tanulmányok

151/1983

MTA Számitástechnikai és Automatizálási Kutató Intézet Budapest

COMPUTER AND AUTOMATION INSTITUTE
HUNGARIAN ACADEMY OF SCIENCES

ON FIRST ORDER MANY-SORTED LOGIC

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Tanulmányok 151/1983

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ISBN 963 311 167 6
ISSN 0324-2951

CONTENTS

ABSTRACT

Many-sorted (heterogeneous) logic is a useful tool for discussing of a great variety of theoretical problems in computer science and artificial intelligence. The purpose of this paper is to give a general and precise system of definitions for the users of many-sorted logic and to be an introduction to a deeper study of many-sorted model theory. A detailed definition of first order many-sorted language is given and a whole section deals with operators defined on heterogeneous models such as submodel, homomorphic image, direct and reduced products. The concept of many-sorted free model is introduced and some well-known theorems of predicate calculus are generalized for first order many-sorted language.

0. INTRODUCTION

Many-sorted (or heterogeneous) logic is widely used in several branches of computer science and artificial intelligence. It is utilized e.g. for a logical foundation of computer-aided problem solving (Gergely - Szőts [11]), for knowledge representation of design (Márkusz [18, 19]) and for definition of semantics of programming languages (Andréka - Németi - Sain [8]). Attempts have been made recently for an implementation of first order many-sorted resolution (Stanford Research Institute) which could increase the effectiveness of mechanical theorem proving. Heterogeneous logic is an aid for experts to discuss a great variety of theoretical problems in computer science (Kamin [15], ADJ [1], Andréka - Németi [7]), as well as to develop theories of data-base systems (Rónyai [34]), to investigate dynamic algebras of programs (Pratt [31, 32, 33], Németi [27, 29]), and semantics of program verification (Andréka - Németi - Sain [8], Németi [28]). Many-sorted logic is a useful tool for a description of connections between initial algebra semantics and algebraic logic, and between initial algebra semantics and cylindric algebras (Andréka - Sain [9], Németi [25]).

There are many possible fields of applications. Computer Science is full of problems which could be discussed in a more natural and simpler way within the framework of many-sorted model theory. Such problems are e.g. Montague's Universal Grammar [23] giving a logical foundation of natural language understanding, and a method of definition of semantics of formal languages based on Montague's work (Márkusz - Szőts [20]). See also Janssen [37]. Many-sorted logics are used in pure mathematics as well (e.g. in Henkin-style higher-order arithmetic, Gödel-Bernays set theory) and there is a need for further development of this logic from their part, too.

Due to this widespread application of many-sorted logic it is time that the concepts used in different papers in different ways were cleared and exactly defined. It means a unique and precise definition of all the basic concepts. First of all a reference work is needed for the users of many-sorted logic. In order to obtain further successful applications, it is necessary to investigate the specialities, possibilities and limits of this new mathematical tool. It needs a more detailed and deeper study of many-sorted model theory, which is not as highly developed as classical predicate calculus. There are only a few works discussing heterogeneous logic and most of them are restricted to the definition of the language (Monk [22], Lugowski [17], Andréka - Németi - - Sain [8]). The first step towards the investigations of some deeper specialities of many-sorted logic is a general and precise system of definitions.

The purpose of this paper is to meet the goals stated above, i.e. to be a reference paper for the users of many-sorted logic and, on the other hand to be an introduction to a detailed study of model theory of many-sorted logic.

In this paper a detailed definition of first order many-sorted language is given and a whole section deals with the operators defined on many-sorted models such as weak and strong sub-model, homomorphic image, direct product, reduced product and ultraproduct. The well-known Łos lemma of classical predicate calculus is also generalized for many-sorted first order language. The connections between the operators defined on the heterogeneous classes of models are discussed and some theorems of axiomatizability are presented. The proofs of most of the theorems in this paper follow from H.Andréka, I.Németi and I.Sain's results on a category theoretical version of abstract model theory [4, 5, 6, 30]. Category theoretical methods are widely used in investigating many-sorted logics (see e.g. [1, 15]). This follows from the nature of this field: in many-sorted logic we have to reformulate and reprove existing notions and theorems in a new environment so that they should fit harmoniously into a new coherent theory about many-sorted algebras. (About this phenomenon see the introduction of [4]. This is just the kind of thing category theoretical logic of [4, 5, 6, 30] was invented for.

One of the specialities of "being many-sorted" is referred to at several places in this paper. In most works published (e.g. Monk [22]) all the models having a sort with an empty universe are excluded. This exclusion essentially restricts the area where many-sorted logic can be used although this area should be extended. In this paper we fail to make this restriction. However, in order to be able to discuss the difference between the class of models defined in Monk [22] and the class of models defined here, we introduce the concept of normal many-sorted models. Normal models are identical with many-sorted models defined in Monk [22]. Non-normal models have at least one sort with an empty universe. Thus, the class of t-type many-sorted models defined here includes both normal and non-normal models, and is often called the class of empty-sorted models. Using classical validity concept for the larger class of empty-sorted models yields logical paradoxes. Thus all the theorems in this paper are stated for the smaller class of normal models. Another paper (Márkusz [21]) will study the special features of the class of empty-sorted models.

There are many further interesting branches of developing the theory of many-sorted algebras. One of these is the investigation of the connections between the usual first-order logic and many-sorted logic. Here we mention a recent result of I.Sain which says that the same relations (remaining within a fixed sequence of sorts) can be defined in a many-sorted model using many-sorted language as in the first-order version of this model using first-order language.

1. NOTATION

Throughout the paper $\overset{d}{=}$ denotes the fact that the concept standing on the left-hand side of the equality symbol is defined by the expression standing on the right-hand side. For example, "$x \overset{d}{=} y$" means that x is equal to y by definition. Similarly, "$\varphi \overset{d}{\Leftrightarrow} \psi$" means that the formula φ is defined by formula ψ, and φ is defined to be true if and only if ψ is true. Troughout the paper "iff" is an abbreviation of "if and only if", Brackets (,) and [,] play the same role and they are used simultaneously.

The following notation is given for arbitrary sets.

$\cup A \overset{d}{=} \{x : (\exists y \in A)\, x \in y\}$.

$\cap A \overset{d}{=} \{x : (\forall y \in A)\, x \in y\}$.

$A \cup B \overset{d}{=} \cup \{A, B\}$.

$A \cap B \overset{d}{=} \cap \{A, B\}$.

$A \sim B \overset{d}{=} \{a \in A : a \notin B\}$.

Natural numbers are used in von Neumann's sense.

0 denotes the empty set.

$a + 1 \overset{d}{=} a \cup \{a\}$.

$\omega \overset{d}{=} \cap \{H : 0 \in H$ and $(\forall n \in H)\, n+1 \in H\}$ and $(\forall n \in \omega)\, n \overset{d}{=} \{0, 1, \dots, n\text{-}1\}$.

$|A|$ denotes the cardinality of the set A.

$Sb\, A \overset{d}{=} \{X : X \subseteq A\}$. $Sb\, A$ is the set consisting of all the subsets of A. $Sb\, A$ is called the power set of the set A.

$(a, b) \overset{d}{=} \{\{a\}, \{a, b\}\}$ is the ordered pair of a and b, where the first member of the pair is a, and the second one is b.

Notation: $(a, b)_0 \overset{d}{=} a$ and $(a, b)_1 \overset{d}{=} b$.

$A \times B \overset{d}{=} \{(a, b) : a \in A$ and $b \in B\}$. $A \times B$ is the Cartesian product of A and B .

$Dom\, A \overset{d}{=} \{a \in \cup\cup A : (\exists b)(a, b) \in A\}$. $Dom\, A$ denotes the domain of the set A.

$Rng\ A \overset{d}{=} \{ b \in \cup\cup A : (\exists a)(a, b) \in A \}$. $Rng\ A$ denotes the range of the set A .

$A \upharpoonright B \overset{d}{=} A \cap (B \times Rng\ A) = \{ (a, b) \in A : a \in B \}$. $A \upharpoonright B$ denotes the restriction of the set A to the set B .

Let f be an arbitrary set. f is a function or a mapping or a sequence iff all the elements of f are ordered pairs and

$\forall a, b, c\ [((a, b) \in f$ and $(a, c) \in f) \to b = c\ ]$.

If f is a function and $i \in Dom\ f$, then there exists exactly one set b such that $(i, b) \in f$.

b is said to be the value of the function f at the argument i and is denoted by

$f(i)$ or

$f\,i$ or

f_i .

${}^A B \overset{d}{=} \{ f \in Sb(A \times B) : f$ is a function, $Dom\ f = A \}$.

${}^A B$ denotes the set of all functions from A into B .

$f : A \to B$ denotes that $f \in {}^A B$.

$f : A \rightarrowtail B$ denotes that $f \in {}^A B$ and f is a one-to-one mapping, i.e.

$[f : A \rightarrowtail B\ ] \overset{d}{\Leftrightarrow} [f : A \to B$ and $(\forall a, b \in A)(f(a) = f(b) \to a = b)]$.

$f : A \twoheadrightarrow B$ denotes that $f \in {}^A B$ and f is a mapping from A onto B, i.e. $Rng\ f = B$.

$f : A \rightarrowtail\!\!\!\!\twoheadrightarrow B$ denotes that $f \in {}^A B$ and f is a one-to-one mapping from A onto B .

Let $p_1, \dots, p_n$ and S be fixed sets.

Let $\tau(x, p_1, \dots, p_n)$ be an expression, which assigns a unique set denoted by $\tau(s, p_1, \dots, p_n)$ to every $s \in S$. Then

$<\tau(s, p_1, \dots, p_n)>_{s \in S} \overset{d}{=} <\tau(s, p_1, \dots, p_n) : s \in S> \overset{d}{=} \{(s, \tau(s, p_1, \dots, p_n)) : s \in S\}$.

That is $<\tau(s, p_1, \dots, p_n)>_{s \in S}$ is a function with domain S.

If $n=0$, i.e. there are no parameters $p_1, \dots, p_n$ then $<\tau(s)>_{s \in S} \overset{d}{=} <\tau_s>_{s \in S}$.

For example, suppose $\tau(s, p) \overset{d}{=} s \cap p$. Then $f \overset{d}{=} <s \cap p : s \in S>$ is a function for every fixed parameter p and S , otherwise f is not defined. That is function f depends on the choice of the parameters p and S .

A further example: Suppose $S \overset{d}{=} \omega$ and $p \in \omega$. Then $g \overset{d}{=} <p + s : s \in \omega>$ is a function $g: \omega \to \omega$ and $(\forall x \in \omega)\ g(x) = x + p$. Obviously function g depends on the choice of the parameter p .

In particular, if f is a function and $Dom\, f = S$, then

$<f_s : s \in S> = f$, .

$<f_s>_{s \in S} \stackrel{d}{=} <f_s : s \in S> = f$.

Let $n \in \omega$; and let f be a sequence of length n.

Sequence f may be given by "enumeration" as follows:

$f = <f_0 , f_1, ..., f_{n-1}>$.

E.g. $f \stackrel{d}{=} <5, 3, 8, 7> = \{ (0, 5), (1, 3), (2, 8), (3, 7) \}$.

That is f is a sequence with length 4, $f : 4 \to \omega$ such that $f(0) = f_0 = 5$, $f(1) = f_1 = 3$, $f_2 = 8$, $f_3 = 7$.

Let $<K_s>_{s \in S}$ and $<L_s>_{s \in S}$ be two sequences with common domain S. Sequence $<K_s>_{s \in S}$ is smaller than or equal to sequence $<L_s>_{s \in S}$ iff $(\forall s \in S)\, K_s \subseteq L_s$,

and we write:

$<K_s>_{s \in S} \leqslant <L_s>_{s \in S}$.

Let f and g be two functions.

$f \circ g \stackrel{d}{=} <f(g(x)) : \quad g(x) \in Dom\, f>_{x \in Dom\, g}$.

Function $f \circ g$ is called the composition of the functions f and g.

$Id_A \stackrel{d}{=} \{ (a, a) : a \in A \}$. Id_A is the identity function on the set A.

$S^+ \stackrel{d}{=} \cup \{ {}^n S : n \in \omega$ and $n \neq 0 \}$. S^+ denotes the set of all finite nonempty sequences of elements of S.

Let A be a function. The direct product of A is as follows:

$PA \stackrel{d}{=} \mathrm{P}_{i \in Dom\, A}\, A_i \stackrel{d}{=} \{ f \in {}^{Dom\, A}(\cup\, Rng\, A) : (\forall i \in Dom\, A)\, f_i \in A_i \}$.

Let "*Sets*" denote the class of all sets.

CONVENTION 0

Throughout the paper each symbol denotes a set unless it is declared to denote a class or a metaclass. All the notations introduced above are used for classes and metaclasses as well as for sets in the usual way.

REMARK 0

Set theory, which is based on the hierarchy of sets – classes – metaclasses, is described e.g. in Herrlich–Strecher [12], where "conglomerate" is used instead of "metaclass". The main point of the hierarchy is

$$Sets \subseteq Metaclasses \text{ such that } Sets \in Metaclasses \text{ and}$$
$$\langle Sets, \in \rangle \models ZFC \text{ and } \langle Metaclasses, \in \rangle \models ZFC \text{ .}$$

The difference between metaclasses and classes is that elements of a metaclass can be metaclasses, classes or sets, while a proper class may have no elements but sets.

CONVENTION 1

1.1 From now on the ordered pairs and the 2-length sequences will not be distinguished. More exactly, $\langle x, y \rangle$ will denote both the ordered pair (x, y) and the function $\{(0, x), (1, y)\}$ for every set x and y, though they are not identical. The reason behind this convention is that it is not so important from the point of view of this paper, which meaning of the symbol $\langle x, y\rangle$ is to be considered. The only requirement is that condition

$$\forall x, y, u, n \ [\langle x, y\rangle = \langle u, w\rangle \Leftrightarrow (x = u \text{ and } y = w)]$$

holds for both meanings, and it obviously holds for both the ordered pairs and the 2-length sequences.

An important consequence of this convention is that $A \times A$ is identical with ${}^{2}A$ for every set A.

This convention (which is improper in princeple) is very wide spread in mathematics, see e.g. Henkin–Monk–Tarski [12] p. 33, or Levy [16] Def. 4.15. p. 58. In these works one can also find the consequences of the convention above, and a technique which helps to avoid false results.

1.2 Let A be a set and $n \in \omega$. Then ${}^{n}A \times A$ is considered to be identical with ${}^{n+1}A$, i.e.

$$ {}^{n}A \times A = {}^{n+1}A \ . $$

Therefore the ordered pair $(\langle s_0, \ldots, s_{n-1} \rangle, s_n)$ is considered to be identical with the sequence $\langle s_0, \ldots, s_{n-1}, s_n \rangle$, and the Cartesian product is considered to be associative:

$$ (A \times B) \times C = A \times (B \times C) \subseteq {}^{3}(A \cup B \cup C) . $$

Hence $A \times A \times A = {}^{3}A$.

DEFINITION 0 (n-ary relation, function)

Let B be a set and $n \in \omega$. By an <u>n-ary relation over B</u> we understand a set $R \subseteq {}^{n}B$, i.e. an n-ary relation is a set of sequences with length n.

By an <u>n-ary function over B</u> we understand a set $f \in {}^{({}^{n}B)}B$. If f is an n-ary function, we write

$$ f : {}^{n}B \to B \ . $$

■

COROLLARY 0

Due to Convention 1, <u>n-ary functions over B</u> are $n+1$-ary relations over B, since

$$ \text{"}n\text{-ary functions over } B\text{"} = {}^{({}^{n}B)}B \subseteq ({}^{n}B) \times B = {}^{n+1}B \ . $$

This corollary is utilized essentially throughout the paper.

2. FIRST ORDER MANY-SORTED LANGUAGE

2.1 MANY-SORTED SIMILARITY TYPE

DEFINTION 1 (many-sorted similarity type)

A set t is said to be a many-sorted (or heterogeneous) similarity type if

$$t \in {}^3(Rng\ t) \quad \text{and}$$
$$t_1 : Dom\ t_1 \to (t_0)^+ \text{ and } t_2 \subseteq Dom\ t_1 \ .$$

■

NOTATION

Generally t_0 is denoted by S and t_2 by H, so

$$t = \langle S, t_1, H \rangle \ .$$

In Definition 1

$t_0 = S$	is called the set of sorts,
t_1	is called arity function,
$t_2 = H$	is the set of function symbols,
$Dom\ (t_1) \sim H$	is the set of relation symbols of the type t.

