik} \right) \\ & - u_g^{ik} \left(\frac{1}{4} g_{ik} \mu_{ab}^j \omega_j + \frac{1}{4} \omega_a \omega_b H_{ik} + \frac{1}{2} g_{i(a} H_{b)}^j \omega_j \omega_k - \frac{1}{2} g_{ik} \lambda_{ab} \right) \\ & \left. + u_\omega^i \mu_{ab}^i + u_\lambda^i - u_\lambda^i g^{ik} g_{ab} \right], \quad (5.10) \end{aligned}$$

$$\{C_{ab}^H, H_T\} = \sqrt{g} \left[\lambda_{ab} - D_{(a} \omega_{b)} + u_g^{ik} \left(\frac{1}{2} \omega_i g_{k(a} \omega_{b)} - \frac{1}{4} \omega^2 g_{ai} g_{bk} \right) + u_\omega^i \omega_{(a} \omega_{b)} \right], \quad (5.11)$$

$$\{C_\lambda^{ab}, H_T\} = \sqrt{g} \left(H^{ab} - u_g^{ab} + u_g^{ik} g_{ik} g^{ab} \right), \quad (5.12)$$

$$\{C_\omega^a, H_T\} = \sqrt{g} \left(D_b H^{ab} - u_g^{ik} \mu_{ik}^a - u_H^{ab} \omega_b \right). \quad (5.13)$$

Solving algebraically for the multiplier functions u_g^{ab} and u_ω^a ,

$$u_g^{ab} = H^{ab} - \frac{1}{2} H_c^c g^{ab}, \quad (5.14)$$

$$\omega^2 u_\omega^a = 2\omega^b (D_{(a} \omega_{b)} - \lambda_{ab}) - \omega_a \left(D^b \omega_b - \lambda_b^b - \frac{1}{8} \omega^2 H_b^b \right). \quad (5.15)$$

The commutators (5.10)-(5.13) also provide secondary constraints. Substituting the expressions (5.14) and (5.15) in (5.11), we get

$$C_{ab}^P \equiv P_a^c P_b^d \sqrt{g} \left(\lambda_{cd} - D_{(c} \omega_{d)} - \frac{1}{4} \omega^2 H_{cd} + \frac{1}{8} \omega^2 g_{cd} H_k^k \right) = 0, \quad (5.16)$$

where $P_a^c = \delta_a^c - \frac{1}{\omega^2} \omega^c \omega_a$ is the projector into the tangent subspace orthogonal to ω , and the secondary scalar constraint is

$$C^s \equiv \sqrt{g} \left(\lambda_c^c - D_c \omega^c + \frac{1}{4} \omega_a \omega_b H^{ab} + \frac{1}{\omega^2} \omega^b \omega^c (D_b \omega_c - \lambda_{bc}) \right) = 0. \quad (5.17)$$

These new constraints do not depend on the momenta.

The conservation of the secondary constraint C^P yields

$$\begin{aligned} \{C_{ab}^P, H_T\} = & -u_g^{ik} \left(\frac{1}{2} g_{ik} C_{ab}^P + \frac{2}{\omega^2} C_{i(a}^P \omega_{b)} \omega_k \right) - \frac{2}{\omega^2} u_i^\omega C_{(a}^{Pi} \omega_{b)} \\ & + \sqrt{g} P_a^P P_b^d \left\{ u_g^{ik} \left(-\frac{1}{4} \omega_i \omega_k H_{cd} + \frac{1}{2} \omega^2 g_{i(c} H_{d)k} + \frac{1}{8} \omega_i \omega_k g_{cd} H_r^r \right. \right. \\ & - \frac{1}{8} \omega^2 g_{ic} g_{dk} H_r^r - \frac{1}{8} \omega^2 g_{cd} H_{ik} \left. \right) - \omega_k D_{(c} u_{d)}^k + \frac{1}{2} \omega^k D_k u_{gc} \\ & + \frac{1}{\omega^2} \omega^k u_{(c}^\omega (D_{d)} \omega_k + D_{|k|} \omega_{d)} - 2\lambda_{d)k} + \frac{1}{2} \omega^2 H_{d)k} \left. \right) + \frac{1}{4} g_{cd} H_r^r u_i^\omega \omega^i \\ & \left. - \frac{1}{2} H_{cd} u_i^\omega \omega^i - D_{(c} u_{d)}^\omega - \frac{1}{4} \omega^2 u_{Hcd} + \frac{1}{8} \omega^2 g_{cd} u_{H^i}^i + u_{cd}^\lambda \right\}. \end{aligned} \quad (5.18)$$

The first three terms containing constraints can be dropped.

All the remaining multiplier functions u_H^{ik} and u_{ik}^λ can be obtained in terms of $u_{H^i}^i$ from (5.10), (5.13) and (5.18). The Poisson bracket $\{C^s, H_T\}$ does not yield any information on u_H and u^λ .

If we next transvect (5.10) with $\omega^a \omega^b$ and subtract from $\{C^s, H_T\}$, we get a lengthy new constraint C_1 involving the Einstein tensor.

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Tidal Forces and Conjugate Point-pairs

J. Kánnár

1. Introduction

The aim of this paper is to give a short review on the recent results in connection with the behaviour of causal geodesic congruences. I am going to consider the following two aspects of this field:

1. Is there any possible way to weaken the so-called energy conditions of the singularity theorems?
2. Is there any way to get restrictions on the behaviour of the curvature tensor along incomplete inextendible causal geodesics (which we considered terminating in singularity)?

Before presenting the results of our investigations [17,18] I am going to recall some notions and some previous results we shall need later.

In a general space-time, which does not possess any kind of symmetries, one cannot choose any preferred reference frame system, so there is no way to measure the accelerations of observers. Only the relative or tidal acceleration of infinitesimally close observers has a physical meaning. Let us consider a congruence of causal geodesics (which, in the timelike case, represents the world-lines of some inertial observers). The behaviour of such a congruence is described by the *Jacobi equation* [13],

$$T^b \nabla_b (T^d \nabla_d Z^a) = -R_{bcd}^a T^b T^d Z^c, \quad (1.1)$$

where T^a is the tangent vector of a geodesic, Z^a is the separation vector of the infinitesimally close geodesics, and R_{bcd}^a denotes the curvature tensor of the space-time. The term appearing on the left-hand side of (1.1) is the tidal acceleration, while on the right-hand side the *tidal force*, originating from the curvature of the space-time is given.[†]

It is convenient to choose an orthonormal (in the null case pseudo-orthonormal) tetrad field along the geodesics with the choice $E_4^a := T^a$, and consider the tensor components in these bases. Then the Jacobi equation takes the form

$$\frac{d^2 Z^i}{dt^2} = -R_{4j4}^i Z^j, \quad (1.2)$$

where $R_{i4j4} = R_{iajb} T^a T^b$ is the matrix of the so-called tidal force tensor (which is a 3×3 or 2×2 matrix according to whether the geodesics are timelike or null ones) and t is an affine parameter of the geodesics so that $T^a = \left(\frac{\partial}{\partial t}\right)^a$.

The second important notion is that of the *conjugate point-pairs*. Two points p and q are conjugate to each other along a given geodesic, if there is a solution of the Jacobi equation, which is zero at the points p and q , but not identically zero over the interval $[p, q]$.

The existence or the lack of conjugate point-pairs plays a crucial role in the behaviour of congruences of causal geodesics. A causal geodesic is maximal, i.e., it is the longest one in its close neighbourhood, if and only if it has no conjugate points.

[†] Our conventions and notations are the same as usual [1, 2, 13]. The first letters of the alphabet will denote abstract indexes, while the letters $i, j, k, l, \dots$ will denote components in a special basis. The space-time metric g_{ab} is considered to be a smooth Lorentz metric of signature $+2$, ∇_a denotes the torsion-free covariant derivative operator determined by the metric.

2. Conditions for the Existence of Conjugate Points

Perhaps the most important application of the techniques related to the descriptions of congruences of causal geodesics is in the proofs of the singularity theorems of general relativity [1, 2, 3, 4]. In these theorems some restrictions on the structure of the space-times (certain causality and initial conditions) assure the existence of maximal causal geodesics. On the other hand some conditions assure the existence of conjugate points for long enough geodesics. These two properties are in contradiction, unless the geodesics under consideration are incomplete, i.e., they cannot be extended into arbitrarily large or small values of their affine parameter.

The conditions assuring the existence of conjugate point-pairs in the first theorems of Hawking and Penrose were the so-called energy conditions, the strong and weak energy conditions. It is said that the *strong energy condition* holds over a space-time, when the inequality

$$R_{ab}T^aT^b \geq 0 \quad (2.1)$$

satisfies for all causal vector T^a . Note, that this quantity is just the trace of the previously introduced tidal force tensor, i.e., $R_{ab}T^aT^b = \sum_{i=1}^n R_{iaib}T^aT^b$ (where n denotes the dimension of the matrix of the tidal-force tensor). Via the Einstein equations this condition gives rise to a restriction on the energy-impulse tensor of the matter fields (which is the explanation for the use of the denomination "energy condition"),

$$T_{ab}T^aT^b \geq \left(\frac{1}{2}T - \frac{1}{8\pi}\Lambda\right)T^aT_a, \quad (2.2)$$

where T is the trace of the tensor T_{ab} and Λ is the cosmological constant. If T^a is a null vector, then this is only the lightlike limes of the *weak energy condition*, namely of the condition $T_{ab}T^aT^b \geq 0$ for all timelike T^a . This means the positivity of the energy density measured by the observer with four velocity T^a . On the other hand, the strong energy condition, which is a stronger restriction on the tensor T_{ab} , can be violated in some physically reasonable cases. For example for massive scalar fields (like the mesons) and for a great enough cosmological constant (as compared to the energy density of the matter) it can be violated. Hence, from the point of view of the singularity theorems it would be useful to change these conditions for weaker ones. Tipler, Roman and Borde have imposed so-called integrate conditions [14–16] on $R_{ab}T^aT^b$. In their conditions only the "average positivity" of the quantity $R_{ab}T^aT^b$ was required, i.e., the integral of this quantity was expected to satisfy some positivity conditions. These results significantly enlarged the validity of the singularity theorems, but some interesting cases were still left out. For example, for vacuum space-times with a positive cosmological constant the "integrate" version of the strong energy condition cannot be satisfied, because the integrand $R_{ab}T^aT^b (= \Lambda T^aT_a)$ is everywhere negative for timelike T^a .

I am going to present some new integral conditions imposed on one of the diagonal elements of the tidal force tensor, which assure the existence of conjugate point-pairs and the same time the trace of the tidal force tensor, $R_{ab}T^aT^b$ might be negative.

In the study of geodesic congruences there exist two methods. One of them is based on the study of the transport equations, while the other one on the study of the Morse index-form.

First I quote the so-called "global" one, which is in a close relationship with the variational calculus. In this method the crucial quantity is the so-called Morse index-form [11, 12],

$$I(X^a, Y^a) := \int_a^b [g_{ab}(T^c \nabla_c X^a) T^d \nabla_d X^b - R_{abcd} T^b T^d X^c Y^a] dt, \quad (2.3)$$

where $(X^a, Y^a) \in V(\gamma) \times V(\gamma)$, where $V(\gamma)$ denotes the real vector space of all continuous piecewise- C^1 vector fields along γ . (In the timelike case the value of the above index-form is the minus of the second variation of the arc length.)

Necessary and sufficient conditions for the existence of conjugate points along causal geodesics can be given in terms of the definiteness of this index-form. To derive our results [18] we applied the so-called timelike and null index lemmas (see Harris [12]).

Timelike Index Lemma: Let $\gamma : [a, b] \rightarrow M$ be a timelike geodesic with no points conjugate to $\gamma(a)$ and X^a be a continuous piecewise- C^1 vector field along γ , perpendicular to γ , with $X^a(a) = 0$. Let, furthermore, J^a be a perpendicular Jacobi field (this is a solution of the Jacobi equation) along γ with $J^a(a) = 0$ and $J^a(b) = X^a(b)$. Then $I(X^a, X^a) \geq I(J^a, J^a)$, and $I(X^a, X^a) = I(J^a, J^a)$ only if $X^a = J^a$.