CONVENTION 2

From now on t denotes a many-sorted similarity type.

NOTATION

Let t be a similarity type and let $r \in Dom\ t_1$.

$$tr \stackrel{d}{=} t(r) \stackrel{d}{=} t_1(r) \ .$$

REMARK 1

If $r \in Dom\, t_1 \sim H$, i.e. r is a relation symbol, then $Dom\,(tr)$ is the number of the arguments of the relation symbol r. For example, let $t = \langle S, t_1, H \rangle$ be a fixed similarity type such that $S = \{ p, q, k \}$, $t_1 = \{ \langle r, \langle q, p, k \rangle \rangle, \langle f, \langle q, k \rangle \rangle \}$, $H = \{ f \}$, $tr = \langle q, p, k \rangle$.

Then $Dom\,(tr) = 3 = \{ 0, 1, 2 \}$, and

$tr(o) = q$, $tr(1) = p$, $tr(2) = k$.

Let $n \overset{d}{=} Dom\,(tr) - 1$. And really r is a ternary relation symbol between sorts q, p and k.

If $f \in H$, i.e. f is a function symbol, then $n \overset{d}{=} Dom\,(tf) - 1$ is the number of the arguments of function symbol f.

2.2 MANY-SORTED MODELS

DEFINITION 2 (t-type model)

Let t be a many-sorted similarity type.

By a many-sorted t-type model we understand a pair $\mathfrak{A} = \langle A, R \rangle$ iff the following (1)-(2) hold:

(1) A is a function such that

$Dom\, A = S$.

(2) R is a function, and conditions (i) – (ii) hold:

(i) $Dom\, R = Dom\, t_1$.

(ii) Let $r \in Dom\, t_1$ be an arbitrary symbol and

$n \overset{d}{=} Dom\,(tr) - 1$. Then:

$R_r \subseteq \mathrm{P}_{i \leq n} A_{tr(i)}$, i.e.

$R_r \subseteq A_{tr(o)} \times \ldots \times A_{tr(n)}$.

Furthermore, if $r \in H$, then

$R_r : \mathrm{P}_{i<n} A_{tr(i)} \rightarrow A_{tr(n)}$, i.e.

$R_r : (A_{tr(o)} \times \ldots \times A_{tr(n-1)}) \rightarrow A_{tr(n)}$, i.e. the relation R_r is a function with domain

$$Dom\,(R_r) = \mathrm{P}_{i<n} A_{(t_1(r))_i} \; .$$

■

By Definition 2 $\mathfrak{A}$ is a t-type many-sorted model iff

$\mathfrak{A} = \langle\langle A_s \rangle_{s \in S} , \langle R_r \rangle_{r \in Dom(t_1)} \rangle$, i.e.

$\mathfrak{A}_0 = \langle A_s \rangle_{s \in S}$ and $\mathfrak{A}_1 = \langle R_r \rangle_{r \in Dom(t_1)}$ and

the conditions (1) and (2) above hold.

NOTATION

Let $\mathfrak{A}$ be an arbitrary t-type model and let $r \in Dom(t_1)$ be an arbitrary symbol. Then R_r is denoted alternatively by $r^{\mathfrak{A}}$, too. Thus

$$\mathfrak{A} = \langle A, R \rangle = \langle \langle A_s \rangle_{s \in S}, \langle R_r \rangle_{r \in Dom(t_1)} \rangle \stackrel{d}{=}$$

$$\stackrel{d}{=} \langle \langle A_s \rangle_{s \in S}, \langle r^{\mathfrak{A}} \rangle_{r \in Dom(t_1)} \rangle .$$

A_s $(s \in S)$ is said to be the universe of the sort s, and

$$\mathfrak{A}_0 = A = \langle A_s \rangle_{s \in S}$$

is said to be <u>the system of universes of the model $\mathfrak{A}$</u>.

DEFINITION 3 (normal t-type model)

Let $\mathfrak{A}$ be a t-type model.

$\mathfrak{A}$ is a <u>normal model iff</u> $(\forall s \in S)\ A_s \neq 0$.

That is $\mathfrak{A}$ is a normal model if and only if there is no sort s such that the corresponding universe A_s is empty.

■

NOTATION

$Mod_t \stackrel{d}{=} \{\mathfrak{A} : \mathfrak{A}$ is a normal t-type model $\}$.

$Mod_t^{o} \stackrel{d}{=} \{\mathfrak{A} : \mathfrak{A}$ is a t-type model $\}$.

Note that Mod_t and Mod_t^{o} are not sets, but proper classes. $Mod_t \subsetneqq Mod_t^{o}$, i.e. Mod_t is a proper subclass of the class Mod_t^{o} .

REMARK 2

In Section 3 we shall define the usual model theoretical concepts (such as submodel, homomorphic image, direct and reduced products) on many-sorted models. The definitions will be given for the larger class Mod_t^o , the propositions and theorems, however, will be given for the smaller class of the normal models Mod_t.

Some theorems proposed in this paper hold only for normal models. We are going to investigate a new validity relation in another paper. This new validity concept will differ from that in Tarski's sense, and the propositions and theorems of this paper will hold for the elements of the class Mod_t^o , as well (see Markusz [21]).

2.3 SYNTAX OF FIRST ORDER MANY-SORTED LANGUAGE

DEFINITION 4 (variables)

Let $t = \langle S, t_1, H \rangle$ be a similarity type and let $v : \omega \times S \rightarrowtail\!\!\twoheadrightarrow Rng\, v$ be a one-to-one function. Let $Rng\, v$ be disjoint from any other set occuring in this paper, e.g. $Dom\,(t_1) \cap Rng\, v = 0$.

Let $\langle i, s \rangle \in \omega \times S$. Then

$$v_i^s \stackrel{d}{=} v\,(\langle i, s \rangle) \;.$$

v_i^s is called the <u>i-th variable of the sort s</u>.

Define

$$V^s \stackrel{d}{=} \{ v_i^s : i \in \omega \} \;.$$

V^s is called the <u>set of variables of the sort s</u>.

Define

$$V \stackrel{d}{=} \bigcup_{s \in S} V^s \;.$$

V is said to be the <u>set of the variables.</u>

■

DEFINITION 5. (set of t-type terms : T_t)

Let $t = \langle S, t_1, H \rangle$ be a similarity type, and let V^s be a set of variables of the sort $s \in S$. Let G be the smallest sequence such that $Dom\, G = S$, and conditions (i)–(ii) hold :

(i) $(\forall s \in S)\; V^s \subseteq G\,(s)$.

(ii) Let $f \in H$ and $n = Dom\,(tf) - 1$.
Suppose $(\forall\, i \in n)\;\; \tau_i \in G\,(tf(i))$. Then

$$f(\tau_0 \,, \dots , \tau_{n-1}) \in G\,(tf\,(n)) \;.$$

Obviously, there exists such a function G, and only one exists.

Let define

$$T_t^s \stackrel{d}{=} G(s)$$

for every $s \epsilon S$.

T_t^s is said to be the set of t-type terms of the sort s.

Let $T_t \stackrel{d}{=} Rng\, G$, i.e. $T_t = \bigcup_{s \epsilon S} T_t^s$.

T_t is called the set of t-type terms.

■

DEFINITION 6 (set of t-type first order formulas : F_t)

The set of t-type atomic formulas is a set Af_t :

$$Af_t \stackrel{d}{=} \{ r(\tau_0, ..., \tau_n) : r \epsilon Dom(t_1) \sim H, \quad n = Dom(tr) - 1 \quad \text{and } \tau_i \epsilon T_i^{tr(i)} \text{ for every } i \leqslant n \} \cup$$

$$\cup \{ (\tau = \sigma) : \tau, \sigma \epsilon T_t^s \text{ for } s \epsilon S \} .$$

The set of t-type first order formulas is the smallest set F_t such that

(i) $Af_t \subseteq F_t$.

(ii) Let $\varphi, \psi \epsilon F_t$ and let $v_i^s \epsilon V$ for any $s \epsilon S$ and $i \epsilon \omega$. Then

$$\{(\varphi \wedge \psi), \neg\varphi, \exists v_i^s \varphi\} \subseteq F_t .$$

■

CONVENTION 3

Let $\varphi, \psi \epsilon F_t$ be arbitrary formulas and let $v_i^s \epsilon$ V be a variable for any fixed $s \epsilon S$ and $i \epsilon \omega$.

Then

$$(\varphi \vee \psi) \stackrel{d}{=} \neg(\neg\varphi \wedge \neg\psi),$$

$$(\varphi \rightarrow \psi) \stackrel{d}{=} (\neg\varphi \vee \psi),$$

$$(\forall v_i^s \varphi) \stackrel{d}{=} (\neg \exists v_i^s \neg\varphi) .$$

2.4 SATISFACTION AND VALIDITY RELATION IN TARSKI'S SENS

DEFINITION 7 (valuation)

Let $\mathfrak{A} \in Mod_t^{\circ}$.

By a valuation of the variables into a model $\mathfrak{A}$ (shortly by a valuation) we understand a sequence of functions $k = \langle k_s \rangle_{s \in S}$ such that

$$(\forall_s \in S)\ k_s : \omega \to A_s \ .$$

That is $k_s \in {}^{\omega}(A_s)$ for every $s \in S$.

Therefore the set of all the valuations of the variables into $\mathfrak{A}$ is

$$\underset{s \in S}{\mathrm{P}}\ ({}^{\omega}(A_s)) \ .$$

■

DEFINITION 8 $(\tau^{\mathfrak{A}}[k])$.

Let $\tau \in T_t$, $\mathfrak{A} \in Mod^{\circ}{}_t$, $k \in \underset{s \in S}{\mathrm{P}}\ ({}^{\omega}(A_s))$.

The meaning of the term τ in the model $\mathfrak{A}$ with respect to the valuation k (notation : $\tau^{\mathfrak{A}}[k]$) is defined by recursion

(i) If τ is a variable $v_i^s \in V^s$ $(s \in S$ and $i \in \omega)$, then

$$v_i^{s\,\mathfrak{A}}[k] \overset{\mathrm{d}}{=} k_s(i) \ . \qquad (k_s(i) \in A_s) \ .$$

(ii) If τ is a term of the form $f(\tau_0, \ldots, \tau_{n-1})$, where $f \in H$, $n = Dom(tf) - 1$ and $(\forall i \in n)\,[\tau_i \in T_t^{tf(i)}$ and $\tau_i{}^{\mathfrak{A}}[k]$ has already been defined], then

$$f(\tau_0, \ldots, \tau_{n-1})^{\mathfrak{A}}[k] \overset{\mathrm{d}}{=} f^{\mathfrak{A}}(\tau_0{}^{\mathfrak{A}}[k], \ldots, \tau_{n-1}{}^{\mathfrak{A}}[k]) \ .$$

■

The concept "satisfaction" in Tarski's sense (notation : $\models$) is a 3-ary relation which connects a class of models, a set of formulas and the corresponding set of valuations. In the case of many--sorted logic, considering the class of models Mod_t^o, the set of formulas F_t and the set of valuations $\underset{s\in S}{\mathrm{P}}({}^{\omega}A_s)$, the satisfaction relation is:

$$\models \subseteq Mod_t^o \times F_t \times \underset{s\in S}{\mathrm{P}}({}^{\omega}A_s) .$$

Let $\mathfrak{A} \in Mod_t^o$, $\varphi \in F_t$, $k \in \underset{s\in S}{\mathrm{P}}({}^{\omega}A_s)^{s\in S}$.

Then $\models < \mathfrak{A}, \varphi, k >$ means, that the valuation k satisfies the formula φ in the model $\mathfrak{A}$, or the formula φ is true in the model $\mathfrak{A}$ with respect to the valuation k. Usually, we write $\mathfrak{A} \models \varphi[k]$ instead of $\models < \mathfrak{A}, \varphi, k >$, i.e.

$$\models \varphi[k] \overset{d}{=} \models < \mathfrak{A}, \varphi, k >$$

(see e.g. Andréka–Gergely–Németi [3] or Monk [22]) .

By convention (sloppily), the symbol $\models$ denotes the validity relation in Tarski's sense, too (see Monk [22]). The validity is a binary relation, defined on a class of models and on a set of formulas. In our case:

$$\models \subseteq Mod_t^o \times F_t .$$

Thus the sequence of symbols $\mathfrak{A} \models \varphi$ means, that the formula φ is valid in the model $\mathfrak{A}$ or $\mathfrak{A}$ is a model of the formula φ.

We define the satisfaction and the validity relation in Tarski's sense for many-sorted logic in details bellow.

DEFINITION 9 (satisfaction: $\mathfrak{A} \models \varphi[k]$)

Let $\varphi \in F_t$, $\mathfrak{A} \in Mod_t^o$, $k \in \underset{s\in S}{\mathrm{P}}({}^{\omega}A_s)$.

"The valuation k satisfies the formula φ in the model $\mathfrak{A}$" (notation: $\mathfrak{A} \models \varphi[k]$) is defined as follows:

1. Atomic formulas

(i) Let $\tau, \sigma \in T_t^s$. Then

$$\mathfrak{A} \models (\tau = \sigma)[k] \overset{d}{\Longleftrightarrow} \tau^{\mathfrak{A}}[k] = \sigma^{\mathfrak{A}}[k] .$$

(ii) Let $r \in Dom(t_1) \sim H$, $n = Dom(tr)\text{-}1$ and $(\forall i \subseteq n)\, \tau_i \in T_t^{tr(i)}$. Then

$$\mathfrak{A} \models r(\tau_0, \dots, \tau_n)[k] \overset{d}{\Longleftrightarrow} \langle \tau_0^{\mathfrak{A}}[k], \dots, \tau_n^{\mathfrak{A}}[k] \rangle \in r^{\mathfrak{A}} .$$

2. Formulas

Let $\varphi, \psi \in F_t$ and $v_i^s \in V^s$.

Suppose $\mathfrak{A} \models \varphi[k]$ and $\mathfrak{A} \models \psi[k]$ has already been defined. Then

(i) $\models \neg \varphi[k] \overset{d}{\Longleftrightarrow} (\mathfrak{A} \models \varphi[k]$ is not true$)$.

(ii) $\models (\varphi \wedge \psi)[k] \overset{d}{\Longleftrightarrow} (\mathfrak{A} \models \varphi[k]$ and $\mathfrak{A} \models \psi[k])$.

(iii) $\models \exists v_i^s\, \varphi[k] \overset{d}{\Longleftrightarrow}$ (there exists a valuation $g \in \underset{s\in S}{\mathbf{P}}({}^{\omega}(A_s))$ such that

$(\forall z \in S \sim \{s\})\, k_z = g_z$ and

$k_s \restriction (\omega \sim \{i\}) = g_s \restriction (\omega \sim \{i\})$ and

$\mathfrak{A} \models \varphi[g])$.

■

DEFINITION 10 (validity: $\mathfrak{A} \models \varphi$)

Let $\mathfrak{A} \in Mod_t^{\circ}$, $\varphi \in F_t$.

<u>The formula φ is valid in the model $\mathfrak{A}$ or $\mathfrak{A}$ is a model of the formula φ</u> iff

$$\mathfrak{A} \models \varphi \overset{d}{\Longleftrightarrow} (\forall k \in \underset{s\in S}{\mathbf{P}}({}^{\omega}(A_s)))\, \mathfrak{A} \models \varphi[k] .$$

■

REMARK 3

If $\mathfrak{A}$ is a non-normal t-type model, i.e. $(\exists s \in S)\, A_s = 0$, then the set of the valuations is empty, i.e.

$$\underset{s\in S}{\mathbf{P}}({}^{\omega}A_s) = 0 .$$

Thus, according to Definition 10, $\mathfrak{A} \models \varphi$ for any $\varphi \in F_t$, since

$$\mathfrak{A} \models \varphi \overset{d}{\Longleftrightarrow} (\forall k \in 0)\ \mathfrak{A} \models \varphi\,[k]\ .$$

Therefore $\mathfrak{A} \models F_t$, e.g. $\mathfrak{A} \models \varphi \wedge \neg \varphi$ as well. Obviously, it contradicts our intuitive imagination, that is why we are going to introduce another validity concept on the class $Mod_t{}^{o}$, which is "good" in the case of $Mod_t{}^{o}$, too. This new validity concept (Mostowski [24]) is said to be the <u>validity relation in Mostowski's sense</u> and is denoted by $\mathrel{|\!\equiv}$.

Thus if $\mathfrak{A} \in Mod_t{}^{o}$ and $\varphi \in F_t$ then $\mathfrak{A} \models \varphi \wedge \neg \varphi$ holds, but $\mathfrak{A} \mathrel{|\!\equiv} \varphi \wedge \neg \varphi$ does not hold.

The definition and the detailed description of this new validity relation is discussed in Markusz [21]. It is easy to see that if $\mathfrak{A}$ is a normal model, then $\mathfrak{A} \models \varphi \wedge \neg \varphi$ does not hold.

2.5 DEFINITION OF FIRST ORDER MANY-SORTED LANGUAGE

DEFINITION 11 (L_t)

Triple $L_t \stackrel{d}{=} \langle F_t, Mod_t, \models \rangle$ is said to be a first order many-sorted language .

■

REMARK 4

4.1 The triple above is said to be a language according to the terminology of abstract model theory (see Sain [35] or Andréka-Németi [4, 6]). The set of formulas denoted by F_t defines the syntax of the language, and the class of models Mod_t and the validity relation define the semantics of the language.

4.2 Many-sorted language L_t is reducible to classical first order predicate calculus (see Monk [22]). Its reducibility is very important from the point of view of the completeness of the language L_t, since many-sorted language L_t is also complete according to Gödel's Completeness Theorem.

4.3 Triple $L_t^{\circ} \stackrel{d}{=} \langle F_t, Mod_t^{\circ}, \models \rangle$ is said to be a first order empty-sorted language. As one can see in Remark 3, this language is much more difficult to work with than language L_t. To avoid these difficulties we define another empty-sorted language

$$L^{\circ}_{Mt} \stackrel{d}{=} \langle F_t, Mod_t^{\circ}, \models \rangle$$

which satisfies our intentions. Note that language L_{Mt} uses validity concept is Mostowski's sense. The comparison of languages L_t° and L°_{Mt} can be found in Markusz [21]. It is worth mentioning that the syntax of all the three languages L_t, L_t° and L°_{Mt} is the same the difference lying in their semantics.

4.4 Validity relation $\models \subseteq Mod_t^{\circ} \times F_t$ induces Galois sorrespondences, namely $Th : Sb\ Mod_t^{\circ} \rightarrow Sb\ F_t$ and $Mod^{\circ} : Sb\ F_t \rightarrow Sb\ Mod_t^{\circ}$, which are called "Theory of" and "Models of", respectively. (See Def. 30, 31.) Note that since Mod_t° is a proper class, $Sb\ Mod_t^{\circ}$ is a metaclass, and therefore Th and Mod° are metafunctions.

PROPOSITION 0

Metafunctions

$Th \quad : \langle Sb\ Mod_t^{\circ}, \subseteq\rangle \to \langle Sb\ F_t, \supseteq\rangle$ and

$Mod^{\circ} : \langle Sb\ F_t, \supseteq\rangle \to \langle Sb\ Mod_t^{\circ}, \subseteq\rangle$

are metahomomorphisms.

3. MANY-SORTED MODEL-THEORETIC CONSTRUCTIONS

3.1 SUBMODEL

DEFINITION 12 (weak submodel)

Let $\mathfrak{A}, \mathfrak{B} \in Mod_t^{\,0}$ **be two models.**

$\mathfrak{B}$ is a weak submodel of $\mathfrak{A}$ (notation: $\mathfrak{B} \in S_w \{\mathfrak{A}\}$ **or** $\mathfrak{B} \subseteq_w \mathfrak{A}$) **iff**

(i) $(\forall s \in S)\, B_s \subseteq A_s$.

(ii) $(\forall r \in Dom\,(t_1))\; r^{\mathfrak{B}} \subseteq r^{\mathfrak{A}}$.

■

EXAMPLE 1 (weak submodel)

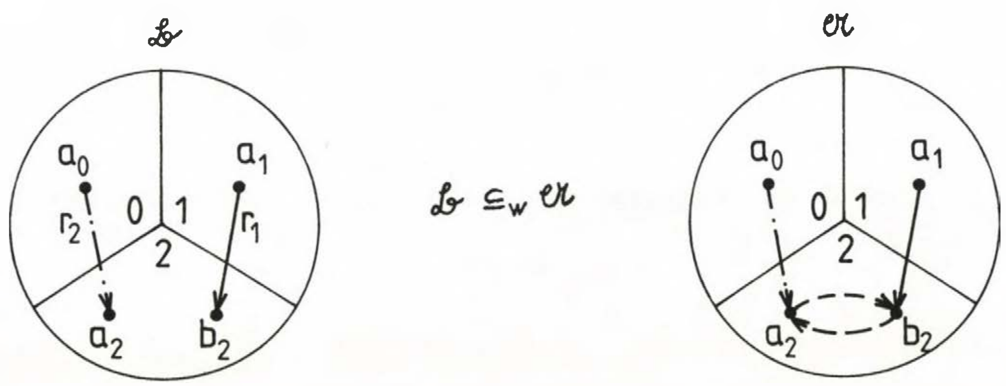

Figure 1.

Let $t = \langle S, t_1, H \rangle$ **be a similarity type, where**

$S = \{0, 1, 2\}$,

$t_1 = \{\langle r_1 \langle 1, 2\rangle\rangle, \langle r_2, \langle 0, 2\rangle\rangle, \langle r_3, \langle 2, 2\rangle\rangle\}$,

$H = 0$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two t-type models as follows:

$$\mathfrak{B} = \langle B, \langle r^{\mathfrak{B}} \rangle_{r \in Dom(t_1)} \rangle , \qquad \mathfrak{A} = \langle A, \langle r^{\mathfrak{A}} \rangle_{r \in Dom(t_1)} \rangle ,$$

$$B = \langle B_o, B_1, B_2 \rangle , \qquad \text{and} \qquad A = \langle A_o, A_1, A_2 \rangle = B .$$

$$B_o = \{ a_o \} , \quad B_1 = \{ a_1 \} , \quad B_2 = \{ a_2, b_2 \} .$$

$$r_1^{\mathfrak{B}} = \{ \langle a_1, b_2 \rangle \} , \qquad r_1^{\mathfrak{A}} = \{ \langle a_1, b_2 \rangle \} ,$$

$$r_2^{\mathfrak{B}} = \{ \langle a_o, a_2 \rangle \} , \qquad r_2^{\mathfrak{A}} = \{ \langle a_o, a_2 \rangle \} ,$$

$$r_3^{\mathfrak{B}} = 0 , \qquad r_3^{\mathfrak{A}} = \{ \langle a_2, b_2 \rangle , \langle b_2, a_2 \rangle \} .$$

CLAIM

$\mathfrak{B} \in S_W \{ \mathfrak{A} \}$.

PROOF

Condition (i) in Definition 12 holds, since $(\forall s \in S)(A_s = B_s)$ **, and so does condition (ii), since** $r_1^{\mathfrak{B}} = r_1^{\mathfrak{A}}$, $r_2^{\mathfrak{B}} = r_2^{\mathfrak{A}}$, $r_3^{\mathfrak{B}} \subset r_3^{\mathfrak{A}}$.

QED .

DEFINITION 13 (strong submodel)

Let $\mathfrak{A}, \mathfrak{B} \in Mod_t^o$.

$\mathfrak{B}$ **<u>is a strong submodel of</u>** $\mathfrak{A}$ **(notation:** $\mathfrak{B} \in S_s \{ \mathfrak{A} \}$ **or** $\mathfrak{B} \subseteq_s \mathfrak{A}$ **) iff**

(i) $(\forall s \in S) \; B_s \subseteq A_s$.

(ii) $(\forall r \in Dom(t_1))[(n = Dom(tr) - 1) \rightarrow (r^{\mathfrak{B}} = r^{\mathfrak{A}} \cap \mathbf{P}_{i \leqslant n} B_{tr(i)})]$.

■

EXAMPLE 2 (strong submodel)

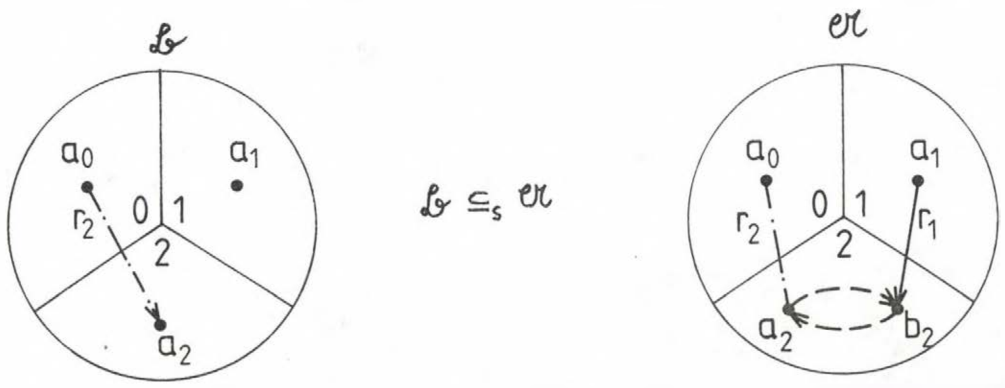

Figure 2.

Let t **denote the same similarity type and let** $\mathfrak{A}$ **denote the same** t**-type model as in Example 1. We define a model** $\mathfrak{B}$ **as follows:**

$$\mathfrak{B} = \langle B, \langle r^{\mathfrak{B}} \rangle_{r \in Dom(t_1)} \rangle \quad \textbf{where} \quad B = \langle B_0, B_1, B_2 \rangle$$

$$B_0 = \{a_0\}, \; B_1 = \{a_1\}, \; B_2 = \{a_2\} .$$

$$r_1^{\mathfrak{B}} = 0, \; r_2^{\mathfrak{B}} = \{\langle a_0, a_2 \rangle\}, \; r_3^{\mathfrak{B}} = 0 .$$

CLAIM

$\mathfrak{B} \in S_s\{\mathfrak{A}\}$.

PROOF

Condition (i) in Definition 13 holds, since

$$B_o = A_o, \quad B_1 = A_1, \; B_2 \subseteq A_2,$$

and so does condition (ii) because

1) $r_1^{\mathfrak{B}} = r_1^{\mathfrak{A}} \cap (B_1 \times B_2)$ since

$r_1^{\mathfrak{B}} = 0$ and

$r_1^{\mathfrak{A}} \cap (B_1 \times B_2) = \{<a_1, b_2>\} \cap (\{a_1\} \times \{a_2\}) = \{<a_1, b_2>\} \cap \{<a_1, a_2>\} = 0$.

2) $r_2^{\mathfrak{B}} = r_2^{\mathfrak{A}} \cap (B_o \times B_2)$ since

$r_2^{\mathfrak{B}} = \{<a_o, a_2>\}$ and

$r_2^{\mathfrak{A}} \cap (B_o \times B_2) = \{<a_o, a_2>\} \cap (\{a_o\} \times \{a_2\}) =$

$= \{<a_o, a_2>\} \cap \{<a_o, a_2>\} = \{<a_o, a_2>\}$.

3) $r_3^{\mathfrak{B}} = r_3^{\mathfrak{A}} \cap (B_2 \times B_2)$ since

$r_3^{\mathfrak{B}} = 0$ and

$r_3^{\mathfrak{A}} \cap (B_2 \times B_2) = \{<a_2, b_2> <b_2, a_2>\} \cap (\{a_2\} \times \{a_2\}) =$

$= \{<a_2, b_2> <b_2, a_2>\} \cap \{<a_2, a_2>\} = 0$.

QED .

REMARK 5

5.1 In Example 2 $\mathfrak{B}$ is a weak submodel of the model $\mathfrak{A}$, but it <u>is not</u> a strong submodel, i.e.

$$\mathfrak{B} \subseteq_w \mathfrak{A} \text{ and } \mathfrak{B} \not\subseteq_s \mathfrak{A}.$$

It is easy to see that condition (ii) in Definition 13 does not hold, since

$r_3^{\mathfrak{B}} \neq r_3^{\mathfrak{A}} \cap (B_2 \times B_2)$ because

$r_3^{\mathfrak{B}} = 0$ and

$$r_3^{\mathfrak{A}} \cap (B_2 \times B_2) = \{ \langle a_2, b_2 \rangle , \langle b_2, a_2 \rangle \} \cap (\{ a_2, b_2 \} \times \{ a_2, b_2 \}) =$$

$$= \{ \langle a_2, b_2 \rangle , \langle b_2, a_2 \rangle \} \cap \{ \langle a_2, a_2 \rangle , \langle b_2, b_2 \rangle , \langle a_2, b_2 \rangle , \langle b_2, a_2 \rangle \} =$$

$$= \{ \langle a_2, b_2 \rangle , \langle b_2, a_2 \rangle \} \neq 0 .$$

5.2 Sloppily, the difference between a weak submodel of a model $\mathfrak{A}$ and a strong one is that if we omit some elements of the universe of $\mathfrak{A}$ and restrict relations to the new universe (i.e. omit exactly the ones which contain any of the omitted elements), then we get a strong submodel of $\mathfrak{A}$. However, if we omit only some relations, or only some elements of the universe A, we get a weak submodel.

For example, in Example 1 we only omitted relation $r_3^{\mathfrak{A}}$ from the model $\mathfrak{A}$ and we have got a weak submodel, while in Example 2, we omitted element b_2 with all the relations on it, and we have got a strong submodel.

PROPOSITION 1

$$\mathfrak{A} \subseteq_s \mathfrak{B} \Rightarrow \mathfrak{A} \subseteq_w \mathfrak{B} .$$

PROOF

The proof is easy by Definition 12, 13.

QED .

3.2 HOMOMORPHIC IMAGE

DEFINITION 14 (homomorphism)

Let $\mathfrak{A}, \mathfrak{B} \in Mod_t^{\circ}$.

By a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$ we understand a sequence of functions $f = \langle f_s \rangle_{s \in S}$ such that

(i) $(\forall s \in S)\, f_s : A_s \rightarrow B_s$.

(ii) $(\forall r \in Dom(t_1))\, (\forall \langle a_0, \ldots, a_n \rangle \in r^{\mathfrak{A}})\, \langle f_{tr(0)}(a_0), \ldots, f_{tr(n)}(a_n) \rangle \in r^{\mathfrak{B}}$.

NOTATION: $f : \mathfrak{A} \rightarrow \mathfrak{B}$ denotes that f is a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$.

■

DEFINITION 15 (isomorphic models)

Let $\mathfrak{A}, \mathfrak{B} \in Mod_t^{\circ}$.

Two models $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic iff there exist two homomorphisms $f : \mathfrak{A} \rightarrow \mathfrak{B}$ and $g : \mathfrak{B} \rightarrow \mathfrak{A}$ such that

$$(\forall s \in S)\, (f_s \circ g_s = Id_{A_s} \text{ and } g_s \circ f_s = Id_{B_s}) .$$

NOTATION: $\mathfrak{A} \cong \mathfrak{B}$ denotes that models $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.

■

PROPOSITION 2

Let $f : \mathfrak{A} \rightarrow \mathfrak{B}$. Then

$$(\forall s \in S)\, (A_s \neq 0 \rightarrow B_s \neq 0) .$$

The proof is trivial. QED .

DEFINITION 16 (weak homomorphic image)

Let $\mathfrak{A}, \mathfrak{B} \in Mod_t^{o}$.

$\mathfrak{B}$ is said to be a weak homomorphic image of the model $\mathfrak{A}$ (notation: $\mathfrak{B} \in H_w\{\mathfrak{A}\}$) iff

(i) there exists a homomorphism $f : \mathfrak{A} \to \mathfrak{B}$.

(ii) $(\forall s \in S)\ Rng\, f_s = B_s$.

■

DEFINITION 17 (strong homomorphic image)

Let $\mathfrak{A}, \mathfrak{B} \in Mod_t^{o}$ and $f : \mathfrak{A} \to \mathfrak{B}$.

$\mathfrak{B}$ is said to be a strong homomorphic image of the model $\mathfrak{A}$ (notation: $\mathfrak{B} \in H_s\{\mathfrak{A}\}$) iff

(i) $\mathfrak{B} \in H_w\{\mathfrak{A}\}$. (i.e. $\mathfrak{B}$ is a weak homomorphic image of $\mathfrak{A}$)

(ii) for every $r \in Dom\,(t_1)$, if $n = Dom\,(tr) - 1$, then
$r^{\mathfrak{B}} = \{ <f_{tr(o)}(a_o), \ldots, f_{tr(n)}(a_n)> : <a_o, \ldots, a_n> \in r \}$.

■

REMARK 6

6.1 Sloppily, a strong homomorphic image of a model $\mathfrak{A}$ is quite alike as "the result of the projection" of the model $\mathfrak{A}$ by the corresponding homomorphism.