If $\gamma : [a, b] \rightarrow M$ happens to be a null geodesic the situation is different because the tangent vector T^a of γ belongs to its own perpendicular space. However, a slightly modified version of the above lemma can be proven even in this case.

Null Index Lemma: Let $\gamma : [a, b] \rightarrow M$ be a null geodesic with no points conjugate to $\gamma(a)$ and X^a be a continuous piecewise- C^1 vector field along γ , perpendicular to γ , with $X^a(a) = 0$. Let, furthermore, J^a be a perpendicular Jacobi field along γ with $J^a(a) = 0$ and $J^a(b) = X^a(b)$. Then $I(X^a, X^a) \geq I(J^a, J^a)$, and $I(X^a, X^a) = I(J^a, J^a)$ only if $X^a - J^a$ is everywhere parallel to T^a .

Whenever $J^a(b) = 0$ is satisfied in the above cases, $J^a \equiv 0$ is held over the given interval, because the geodesic segment does not contain conjugate point-pairs. So the conditions would be simpler in both lemmas, $I(X^a, X^a) > 0$ would be true, for all not identically zero (in the null case not parallel to T^a) vector fields X^a over the interval $[a, b]$. In our investigations this simpler case will occur.

By the application of the above lemmas and a particular choice of vector fields sufficient conditions can be settled for the existence of conjugate point-pairs along causal geodesics. I am going to give only a brief review of the main results and some crucial aspects of their proofs.

Let us consider special vector fields $X^a := fE^a$ in the index-form, where f denotes a continuous piecewise- C^2 function over a given interval $[a, b]$, and E^a denotes one of the unit spacelike vector fields parallelly propagated along and perpendicular to γ . With the vector field $X^a = fE^a$ the index-form takes the value

$$I(X^a, X^a) = \int_a^b (f'(t))^2 dt - \int_a^b (f(t))^2 R_{1414}(t) dt, \quad (2.4)$$

where the prime denotes differentiation with respect to t . The components of the curvature tensor are taken in a parallelly propagated orthonormal (in the null case pseudo-orthonormal) tetrad field, $\{E_i^a\}$, satisfying $E_1^a := E^a$ and $E_4^a := T^a$.

It is worth noting that the tidal force component $R_{1414}(t)$ has the following geometrical interpretation: It is equal in the timelike case to the sectional-curvature, in the null case to the null-curvature of the two-dimensional subspace of $T_{\gamma(t)}M$ generated by the basis $\{E^a, T^a\}$.

Our strategy is the following: First we try to eliminate the term f^2 from the second part of the above expression. Then *via* the positive definiteness of the index-form over the introduced vector fields (which are vanishing at the endpoints of the interval) we can get restrictions on the tidal force component, $R_{1414}(t)$. To do this we select a special piecewise- C^2 function, f_δ . Let the function $f_\delta : [a, b] \rightarrow \mathbb{R}$ be defined as follows:

$$f_\delta(t) := \begin{cases} \frac{(t-a)}{\delta}, & \text{if } a \leq t \leq a + \delta; \\ 1, & \text{if } a + \delta \leq t \leq b - \delta; \\ \frac{(b-t)}{\delta}, & \text{if } b - \delta \leq t \leq b. \end{cases}$$

With the help of this special function f_δ the second term of the above index-form can be converted into the form $\int_{t_1}^{t_2} R_{1414}(t)dt$, where $t_1 \in [a, a + \delta]$ and $t_2 \in [b - \delta, b]$. The function f_δ is chosen so that the value of the first term does not depend on the limits of the integral.

By the application of the quoted index lemmas one can prove the following proposition.

Proposition 2.1. *Let $\gamma : [\alpha, \beta] \rightarrow M$ be a causal geodesic in M , and $t_1, t_2 \in (\alpha, \beta)$ and $0 < \delta \leq \frac{t_2 - t_1}{2}$ so that $[t_1 - \delta, t_2 + \delta] \subset [\alpha, \beta]$. Suppose that there exists a vector field E^a along γ so that the tidal force component (in the orthonormal resp. pseudo-orthonormal bases introduced earlier) $R_{1414}(t)$ does not change sign in each of the intervals $[t_1 - \delta, t_1 + \delta]$ and $[t_2 - \delta, t_2 + \delta]$. Furthermore, suppose that*

$$\int_{t_1}^{t_2} R_{1414}(t)dt \geq \frac{2}{\delta} \quad (2.5)$$

is satisfied. Then there exists a conjugate point-pair along γ on the segment between $\gamma(t_1 - \delta)$ and $\gamma(t_2 + \delta)$.

Both sides of the above inequality are equal to one term in the index-form taking over the interval $[a, b]$, where $a \in [t_1 - \delta, t_1]$ and $b \in [t_2, t_2 + \delta]$. The right-hand side of the previous inequality, $\frac{2}{\delta}$, is the value of the first term in the index-form, and the integral is equal to the second one with the function f_δ . If the inequality holds, then $I(X^a, X^a) \leq 0$ is satisfied. Since $X^a \neq 0$ (resp. in the null case not parallel to T^a), the index lemmas imply the existence of conjugate point-pairs along the given geodesic.

It can be shown that the function f_δ is the optimal element of a greater function class, F_δ ,

$$F_\delta(t) := \begin{cases} g, & \text{if } a \leq t \leq a + \delta; \\ 1, & \text{if } a + \delta \leq t \leq b - \delta; \\ h, & \text{if } b - \delta \leq t \leq b, \end{cases}$$

where $|F_\delta| \leq 1$, and g, h are C^2 functions satisfying $g(a) = h(b) = 0$ and $g(a + \delta) = h(b - \delta) = 1$. The lower bound $\frac{2}{\delta}$ appearing in the inequality cannot be improved within this class.

As a direct consequence of the above proposition we can get a sufficient condition for assuring the existence of conjugate point-pairs along complete causal geodesics.

Corollary 2.2. *Let $\gamma : (-\infty, +\infty) \rightarrow M$ be a complete causal geodesic in M . Suppose that there exists a special choice of the basis vector field E_1^a along γ so that the tidal force*

$R_{1414}(t)$ is non-negative along γ while being positive somewhere. Then there exists conjugate point-pair along γ .

The proof is trivial, because in this case the integral of the given tidal force is positive, so we can choose a δ and an interval $[t_1, t_2]$ satisfying the earlier proposition.

It turned out that the conditions of the above convergence condition for complete causal geodesics can be further weakened. Using the second usual techniques, studying the so-called transport equations one can get even weaker restrictions. The transport equations are derived from the Jacobi equation, and they are ordinary differential equations for the deformation, θ_{ab} , and the vorticity tensor, ω_{ab} , of the congruence. With the assumption that the congruence is vorticity free, the transport equations take the following simple form,

$$\frac{d}{dt}\theta_{ij} = -R_{i4j4} - \sum_{k=1}^n \theta_{ik}\theta_{kj}, \quad (2.6a)$$

$$\frac{d}{dt}\theta = -R_{44} - 2\sigma^2 - \frac{1}{n}\theta^2, \quad (2.6b)$$

where θ denotes the trace of the deformation tensor, and $\sigma^2 = \frac{1}{2}\sum_{i,j}\sigma_{ij}\sigma_{ij}$, where σ_{ij} denotes the shear tensor, $\sigma_{ij} = \theta_{ij} - \frac{1}{n}\theta$. For the diagonal components of the deformation tensor θ_{ab} , then we have

$$\frac{d}{dt}\theta_{ii} = -R_{i4i4} - \sum_{k=1, k \neq i}^n \theta_{ik}^2 - \theta_{ii}^2. \quad (2.6c)$$

Using these notations two points, p and q , are said to be conjugate along a given geodesic if and only if there is such a continuous solution for the trace of the deformation tensor, θ , on this segment, whose limeses are $+\infty$ at p and $-\infty$ at q .

Studying the properties of the solutions of the above equations one can generate conditions, which assure the existence of such solutions for the deformation which have conjugate point-pairs. Previously the authors have studied in the first place the solutions of the transport equation for the trace, see (2.6b). It is easy to see that whenever the quantity $R_{44}(= R_{ab}T^aT^b)$ is greater than or equal to zero, and the matrix of the tidal force tensor R_{i4j4} is not identically zero along the given complete geodesic, then the solutions for θ will have conjugate point-pairs. On the other hand we have studied the transport equations for the diagonal components of the deformation, see (2.6c). Because of the similarity of these equations to that for the trace, in form of restrictions on a particular tidal force we have been able to derive similar conditions assuring the existence of a solution for θ with a conjugate point-pair. I am going to demonstrate only the most general result, because the details are fairly complicated and lengthy. We have proven the following statement which assures the existence of conjugate point-pairs along complete causal geodesics under a very weak restriction [17]. Only a recurrent positivity of the integral of a given tidal force component is required.

Proposition 2.3. Let γ be a complete causal geodesic with affine parameter t and suppose that there is such a unit vector E_1^a orthogonal to and parallelly propagated along γ that for any $\epsilon > 0$ there is a $b > 0$ and that for any $t_1 < t_2$ there is such a pair of intervals I, J of length $\geq b$ that $I < t_1 < t_2 < J$ and

$$\int_{t'}^{t''} R_{1414}dt \geq -\epsilon, \quad \forall t' \in I; \quad \forall t'' \in J. \quad (2.7)$$

Then, if one of the functions $R_{14i4}(t)$ is not identically zero, then there is a pair of conjugate points along γ . (Here the index i takes the values from 1, 2, 3 in the timelike, 1, 2 in the null case.)

Consider now space-times, where the tidal forces cannot have infinite negative oscillations. Over this class of space-times the above statement is, in mathematical sense, generalization of the integrate energy condition of Borde [16]. His theorem has almost the same form. The main distinction is that instead of a given diagonal component of the tidal force tensor he used its trace. The similarity of the statements is a consequence of the analogous form of the transport equation for the diagonal components of the deformation tensor, see (2.6c), and the equation for its trace (Raychaudhuri equation), see (2.6b).

3. On the Divergence Rate of the Components of the Tidal Force Tensor along Maximal Causal Geodesics

Considering the relationship between the behaviour of tidal forces and the existence of the conjugate point-pairs the lack of the latter gives rise to restrictions on the possible behaviour of the tidal forces. So we can study how these quantities can behave "near to a singularity" along maximal geodesics. More exactly, we can give bounds to the possible divergency rate of the components of the tidal force tensor along maximal incomplete causal geodesics running into singularities.

The singularity theorems guarantee only the existence of incomplete causal geodesics [1–4]. (Although a series of theorems guarantee the maximality of some incomplete causal geodesics, as well [6].) The incompleteness of causal geodesics does not imply that some of the physical quantities blow up along them. It was shown earlier that there is an upper bound to the rate of the divergency of the trace of the tidal force tensor (in the terminology of the previous authors this is the Ricci-part of the curvature) [5, 7–10]. More precisely, they have proven that along a maximal incomplete geodesic $\gamma : (-\infty, t_0) \rightarrow M$ the quantity $R_{ab}T^aT^b$ cannot diverge faster than $\frac{1}{(t-t_0)^2}$ to $+\infty$. The lower bound here is the strong energy condition, which was applied in their investigations.

In this section I am going to present the consequences of the earlier results, which describe the behaviour of the components of the tidal force tensor along maximal incomplete causal geodesics. Considering the reversal of Prop. 2.1 with the special choice $\delta := (t_2 - t_1)/2$, we can get a restriction for the integral of the tidal forces along maximal causal geodesics. By the application of these results for a final segment of a maximal causal geodesic, we get the following restriction on the possible divergency rate of a diagonal component of the tidal force tensor [18].

Proposition 3.1. Let $\gamma : (-\infty, t_0) \rightarrow M$ a future incomplete maximal causal geodesic. Then for all choice of the basis vector field E_1^a along γ for which there is an interval $[t^, t_0)$ so that the tidal force component $R_{1414}(t)$ does not change sign on $[t^*, t_0)$ we have that*

$$\liminf_{t \rightarrow t_0} \{(t - t_0)^2 R_{1414}(t)\} \leq 9. \quad (3.1)$$

Whenever the above inequality is not satisfied there exists a conjugate point-pair along γ , so it cannot be maximal.