6.2 In the case of a weak homomorphic image of a model $\mathfrak{A}$ there may be some relation which does hold on some elements of the homomorphic image, but do not hold on the corresponding elements of the pre-model $\mathfrak{A}$.

EXAMPLE 3 (weak homomorphic image)

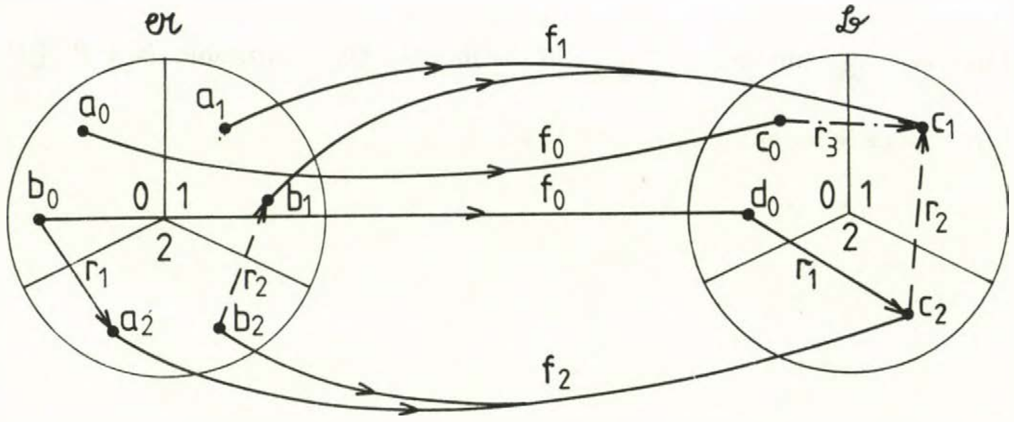

Figure 3.

Let $t = \langle S, t_1, H\rangle$ be a similarity type, where

$S = \{0, 1, 2\}$,

$t_1 = \{\langle r_1, \langle 0, 2\rangle\rangle, \langle r_2, \langle 2, 1\rangle\rangle, \langle r_3, \langle 0, 1\rangle\rangle\}$,

$H = 0$.

Define models $\mathfrak{A}$ and $\mathfrak{B}$ as follows:

$$\mathfrak{A} = \langle A, \langle r^{\mathfrak{A}}\rangle_{r \in Dom(t_1)}\rangle \quad , \quad \mathfrak{B} = \langle B, \langle r^{\mathfrak{B}}\rangle_{r \in Dom(t_1)}\rangle$$

$$A = \langle A_0, A_1, A_2\rangle \ , \qquad B = \langle B_0, B_1, B_2\rangle \ ,$$

$$A_0 = \{a_0, b_0\} \ , \ A_1 = \{a_1, b_1\} \ , \ A_2 = \{a_2, b_2\}.$$

$$B_0 = \{c_0, d_0\} \ , \ B_1 = \{c_1\} \ , \ B_2 = \{c_2\} \ .$$

$$r_1^{\mathfrak{A}} = \{\langle b_0, a_2\rangle\} \ , \qquad r_1^{\mathfrak{B}} = \{\langle d_0, c_2\rangle\} \ ,$$

$$r_2^{\mathfrak{A}} = \{\langle b_2, b_1\rangle\} \ , \qquad r_2^{\mathfrak{B}} = \{\langle c_2, c_1\rangle\} \ ,$$

$$r_3^{\mathfrak{A}} = 0 \ . \qquad r_3^{\mathfrak{B}} = \{\langle c_0, c_1\rangle\} \ .$$

Let the homomorphism $f = \langle f_0 , f_1 , f_2\rangle$ be defined as follows: $f : \mathfrak{A} \to \mathfrak{B}$ such that

$f_0(a_0) = c_0$, $f_0(b_0) = d_0$,

$f_1(a_1) = f_1(b_1) = c_1$, $f_2(a_2) = f_2(b_2) = c_2$.

CLAIM

$\mathfrak{B} \in H_w\{\mathfrak{A}\}$, but $\mathfrak{B} \notin H_s\{\mathfrak{A}\}$.

PROOF

1) $\mathfrak{B} \in H_w\{\mathfrak{A}\}$.

Conditions (i) and (ii) in Definition 16 holds trivially, therefore $\mathfrak{B} \in H_w\{\mathfrak{A}\}$.

2) $\mathfrak{B} \notin H_s\{\mathfrak{A}\}$.

In the definition of strong homomorphic images (Definition 17) condition (ii) does not hold, since

$$<f_0(a_0), f_1(a_1)> = <c_0, c_1> \in r^{\mathfrak{B}}, \text{ but}$$

$$<a_0, a_1> \notin r^{\mathfrak{A}}, \text{ so}$$

$$\mathfrak{B} \notin H_s\{\mathfrak{A}\}.$$

QED .

REMARK 7

We construct a strong homomorphic image of the model $\mathfrak{A}$ by modification of the model $\mathfrak{B}$ in Example 3. Let omit the relation $r_3^{\mathfrak{B}} = \{<c_0, c_1>\}$ from the model $\mathfrak{B}$, more exactly let $\mathfrak{L}$ be a model, which is the same as the model $\mathfrak{B}$ in Example 3, but $r_3 = 0$. Then

$$\mathfrak{L} \in H_s\{\mathfrak{A}\} .$$

3.3 DIRECT PRODUCT

DEFINITION 18 (direct product)

Let I be an arbitrary set and $\mathfrak{A} \in {}^I Mod_t^{\circ}$, i.e. $\mathfrak{A} = <\mathfrak{A}_i>_{i\in I}$.

Let us denote the universe of the sort s of the model $\mathfrak{A}_i$ by $A_{i,s}$. By the direct product of the models $\mathfrak{A}_i\,(i\in I)$ we understand a t-type model $\mathfrak{B} = <<B_s>_{s\in S}, <r^{\mathfrak{B}}>_{r\in Dom(t_1)}>$ such that

(i) $(\forall s \in S)\, B_s \stackrel{d}{=} \mathrm{P} < A_{i,s} : i \in I >$.

(ii) $\forall r \in Dom\,(t_1)$ if $n = Dom\,(tr) - 1$ then

$\forall b \in (B_{tr(o)} \times \ldots \times B_{tr(n)})\,[b \in r^{\mathfrak{B}} \Leftrightarrow (\forall i \in I) <b_o(i), \ldots, b_n(i)> \in r^{\mathfrak{A}_i}]$.

The direct product of the models $\mathfrak{A}_i\,(i\in I)$ is usually denoted by $\mathrm{P}\,\mathfrak{A}$ or $\mathrm{P}_{i\in I}\, \mathfrak{A}_i$.

■

NOTATION

The direct product of the systems of universes of t-type models $\mathfrak{A}_i\,(i\in I)$ is denoted by

$$\mathrm{P}\,A \stackrel{d}{=} \mathrm{P} <A_i>_{i\in I} \stackrel{d}{=} <\mathrm{P}<A_{i,s}>_{i\in I} : s\in S> \stackrel{d}{=} (\mathrm{P}\,\mathfrak{A})_o \quad .$$

EXAMPLE 4 (direct product of many-sorted models)

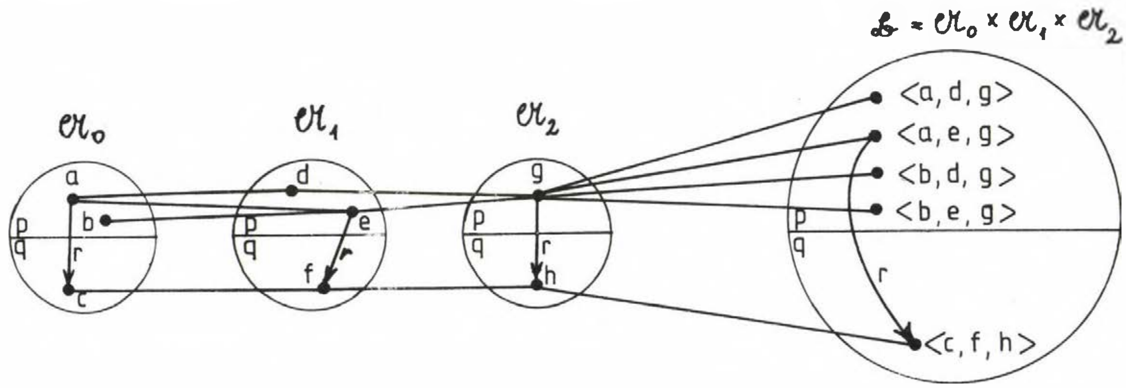

Figure 4.

Let $I = 3 = \{0, 1, 2\}$, and let $t = <s, t_1, H>$ be a similarity type where

$$S = \{p, q\}, \quad t_1 = \{<r, <p, q>>\}, \quad H = 0.$$

$$\mathfrak{A}_0 = <<A_{0,s}>_{s\in S}, <r^{\mathfrak{A}_0}>_{r\in Dom(t_1)}>$$

$$<A_{0,s}>_{s\in S} = \{<p, A_{0,p}>, <q, A_{0,q}>\} \quad \text{where}$$

$$A_{0,p} = \{a, b\}, \quad A_{0,q} = \{c\}.$$

$$r^{\mathfrak{A}_0} = \{<a, c>\}.$$

We define the models $\mathfrak{A}_1$ and $\mathfrak{A}_2$ in a similar way:

$$A_{1,p} = \{d, e\}, \quad A_{1,q} = \{f\} \quad \text{and} \quad r^{\mathfrak{A}_1} = \{<e, f>\}.$$

$$A_{2,p} = \{g\}, \quad A_{2,q} = \{h\} \quad \text{and} \quad r^{\mathfrak{A}_2} = \{<g, h>\}.$$

The system of universes of the model $\mathfrak{B} = \mathfrak{A}_0 \times \mathfrak{A}_1 \times \mathfrak{A}_2$ is as follows:

$$\mathfrak{B}_0 = B = <B_s : s\in S> = <\mathbf{P} <A_{i,s}>_{i\in I} : s\in S> =$$

$$= \{<p, \{A_{0,p} \times A_{1,p} \times A_{2,p}\}>, <q, \{A_{0,q} \times A_{1,q} \times A_{2,q}\}>\} =$$

$$= \{<p, \{<a, d, g>, <a, e, g>, <b, d, g>, <b, e, g>\}>, <q, \{<c, f, h>\}>\}.$$

CLAIM

1) $<<a, e, g>, <c, f, h>> \in r^{\mathfrak{B}}$.

2) $<<a, d, g>, <c, f, h>> \notin r^{\mathfrak{B}}$.

PROOF

1) $<a, c> \in r^{\mathfrak{A}_0}$, $<e, f> \in r^{\mathfrak{A}_1}$, $<g, h> \in r^{\mathfrak{A}_2}$, therefore condition (ii) of Definition 18 holds.

2) $< d, f > \notin \mathfrak{A}_1$, so the condition (ii) mentioned above does not hold.

QED .

3.4 REDUCED PRODUCT

DEFINITION 19 (pre-filter, filter)

Let I be an arbitrary set. We recall that SbI is the set of all subset of I. A pre-filter D over I is defined to be a set $D \subseteq SbI$ such that

$$(\forall X, Y \in D)(X \cap Y) \in D \quad .$$

A pre-filter D over I is said to be a proper pre-filter over I iff

$$0 \notin D \ .$$

Recall that a filter D over I is nothing but a pre-filter over I such that

$$(\forall X, Y \in SbI)\,[((X \subset Y) \wedge (X \in D)) \rightarrow Y \in D]$$

holds.

■

DEFINITION 20 (ultrafilter)

Let I be a set. A set $U \subseteq SbI$ is said to be an ultrafilter over I iff U is a nonempty, maximal proper pre-filter over I.

■

REMARK 8

Let I be a set. A set $U \subseteq SbI$ is an ultrafilter over I iff U is nonempty and

(i) $0 \notin U$.

(ii) $(\forall X, Y \in U)(X \cap Y) \in U$.

(iii) $(\forall X, Y \in SbI)[((X \subset Y) \wedge (X \in U)) \rightarrow Y \in U]$.

(iv) $(\forall Y \in SbI)(Y \in U$ or $(I \sim Y) \in U)$.

In our case the straightforward generalization of the definition of reduced product, which is based on the equivalence classes on the direct product is not satisfactory. We shall use other equivalence classes on the set $(\mathrm{P}^D A)$ defined bellow instead.

DEFINITION 21 $(\mathrm{P}^D A)$

Let I be an arbitrary set, let $D \subseteq SbI$ be a pre-filter over I, and let $\mathfrak{A} \in {}^I Mod_t^{\,o}$, i.e. $\mathfrak{A} = <\mathfrak{A}_i>_{i \in I}$. Define $\mathrm{P}^D A$ to be a set

$$\mathrm{P}^D A \overset{d}{=} <\mathrm{P}^D A_s : s \in S> \quad \text{such that for every } s \in S$$

$$\mathrm{P}^D A_s \overset{d}{=} \{ f_s : (\exists Y \subseteq D) [f_s \in {}^Y Rng f_s \text{ and } (\forall i \in Y) f_s(i) \in A_{i,s}] .$$

■

DEFINITION 22 (equivalence relation $\equiv_D$)

Let $t = <S, t_1, H>$ be a similarity type. Let I be an arbitrary set, let D be a pre-filter over I. Let $\mathrm{P}^D A = <\mathrm{P}^D A_s : s \in S>$ be a set defined in Definition 21, and let $f_s, g_s \in \mathrm{P}^D A_s$ for any $s \in S$. Then

$$(\forall s \in S)(f_s \equiv_D g_s) \overset{d}{\Longleftrightarrow} Dom(f_s \cap g_s) \in D .$$

■

REMARK 9

It is easy to se that the relation $\equiv_D$ is an equivalence relation over $\mathrm{P}^D A$, since it is symmetrical, reflexive and transitive due to the definition of pre-filter.

NOTATION

Let $t = <S, t_1, H>$ be a similarity type and $s \in S$. Let us denote the equivalence class of the function $f_s \in P^D A_s$ by $\overline{f_s}$.

The set of all equivalence classes of the relation $\equiv_D$ of the sort s is denoted by

$$P^D A_s/D \stackrel{d}{=} \underset{i\in I}{P} A_{i,s}/D \stackrel{d}{=} \{ \overline{f_s} : f_s \in P^D A_s \} .$$

$$P^D A/D \stackrel{d}{=} \underset{i\in I}{P^D} A_i/D \stackrel{d}{=} <P^D A_s/D : s \in S> .$$

The set $P^D A/D$ is said to be the reduced product of A modulo D, where $A = <A_i>_{i\in I}$, I is a set, and D is a pre-filter over I.

REMARK 10

10.1 Note that there is a significant difference between the direct product of systems of universes $A_i\ (i\in I)$ (denoted by PA) and the set $P^D A$ defined in Definition 21.

$PA \stackrel{d}{=} <P<A_{i,s}>_{i\in I} : s \in S>$ and $P^D A = <P^D A_s : s \in S>$

where for every $s \in S$

$P<A_{i,s}>_{i\in I} \stackrel{d}{=} \{ f_s : [f_s \in {}^I(Rng\, f_s)$ and $(\forall i \in I)\, f_s(i) \in A_{i,s}\ ] \}$

$P^D A_s \stackrel{d}{=} \{ f_s : (\exists\, Y \in D)\, [\, f_s \in {}^Y(Rng\, f_s)$ and $(\forall i \in I)\, f_s(i) \in A_{i,s}\ ] \}$

where I is an arbitrary set, $D \subseteq SbI$ is a pre-filter and $\mathfrak{A} \in {}^I Mod_t^{\,o}$.

That is the significant difference is that each element of the direct product of the sort s is a sequence with length $|I|$, the elements of $P^D A_s$, however, are sequences with length depending on the elements of the pre-filter D. In the case of a finite pre-filter (see e.g. Example 5) the lengths of the sequences $P^D A_s$ may be different.

Moreover $(\forall s \in S)\ |P^D A_s| \geq |P<A_{i,s}>_{i\in I}|$.

10.2 Note that the same idea is used to define the equivalence relation $\equiv_D$ both in this paper (for $P^D A$) and in classical model theory (for reduced product).

DEFINITION 23 (many-sorted reduced product)

Let I be an arbitrary set, let $D \subseteq SbI$ a pre-filter over I and let $\mathfrak{A} \in {}^I Mod_t^{\circ}$. Let $P^D A = \langle P^D A_s : s \in S \rangle$ be the set defined in Definition 21.

By the reduced product of $\mathfrak{A}$ modulo D we understand a t-type model

$$\mathfrak{B} = \langle \langle B_s \rangle_{s\in S} \;,\; \langle r^{\mathfrak{B}} \rangle_{r\in Dom(t_1)} \rangle$$

such that

(i) $(\forall s\in S)\, B_s \overset{d}{=} \{ \overline{f_s} : f_s \in P^D A_s \} \overset{d}{=} P^D A_s / D$.

(ii) For every $r \in Dom(t_1)$ if $n = Dom(tr)\text{-}1$, then

$$\forall b \in (B_{tr(o)} \times \ldots \times B_{tr(n)}) \,[\, b \in r^{\mathfrak{B}} \overset{d}{\Longleftrightarrow}$$

$$\overset{d}{\Longleftrightarrow} (\exists\, Y \in D)(\forall i \in Y) \langle b_o(i), \ldots, b_n(i) \rangle \in r^{\mathfrak{A}_i} \,] \;.$$

The many-sorted reduced product of $\mathfrak{A}$ modulo D is denoted by $P\mathfrak{A}/D$ or $\underset{i\in I}{P}\, \mathfrak{A}_i / D$.

■

EXAMPLE 5 (many-sorted reduced product)

Let $I = 4 = \{0, 1, 2, 3\}$.

Let $D = \{\{0, 1\}, \{1\}, \{1, 2\}\}$ a proper pre-filter over I, and let the similarity type t fix as follows: $t = \langle S, t_1, H \rangle$, where $S = \{p, q\}$, $t_1 = \{\langle r, \langle p, q \rangle \rangle\}$, $H = 0$.

Models $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3 \in Mod_t^{\circ}$ can be seen in Figure 5. Note that $\mathfrak{A}_3$ is a non--normal model, since $A_{3,p} = 0$. The reduced product of the models $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3$ modulo D is a t-type model

$$\mathcal{B} = \langle \langle B_s \rangle_{s \in \{p,q\}}, \langle r^{\mathcal{B}} \rangle \rangle$$

such that

$$B = \langle B_s \rangle_{s \in \{p,q\}} = \{ \langle p, B_p \rangle, \langle q, B_q \rangle \} = \{ \langle p, \{X\} \rangle, \langle q, \{Z, W\} \rangle \} ,$$

where X, Z, W are equivalence classes. E.g. the equivalence class of the function $\langle a, d \rangle \in {}^{\{0,1\}}\mathrm{P}^D A_p$ is denoted by X, or the equivalence class of the function $\{\langle 1, c \rangle, \langle 2, g \rangle\} \in {}^{\{1,2\}}\mathrm{P}^D A_q$ is denoted by W. (See Fig.5.)

CLAIM

$\langle X, Z \rangle \in r^{\mathcal{B}}$ and $\langle X, W \rangle \notin r^{\mathcal{B}}$, therefore

$$r^{\mathcal{B}} = \{ \langle X, Z \rangle \} .$$

The proof is easy by Definition 23.

REMARK 11

If we had defined the reduced product in the usual way i.e. by the equivalence classes on the set of the direct products, the reduced product $\mathcal{B}'$ of the same models $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$ modulo D would have been <u>entirely different:</u>

$$\mathcal{B}' = \langle \langle B'_s \rangle_{s \in \{p,q\}}, \langle r^{\mathcal{B}'} \rangle \rangle \quad \text{where}$$

where

$$B_p = 0 , \quad B_q = \{ V, Q \} , \quad r^{\mathcal{B}'} = 0 .$$

(See Figure 6.)

This example shows that when defining the new concepts one should take into consideration the specialities of the empty-sorted model class in order to make easy the generalization of the classical theorems (e.g. Łoś lemma).

DEFINITION 24 (ultraproduct)

Let I be an arbitrary set and let $\mathfrak{A} \in {}^I Mod_t^{\circ}$. Let U be an ultrafilter over I . The ultraproduct of $\mathfrak{A}$ modulo U is a reduced product of $\mathfrak{A}$ modulo U . The ultraproduct of modulo U is denoted by

$$\mathrm{P}\,\mathfrak{A}/U \quad \text{or} \quad \mathop{\mathrm{P}}_{i \in I} \mathfrak{A}_i/U \ .$$

■

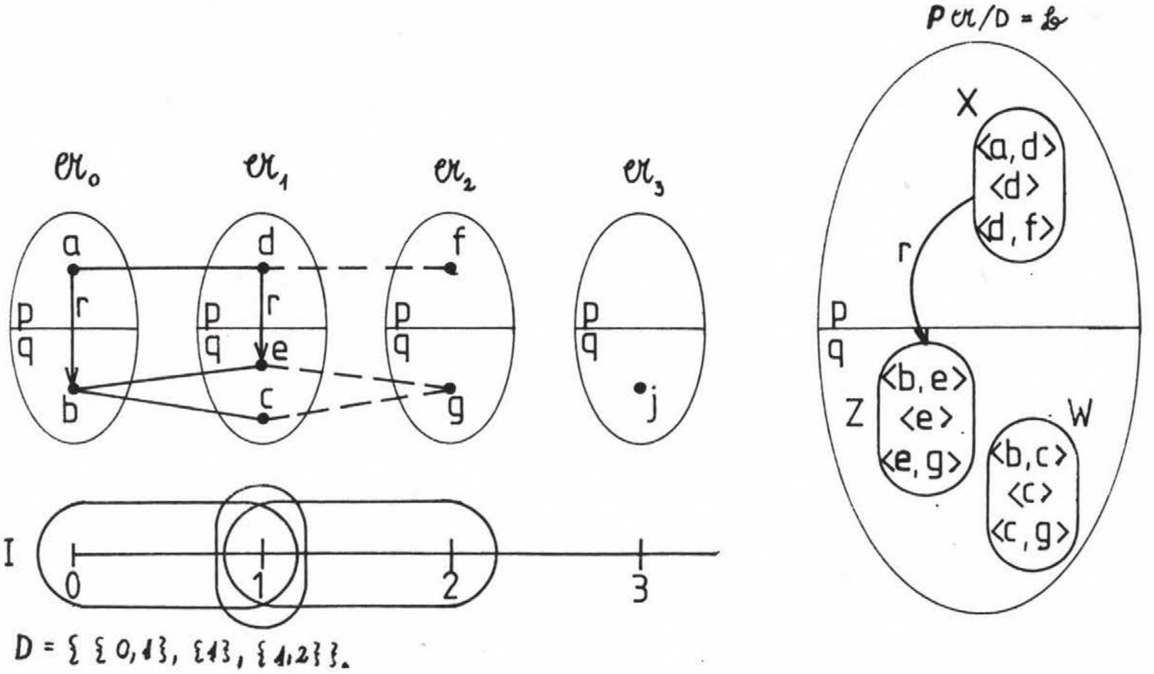

Figure 5.

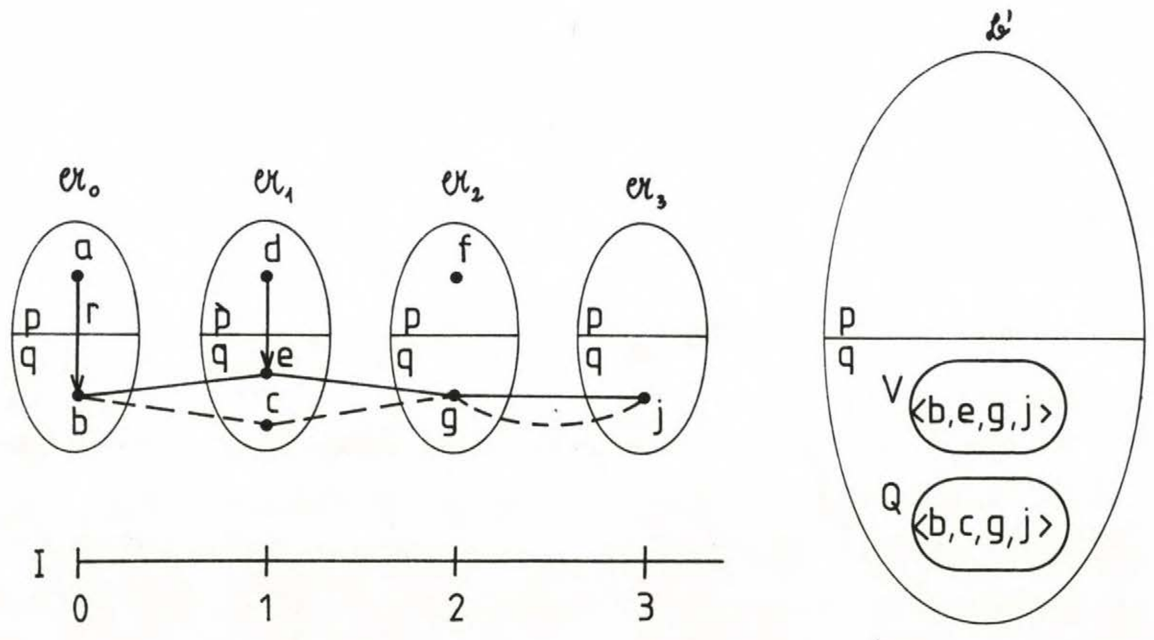

Figure 6.

3.5 ŁOŚ LEMMA

THEOREM 3 (generalizetion of Łoś lemma)

Let I be an arbitrary set, $\mathfrak{A} \in {}^I Mod_t$, let U be an ultrafilter over I and let $\varphi \in F_t$ be an arbitrary formula. Then the following propositions (i) and (ii) hold:

(i) $\mathbf{P}\,\mathfrak{A}/U \models \varphi \Longleftrightarrow (\exists Y \in U)(\forall i \in Y)\ \mathfrak{A}_i \models \varphi$.

(ii) **Let** $k \in \underset{i\in I}{\mathbf{P}}\ \underset{s\in S}{\mathbf{P}} ({}^{\omega}A_{i,s})$, **i.e. let** $k_i \in \underset{s\in S}{\mathbf{P}} ({}^{\omega}A_{i,s})$ **be a valuation into** $\mathfrak{A}_i$, **i.e.**

$$(\forall i \in I)(\forall s \in S)\ k_{i,s} \in {}^{\omega}A_{i,s} .$$

Let $(\forall s \in S)\ \overline{k_s} : \omega \to \underset{i\in I}{\mathbf{P}} A_{i,s}/U$ **such that**

$$\overline{k_s} \overset{d}{=} \langle\langle k_{i,s}(n) : i \in I\rangle/U : n \in \omega\rangle .$$

Let $\bar{k} \overset{d}{=} \langle \overline{k_s} : s \in S\rangle$ **be a valuation into** $\mathbf{P}\,\mathfrak{A}/U$. **Then**

$$\mathbf{P}\,\mathfrak{A}/U \models \varphi[\bar{k}] \Longleftrightarrow (\exists Y \in U)(\forall i \in Y)\ \mathfrak{A}_i \models \varphi[k_i] .$$

PROOF

(i) **The proof is based on the following lemma:**

<u>*Lemma 3.1*</u>

The ultraproducts in class Mod_t defined in this paper are equivalent to the ultraproducts of cathegorie theory (see Andréka–Németi [6] and Sain–Hien [36]). They are equivalent to the universal ultraproducts, too, see Sain-Hien [36]. Thus $\mathbf{P}\,\mathfrak{A}/F$ is equivalent to the reduced product of the sequence of models $\mathfrak{A}$ modulo F for every filter F and for every sequence of models $\mathfrak{A} \in {}^{\cup F}Mod_t$. Lemma 3.1 is easy to prove. A detailed analogue proof can be found in Andréka–Németi–Burmister [2].

QED . (Lemma 3.1)

(ii) The proof of Theorem 3 follows immediately from Lemma 3.1 and from the results of Andréka–Németi: "Los lemma holds in every category" [6]. For this it is sufficient to check whether the validity relation $\models \subseteq Mod_t \times F_t$ defined in this paper corresponds to that used in Andréka–Németi [6].

QED . (Theorem 3.)

3.6 OPERATORS ON CLASSES OF MODELS

We define metafunctions H_w, H_s, S_w, S_s, P, P^r, Up and Uf, each of them is a metafunction from $Sb\ Mod_t^{\circ}$ into $Sb\ Mod_t^{\circ}$. It is known, that the difference between a function and a metafunction is that a metafunction can be not only a set or a class, but also a metaclass.

DEFINITION 25 (operator)

Let Q be a metaclass. Q is said to be an operator iff $Q : Sb\ Mod_t^{\circ} \to Sb\ Mod_t^{\circ}$.

■

DEFINITION 26 (closure operator)

Let A be a metaclass. Metafunction $Q : SbA \to SbA$ is said to be a closure operator iff

(i) $(\forall X \subseteq A)\ X \subseteq QX$.

(ii) $(\forall X \subseteq A)\ QQX = QX$.

(iii) $(\forall X, Y \subseteq A)\ (X \subseteq Y \to QX \subseteq QY)$.

■

DEFINITION 27 (operators on classes of models)

Let $K \subseteq Mod_t^{\circ}$ be a class.

1. Weak homomorphic image

$H_w K = \{ \mathfrak{B} \in Mod_t^{\circ} : (\exists \mathfrak{A} \in K)\ \mathfrak{B} \in H_w \{ \mathfrak{A} \} \}$.

2. Strong homomorphic image

$H_s K = \{ \mathfrak{B} \in Mod_t^{\circ} : (\exists \mathfrak{A} \in K)\ \mathfrak{B} \in H_s \{ \mathfrak{A} \} \}$.

3. Weak submodel

$S_w K = \{ \mathfrak{B} \in Mod_t^{\circ} : (\exists \mathfrak{A} \in K)\ \mathfrak{B} \in S_w\{\mathfrak{A}\}\}$.

4. Strong submodel

$S_s K = \{ \mathfrak{B} \in Mod_t^{\circ} : (\exists \mathfrak{A} \in K)\ \mathfrak{B} \in S_s\{\mathfrak{A}\}\}$.

5. Direct product

$PK = \{ \mathfrak{B} \in Mod_t^{\circ} : (\exists \text{ set } I)(\exists \mathfrak{A} \in {}^I K)\ P\mathfrak{A} \cong \mathfrak{B} \}$.

$P^+ K = (PK \sim P0) \cup K$.

6. Reduced product

$P^r K = \{ \mathfrak{B} \in Mod_t^{\circ} : (\exists \text{ set } I)(\exists \mathfrak{A} \in {}^I K)(\exists D \subseteq SbI)$
$[((\forall i \in D)\ \mathfrak{A}_i \in K \wedge D \text{ is a filter}) \rightarrow P\mathfrak{A}/D \cong \mathfrak{B}]\}$.

7. Ultraproduct

$Up\, K = \{ \mathfrak{B} \in Mod_t^{\circ} : (\exists \text{ set } I)(\exists \mathfrak{A} \in {}^I K)(\exists U \subseteq SbI)$
$[((\forall i \in D)\ \mathfrak{A}_i \in K \wedge U \text{ is an ultrafilter}) \rightarrow P\mathfrak{A}/U \cong \mathfrak{B}]\}$.

8. Ultrafactor

$Uf\, K = \{ \mathfrak{B} \in Mod_t^{\circ} : K \cap Up\{\mathfrak{B}\} \neq 0 \}$.

PROPOSITION 4

For any similarity type t

$H_w Mod_t \subseteq Mod_t$, $H_s Mod_t \subseteq Mod_t$, $P\, Mod_t \subseteq Mod_t$, $P^r Mod_t \subseteq Mod_t$, $Up\, Mod_t$, $Uf\, Mod_t \subseteq Mod_t$, **but there is a similarity type** t **such that**

$$S_s Mod_t \not\subseteq Mod_t \quad \text{and} \quad S_w Mod_t \not\subseteq Mod_t \ .$$

Moreover, if a similarity type t does not contain any constant symbol, then

$$\mathbf{S}_s\, Mod_t = Mod_t^{\circ} \quad \text{and} \quad \mathbf{S}_w\, Mod_t = Mod_t^{\circ}$$

The proof is trivial.

QED .

Knowing Proposition 4 we define operators $\mathbf{S}_s^+$ and $\mathbf{S}_w^+$ as follows:

DEFINITION 28 ($\mathbf{S}_s^+$, $\mathbf{S}_w^+$)

Let us define metafunctions

$$\mathbf{S}_s^+ : Sb\, Mod_t \rightarrow Sb\, Mod_t \text{ and}$$

$$\mathbf{S}_w^+ : Sb\, Mod_t \rightarrow Sb\, Mod_t$$

as follows:

$$\mathbf{S}_s^+ \overset{d}{=} \langle Mod_t \cap \mathbf{S}_s K \;:\; K \subseteq Mod_t \rangle \;.$$

$$\mathbf{S}_w^+ \overset{d}{=} \langle Mod_t \cap \mathbf{S}_w K \;:\; K \subseteq Mod_t \rangle \;.$$

■

PROPOSITION 5

$\mathbf{S}_s^+ Mod_t \subseteq Mod_t$ and $\mathbf{S}_w^+ Mod_t \subseteq Mod_t$.