It might seem to be a very strong restriction that the components of the tidal force tensor are expected to keep their sign over certain intervals. However the same type of restriction was also applied in the earlier works. More precisely, then the investigated quantity, the Ricci part of the curvature $R_{ab}T^aT^b$ (which is the trace of the tidal force tensor) was required to satisfy the strong energy condition, namely to be nonnegative.

It is important to note that so far we have *not* used any energy condition type assumption along our investigations for the above upper bound results. To derive the "dual" lower bound results we have to impose certain restrictions on the trace of the tidal force tensor $R_{ab}T^aT^b$, although this restriction is significantly weaker than the strong energy condition.

It is said that the *limes condition* is satisfied along the future incomplete causal geodesic $\gamma : (-\infty, t_0) \rightarrow M$ whenever

$$\lim_{t \rightarrow t_0} \{(t - t_0)^2 R_{ab}T^aT^b(t)\} \geq 0. \quad (3.2)$$

This condition is satisfied whenever $R_{ab}T^aT^b(t)$ does not diverge too quickly to $-\infty$ and, even more, if it has a lower bound along γ .

With the application of the earlier upper bound results and the limes condition one can get similar quadratic lower bounds to the divergence rates of the diagonal, and quadratic upper and lower bounds to the divergence of the off-diagonal elements along maximal incomplete geodesics.

4. Final Remarks

Clearly, the restrictions on the tidal force component $R_{1414}(t)$ involved in the conditions assuring the existence of conjugate point-pairs play the same role as the energy conditions and the so-called generic condition together in the singularity theorems of Hawking and Penrose. Consequently, generalizations of the Hawking-Penrose singularity theorems can be given.

The presented statements are generalisations of the previous upper bound results and show that neither of the diagonal or off-diagonal components of the tidal force tensor that do not change sign on some final segment of γ can blow up faster than $(t - t_0)^{-2}$ along maximal incomplete causal geodesics.

The applied limes condition is definitely weaker than the energy conditions appearing in the earlier examinations and is fitting to the quadratic divergence results.

Acknowledgments

The above is a summary of the results of my thesis at the L. Eötvös University (Budapest) under the guidance of Dr. István Rácz. I would like to thank him for his help during my work and during the preparation of this paper.

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Part VI

Beyond General Relativity

Physical Versions of Gravity Theories in Higher Dimensions

L.M. Sokolowski

We argue that there is a purely theoretical criterion allowing for discriminating between various gravity theories including scalar fields. Contrary to some claims that in the absence of experimental evidence any coupling of a scalar to the metric field is admissible, we show that some couplings are theoretically excluded and that it is possible to determine the physical field variables (if there are any), i.e. the empirically measurable quantities. The criterion works for theories wherein one can define the stress tensor for all fields except for the metric field and states that a gravity theory is viable if there is a transformation to these field variables for which the kinetic energy density for the fields is positive definite. For the most important gravity theories the criterion uniquely identifies the physical variables. Some other theories, e.g. Kaluza-Klein and Gauss-Bonnet theory, can be discarded on its basis.

1. Introduction

The title may be slightly misleading. It suggests that the problem refers exclusively to multidimensional theories while this is not so. Actually the very similar troubles are encountered in four dimensions and the inclusion of the extra dimensions just makes them more conspicuous and clear-cut. Therefore let me start with a gravity theory in any number of dimensions $d > 4$.

Assuming that the extra dimensions have a physical meaning, a gravity theory in the total space-time should satisfy two stringent conditions, namely both the full d -dimensional theory and its low-energy four-dimensional sector consisting of ground state solutions and their small excitations must be viable, i.e. physically acceptable theories. It sounds trivially, but the consequences of the conditions are not trivial. Let us consider Kaluza-Klein cosmology as an example. The dimensionality is $d = 4 + n$ and the torus T^n is taken, for simplicity, as the internal n -dimensional compact space (for other spaces the results are essentially the same). The metric of the total space-time is block-diagonal (the off-diagonal terms, depending on the gauge fields, are set zero) and the internal torus has an average radius $Le^{F(x)}$ (we choose the exponential function for convenience), so that

$$\hat{g}_{AB} = \begin{pmatrix} \hat{g}_{\mu\nu}(x) & 0 \\ 0 & L^2 e^{2F(x)} \delta_{ik} \end{pmatrix},$$

$A, B = 0, \dots, d-1$, $\mu, \nu = 0, \dots, 3$, $i, k = 4, \dots, d-1$. L is a characteristic scale length for the internal torus and the scalar function $F(x^\mu)$ is referred to as the *dilaton* field. We assume pure Einstein gravity with the cosmological term in the total space-time (adding any matter would not alter the conclusions) and dimensionally reduce it to the 4-dimensional external world. The problem is that *there is no derived from the first principles prescription for how to construct the metric $g_{\mu\nu}$ of the external space-time M in the effective theory out of the total metric $\hat{g}_{AB}$* . There is only a consensus that the definition should be based on the four-dimensional piece $\hat{g}_{\mu\nu}$ of $\hat{g}$. Let us naively assume, as it has been done in most papers

on Kaluza-Klein cosmology, that $g_{\mu\nu} \equiv \hat{g}_{\mu\nu}$, then the Einstein field equations give rise to the following equations of motion for the system (metric, dilaton) in M ,

$$G_{\mu\nu}(g) = 8\pi G T_{\mu\nu}(g, F) = 8\pi G \left\{ \frac{1}{8\pi G} e^{nF} [nF_{,\mu} F_{,\nu} + nF_{;\mu\nu} - \left(n\Box F + \frac{n(n+2)}{2} F_{,\alpha} F^{,\alpha} + \Lambda \right) g_{\mu\nu}] + \frac{1}{8\pi G} (1 - e^{nF}) G_{\mu\nu} \right\}, \quad (1.1a)$$

$$\Box F = -nF_{,\mu} F^{,\mu} - \frac{2}{n+2} \Lambda. \quad (1.1b)$$

Clearly the system $(g_{\mu\nu}, F)$ represents a version of a scalar-tensor theory of gravity. The stress tensor for the dilaton acts as a source for the tensor part of gravity. By investigating this tensor one finds [1] that the system has physically unacceptable features:

- (i) kinetic energy for F is *indefinite* in general (and is negative e.g. for static dilaton in Minkowski space),
- (ii) the energy spectrum is unbounded from below,
- (iii) as a consequence there is *no* translationally invariant stable ground state solution.

There is no doubt that this theory of gravity is physically meaningless. (It took some time to recognize this since in cosmology one usually envisages homogeneous solutions while the "inherent instability" of the theory due to the energy indefiniteness becomes evident only when one allows for spatially inhomogeneous perturbations.)

This does not, however, imply that Kaluza-Klein cosmology as a whole is meaningless too. The scalar-tensor gravity can easily be rendered a viable physical system by assuming that the physical metric of M is conformally related to the previous one, $g_{\mu\nu} = \Omega^2(F) \hat{g}_{\mu\nu}$. For a unique choice of the conformal factor, $\Omega^2 \equiv e^{-nF}$, the equations of motion (1.1) are replaced by (see [1])

$$G_{\mu\nu}(g) = \frac{n(n+2)}{2} \left(F_{,\mu} F_{,\nu} - \frac{1}{2} F_{,\alpha} F^{,\alpha} g_{\mu\nu} \right) - \Lambda e^{-nF} g_{\mu\nu}, \quad (1.2a)$$

$$\Box F = -\frac{2}{n+2} \Lambda e^{-nF} \quad (1.2b)$$

and one gets an acceptable theory of gravity for the dilaton turns out to be just an ordinary nonlinear scalar field minimally coupled to the metric. This result is by no means accidental as it is linked to the concept of gravitational energy in the total space-time. Bombelli et al. [2] introduced three expressions (different in general) for gravitational energy in five-dimensional Kaluza-Klein space-time and related them to the notion of gravitational energy in reduced four-dimensional external worlds. The effective metrics for the latter are all conformally related, $g_{\mu\nu} = \Omega^2 \hat{g}_{\mu\nu}$, with various conformal factors. They show that this conformal ambiguity can be removed if the energy in $d=5$ equals that in $d=4$. They prove

Theorem: *There is a unique choice of Ω^2 giving $g_{\mu\nu}$ for the reduced theory for which:*

- (1) *the five-dimensional gravitational energy is given by the usual four-dimensional ADM integral expressed in terms of the reduced metric,*
- (2) *the energy of the Gross-Perry-Sorkin monopole [3] computed by means of the four-dimensional Freud superpotential is positive and equals to its mass as found from the three expressions for gravitational energy in $d = 5$,*
- (3) *the gravitational energy is additive in four dimensions.*

This holds for $\Omega^2 = e^{-F}$ ($n = 1$).

One thus sees that the problem of finding out the proper field redefinitions yielding physical variables is crucial for establishing whether a given theory is tenable or not. There are many various gravity theories in four dimensions and one inevitably encounters the problem of which of these are really different (instead of being different versions in different variables of the same theory) and which versions of the theories are the physical ones (and what is actually meant by "physical", the term being clearly abused in the literature). There are still some issues in the problem and first of all the experimental data are at present insufficient to settle the matter. It is my aim to show that a criterion of energy definiteness may considerably help. Let me first discuss the two best known cases.

2. Jordan-Brans-Dicke Theory

After many years of being almost dead the theory has unexpectedly revived quite recently due to the concept of "extended inflation" and its modifications (see e.g. [4]). In its original formulation the theory is based on the action

$$S = \int d^4x \sqrt{-g} \left[\frac{-1}{16\pi} R(g)\phi + \frac{\omega}{16\pi} g^{\mu\nu} \phi^{-1} \phi_{,\mu} \phi_{,\nu} + L(\text{matter}) \right]. \quad (2.1)$$

The dynamical variables $g_{\mu\nu}$ and ϕ form the so-called Jordan conformal frame. The field equations resulting from (2.1) are

$$\begin{aligned} G_{\mu\nu}(g) &\equiv T_{\mu\nu}(g, \phi) + \frac{8\pi}{\phi} T_{\mu\nu}(\text{matter}) = \\ &= \frac{\omega}{\phi^2} \left(\phi_{,\mu} \phi_{,\nu} - \frac{1}{2} \phi_{,\alpha} \phi^{,\alpha} g_{\mu\nu} \right) + \frac{1}{\phi} \phi_{;\mu\nu} - \frac{1}{\phi} g_{\mu\nu} \square \phi + \frac{8\pi}{\phi} T_{\mu\nu}(\text{matter}), \end{aligned} \quad (2.2a)$$

$$\square \phi = \frac{8\pi}{3 + 2\omega} T^\mu_\mu(\text{matter}). \quad (2.2b)$$

The stress tensor $T_{\mu\nu}(g, \phi)$ for the Brans-Dicke scalar is indefinite in general and is a homogeneous function of zeroth order, $T_{\mu\nu}(g, \lambda\phi) = T_{\mu\nu}(g, \phi)$. These peculiarities have convinced most authors working in inflationary cosmology to abandon Jordan conformal frame and to make a conformal rescaling (without giving a deeper physical motivation for doing so) to the Einstein conformal frame:

$$\bar{g}_{\mu\nu} \equiv \Omega^{-2} g_{\mu\nu}, \quad \Omega^2 = \frac{1}{G\phi}, \quad \psi \equiv \psi_0 \ln(G\phi), \quad \psi_0 = \text{const.} \quad (2.3)$$

After these field redefinitions the action in Einstein frame

$$S = \int d^4x \sqrt{-\bar{g}} \left[\frac{-1}{16\pi G} \bar{R}(\bar{g}) + \frac{1}{2} \bar{g}^{\mu\nu} \psi_{,\mu} \psi_{,\nu} + L(\bar{g}, \psi, \text{matter}) \right] \quad (2.4)$$

describes Brans-Dicke field ψ minimally coupled to $\bar{g}_{\mu\nu}$ and interacting with matter.

3. The Conformally Invariant Scalar Field

The field was intensively studied in the hope that its properties might be more suitable for quantum field theory in curved space-times than those of the ordinary scalar field. The hopes failed soon and the cause for that was revealed by a theorem proven by J. Bekenstein [5]. Bekenstein's theorem is surprisingly little known so it is plausible to present it in full extent.