The proof is trivial by Definition 28.

QED .

REMARK 12

We investigate bellow the connection between the operators we have already defined and the operators obtained by composing several of them. We also investigate which operators are closure operators.

Frequently, we omit the symbol of the composition $\circ$, i.e. instead of $H_w \circ S_w$ we write $H_w S_w$, and instead of $H_s \circ S_w \circ P$ we write $H_s S_w P$.

DEFINITION 29 ($Q_1 \leqslant Q_2$)

Let Q_1 and Q_2 be two operators over Mod_t° . Then
$Q_1 \leqslant Q_2 \Leftrightarrow$ (for every similarity type t) $(\forall X \subseteq Mod_t)\ Q_1 X \subseteq Q_2 X$.

■

EXAMPLE 6

CLAIM

(i) $(\forall Q \in \{ H_w , H_s , S_w , S_s , P , P^r , Up , Uf \})\ Q \geqslant Id_{(Sb\ Mod_t)}$.

(ii) $H_w \geqslant H_s$, moreover $H_w > H_s$.

(iii) $S_w \geqslant S_s$, moreover $S_w > S_s$.

The proof is easy.

QED .

EXAMPLE 7

CLAIM

$\mathbf{S}_s \nleq \mathbf{H}_s$ and $\mathbf{S}_s \ngeq \mathbf{H}_s$, i.e. relation $\leqslant$ does not hold between operators $\mathbf{S}_s$ and $\mathbf{H}_s$.

PROOF

1) $\underline{\mathbf{S}_s \nleq \mathbf{H}_s}$

It is sufficient to show that

$\exists t\, (\exists X \subseteq Mod_t)\, \mathbf{S}_s X \nsubseteq \mathbf{H}_s X$.

Let the similarity type t be the following: $t = <S, t_1, H>$, where $S = \{p\}$, $t_1 = \{ <r, <p, p>> \}$ and $H = 0$, i.e. t is a one-sorted type.

It is sufficient to show that there exsist a model $\mathfrak{A} \in Mod_t$, and a model $\mathfrak{B} \in \mathbf{S}_s \{\mathfrak{A}\}$ such that $\mathfrak{B} \notin \mathbf{H}_s \{\mathfrak{A}\}$.

Let $\mathfrak{A}$ be the following:

$\mathfrak{A} = <A, R> = < \{<p, A_p>\} \;, \{<r, r^{\mathfrak{A}}>\} >$

where

$A_p = \{a, b\}$ and $r^{\mathfrak{A}} = \{ <a, b> \}$.

Let $\mathfrak{B}$ be a strong submodel of the model $\mathfrak{A}$ such that

$B_p = \{a\}$ and $r^{\mathfrak{B}} = 0$.

(See Figure 7.)

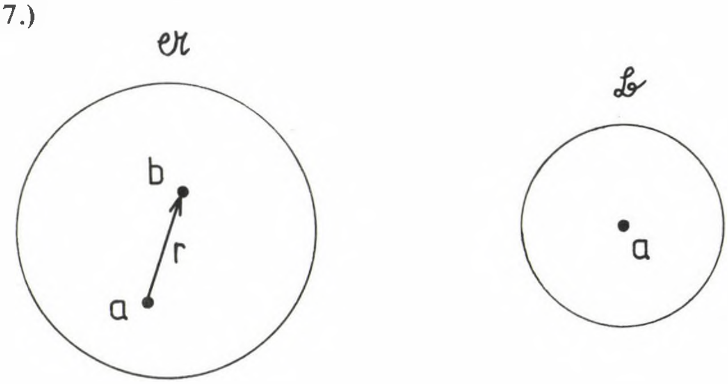

Figure 7.

Then $\mathfrak{B} \notin \mathbf{H}_s\{\mathfrak{A}\}$ since $r^{\mathfrak{B}} = 0$ and, by the definition of strong homomorphic image $r^{\mathfrak{B}}$ should be equal to $\{<a, a>\}$.

2) $\underline{\mathbf{S}_s \nleqslant \mathbf{H}_s}$

It is sufficient to show that

$\exists\, t\, (\exists\, X \subseteq Mod_t)\, \mathbf{H}_s X \nsubseteq \mathbf{S}_s X$.

Let t be the same similarity type and let $\mathfrak{A}$ be the same model as in (1) above. Define a model $\mathfrak{L} \in \mathbf{H}_s\{\mathfrak{A}\}$ such that $C_p = \{a\}$ and $r^{\mathfrak{L}} = \{<a, a>\}$. (See Figure 8.) It is easy to see, that $\mathfrak{L}$ is not a strong submodel of $\mathfrak{A}$.

QED .

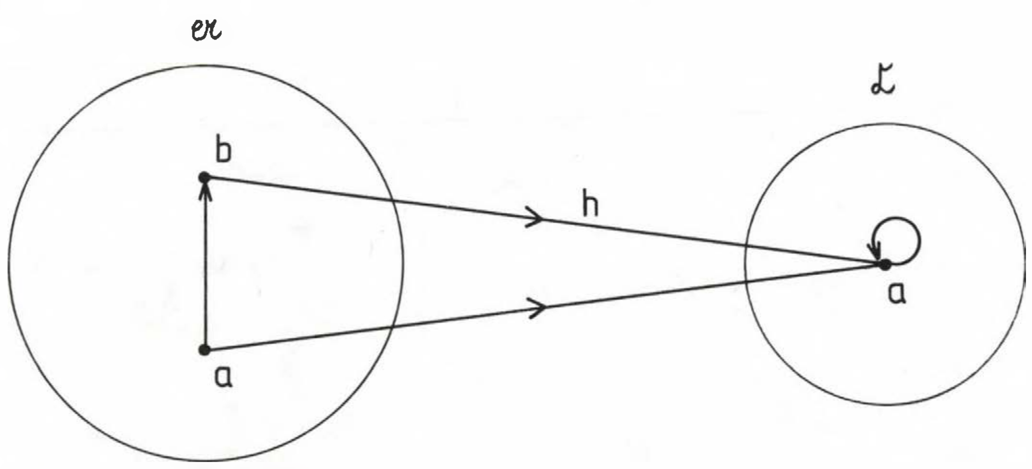

Figure 8.

Figure 9 summarizes the propositions we have about the operators $\mathbf{H}_s$, $\mathbf{S}_s$, $\mathbf{P}$, $\mathbf{Up}$ and about their compositons. Figure 9 also represents a partial ordered set (poset), where the elements of the set are operators and the partial ordering on them is the relation $\leqslant$. If $Q_1 \leqslant Q_2$ then operator Q_2 is drawn upper than operator Q_1 , and they are connected with a line. The fact that an operator is a closure operator is denoted by a circle around its symbol. For every operator Q , $(Q \restriction Mod_t)$ is taken into consideration instead of Q . Figure 9 is reasonably simple because it does not illustrate operators $\mathbf{H}_w$ and $\mathbf{S}_w$.

The proofs of all the propositions illustrated in Figure 9 follow immediately from Corollary 7 and Corollary 13 of Németi–Sain [30] page 568.

REMARK 13

The poset in Figure 9 is said to be a poset generated by the set $\{ H_S , S_S , P , Up \}$, where the ordering is the relation $\leqslant$.

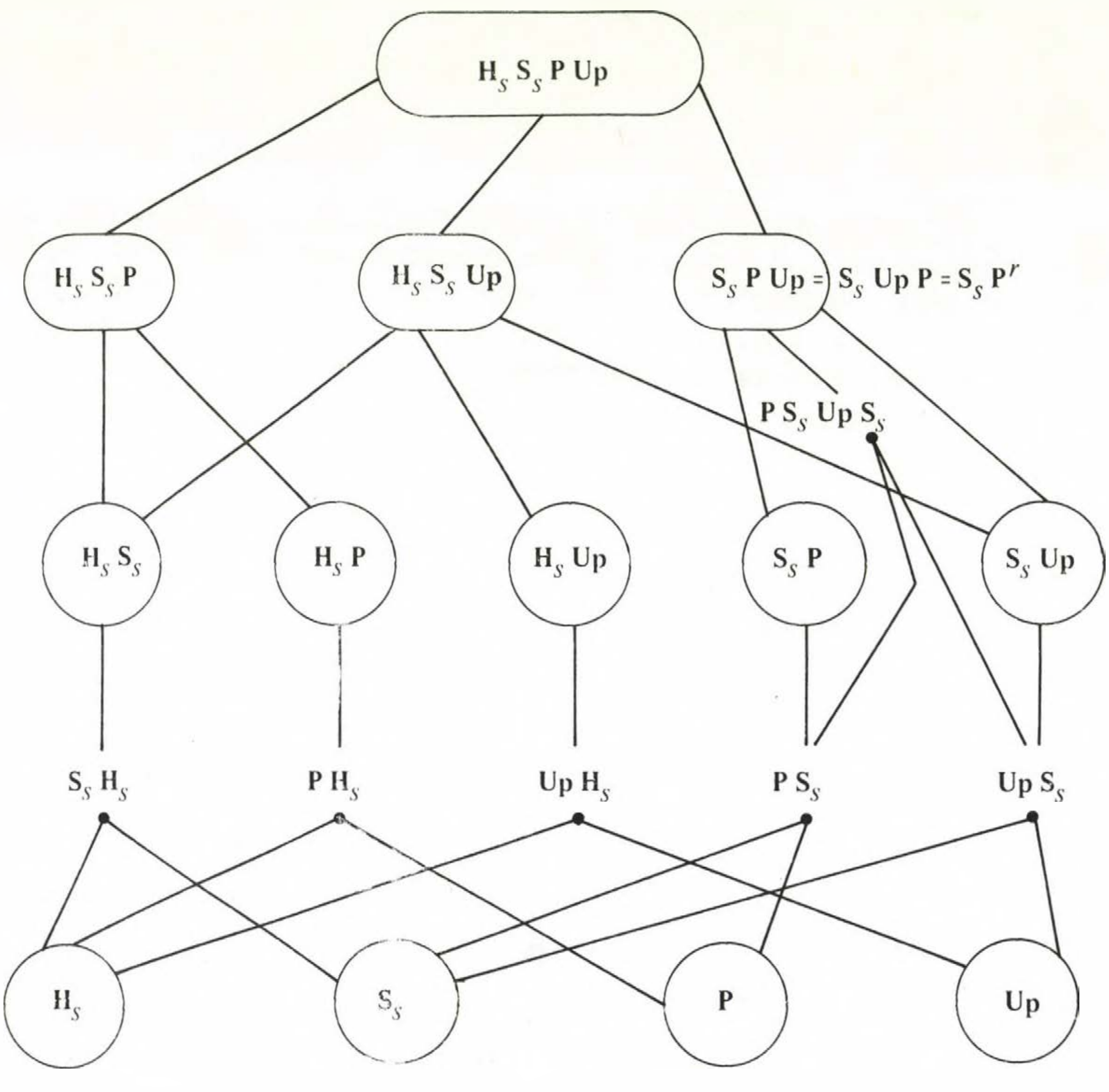

Figure 9.

THEOREM 6

Let Op be the partially ordered semigroup generated by $\{ H_w , S_w , H_s , S_s , P , P^r , Up \}$.

Let $L \subseteq Op$ be a subsemigroup with $P , Up \in L$. Then L is not a lattice.

PROOF

We shall prove that sup (P, Up) does not exist. The proof will use the following lemma:

LEMMA 6.1

(i) $UpP \not\leqslant PUp$.

(ii) $UpP \not\geqslant PUp$.

(iii) ($\exists$ class K of similar algebras) ($UpP\,K \not\subseteq PUp\,K$ and $UpP\,K \not\supseteq PUp\,K$) .

The proof of Lemma 6.1 is presented separately after the proof of Theorem 6. See Fig.10.

By Lemma 6.1 we know $PUp \not\leqslant UpP \not\leqslant PUp$. Let $Q \in L$ be an upper bound of P and Up , i.e.: $P \leqslant Q \geqslant Up$. If P^r occurs in Q then by $UpP \leqslant P^r \leqslant Q$ we have $Q \not\leqslant PUp$ thus $Q \neq sup$ (P, Up) , see Fig. 10.

Therefore we may assume that P^r does not occur in Q .

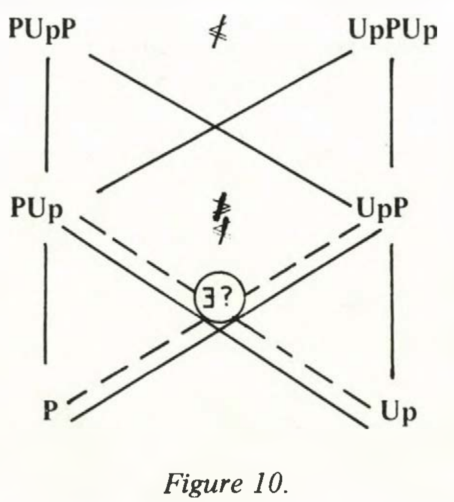

Figure 10.

CASE 1

The letters P and Up do occur in Q. If they occur in the order " ... P ... Up ... " $= Q$ then PUp $\leqslant Q$. Hence $Q \leqslant$ UpP proving that $Q \neq sup$ (P, Up) .

If they occur in the reversed order then UpP $\leqslant Q$ and thus $Q \nleqslant$ PUp proving that $Q \neq sup$ (P, Up) in Case 1.

CASE 2

Up does not occur in Q. If only P occurs in Q then $Q =$ P $\ngeqslant$ Up thus Q is not supremum. Assume therefore that some other generator of Op occurs in Q . Since P$'$ does not occur in Q we have S_i or H_i occurs in Q for $i \in \{ s, w \}$.

Let the similarity type t be empty and let $K \stackrel{d}{=} \{\mathfrak{A} \in Mod_t : |A| \geqslant \omega \}$. Assume S_i occurs in Q . Then $S_s \leqslant S_i \leqslant Q$.

Clearly $K =$ PUp $K \overset{<}{\neq} S_s K \leqslant QK$. Similarly if H_i occurs in Q then $H_s \leqslant H_i \leqslant Q$ and $H_s K \overset{>}{\neq} K$. Thus in both cases $Q \nleqslant$ PUp . Hence $Q \neq sup$ (P, Up) in Case 2.

CASE 3

P does not occur in Q . The proof is exactly the same as in Case 2 but the roles of P and Up reversed, however, note that without any computation P$\{2\} \nsubseteq H_w S_w$ Up $\{2\}$.

Since there are no other alternatives Cases 1-3 prove that Q is not sup (P, Up) .

QED . (Theorem 6.)

LEMMA 6.1

(i) UpP $\nleqslant$ PUp .

(ii) UpP $\ngeqslant$ PUp .

(iii) ($\exists$ class K of similar algebras) (UpP $K \nsubseteq$ PUp K and UpP $K \nsupseteq$ PUp K) .

PROOF

Here we use the notations of Henkin–Monk–Tarski [12] and Henkin–Monk–Tarski–Andréka–Németi [13] without recalling them. For any set b **we let** $\bar{b} \stackrel{d}{=} \langle b : i \in \omega \rangle$.
Let $\mathfrak{L} \stackrel{d}{=} \langle 2, \cup, {}_2{\sim} \rangle$ **be the two element Boolean Algebra (abbreviation:** BA**) .**

PROOF of (i)

PUp$\mathfrak{L}$ = **P**$\mathfrak{L}$, **hence** ($\forall \mathfrak{B} \in$ **PUp**$\mathfrak{L}$) $\mathfrak{B}$ **is a complete lattice.**
But ($\exists \mathfrak{A} \in$ **UpP**$\mathfrak{L}$) $\mathfrak{A}$ **is not a complete lattice .**
Bellow we present a different proof in detail:
Let $\mathfrak{P} \stackrel{d}{=} {}^{\omega}\mathfrak{L}$. **Then** $At\mathfrak{P} = \{ 0_1^i : i \in \omega \}$. **We note** $P = Uv(\mathfrak{P})$.