Theorem: *If $g_{\mu\nu}$, $F_{\mu\nu}$ and ϕ form a solution to coupled Einstein-Maxwell-conformally-invariant-scalar field equations*

$$G_{\mu\nu}(g) = 8\pi G(1 - \zeta^2\phi^2)^{-1} [t_{\mu\nu}(g, F) + \Theta_{\mu\nu}(g, \phi)], \quad (3.1a)$$

$$t_{\mu\nu} = \frac{1}{4\pi} (F_{\mu\alpha}F_{\nu}^{\alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}), \quad \zeta^2 \equiv \frac{4\pi}{3}G, \quad (3.1b)$$

$$\Theta_{\mu\nu}(g, \phi) \equiv \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha} - \frac{1}{6}(\phi^2)_{;\mu\nu} + \frac{1}{6}g_{\mu\nu}\square(\phi^2), \quad (3.1c)$$

$$\square\phi = \frac{1}{6}R\phi \quad \text{and} \quad F_{\nu}^{\mu\nu} = 0, \quad (3.1d)$$

then the rescaling

$$g_{\mu\nu} = \cosh^2(\zeta\psi)\bar{g}_{\mu\nu} \quad (3.2a)$$

and field redefinitions

$$\phi = \frac{1}{\zeta} \tanh(\zeta\psi) \quad \text{and} \quad F_{\mu\nu} = \bar{F}_{\mu\nu} \quad (3.2b)$$

yield the corresponding solution to Einstein-Maxwell-ordinary-scalar field equations,

$$\bar{G}_{\mu\nu}(\bar{g}) = 8\pi G \left[t_{\mu\nu}(\bar{g}, \bar{F}) + \psi_{,\mu}\psi_{,\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta} \right], \quad (3.3a)$$

$$\bar{\square}\psi = 0 \quad \text{and} \quad \bar{F}_{\nu}^{\mu\nu} = 0. \quad (3.3b)$$

The stress tensor $\Theta_{\mu\nu}(g, \phi)$ is indefinite so that the stability of a background solution is uncertain. On the other hand the stress tensor in (3.3a) in Einstein conformal frame is positive definite (actually it satisfies the Dominant Energy Condition). This ensures that the ground state solution $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ and $\psi = \text{const.}$ is stable against any perturbations. This in turn implies, *via* transformations (3.2), that the corresponding background solution in Jordan conformal frame, $g_{\mu\nu} = \eta_{\mu\nu}$ and $\phi = \text{const.}$ is stable too and no unphysical instabilities can occur in that frame. The same holds for stability of the flat background in both the frames in JBD theory.

The conformal equivalence raises the question: which frame is physical — Jordan's or Einstein's? Or in other terms: what are the "physical variables" for gravity and scalar field? The problem can be posed in a more generic setting. Consider a metric gravity interacting with any matter denoted collectively by ϕ . An action functional $S[g, \phi]$ gives rise to field equations

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \quad \text{and} \quad \frac{\delta S}{\delta \phi} = 0, \quad (3.4)$$

Field redefinitions

$$g'_{\mu\nu} = g'_{\mu\nu}(g, \phi), \quad \phi' = \phi'(g, \phi) \quad (3.5)$$

and the transformed action $S'[g', \phi'] \equiv S[g, \phi]$ give rise to new field equations,

$$\frac{\delta S'}{\delta g'^{\mu\nu}} = 0 \quad \text{and} \quad \frac{\delta S'}{\delta \phi'} = 0. \quad (3.6)$$

Mathematically both theories are equivalent as they describe the same physical system in different names, (g, ϕ) or (g', ϕ') . Equivalence means that the space of solutions to (3.4) in variables (g, ϕ) is isomorphic to the space of solutions to (3.6) in (g', ϕ') and the isomorphism is given by (3.5). (Warning: in most practically relevant cases the mapping (3.5) is only a *local* isomorphism, e.g. the transformation (3.2) maps ϕ to ψ only for $|\phi| \leq \zeta^{-1}$. Thus the equivalence is not global in general. Little is known on the impact of this fact on the problem of physical equivalence of the theories and identification of physical variables.) Are then both theories also physically equivalent?

The problem has become particularly relevant in the light of a recent discovery by Magnano et al. [6] that Einstein General Relativity is in a sense a "Hamiltonian image" for any nonlinear metric theory of gravity with a Lagrangian

$$L_i = \sqrt{-g} f(g_{\mu\nu}, R(g), \text{matter}). \quad (3.7)$$

Here f is a smooth function of matter fields, the metric tensor and its first derivatives and of scalar curvature (no explicit dependence on Ricci and Weyl tensors). A Legendre transformation to "momenta" canonically conjugated to the metric connection maps the system $(g_{\mu\nu}, \text{matter})$ to $(\bar{g}_{\mu\nu}, \text{matter}, F)$, F being a new additional dynamical variable, a scalar field. The transformation, actually being a conformal rescaling of the metric g , converts (after dropping divergence terms) L_i to the standard Einstein-Hilbert Lagrangian,

$$L_f = \sqrt{-\bar{g}} \left[\bar{R}(\bar{g}) - \frac{3}{2} \bar{g}^{\alpha\beta} F_{,\alpha} F_{,\beta} + L(\bar{g}, F, \text{matter}) \right]. \quad (3.8)$$

Mathematically all nonlinear metric theories of gravity are equivalent to General Relativity plus scalar field and provide no new physical effects in comparison to the latter. The differences between various original Lagrangians (3.7) are reflected in the potential term for F in $L_f(\bar{g}, F, \text{matter})$ and in various couplings of F to matter. This generic result comprises previous results for L being a polynomial function of R [7]. The same holds for dimension $d > 4$ [8] and for supergravity [9]. (The theorem applies for arbitrary functions $f(R)$, thus almost nothing can be stated about the reversibility of the Legendre transformation except that it provides an isomorphic map of an open subset in the space of solutions.)

At present it seems that the result of Magnano et al. cannot be further generalized:

- (1) If L_i depends explicitly on Ricci tensor, e.g. via the $R_{\alpha\beta} R^{\alpha\beta}$ scalar, the appropriate Legendre transformation gives rise to a new metric being proportional to $R_{\alpha\beta}(g)$, thus severe problems with non-degeneracy and correct signature appear;
- (2) If L_i depends on Weyl tensor and does not on Ricci tensor, one cannot use a Legendre transformation to define a new metric and it is unclear how to properly deal with this case.

The conclusion to the effect that all metric gravity theories are just "already unified" versions of General Relativity minimally coupled to nonlinear ordinary scalar field (with various potentials) has recently been a matter of debate [10]. According to Brans mathematical equivalence in this case does not entail a physical equivalence and JBD theory is *not* "nothing but" General Relativity + scalar. He claims that metric $g_{\mu\nu}$ in Jordan conformal frame has direct empirical operational meaning — all test particles move along geodesic lines of that metric and any matter stress tensor is conserved with respect to it rather than being conserved in Einstein conformal frame.

There is no doubt that in general mathematical equivalence does not imply physical equivalence. The simplest example is provided by classical Hamiltonian mechanics. The canonical transformation $P_i = q_i$, $Q_i = -p_i$, being a mere renaming of positions and momenta, clearly shows that mathematically there is no significant difference between them. On the other hand, if one has to operationally construct a Hamiltonian $H(q, p)$ for a given physical system, the distinction acquires a fundamental significance. For one should first clearly define which variables are positions and which are momenta of the system and then determine by experimental and/or theoretical methods its Hamiltonian in terms of these variables; in this procedure one should not confuse kinetic and potential energies. Variables which have been used for determining H deserve to be named "physical variables". Once physical variables have been identified (in mechanics they need not be unique) one can perform arbitrary canonical transformations and solve the Hamiltonian equations in the variables which are most suitable for this aim. Physical interpretation of the solution, however, usually requires making the reverse transformation to the physical variables.

The above comment sounds almost trivial if applied to classical mechanics alone. It becomes less trivial when applied to other theories, e.g. to nonrelativistic quantum mechanics. The well-known troubles with canonical quantization in curvilinear coordinates clearly point to the necessity of discrimination between physical variables (Cartesian coordinates) and those variables in which one is able to solve Schrödinger equation for a given problem. In a field theory one identifies the physical variables by experimental investigating interactions of the field with other forms of matter. In this sense the physical variable for electromagnetism and its coupling to charged particles have been identified beyond any doubt. In the case of gravity one should follow the same way: to study gravitational interactions of all known forms of matter, e.g. behaviour of light, charged and neutral massive test particles, atomic clocks, rigid rods etc. If the variety of effects under study is large enough one can reliably identify a physical version of gravity theory expressed in terms of physical variables. One should experimentally determine whether test particles follow geodesic lines and matter stress tensor is conserved in Jordan or in Einstein conformal frame.

The difficulty lies in the fact that the presently available empirical data are too scarce for this purpose. It is still impossible to test the Brans' claim that we should attach the physical meaning to Jordan frame. The problem is furthermore obscured by the fact that there is *no* experimental evidence at all on interaction of a scalar field with known matter. It might therefore seem that for the system (gravity, scalar field) almost any coupling is admissible and any Legendre transformation can be performed *without* changing the physics. However this is not so and in the absence of any observational data one can appeal to a theoretical criterion discriminating various forms of interactions on energetic grounds. It reads: *requirement of positive definiteness for kinetic part of the total field energy uniquely determines the physical version (if there is any) of a theory for (g, ϕ) among all "canonically equivalent" ones.*

I stress that this criterion actually applies to any relativistic field theory. It needs not to apply to nonrelativistic theories, a notable example being provided by classical Newtonian gravity with its negative field energy and positive mass. In the case of gravity the criterion works in a restricted class of theories — no Weyl tensor in the Lagrangian. Before applying the criterion to gravity coupled with a scalar field I will show that one can exclude some forms of interaction of scalar fields with ordinary matter on the basis of energy considerations. Take as an example a massless complex scalar field *incorrectly* coupled to electromagnetic field in Minkowski space

$$L = \frac{-1}{16\pi} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi \overline{(D^\mu \phi)}, \quad D_\mu \phi \equiv (\partial_\mu + ieA_\mu)\phi. \quad (3.9)$$

This theory is mathematically acceptable, e.g. the Cauchy problem for the field equations

$$\partial^\nu F_{\mu\nu} - 8\pi \Re(ie\bar{\phi} D_\mu \phi) = 0 \quad \text{and} \quad D_\mu D^\mu \phi = 0 \quad (3.10)$$

is well posed in the sense that it has a unique solution for given initial data. On the other hand the coupling is regarded in physics as incorrect since the energy-momentum tensor for the system,

$$T_{\mu\nu}(F, \phi) = \frac{1}{4\pi} (F_{\mu\alpha} F_\nu^\alpha - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}) - \frac{1}{2} [D_\mu \phi \overline{(D_\nu \phi)} + D_\nu \phi \overline{(D_\mu \phi)} - g_{\mu\nu} D_\alpha \phi \overline{(D^\alpha \phi)}], \quad (3.11)$$

is *indefinite*. Owing to it the system has no stable ground state. The solution $\phi = F_{\mu\nu} = 0$ has vanishing total energy $E = 0$, but it is neither ground nor stable solution. Although it has not been explicitly shown by finding such solutions to (3.10), one expects that there exist perturbation modes about this state, growing in time, for which $E = 0$ (total energy is conserved) and energy is pumped out from the scalar field to the electromagnetic one without bound. Such a self-exciting system, radiating away infinite amounts of electromagnetic energy, is clearly unphysical.

Returning to a scalar field coupled to a metric gravity, if the equations of motion are of second order one can unambiguously define a stress tensor for the scalar by writing them in the form of Einstein's field equations, $G_{\mu\nu}(g) = T_{\mu\nu}(g, \phi)$ (here we set $8\pi G = c = 1$). In general the stress $T_{\mu\nu}$ is *not* positive definite what entails that the Positive-Energy Theorem at Null Infinity [11] does *not* hold. Then one should expect, as in the above example, that there are perturbations $\delta g_{\mu\nu}$ and $\delta \phi$ about the "ground state solution" $g_{\mu\nu} = \eta_{\mu\nu}$ and $\phi = 0$ such that while the ADM energy is conserved and vanishes, the Bondi-Sachs mass remaining in the radiating system decreases to minus infinity. Undoubtedly any gravity theory allowing for *classical* instability of Minkowski space-time due to this self-excitation effect is unphysical and should be rejected.

As we have seen above, the Dicke transformation, Bekenstein transformation and the Magnano et al. theorem show that this unphysical decay of Minkowski space *cannot* occur in the presence of Brans-Dicke scalar, conformally invariant field and in the restricted nonlinear metric theories of gravity respectively. In each of these cases the appropriate transformation maps the original system (whose physical meaning is a priori unclear) to (General Relativity + minimally coupled scalar) having a stable ground state. Physically this is a transition (energy is not invariant under these transformations) to a system with *definite* (positive) energy.

The following axiom is experimentally well established: *all known classical matter satisfies the Weak Energy condition*. (In quantum field theory the energy density for a coherent superposition of state vectors with various particle numbers can be negative at some points [12].) Should (yet undiscovered) scalar fields be the exception? Our conclusion is therefore: physical variables g and ϕ (and their coupling) are *uniquely* determined by the requirement that the system has *positive definite* kinetic energy density.

Physical variables may not exist! Then the theory is regarded as unphysical. In four dimensions for example, it is known that the $aR + bC_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}$ theory (a, b constants) has indefinite gravitational energy and is unstable [13]. In $d > 4$ it was shown [14] that the Einstein-Gauss-Bonnet theory in Kaluza-Klein framework (that is, with compactified higher dimensions) gives rise to clearly unphysical results in its four-dimensional low-energy sector and should be discarded on these grounds.

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Signal for Compactification in Galaxy Redshifts?

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Abstract : Galaxy redshift distributions of recent surveys show expressed quasiperiodic peaks. In $k = 0$ cosmologies the periodicity is the strongest at $q_0 = -0.5$ (vacuum/dust ratio 2/1), which is not impossible with Λ term. Such a periodicity gets a natural explanation in a Universe of multiply connected topology.

1. Introduction

Broadhurst, Ellis, Koo and Szalay [1] reported an apparent regularity in the galaxy distribution with a characteristic scale of 128 Mpc. This is a sequence of peaks in a sample extending to approximately 2000 Mpc. The regularity was observed in very deep narrow pencil-beam surveys in the two Galactic polar fields. It seems to be a periodicity extending over 8-10 periods. Such a long range periodicity is not interpretable in any standard cosmology (because of the horizon problem, for example).

Together with Dr. Paál who unfortunately died this March, we have started to reexamine this topic. We shall argue here that the situation may not be so bad after all, provided that one allows topologically nontrivial, compactified Universes. No physical mechanism is needed in these for periodicities to be seen [2, 3].

2. Examination of Pencil-beam Data

We shall reexamine the data of the pencil-beam surveys as presented by Broadhurst et. al. Redigitalizing their histograms into steps of size $\Delta z = 0.005$ gives us our starting point. There are 13 well separated significant peaks, 4 in the southern and 9 in the northern direction. In addition, the selection function of the surveys was very low in our neighbourhood, so that the big gap between the initial southern and northern peaks is clearly an artefact.

There seems to be two missing peaks. The picture can be described in first approximation as a sequence of peaks from serial number -5 to $+9$, at the average position of peaks. Peaks numbered -1 and 0 are absent due to selection.

Broadhurst et al. [1] noticed the remarkable regularity of this histogram by marking equidistant comoving steps according to a dust model with $q_0 = 0.5$, rather near to the peaks. However, the redshift z is not the proper parameter for understanding this global cosmic regularity. If any such regularity exists amongst the data, it can be expected in the spatial positions (relative to each other) rather than in the redshifts relative to us, and so we shall henceforth work directly in the comoving coordinate, X . Our Figure 1a is the histogram for the numbers of galaxies vs. X in a dust model with $q_0 = 0.5$, as in the comparison by Broadhurst et al.

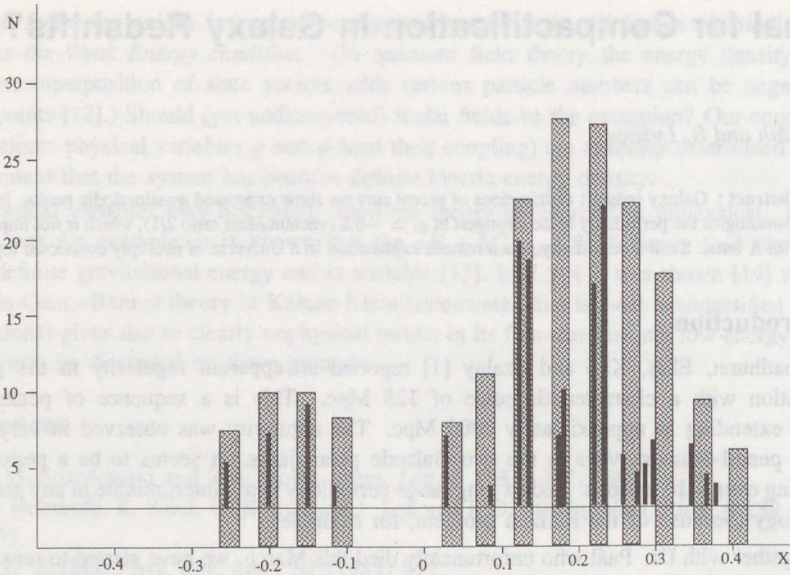


Figure 1a. Schematic diagram of the numbers of galaxies N in bins as a function of the comoving distance coordinate X for the flat dust model, $q_0 = +0.5$. For details, see the text

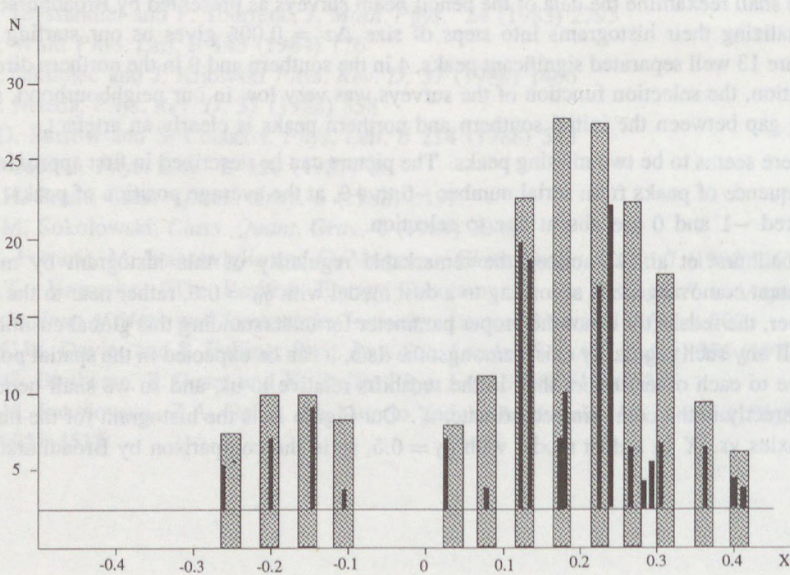


Figure 1b. As Figure 1a for the flat model, but for $q_0 = -0.5$

In this sample, then, the average spread of peak separation is somewhat less than 25%, so that it is possible to divide the available X values into two sets of disjoint intervals of equal sizes, one in which the galaxies are expected (for whatever reason), and one in which they are not. The preferred intervals are shown in Figure 1a.

The comparison of the expected and observed patterns of peaks shows some similarity in heights, but reveals the interesting fact that while for $-5 \leq i \leq +5$ all peaks are inside the preferred region, from $i = 6$ none of them lies there. Two of the preferred intervals even turn out to be absolutely empty.

This may not be a sign of lack of regularity, but rather of a different regularity. Regularity means rule, and it is improbable that it can be accidental. If fitting one model of the universe to the data gives a more regular result than fitting another, then one can argue that the former must give a better description of the world. Figure 1b clearly displays more regularity. We shall see later the model that leads to this.

The simplest regularity is equidistancy, and such was conjectured earlier. An equidistant sequence means $x_i = a(i + b)$, and so the deviation from equidistancy is measured by

$$\sigma^2 = \min_{a,b} (\sum_i (a^{-1}x_i - (i + b))^2). \quad (2.1)$$

This is a two-parameter fitting problem, with the relative deviation being given by $\sigma/\sqrt{\text{number of points}}$.

Broadhurst et al. chose $q_0 = +0.5$, corresponding to $k = 0$ for dust. The best fit in (2.1) for the pencil-beam surveys then yields

$$a = 0.0413, \quad b = -0.399, \quad \sigma = 0.530. \quad (2.2)$$

This value for a corresponds to $124 \text{ h}^{-1} \text{ Mpc}$, with a confidence interval of $\pm 5 \text{ h}^{-1} \text{ Mpc}$ at the 96% level. Trying dust with other q_0 values gives a function $\sigma^2 = \sigma^2(q_0)$. The lower σ , the higher the regularity. This function is displayed in Figure 2a, and is monotonously decreasing for $1 \geq q_0 \geq 0$. This result suggests a very regular Universe at some $q_0 < 0$, something which is impossible for dust.

In spite of all this, no clear evidence exists against a Λ term (interpretable as vacuum energy, for example). The Einstein-Hilbert equations can be integrated analytically for $k = 0$, using a dust + Λ model. Calculating the best fits for (2.1), one now obtains Figure 2b. The minimum is not too sharp, but does, rather remarkably, appear almost precisely at $q_0 = -0.5$. This is a rather round value, but we shall accept it and leave any ideas about "equipartition" or accident to the future. The best fit is given by

$$q_0 = -0.5, \quad a = 0.0466, \quad b = -0.366, \quad \sigma = 0.220. \quad (2.3)$$

The value for a corresponds to $140 \text{ h}^{-1} \text{ Mpc}$ ($\pm 4 \text{ h}^{-1} \text{ Mpc}$ at 96% level confidence). Figure 1b corresponds to this fit.

We used 13 peaks. The parameters a and b can always be fitted, and one can accept the value $q_0 = +0.5$ as the trivial value of this parameter, but as soon as we want to determine the best values of q_0 we have one more parameter to be fitted. After fitting the new parameter Λ (or q_0 if appropriate), σ^2/N drops to half of its original value. This decrease is significant in χ^2 -type tests (for further quantitative arguments see [4]).