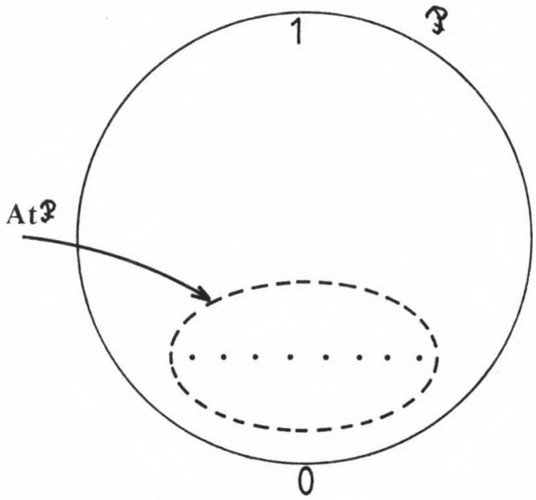

Figure 11.

Thus $|At\mathfrak{P}| = \omega$ **and** $|P| = 2^{\omega}$. **Let** F **be any nonprincipal ultrafilter over** ω . **Let** $\mathfrak{B} \stackrel{d}{=} {}^{\omega}\mathfrak{P}/F$.

Then $|At\mathfrak{B}| = 2^{\omega} = |B|$ **because** $At\mathfrak{B} = (At\mathfrak{P})/\bar{F}^{(\mathfrak{P})}$ **by Łos lemma, and because by 0.3.73 of Henkin–Monk–Tarski [12] for any countable set** $H \subseteq P$ **we have** $|{}^{\omega}H/\bar{F}| = 2^{\omega}$.
Further $2^{\omega} \leqslant |P| \leqslant |{}^{\omega}P/\bar{F}| \leqslant |{}^{\omega}P| = (2^{\omega})^{\omega} = 2^{\omega}$.

So we have

(*) $\mathfrak{B} \in \mathrm{UpP}\,\mathfrak{L}$ and $|At\,\mathfrak{B}| = 2^{\omega} = |B|$.
Clearly $\mathrm{Up}\,\mathfrak{L} = \mathrm{I}\,\mathfrak{L}$ since $|C| < \omega$ and the similarity type of $\mathfrak{L}$ is finite too.

Let $\mathfrak{A} \in \mathrm{PUp}\,\mathfrak{L}$. Then $\mathfrak{A} \in \mathrm{P}\,\mathfrak{L}$ by the above. Thus $\mathfrak{A} \cong {}^{I}\mathfrak{L}$ for some set I .
Well, ${}^{I}\mathfrak{L} \cong \mathfrak{S}bI \stackrel{d}{=} \langle SbI, \cup, I\!\sim\, \rangle$ and obviously $At\ \mathfrak{S}bI = \{ \{i\} : i \in I \}$. Then $|At\,{}^{I}\mathfrak{L}| = |I|$ and $|At\,{}^{I}\mathfrak{L}| = |I| < 2^{|I|} = |{}^{I}C|$.

Hence by (*) $\mathfrak{B} \not\cong {}^{I}\mathfrak{L}$. This proves

(**) $(\forall\, \mathfrak{A} \in \mathrm{PUp}\,\mathfrak{L})\ |At\,\mathfrak{A}| < |A|$.

By (*) and (**) we proved

(***) $\mathrm{UpP}\,\mathfrak{L} \not\subseteq \mathrm{PUp}\,\mathfrak{L}$. This also proves that

($*^4$) PUp is not closure operator, in particular

$$(\mathrm{PUp})\,(\mathrm{PUp}) \not\leqslant \mathrm{PUp}\ .$$

QED (i) .

PROOF OF (ii) and (iii)

Here α, β always denote ordinals. Notation: Let $\mathfrak{A}$ be similar to CA_{α}'s and $x \in A$.

Then $\Delta^{\mathfrak{A}}(x) \stackrel{d}{=} \Delta(x) \stackrel{d}{=} \{ i \in \alpha : c_i x \neq x \}$.
$K \stackrel{d}{=} \{ \mathfrak{A} \in Bo_{\omega} : |A| = 2$ and $|\omega \sim \Delta^{\mathfrak{A}}(0)| < \omega \}$.
Thus $(\forall\, \mathfrak{A} \in K)(\exists\, n \in \omega)\, \Delta^{\mathfrak{A}}(0) \supseteq \omega \sim n$ that is $\mathfrak{A} \models \{ c_i(0) = 1 : n \leqslant i \in \omega \}$.
Let $(\forall n \in \omega)\ \mathfrak{B}_n$ be such that $\mathfrak{B}_n \models \Delta(0) = \omega \sim n$.
Taking an ultraproduct of $\langle \mathfrak{B}_n : n \in \omega \rangle$ we have

($*^5$) $(\exists\, \mathfrak{U} \in \mathrm{Up}\,K)\ \Delta^{\mathfrak{U}}(0) = 0$. That is $\mathfrak{U} \models \{ c_i 0{=}0 : i \in \omega \}$. Let $\mathfrak{N} \stackrel{d}{=} {}^{\omega}\mathfrak{U}$. Then

($\star^6$) $\mathfrak{N} \in \mathbf{PUp}\,K$ and $\Delta^{\mathfrak{N}}(0) = 0$ and $|At\,\mathfrak{N}| = \omega$.

Let $\mathfrak{A} \in \mathbf{P}K$. Then $|\omega \sim \Delta^{\mathfrak{A}}(0)| < \omega$ since $\mathfrak{A} \cong \mathbf{P}_{i\in I}\, \mathfrak{B}_i$ for some $\mathfrak{B} \in {}^I K$ and $(\exists\, n \in \omega)(\exists\, i \in I)\ \mathfrak{B}_i \models \{ c_j 0{=}1 : j > n \}$ and hence $\mathfrak{A} \models \{ c_j 0{\neq}0 : j > n \}$ because if $j > n$ then $c_j < 0 : k \in I > = < c_j 0^{(\mathfrak{B}_k)} : k \in I >$ and $c_j 0^{(\mathfrak{B}_i)} \neq 0$ thus $c_j < 0 : k \in I > \neq < 0 : k \in I >$.

Thus

($\star^7$) $(\forall\, \mathfrak{A} \in \mathbf{P}K)(\exists\, n \in \omega)(\forall i > n)\ \mathfrak{A} \models c_i 0{\neq}0$.

That is $\Delta^{\mathfrak{A}}(0) \supseteq \omega \sim n$.

Let $\mathfrak{A} \in \mathbf{UpP}K$ and assume $\Delta^{\mathfrak{A}}(0) = 0$. Then by ($\star^7$) $\mathfrak{A} \cong \mathbf{P}\mathfrak{L}/F$ for some $\mathfrak{L} \in {}^I\mathbf{P}K$ and ultrafilter F over I. Then F is countably incomplete (i.e. is not ω_1-complete) because every ω_1- complete ultraproduct preserves the ω_1 - ary formula

$$\mathbf{P}K \models \bigvee_{i<\omega} c_i(0) \neq 0$$

(for completeness we include an explicite proof: let $Z_n \triangleq \{ i \in I : \mathfrak{L}_i \models c_n 0 = 0 \}$ for all $n \in \omega$, then $\forall n\, Z_n \in F$ but $\bigcap_{n\in\omega} Z_n \notin F$). Then by Exesice 4.3.14 on page 210 of Chang-Keisler |10| for any family $< H_i : i \in I >$ we have

($\star^8$) $|\mathbf{P}_{i\in I} H_i / F| \neq \omega$.

By Łos lemma we have $At\,(\mathbf{P}\,\mathfrak{L}\,/F) = \mathbf{P}_{i\in I}(At\,\mathfrak{L}_i)\,/\,F$.

Hence by ($\star^8$) above, $|At\,(\mathbf{P}\mathfrak{L}\,/F)| \neq \omega$. Thus $|At\,\mathfrak{A}| \neq \omega$. Hence by ($\star^6$) we have $\mathfrak{A} \not\cong \mathfrak{N}$.

By the choice of $\mathfrak{A}$ we proved $\mathfrak{N} \notin \mathbf{UpP}K$. This by ($\star^6$) proves $\mathbf{PUp}K \not\subseteq \mathbf{UpP}K$. This proves (ii) .

QED (ii) .

By replacing $\{\mathfrak{L}\}$ with K in the proof of (i), we also obtain

$$\mathbf{PUp}K \not\supseteq \mathbf{UpP}K$$

(since $(\forall\, \mathfrak{B} \in \mathbf{PUp}K)\ |At\,\mathfrak{B}| < |B|$) . This proves (iii) .

QED (Lemma 6.1)

REMARK 14

In the proofs of (i), (ii) of Lemma 6.1 it would be enough to consider instead of BA's $< A, +, - >$ additive semilattices $< A, + >$ and instead of Bo_α's $< A, +, -, 0, c_i, d_{ij} >_{ij\in\alpha}$ only bounded semilattices $< A, +, 0, c_i >_{i\in\omega}$ with infinitely many (arbitrary) constants $c_i (i \in \omega)$ (that is with $c_i \in A$ and $\mathfrak{A} \models 0+x=x$) .

Let the similarity type of K consists of one binary operation $+$ and infinitely many constants $d_i (i \in \omega)$.

Let $K \stackrel{d}{=} \{ < 2, \{ < +, \cup >, < d_i, 0 >, < d_j, 1 > : i < n \leqslant j \in \omega \} > : n \in \omega \}$.

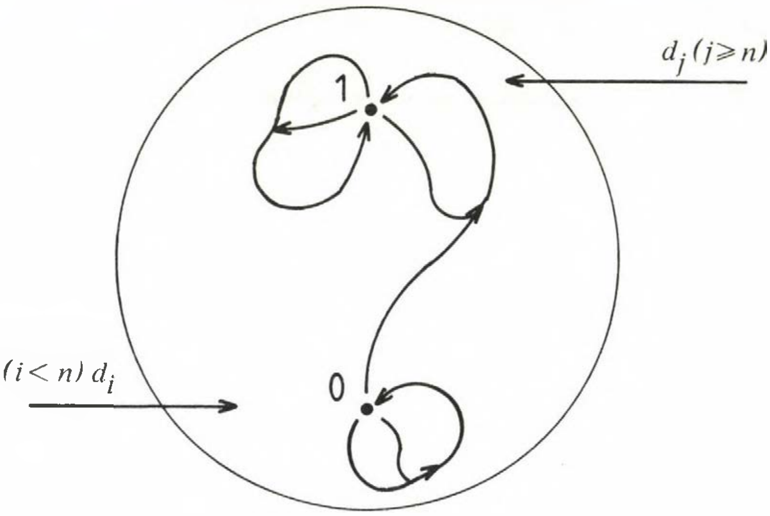

Figure 12.

That is $\mathfrak{A} \in K \Longleftrightarrow (\exists\, n \in \omega)\ \mathfrak{A} = < 2, \cup, \underbrace{0, \dots, 0}_{\text{n times}}, 1, 1, \dots 1, \dots >$.

LEMMA 6.2

$$\mathbf{PUp}K \not\subseteq \mathbf{UpP}K \not\subseteq \mathbf{PUp}K\ .$$

PROOF

The proof is the same as that of Lemma 6.1 above. (Hint: replace the constant d_i by the constant term $c_i(0)$ of Bo_ω's).

QED. (Lemma 6.2)

The following theorem 6.3 is due to I.Németi, see Németi [26].

THEOREM 6.3

UpPUp is not a closure operator, moreover PUpP $\not\leq$ UpPUp.

PROOF

We shall use the notations of HMT [12] and HMTAN [13] without further warning.
Let $\mathfrak{L} \stackrel{d}{=} \langle 2, \cup, {}_2{\sim}, 0 \rangle \in BA$ be the two-element BA. (We note that the poset $\langle 2, \subseteq \rangle$ would be amply enough for our purposes but then we could not use the notations of HMT [12]).

CLAIM

$$\mathbf{PUpP}\mathfrak{L} \not\subseteq \mathbf{UpPUp}\mathfrak{L}.$$

First version of the proof: Let $\mathfrak{N} \stackrel{d}{=} {}^\omega\mathfrak{L}$. Then $|At\,\mathfrak{N}| = \omega < 2^\omega = |N|$ and $\mathfrak{N} \in \mathbf{P}\mathfrak{L}$. Let F be any nonprincipal ultrafilter on ω and $\mathfrak{Q} \stackrel{d}{=} {}^\omega\mathfrak{N}/F$. Then $\mathfrak{Q} \in \mathbf{UpP}\mathfrak{L}$, and $|At\,\mathfrak{Q}| = 2^\omega = |Q|$ (by 0.3.73 of HMT [12] since $At\,\mathfrak{Q} = ({}^\omega At\,\mathfrak{N})/\overline{F}^{\mathfrak{N}}$). Let $\mathfrak{A} \stackrel{d}{=} \mathfrak{N} \times \mathfrak{Q}$ and $b \stackrel{d}{=} \langle 1^{\mathfrak{N}}, 0^{\mathfrak{Q}} \rangle$ and $d \stackrel{d}{=} \langle 0^{\mathfrak{N}}, 1^{\mathfrak{Q}} \rangle$. Then $b, d \in A$. For any $a \in A$ we let $Iga \stackrel{d}{=} Ig^{(\mathfrak{A})}\{a\}$. (Recall from HMT [12] that $Iga = \ x \in A : x \leqslant a\ $.) Now,

(☆) $|(At\,\mathfrak{A}) \cap Igb| = |At\,\mathfrak{N}| = \omega$ and $|At\,\mathfrak{A} \cap Igd| = 2^{\omega} = |A|$

which holds since $At\,\mathfrak{A} \cap Igb = \{ <a, 0> \in A : a \in At\,\mathfrak{N} \}$.

Assume $\mathfrak{A} \in$ UpPUp $\mathcal{L}$. Since Up $\mathcal{L}$ = I $\mathcal{L}$, this means $\mathfrak{A} \in$ UpP $\mathcal{L}$. Then there are I, an ultrafilter F on I and $\mathfrak{B} \in {}^{I}\mathrm{P}\mathcal{L}$ such that $\mathfrak{A} \cong \mathrm{P}\mathfrak{B}/F$. Hence there is $h \in Is(\mathfrak{A}, \mathrm{P}\mathfrak{B}/F)$. $(\forall x \in A)$ we let $x^{+} \stackrel{d}{=} h(x)$. Let $\mathfrak{R} \stackrel{d}{=} \mathrm{P}\mathfrak{B}/F$ and $k \in a^{+}$ be fixed. By Łos lemma

(☆☆) $At\,\mathfrak{R} \cap Ig\{a^{+}\} = \mathrm{P}_{i \in I}(At\,\mathfrak{B}_i \cap Ig\{k_i\})/\overline{F}^{\mathfrak{B}}$.

Since $|R| \leqslant 2^{\omega}$, there is $Z \in F$, $(\forall i \in Z)\ |B_i| \leqslant 2^{\omega}$. Thus we may assume $(\forall i \in I)\ |B_i| \leqslant 2^{\omega}$. By $\mathfrak{B}_i \in \mathrm{P}\mathcal{L}$ then we conclude $|At\,\mathfrak{B}_i| \leqslant \omega$ for all $i \in I$ (since $(\forall \mathfrak{P} \in \mathrm{P}\mathcal{L})(\exists H)\ \mathfrak{P} \cong {}^{H}\mathcal{L}$ thus $|P| = 2^{|At\,\mathfrak{P}|}$). Thus $|At\,\mathfrak{R}| \leqslant |{}^{I}\omega/F|$ (because $At\,\mathfrak{R} = \mathrm{P}_{i \in I} At\,\mathfrak{B}_i/F$). By $2^{\omega} \leqslant |At\,\mathfrak{C}| \leqslant |At\,\mathfrak{A}| = |At\,\mathfrak{R}|$ we conclude $|{}^{I}\omega/F| \geqslant 2^{\omega}$. This is a property of F. By 4.2.8 on p.182 (or equivalently 4.2.4) of Chang-Keisler [10] then F is not ω_1 - complete. Thus by Ex. 4.3.14 on p. 210 of Chang-Keisler [10] $|\mathrm{P}_{i \in I} H_i/F| \neq \omega$ for any system $< H_i : i \in I >$ of sets. In particular, $|\mathrm{P}_{i \in I}(At\,\mathfrak{B}_i \cap Igk_i)/\overline{F}^{\mathfrak{B}}| \neq \omega$, which contradicts (☆) + (☆☆). This contradiction proves that our assumption $\mathfrak{A} \in$ UpPUp $\mathcal{L}$ was wrong. Thus PUpP $\mathcal{L} \nsubseteq$ UpPUp $\mathcal{L}$ is proved.

QED. (First version of proof.)

Second version of the proof. We include this version in the hope that it might be useful in attacking the problem of deciding whether $(\exists n \in \omega)$ [(PUp)^{n} is a closure operator].