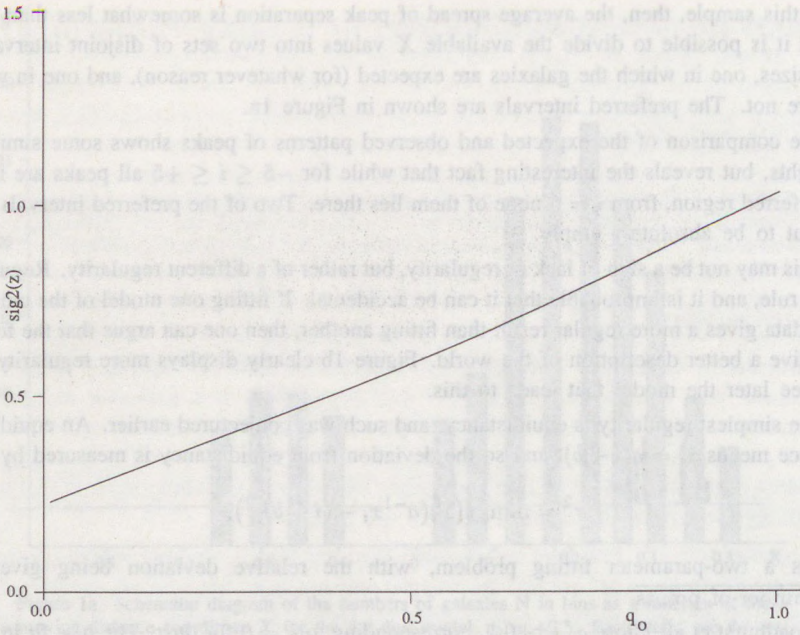


Figure 2a. Mean squared deviation σ^2 (in z) of the peaks from the best equidistant set, as a function of q_0 in dust models

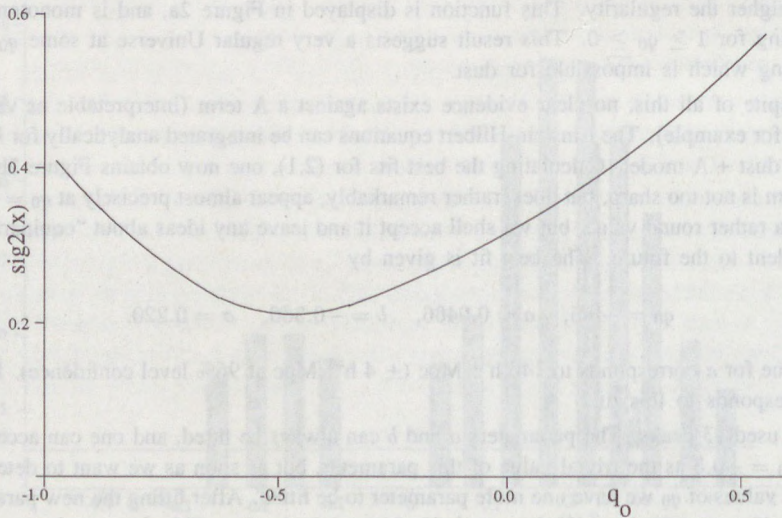


Figure 2b. As Figure 2a, but in x , and for a flat model family with vacuum energy and dust

The minimum at $q_0 = -0.5$ is rather wide, but one finds that $q_0 < 0.3$ with cca. 95% probability. As for the significance of the regularity, Figure 3 shows that the transformed autocorrelation function,

$$\rho = \frac{1}{2} \ln \frac{1+r}{1-r}, \quad (2.4)$$

has an approximately Gaussian distribution. The curve clearly shows peaks at the 3σ level and above, and therefore the regularities seem very significant indeed.

Now that the equidistancy of the peaks is better expressed, explaining it with the standard theory would be even more difficult. The alternative explanation is to assume a topologically nontrivial, compactified Universe. The Einstein-Hilbert equations are compatible with infinitely many topologically different solutions, locally equivalent but globally different (see [5] and references therein, and [6]), and therefore the actual topology ought to be prescribed. In the absence of a prescription one usually works with a trivial topology, and then the observations of a Universe of nontrivial topology interpreted using such a trivial one automatically result in an apparent crystal-like distribution of matter. *A priori* information about the true topology of the Universe is not available, so an apparent periodicity always allows alternative conclusions, either true periodicity on a simple topology or no periodicity on a multiply-connected topology.

The apparent length is 140 Mpc, so that one expects repeated "ghost images" of some celestial objects at a distance $L \approx 140$ Mpc. This is impossible, due to a rigorous observational constraint on the cell size [6] based on the absence of the repeated image of the Serpens-Virgo supercluster.

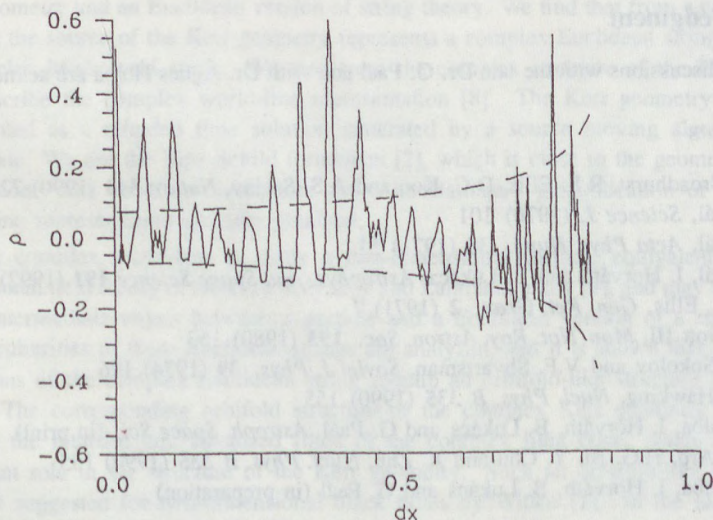


Figure 3. The transformed correlation coefficient of (2.4) vs. the separation dx . Dashed lines show the 1σ level

However, note that even such constraints could be satisfied by two or three walls. Even and odd maxima have unequal autocorrelation, which would just be a sign for two walls in

the cell. Then one is in agreement with the lower bound of [6]; the higher value of [7] is not obligatory because it depends on the fine details of the classification of clusters. The walls must have special locations, but there is a mechanism to generate equidistancy. A single primary wall in the compactified topology is a source of local gravity disturbing the density perturbations everywhere except the gravitationally neutral antipodal region, where, therefore, a secondary, smaller condensation is expected.

Finally, we note that a model with nontrivial topology (which may or may not be likely from a philosophical viewpoint) does have some advantages. For example,

1. In a fully compactified Universe the long-range homogenization is easier than in the usual inflationary models, since the largest physical distance is the size of the fundamental cell, and propagating physical effects occur again and again. In a cell of cca. 300 Mpc size, 10^{10} years may be sufficient to homogenize.
2. Irregularities do not have to (and cannot) be pushed far outside the observable region, because there is no outside at all.
3. If $\Lambda > 0$ corresponds to a vacuum density, then quantum gravitational considerations prohibit tunneling and wormhole connection to other Universes [8]. This Universe remains a totality even after quantum gravitational extension.

This study has been extended quite recently to $k \neq 0$ [9]. One then gets $\sigma^2 = \sigma^2(\Lambda, \Omega)$ with significant regularity on a diagonal strip in the (Λ, Ω) plane. For a very wide range of Ω the minimal value of σ is practically independent of Ω . Therefore "good regularity" appears at any value of k . Quasar data seem to confirm the existence of a "most regular" region of the (Λ, Ω) plane [10, 11], and so it seems that the tendency for periodicity is not an accidental property of a special sample.

Acknowledgment

Extended discussions with the late Dr. G. Paál and with Dr. Ágnes Holba are acknowledged.

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String-like Structures in Complex Kerr Geometry

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Abstract: The Kerr geometry is represented as being created by a source moving along an analytical complex world-line. The equivalence of this complex world-line and an Euclidean version of complex strings is discussed. It is shown that the complex Kerr source satisfies the corresponding string equations. The boundary conditions of the complex Euclidean strings require an orbifold-like structure of the world-sheet. The related orbifold-like structure of the Kerr geometry is discussed.

1. Introduction

Much attention has been paid recently to the relations between two-dimensional black holes and strings [1]. In this paper we shall consider four-dimensional Kerr geometry and derive some string-like structures for its complex source. The Kerr solution [2] is well known as the field of a rotating black hole. However, for the case of a large angular momentum $|a| = L/m \geq m$ its horizons are all absent and a naked ring-like singularity appears, along which space branches into two sheets. This naked singularity has many unpleasant manifestations, and must be hidden inside a rotating disk-like source [3, 4]. The solution with $|a| \gg m$ displays some remarkable features, suggesting certain relationships with spinning elementary particles [3–6]. The real structure of the Kerr metric source has itself some quite exotic properties [4] and contains a string-like closed vortex line resembling the superconducting strings of Nielsen and Olesen and of Witten [7].

The aim of this paper is to show some connections between the complex structure of the Kerr geometry and an Euclidean version of string theory. We find that from a complex point of view the source of the Kerr geometry represents a complex Euclidean string propagating in complex Minkowski space. We start from the complex structure of the Kerr geometry and describe the complex world-line representation [8]. The Kerr geometry can then be represented as a retarded time solution generated by a source moving along a complex world-line. We use the Kerr-Schild formalism [2], which is close to the geometrical twistor description. The necessary conditions for a nonstationary generalization of the complex world-line representation are also obtained.

The complex world-line is really a two-dimensional surface, equivalent to the two-dimensional field theory of Belavin, Polyakov and Zamolodchikov [9], and may be considered as an intermediate object between a particle and a Euclidean version of a string [10–12]. The peculiarities of these Euclidean strings are analyzed, and it is shown that the boundary conditions of the complex Euclidean string require an orbifold-like structure of the world-sheet. The corresponding orbifold structure of the complex Kerr geometry is discussed. It links the retarded and advanced folds of the complex light cone, which plays a very important role in the structure of the Kerr geometry. Such an orbifold-like structure was recently suggested for two-dimensional black holes by Witten [1]. In the last section we shall briefly discuss some generalizations.

2. The Complex World-line Representation of the Kerr Geometry

We use the Kerr-Schild form of the Kerr-Newman metric [2]

$$g_{ik} = \eta_{ik} + h k_i k_k, \quad (2.1)$$

where $\eta_{ik} = \text{diag}(-1, 1, 1, 1)$ is the auxiliary Minkowski metric in Cartesian coordinates (x, y, z, t) , and $h = (2mr - e^2)/(r^2 + a^2 \cos^2 \theta)$. The null vector field $k_i(x)$ is a geodesic and shear-free principal null congruence and may be represented in the form

$$K(x) = k_i dx^i = P^{-1}(du + \bar{Y} d\xi + Y d\bar{\xi} - Y\bar{Y} dv), \quad (2.2)$$

where the coordinates used are null,

$$u = (z + t)/\sqrt{2}, \quad v = (z - t)/\sqrt{2}, \quad \xi = (x + iy)/\sqrt{2}, \quad \bar{\xi} = (x - iy)/\sqrt{2},$$

and Y is a complex projective spinor coordinate, $Y = \psi^1/\psi^0$. The world-line representation was first introduced by Lind and Newman [8]. It shows that the Kerr-Newman solution may be considered as a retarded time solution generated by a "complex point" source [6, 8]. This source propagates in complex Minkowski space CM^4 along a complex "world-line", $X_o^i(\tau)$ ($i = 0, 1, 2, 3$), parametrised by a complex time parameter $\tau = t + i\sigma = X_o^0(\tau)$. An important role in this retarded time construction is played by the complex light cones $\mathcal{K}$ whose apexes lie on the complex "world-line" $X_o(\tau)$.

2.1. The complex light cone

The complex light cone $\mathcal{K}$, written in spinor form as

$$\mathcal{K} = \{x : x = X_o^i(\tau) + \psi_L^A \sigma_{AA}^i \tilde{\psi}_R^{\dot{A}}\}, \quad (2.3)$$

may be split into two families of null planes: "left" ($\psi_L = \text{const.}; \tilde{\psi}_R\text{-var.}$) and "right" ($\tilde{\psi}_R = \text{const.}, \psi_L\text{-var.}$). These are the only two-dimensional planes which are wholly contained in the complex null cone. The rays of the principal null congruence K of the Kerr geometry are the tracks of these complex null planes (right or left) on the real slice of Minkowski space. The light cone equation in the Kerr-Schild metric coincides with the corresponding equation in Minkowski space because the null directions k^i are null in both metrics, g_{ij} and η_{ij} . This splitting of the light cone on the null planes has a close connection with twistors. By introducing the projective parameter $Y = \psi^1/\psi^0$ the complex light cone equation,

$$(\xi - \xi_o)(\bar{\xi} - \bar{\xi}_o) = -(u - u_o)(v - v_o),$$

splits into two linear equations[†],

$$\begin{aligned} \xi - \xi_o &= Y(v - v_o), \\ u - u_o &= -Y(\bar{\xi} - \bar{\xi}_o), \end{aligned} \quad (2.4)$$

describing the "left" complex null planes (the null rays in the real space). Another splitting

$$\begin{aligned} u - u_o &= -\bar{Y}(\xi - \xi_o), \\ \bar{\xi} - \bar{\xi}_o &= \bar{Y}(v - v_o) \end{aligned} \quad (2.5)$$

[†] It is a generalization of the Veblen and Ruse construction [13] which has been used for a geometric representation of spinors.

gives the "right" null planes. The three parameters

$$Y, \quad \lambda_1 = u + Y\bar{\xi}, \quad \text{and} \quad \lambda_2 = \xi - Yv \quad (2.6)$$

are indeed projective twistor coordinates.[†] The equations for the "left" null planes of the complex light cone with apex at the point X_o take the form

$$\lambda_1 = \lambda_1^o = u_o + Y\bar{\xi}_o, \quad \lambda_2 = \lambda_2^o = \xi_o - Yv_o. \quad (2.7)$$

The "left" null planes of the complex light cones form a complex congruence or regulus which generates all the rays of the principal null congruence K in real space. The ray with polar direction θ, ϕ is the real track of the "left" plane corresponding to $Y = \exp(i\phi) \tan(\theta/2)$ and belonging to the cone which is placed at the point X_o corresponding to $\sigma = a \cos(\theta)$. The parameter $\sigma = \text{Im}(\tau)$ has a meaning only in the range $-a \leq \sigma \leq a$ where the cones have real slices.

Thus, the complex world-line $X_o(t, \sigma)$ represents a restricted two-dimensional surface, or strip, in complex Minkowski space, and is really a world-sheet. It may be considered as a complex open string with a Euclidean parametrization, $\tau = t + i\sigma$, $\bar{\tau} = t - i\sigma$, and with end points $X_o(t, \pm a)$.

2.2. Complex world-lines and open Euclidean strings

A complex world-line, or world-sheet, is an intermediate object between particle and string [10, 11].[‡] The complex source of the Kerr-Newman solution corresponds to a straight analytical world-line in complex Minkowski space CM^4 ,

$$X_o^i(\tau) = X_o^i(0) + U_o^i \tau. \quad (2.8)$$

In the Euclidean version, the general string solution has to be a sum of the left and right modes, taking the form

$$X^i(t, \sigma) = X_L^i(\tau) + X_R^i(\bar{\tau}). \quad (2.9)$$

It seems most natural to use the complex conjugate world-line as the right mode, leading to the real string solution $X^i(t, \sigma)$. However, the string solutions have also to satisfy the constraints

$$\tau = t + i\sigma, \quad \bar{\tau} = t - i\sigma, \quad \partial_\tau = (\partial_t - i\partial_\sigma)/2, \quad \partial_{\bar{\tau}} = (\partial_t + i\partial_\sigma)/2, \quad (2.11)$$

$$(\partial_\tau X)^2 = 0; \quad (\partial_{\bar{\tau}} X)^2 = 0, \quad (2.10)$$

where and the boundary conditions for $\sigma = \pm a$ which are

$$\text{Re} \left[\frac{\partial L_e}{\partial (\partial_\tau X^i)} \right]_{\sigma=\pm a} = 0. \quad (2.12)$$

This equation puts certain strong restrictions on the solutions, with the result that the real solutions for the open Euclidean strings are absent. Nevertheless, one can find a consistent formulation of the complex closed Euclidean strings, in which the complex analytical straight world-lines form closed strings satisfying the corresponding full system of string equations.

[†] In the standard twistor form $(1, Y, \lambda_1, \lambda_2) = (Z^0, Z^1, Z^2, Z^3)/Z^0$, where $Z = (\psi^B, \pi_A)$, $\pi_A = X_{AB}\psi^B$.

[‡] The corresponding complex Euclidean strings have also been considered in recent papers of Ooguri and Vafa [12], where they were called "hyperbolic".

3. Closed Complex Euclidean Strings with an Orbifold-like World-sheet

One can consider Euclidean strings in complex Minkowski space CM^4 as complex objects with a Hermitian Lagrangian,

$$L = -\eta_{i\bar{j}}(\partial_\tau X^i \partial_{\bar{\tau}} \bar{X}^{\bar{j}} + \partial_{\bar{\tau}} X^i \partial_\tau \bar{X}^{\bar{j}}). \quad (3.1)$$

The general solution is given as a sum of the left and right modes,

$$X^i(t, \sigma) = X_L^i(\tau) + X_R^i(\bar{\tau}), \quad (3.2)$$

which are not necessarily complex conjugates of each other. The constraints take the form

$$\eta_{i\bar{j}} \partial_\tau X_L^i \partial_{\bar{\tau}} \bar{X}_R^{\bar{j}} = 0, \quad \eta_{i\bar{j}} \partial_{\bar{\tau}} X_L^i \partial_\tau \bar{X}_R^{\bar{j}} = 0. \quad (3.3)$$

An oscillator expansion of the Euclidean string is

$$\begin{aligned} X_L &= X_L(0) + P\tau + \omega^{-1} \sum (1/n) \alpha_n \exp\{-in\omega\tau\}, \\ X_R &= X_R(0) + P\bar{\tau} + \omega^{-1} \sum (1/n) \tilde{\alpha}_n \exp\{-in\omega\bar{\tau}\}, \end{aligned} \quad (3.4)$$

where a periodic time dependence is proposed with a period $T = 2\pi/\omega$. The expansion contains hyperbolic basis functions which are not orthogonal over the string length. For the parameter σ it is convenient to use an interval $[-a, a]$. The complex Euclidean strings can not be open, since the boundary conditions for open strings,

$$(\partial_\sigma X)_{|\sigma=\pm a}^2 = 0,$$

lead only to a trivial solution with $\omega = 0$.

Attempts to introduce boundary conditions for closed complex Euclidean strings meet obstacles too, because it is impossible to introduce the same boundary conditions for both the real and the imaginary part of a string. However, the problem may be resolved when the world-sheet admits an orbifold-like structure [11,14,15]. The general idea is very simple: it is necessary to modify the world-sheet to include twisting boundary conditions for the imaginary part of the complex string. A more precise formulation is given by the following. To construct an orbifold it is necessary to double the interval $\Sigma = [-a, a]$ for the parameter $\sigma = \text{Im}(\tau)$. Two copies of the interval, $\sigma_+ \in \Sigma_+ = [-a, a]$ and $\sigma_- \in \Sigma_- = [-a, a]$, must be joined to form an *oriented* circle, $S^1 = \Sigma_+ \cup \Sigma_-$, which may be parametrised by a periodic coordinate $\theta \sim \theta + 2\pi$, and represents a covering space for the original interval Σ . For an explicit correspondence one can use the projection S^1 on Σ ,

$$\sigma = a \cos \theta. \quad (3.5)$$

The original string is then parametrised by $0 \leq \theta \leq \pi$, and $\pi \leq \theta \leq 2\pi$ covers the string a second time in the opposite direction.

A group $G = \mathbb{Z}_2$ acts on the circle as $g\theta = \theta + \pi$, $g \in G$, $g^2 = 1$, and induces the transformation

$$g\sigma_+ = -\sigma_-, \quad g\sigma_- = -\sigma_+, \quad g\tau_+ = \bar{\tau}_-, \quad g\tau_- = \bar{\tau}_+. \quad (3.6)$$

The string modes $X_L(\tau_L)$ and $X_R(\bar{\tau}_R)$ can be extended on the circle for $\pi \leq \theta \leq 2\pi$ to form a closed string by means the well-known kind of extrapolation [15]

$$X_L(g\tau_+) = X_R(\bar{\tau}_+), \quad X_R(g\bar{\tau}_+) = X_L(\tau_+). \quad (3.7)$$

The group G is a symmetry of the theory, and therefore acts on the string variables

$$\tilde{g}X_L(\tau_{\pm}) = X_L(g\tau_{\pm}) = X_R(\bar{\tau}_{\pm}), \quad \tilde{g}X_R(\bar{\tau}_{\pm}) = X_R(g\bar{\tau}_{\pm}) = X_L(\tau_{\pm}).$$

One can see that the group acts by interchanging the left and right modes. The time coordinate t may be transformed by projection onto the circle S^1 so that the world-sheet forms the torus $T^2 = S^1 \times S^1$. The action of the time-reversing element h of Z_2 is $ht = -t$, $h^2 = 1$. The elements g and h commute with each other and act together as

$$gh\tau_L = -\tau_R, \quad gh\tau_R = -\tau_L, \quad h\tau_L = -g\tau_R, \quad h\tau_R = -g\tau_L.$$

The quotient space T/Z_2 is the orbifold. The action of g on the orbifold is defined by the projection (3.5)

$$g\sigma = -\sigma, \quad g\tau = \bar{\tau}, \quad \tilde{g}X(\tau) = X(g\tau) = X(\bar{\tau}). \quad (3.8)$$

The actions of gh and h are

$$gh\tau = -\tau, \quad gh\bar{\tau} = -\bar{\tau}, \quad h\tau = -g\tau, \quad h\bar{\tau} = -g\bar{\tau}. \quad (3.9)$$

The Hilbert space of the string solutions H on the orbifold have to be decomposed on two subspaces: the H_+ space of the even functions,

$$X_{ev}(t, \sigma) = (X + \tilde{g}X)/2, \quad \tilde{g}X_{ev} = X_{ev}, \quad (3.10)$$

and the H_- space of the odd functions,

$$X_{odd}(t, \sigma) = (X - \tilde{g}X)/2, \quad \tilde{g}X_{odd} = -X_{odd}. \quad (3.11)$$

As a result the even solutions are closed on the orbifold and have the form

$$X_{ev} = (X_L(\tau) + X_R(\bar{\tau}) + X_L(\bar{\tau}) + X_R(\tau))/2, \quad (3.12)$$

whereas the odd solutions,

$$X_{odd} = (X_L(\tau) + X_R(\bar{\tau}) - X_L(\bar{\tau}) - X_R(\tau))/2, \quad (3.13)$$

form closed strings on the orbifold with twisted boundary conditions.

A general complex string solution on the orbifold may be represented as the sum of the even and odd parts,

$$X(t, \sigma) = X_{ev} + X_{odd} = (X + \tilde{g}X)/2 + (X - \tilde{g}X)/2. \quad (3.14)$$

The gh transformations do not mix the analytic and anti-analytic parts of the functions on the orbifold, but the functions may be decomposed into even and odd parts with respect to the gh transformations,

$$X(\tau) = X_{gh+}(\tau) + X_{gh-}(\tau).$$

In the terms of the gh decomposition the general even and odd solutions for a closed string are

$$X_{cv} = 2\text{Re}X_{gh+} + 2i\text{Im}X_{gh-}, \quad X_{od} = 2\text{Re}X_{gh-} + 2i\text{Im}X_{gh+}. \quad (3.15)$$

The complex constraints (3.3) then take the form

$$g_{ij}[\partial_\tau \text{Re}X^i \partial_\tau \text{Re}X^j + \partial_\tau \text{Im}X^i \partial_\tau \text{Im}X^j] = 0, \quad (3.16)$$

which is a generalization of the real constraints, taking into account that the imaginary components are the independent degrees of freedom.

These constraints admit an interesting class of solutions with $X_R = 0$,

$$X^i(t, \sigma) = X_L^i(\tau), \quad (3.17)$$

which can be seen from the complex form of constraints (3.3). These solutions contain only left modes like the Schild null strings or the bosonic sector of the heterotic string, and correspond to analytical world-sheets. In spite of the absence of a right mode the solutions are nontrivial and contain both even and odd parts. It is interesting that the straight analytical world-line corresponding to the stationary Kerr solution satisfies the constraints as well as the arbitrary analytical complex world-lines.[†] Thus, the world-line corresponding to the complex source of the Kerr solution satisfies the string equations on the orbifold.

4. Orbifold-like Structure of the Complex Kerr Geometry

To obtain a real cut of the complex Kerr-Schild space for every τ we shall use a null vector

$$\mathcal{M}^i = \psi \sigma^i \tilde{\psi} \quad (4.1)$$

of the complex light cone, linking points of the complex world-line $X_o(\tau)$ with points of the real slice. If a cone has a real slice, then for every normalised spinor ψ (with $\psi\tilde{\psi} = 1$) there exists a spinor

$$\tilde{\psi} = \bar{\sigma}^o \sigma_a (X_o^a - \bar{X}_o^a) \tilde{\psi}, \quad (a = 1, 2, 3), \quad (4.2)$$

such that $X_o + \mathcal{M}$ is a point on the real slice, i.e.

$$X_o + \mathcal{M} = \bar{X}_o + \bar{\mathcal{M}}. \quad (4.3)$$

The explicit form of $\mathcal{M}$ is

$$\mathcal{M}^i = (X_o^i - \bar{X}_o^i)/2 + i\epsilon^{ijk} k_k (X_{oj} - \bar{X}_{oj})/2, \quad (4.4)$$

where $k_i = \psi \sigma_i \tilde{\psi}$ is the real vector of the principal null congruence. One can also add to $\mathcal{M}$ a real vector directed along the null ray rk^i . The null vectors k^i and $\mathcal{M}^i$ belong to the "left" null plane.

One can find even and odd structures in the Kerr geometry. The real "center" of the Kerr geometry, $X_{rc}(t, \sigma) = (X_{oL}(\tau) + \bar{X}_{oL}(\bar{\tau}))/2$, is an even object. The imaginary part of the world-line $X_{im}(t, \sigma) = (X_{oL}(\tau) - \bar{X}_{oL}(\bar{\tau}))/2$ has odd parity with respect to complex time, but it also contains an even constant $X_{oL}(0)$. The imaginary degrees of freedom in the Kerr geometry may be included in a compactification scheme based on the constrained

[†] In the case of real strings the stationary solutions do not satisfy the constraints.

CP^3 sigma-model [16]. The light cones, which are adjoined to every point of the complex world-line, can be considered as an "internal" space of the Kerr string in analogy with the compactification in string models.

One can introduce relative coordinates for the real points of the Kerr-Schild space,

$$Z^i = X^i - X_o^i, \quad i = 0, 1, 2, 3, \quad (4.5)$$

which include the imaginary part of the complex world-line $\text{Im}Z^i = -\text{Im}X_o^i$ and satisfy the light cone constraints

$$Z_i Z^i = 0. \quad (4.6)$$

In the constrained sigma-model the Z^i are considered as homogeneous coordinates for complex projective space CP^3 ,

$$\xi^\alpha = Z^\alpha / Z^0, \quad \alpha = 1, 2, 3. \quad (4.7)$$

The space may be endowed with a Kähler metric in the standard manner,

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \mathcal{M}^{fs}. \quad (4.8)$$

It is a metric on the compact space of the complex null rays, defined by the Fubini-Study potential

$$\mathcal{M}^{fs} = \ln(1 + \xi\bar{\xi}). \quad (4.9)$$

For every real point of the Kerr-Schild space X^i one can find both the "left" retarded time τ_L and a point of the complex world-line $X_o(\tau_L)$ defined by the intersection of the world-line with the "left" null plane (2.4). This plane is spanned by the null vectors $\mathcal{M}^i$ and k^i . The "right" plane (2.5) contains the vector $\mathcal{M}^i$, but it also defines the congruence $l^i = \tilde{\psi}\sigma^i\tilde{\psi}$ through the equation $\tilde{Y} = \tilde{\psi}^1/\tilde{\psi}^0$. It should be mentioned that the "right" and "left" planes change places under the action of the gh-transformation $(v - v_o) \longleftrightarrow (u - u_o)$ and a redefinition $\tilde{Y} = -1/Y$. On the other hand the gh-transformations change the retarded and advanced folds of the complex light cone, as can be seen from the alternative splitting of the complex light cone,

$$t - \tau = \pm\tilde{r}, \quad gh\tilde{r} = -\tilde{r} \quad (4.10)$$

where $(\tilde{r})^2 = (x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2$. When t is real the transformation g acts on $\tilde{r}$ as complex conjugation. These transformations induce the orbifold structure on the CP^3 by defining an equivalence between positive and negative folds relating Z^0 to $-Z^0 = ghZ^0$, and by introducing two new folds of the orbifold corresponding to $\pm\tilde{Z}^0 = h(\pm Z^0)$. This orbifold may be described in CP^3 by the quartic equation

$$[Z_1^2 + Z_2^2 + Z_3^2 - (\bar{\tau}/\tau)Z_0^2](Z_1^2 + Z_2^2 + Z_3^2 - Z_0^2) = 0. \quad (4.11)$$

In the limiting case $a = 0$ the factor $\bar{\tau}/\tau = 1$ and the equation corresponds to a peculiar case of the $K3$ surface, a known example of the manifolds [14,18]. Notice that the complex light cone plays an important role in the structure of the Calabi-Yau spaces. It is a "conifold" [17], or a singular limit of many Calabi-Yau spaces. The procedure of "small resolution" of the singularities is produced by introducing the extra CP^1 space $Y = \psi^1/\psi^0$ and splitting the constraints [17]. One can note, that this is an exact copy of the above (see (2.4), (2.5)) splitting of the complex light cone on the left (or right) twistor null planes. It not only removes the singularity but also annihilates the multiple connectedness caused by the group of discrete symmetry.

5. Generalisations

We will discuss two generalisations of the previous sections:

5.1. A nonstationary generalization of the complex world-line representation

The complex Kerr source corresponds to a straight analytical world-line, but from the point of view of string excitations the oscillating world-lines are very interesting. The problem of the existence of corresponding solutions is open. One of the problems involved in obtaining the nonstationary generalizations of the Kerr-Schild solutions is the question of the existence of the corresponding principal null congruences (which have to be both geodesic and shear-free).

In this section we shall briefly discuss the nonstationary generalization of the congruences, following [20]. According to Kerr's theorem [2,19], geodesic and shear-free congruences are defined by (2.2), where the function $Y(X)$ is a solution of the equation

$$F(Y, \lambda_1, \lambda_2) = 0, \quad (5.1)$$

F being an analytical function. The singularities of the solutions occur as caustics in Y , satisfying the equations $F = 0$, $\partial_Y F = 0$. In the case when the singularities are confined to a bounded region [6,21] the F is, at most, quadratic in Y and the equation $F(Y) = 0$ may be solved explicitly, giving an explicit form for the principal null congruence. The function F may be represented in a form

$$F = (\partial_\tau \lambda_1^0)(\lambda_2 - \lambda_2^0) - (\partial_\tau \lambda_2^0)(\lambda_1 - \lambda_1^0), \quad (5.2)$$

which is convenient for the nonstationary generalizations because the parameters λ_A^0 and $\partial_\tau \lambda_A^0$ are functions of the complex world-line parameters $X_o(\tau)$ and therefore depend on τ .

Following the general scheme described in [2], one can show that the shear-free and geodesic conditions for the congruence

$$(\partial_{\bar{\xi}} - Y\partial_u)Y = 0, \quad (\partial_v + Y\partial_{\bar{\xi}} + \bar{Y}\partial_{\bar{\xi}} - Y\bar{Y}\partial_u)Y = 0 \quad (5.3)$$

lead to a differential equation,

$$Z^{-1}(\bar{Y} - \phi)dY = \phi(d\xi - Ydv) + (du + Yd\bar{\xi}), \quad (5.4)$$

where $Z = (\partial_{\bar{\xi}} - Y\partial_u)Y$ is the complex expansion of the congruence, and ϕ is an arbitrary solution of the equation

$$(\partial_{\bar{\xi}} - Y\partial_u)\phi = 0. \quad (5.5)$$

By using the form (5.2) one can obtain the following equations which are necessary for the congruence to be both geodesic and shear-free

$$(\partial_{\bar{\xi}} - Y\partial_u)\tau = 0, \quad (5.6)$$

$$\lambda_1 = \lambda_1^0(\tau), \quad \lambda_2 = \lambda_2^0(\tau). \quad (5.7)$$

These equations may be considered as constraints on the retarded time parameter τ . Equations (5.7) are equivalent to (2.4) and imply that the points X and $X_o(\tau)$ belong to the same "left"

null plane of the complex light cone. Consequently, the retarded time $\tau = \tau_L(X)$ has to be defined by the point of intersection of the "left" null plane with the world-line X_o , or by the "left" projection of the X on the world-line along the corresponding "left" null planes. Thus, in the nonstationary case the complex world-line representation acts *via* the "left" retarded time τ_L , which must be used in the parameters of equation (5.2).

Equation (5.6) gives an extra necessary condition on the world-line $X_o(\tau)$. It has to be an analytical function of τ . Remarkably, this condition coincides with the necessary condition on the world-line to be a solution of the complex Euclidean string equations. In the real space these nonstationary solutions for the congruence K correspond to a moving singular ring, the displacement and orientation of which depend on the real and imaginary parts of the complex world-line $X_o(\tau_L)$.

5.2. Super-extension

One can extend the complex time parameter on superspace by introducing the (2,0) world-sheet supersymmetry. The complex (2,0)-supertime coordinates [22] $\tau, \bar{\tau}, \xi^+, \bar{\xi}^+$ may be united in the chiral superfield,

$$T = \tau + \eta_+(\tau, \bar{\tau})\xi^+ + i\xi^+\bar{\xi}^+, \quad (5.8)$$

and then the analytical complex world-line turns out to be the chiral superfunction of supertime,

$$\mathcal{X}^i(T) = x^i(\tau) + \eta_+^i(\tau, \bar{\tau})\xi^+ + i\xi^+\bar{\xi}^+\partial_\tau x^i(\tau). \quad (5.9)$$

The (2,0)-supercovariant derivatives are

$$D_+ = \partial_{\xi^+} + i\bar{\xi}^+\partial_{\bar{\tau}}; \quad \bar{D}_+ = \partial_{\bar{\xi}^+} + i\xi^+\partial_\tau. \quad (5.10)$$

The superfield action corresponding to (3.1) takes the form

$$\begin{aligned} S &= \frac{-1}{4\pi\alpha'} \int d\tau d\bar{\tau} d\xi d\bar{\xi} \eta_{ij} D_+ \mathcal{X}_o^i \overline{(D_+ \mathcal{X}_o^j)} \\ &= \frac{-1}{\pi\alpha'} \int d\tau d\bar{\tau} [(\partial_\tau x_{oi})(\overline{\partial_\tau x_o^i}) - i(\bar{\eta}_{i+}\partial_\tau \eta_+^i - \overline{\bar{\eta}_{i+}\partial_\tau \eta_+^i})], \end{aligned} \quad (5.11)$$

and gives rise to the Klein-Gordon action for x_o^i and a Dirac action for η_+^i .

There exists a hint that the problem of the nonstationary generalization of the Kerr-Newman solution and the problem of its superextension are related problems. It is known that the dynamical extension of the Kerr-Newman solution runs into many serious problems. One of the obstacles is connected with the radiative character of such solutions and in consequence with the corresponding violation of the flat asymptotic behavior of the space-time, since the stress tensor does not decrease rapidly enough at the large distances. One can hope that the corresponding supersolutions could be asymptotically flat, because in the vacuum region, at large distances from source, the supersymmetry could be restored. This ought to guarantee both the vanishing of the stress tensor and the asymptotically flat behavior at infinity, in spite of the radiative character of nonstationary solutions.

Acknowledgment

In conclusion it is pleasure for me to thank Professors R. Kerr and Z. Perjés for the kind hospitality in the Central Research Institute for Physics of Budapest and for the very important discussions. I am thankful also for useful conversation to V. N. Ponomariov, A. O. Barvinsky, O. A. Soloviov, D. P. Sorokin.

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This volume contains the material of the lectures given at the Fourth Hungarian Relativity Workshop, which was organized as part of a triennial series of meetings. As the earlier volumes, it surveys some developing directions in general relativity. The material is arranged following the layout of the previous volumes.

General Relativity today is an intensively researched area of theoretical physics. In the years following Einstein's own work, the theory has been enriched by deep new ideas. New ingredients are the theories of relativistic causality, black holes and canonical gravity. Part I of the volume contains the contributions on the generic properties of space-time, and Part II is devoted to the relativistic gravitational fields of rotating bodies. A special chapter of this is the theory of black holes, and it attracts more and more researchers worldwide. Part III deals with various properties of relativistic spinors.

Throughout the world, observatories are being built for the detection of gravitational radiation. In the U.S. the laser interferometric gravitational observatory (LIGO) is today the biggest endeavour in experimental physics. The 1993 Nobel Prize in physics was awarded for the precise observation of the gravitational radiation of a double pulsar. The subject is treated in Part IV. Part V is devoted to the dynamics of gravitating systems.

This volume is probably the first English-language publication in Hungary on general relativity. The existing works (the translation of Landau and Lifshitz: *Theoretical Physics II. Classical Fields* and K. Novobátzky: *The Theory of Relativity*) cannot be found in bookshops any longer, and lag behind the current state of the theory by some thirty years.

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