The following Lemma 6.3.1 will be useful:

LEMMA 6.3.1

Let $\mathfrak{A} \in BA$ and F be a countably incomplete ultrafilter on I. Let $\mathfrak{B} \stackrel{d}{=} {}^{I}\mathfrak{A}/F$. Let $X \subseteq B$ with $|X| = \omega$ and with

(☆) $(\forall Y \subseteq_{\omega} X)(\exists a \in X)\ a \nleq \Sigma Y$.

Then ΣX does not exist in $\mathcal{B}$.

PROOF

Let $Z \in {}^{\omega}F$ with $(\forall i \in j \in \omega)\ Z_i \supset Z_j$ and $\cap\ Rng\ Z \in F$. Exists since F is countably incomplete.

Let $X \stackrel{d}{=} \{ x_i : i \in \omega \}$. Assume $p/F = \Sigma X$ for some $p \in {}^I A$. We may assume $0 \notin Rng\ p$ since $(|X| \geqslant 2 \Rightarrow p/F \neq 0)$. For every $n \in \omega$, let $q_n \in x_o + \ldots + x_n$ be such that $(\forall i \in I)\ q_n(i) < p(i)$ and $q_n < q_{n+1}$ (in ${}^I\mathfrak{A}$). These exist by the following:

Let $(\forall n \in \omega)\ [\ t_n \in \sum_{i \leqslant n} x_i$ and $W_o \stackrel{d}{=} \{ i \in I : t_o(i) \nless p_i \}$ and

$W_{n+1} \stackrel{d}{=} W_n \cup \{ i \in I : t_{n+1}(i) \nless p_i$ or $t_n(i) \nleqslant t_{n+1}(i) \}\ ]$.

Then $W_n \notin F$ by $x_o + \ldots + x_n \leqslant x_o + \ldots + x_{n+1} \underset{(\star)}{<} p/F$ by $(\star)$.

Let $q_o \stackrel{d}{=} t_o[\ W_o / 0\ ]$. Then $q_o \in x_o$ and $q_o(i) < p(i)$ for all $i \in I$. Define $q_{n+1} \stackrel{d}{=}$
$\stackrel{d}{=} t_{n+1}[\ W_{n+1} / q_n\ ]$ for all $n \in \omega$ (by induction).

Let $i \in I$. Then $q_o(i) < p(i)$ since if $t_o(i) \nless p_i$ then $i \in W_o$ hence $q_o(i) = 0 < p(i)$ since we may assume $0 \notin Rng\ p$. Assume the same for q_n. Then, if $t_{n+1}(i) \nless p(i)$ then $i \in W_{n+1}$ hence $q_{n+1}(i) = q_n(i) < p(i)$. By induction then $(\forall n \in \omega)\ q_n(i) < p_i$ as desired. If $i \in W_{n+1}$ then $q_{n+1}(i) = q_n(i)$ else if $i \notin W_{n+1}$ then

$$q_n(i) \underbrace{=}_{(W_n \subseteq W_{n+1})} t_n(i) \leqslant t_{n+1}(i) = q_{n+1}(i)$$

since $i \notin W_n$ either. Thus $q_n \leqslant q_{n+1}$.

Let $f \in {}^I A$ be defined as follows . $(\forall i \in I)\ (\forall n \in \omega)\ [\ i \in Z_n \sim Z_{n+1} \Rightarrow f_i \stackrel{d}{=} q_n(i)\ ]$. This completely defines f since $I = \cup \{ Z_n \sim Z_{n+1} : n \in \omega \}$ is a disjoint union (by $\forall n\,(Z_n \supseteq Z_{n+1})$). Let $n \in \omega$ and $i \in Z_n$. Then $(\exists m \geqslant n)\ i \in Z_m \sim Z_{m+1}$ thus $f_i = q_m(i) \geqslant q_n(i)$. Hence $f/F \geqslant q_n/F$ (by $Z_n \in F$).

Let $i \in I$. Then $\exists n\,(i \in Z_n \sim Z_{n+1})$. Thus $f_i = q_n(i) < p_i$ proving $f/F < p/F$.

By $X = \{ q_n/F : n \in \omega \}$ we have that f/F is an upper bound of X. Hence p/F is not the least upper bound of X.

QED . (Lemma 6.3.1)

Let $\mathfrak{L}$, $\mathfrak{N} \stackrel{d}{=} {}^{\omega}\mathfrak{L}$, and $\mathfrak{A} = {}^{\omega}\mathfrak{N}/F$ be as in the first version of the proof. Let $\mathfrak{G} \stackrel{d}{=} {}^{\omega}\mathfrak{A}$.
Then $\mathfrak{G} \in \mathrm{PUpP}\mathfrak{L}$.

Let $(\forall i \in \omega)\ a_i \stackrel{d}{=} \langle 0 : n \in \omega \rangle_1^i$ where 0 and 1 are understood in $\mathfrak{A}$. Thus $a_i \in G$ for $i \in \omega$. Now $\Sigma \{ a_i : i \in \omega \} = 1^{\mathfrak{G}}$ since for any $f \in G$, $[f \geq a_i \Leftrightarrow f_i = 1]$ thus $(\forall i \in \omega)\ f \geq a_i \Rightarrow f = 1^{\mathfrak{G}}$.

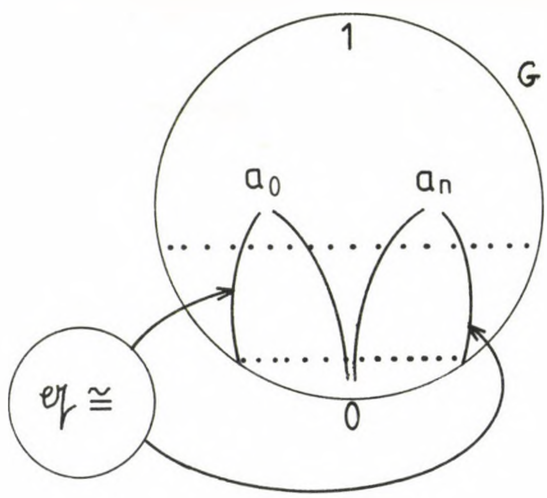

Figure 13.

Assume $\mathfrak{G} \in \mathrm{UpPUp}\,\mathfrak{L}$. Then $\mathfrak{G} \in \mathrm{UpP}\,\mathfrak{L}$ too. Thus ($\exists$ ultrafilter F on I) ($\exists \mathfrak{B} \in {}^{I}\mathrm{P}\mathfrak{L}$) $\mathfrak{G} \cong \mathfrak{R} \stackrel{d}{=} \mathrm{P}\mathfrak{B}/F$. Since $|At\ \mathfrak{R}| = 2^{\omega} = |R|$, exactly as we did in the first version, we conclude from this that F is countably incomplete. Then, by Lemma 6.3.1, we have $(\forall X \subseteq_{\omega_1} R)$ [$(\forall Y \subseteq_{\omega} X)\ \Sigma Y \neq \Sigma X \Rightarrow \Sigma X$ does not exist in $\mathfrak{R}$].
But then $\Sigma \{ a_i : i \in \omega \} = 1^{\mathfrak{G}}$ is a contradiction.
Actually we can derive a "stronger contradiction" (that might be relevant for the harder $(\mathrm{PUp})^n$ problem) as follows: Let $H \stackrel{d}{=} {}^{\omega}\{ 0^{\mathfrak{A}}, 1^{\mathfrak{A}} \}$. Then $H \subseteq G$. Let $\mathfrak{M} \stackrel{d}{=} \mathfrak{Sg}^{(\mathfrak{G})} H$. Now by $H \in Su\ \mathfrak{G}$ we have $M = H$.
Then $\mathfrak{M} \in \mathrm{P}\mathfrak{L}$ is a complete BA with $At\ \mathfrak{M} = \{ a_i : i \in \omega \}$. Further

$(\forall i \in \omega)\ |At\,\mathfrak{A} \cap Ig\,\{a_i\}| = 2^\omega$ since $\mathfrak{Rl}_{(a_i)}\mathfrak{A} \cong \mathfrak{A}$.

Anyway, we have a "stronger condition" since $\mathfrak{M} \subseteq \mathfrak{A}$ is a complete BA with $|M| = 2^\omega$!
This proves $\mathfrak{A} \notin$ UpPUp $\mathfrak{L}$.

Note that $\mathfrak{A} \supseteq \mathfrak{M} \cong {}^\omega\mathfrak{L}$. Actually $\mathfrak{M} = {}^\omega\, \mathfrak{Sg}^{(\mathfrak{A})}\{0^{\mathfrak{A}}\}$.

QED . (Second version.)

QED . (Theorem 6.3.)

PROPOSITION 7

Let $t = \langle S, t_1, H \rangle$ be a similarity type such that $|Dom\, t_1 \sim H| < \omega$, i.e. the set of relation symbols is finite. Then the following propositions (i)–(ii) hold:

(i) $H_s\, S_s\, P \geqslant H_s\, S_s\, Up$.

(ii) $H_s\, S_s\, P\, Up = H_s\, S_s\, P$.

PROOF

The proof is based on Andréka–Németi [4]. See also Németi–Sain [30] page 573.

QED .

OPEN PROBLEM 8

Is the poset generated by the set $\{H_s, S_s, P, S_sUp, Up\,S_s\}$ a lattice in one-sorted universal algebras (where the similarity type t has only functions) ? Describe this poset.

■

3.7 THEOREMS OF AXIOMATIZABILITY

DEFINITION 30 (Th)

Metafunction $Th : Sb(Mod_t^{\circ}) \rightarrow Sb(F_t)$ **is defined as follows:**

$$(\forall K \subseteq Mod_t^{\circ})\ Th(K) \overset{d}{=} \{ \varphi \in F_t : K \models \varphi \} .$$

■

DEFINITION 31 (Mod°)

Metafunction $Mod^{\circ} : Sb(F_t) \rightarrow Sb(Mod_t^{\circ})$ **is defined as follows**

$$(\forall T \subseteq F_t)\ Mod^{\circ}(T) \overset{d}{=} \{ \mathfrak{A} \in Mod_t^{\circ} : \mathfrak{A} \models T \} .$$

■

DEFINITION 32 (Mod Y)

Let $Y_t \subseteq F_t$ **and let** $K \subseteq Mod_t^{\circ}$ **. We define metafunctions**

$$Y : Sb(Mod_t^{\circ}) \rightarrow Sb(Y_t)$$

and

$$Mod : Sb(F_t) \rightarrow Sb(Mod_t) ,$$

and their composition

$$Mod\ Y : Sb(Mod_t^{\circ}) \rightarrow Sb(Mod_t) ,$$

as follows:

(i) $Y : Sb(Mod_t^{\circ}) \rightarrow Sb(Y_t)$ **such that**

$$(\forall K \subseteq Mod_t^{\circ})\ Y(K) \overset{d}{=} Y_t \cap Th(K) .$$

(ii) $(\forall T \subseteq F_t)\, Mod(T) \overset{d}{=} Mod^{\circ}(T) \cap Mod_t$.

(iii) $Mod\ Y = Mod \circ Y$ i.e. $Mod\ Y(K) = Mod(Y(K))$.

■

NOTATION

If $Y_t = F_t$, then $Th = Y$, i.e. $Th \overset{d}{=} F : Sb(Mod_t^{\circ}) \to Sb(F_t)$. Moreover $Th\ 0 = F_t$.

DEFINITION 33

Let us define the set of formulas Eq_t, Af_t, Qeq_t, Qaf_t, Ude_t, Uda_t, Uhf_t, $Unv_t \subseteq F_t$ as follows :

$Eq_t \overset{d}{=} \{<\tau, \sigma> : \tau, \sigma \in T_t\}$. (equalities)

$Af_t \overset{d}{=} \{R(\tau_o, \dots, \tau_{t(R)-1}) : R \in Dom\ t_1 \sim t_2$ and $\tau_o, \dots, \tau_{t(R)-1} \in T_t\} \cup$

$\cup\ \{(\tau = \sigma) : \tau, \sigma \in T_t\}$. (atomic formulas)

$Qeq_t \overset{d}{=} \{(\bigwedge_{i<n} e_i \to e_n) : n \in \omega$ and $(\forall i \leq n)\, e_i \in Eq_t\}$.

(quasi-equalities)

$Qaf_t \overset{d}{=} \{\bigwedge_{i<n} R_i(\tau_{i,o}, \dots, \tau_{i,t(R)-1}) \to R_n(\sigma_o, \dots, \sigma_k) :$

$n, k \in \omega$, $(\forall i \leq n)\, R_i \in Dom\ t_1 \sim t_2$ and

$(\forall i \in n)\, (\forall j \in t(R_i))\, \tau_{i,j} \in T_t$ and $(\forall i \leq k)\, \sigma_i \in T_t\}$.

(quasi atomic formulas)

$Ude_t \overset{d}{=} \{\bigvee_{i<n} e_i : (\forall i < n)\, e_i \in Eq_t$ and $n \in \omega\}$.

(universal disjunction of equalities)

$Uda_t \stackrel{d}{=} \{ \bigvee_{i<n} a_i : (\forall i < n)\, a_i \in Af_t$ and $n \in \omega \}$.

(universal disjunction of atomic formulas)

$Uhf_t \stackrel{d}{=} \{ \bigvee_{i<n} \theta_i :$ at most one of the formulas θ_i is an atomic formula $(\forall i < n)\, \theta_i$ is an atomic formula or negation of atomic formula , $n \in \omega \}$.

(universal Horn formulas)

$Unv_t \stackrel{d}{=} \{ \varphi \in F_t : \varphi$ is a formula without quantifier $\}$.

(universal formulas)

■

REMARK 14

14.1 Let $\varphi = \theta_o \vee \ldots \vee \theta_m$ be an universal Horn formula. If $m = 1$, then φ is either an atomic formula or the negation of an atomic formula. If $m > 1$ and θ_m is an atomic formula, say, then φ is equivalent to a formula

$$(\psi_o \wedge \psi_1 \wedge \ldots \wedge \psi_{m-1}) \rightarrow \theta_m ,$$

where each ψ_i , $i<m$, is also an atomic formula.

14.2 The difference between the quasi atomic formulas (Qaf_t) and the universal Horn formulas (Uhf_t) is that in a quasi atomic formula

$$\lambda = \theta_o \vee \ldots \vee \theta_m$$

there is exactly one $i \leqslant m$ such that θ_i is an atomic formula, the rest are negations of atomic formulas.

REMARK 15

15.1 According to Definitions 32, 33

$$Eq : Sb\,(Mod_t^{\,o}) \to Sb\,(F_t)$$

denotes a metafunction, and similarly, Af, Qeq, Qaf, Ude, Uda, Uhf, Unv are also metafunctions. That is metafunction

$$Mod^o\,Eq : Sb\,Mod_t^{\,o} \to Sb\,Mod_t^{\,o}$$

is an operator over $Mod_t^{\,o}$. Similarly, metafunctions $Mod^o Af$, $Mod^o Qeq$, $Mod^o Qaf$, $Mod^o Uda$, $Mod^o Unv$ are operators over $Mod_t^{\,o}$. Thus
$(\forall Q \in \{ Eq, Af, Qeq, Qaf, Ude, Uda, Uhf, Unv \})$

$$Mod\,Q : Sb\,Mod_t \to Sb\,Mod_t \;.$$

15.2 We describe the operators

$$Mod\,Q : Sb\,Mod_t \to Sb\,Mod_t$$

above in a pure algebraic way using the so called <u>axiomatizability theorems</u>. The axiomatizability theorems are based on the following scheme:

Let $Y \subseteq F_t$ be a set of formulas, let $K \subseteq Mod_t$ be a class of models and let $C : Sb\,Mod_t \to Sb\,Mod_t$ be a closure operator over Mod_t. Then

$$(\forall K \subseteq Mod_t)\; CK = Mod\,Y\,(K)\,, \text{ i.e. } \; C = Mod\,Y \;.$$

15.3 The so called <u>preservation theorems</u> are based on the following scheme:

Let $Y \subseteq F_t$, $K \subseteq Mod_t$ and $C : Sb\,Mod_t \to Sb\,Mod_t$ be the same as in 15.2. Then

$$\varphi \in Y \iff (\forall K \subseteq Mod_t)\,[\,(K \models \varphi) \Rightarrow (CK \models \varphi)\,] \;.$$

CLAIM

Every axiomatizability theorem implies a preservation theorem and the converse is not true. More precisely

$$[\; \forall K \subseteq Mod_t)\; CK = Mod\,Y\,(K) \;\; \overset{\Rightarrow}{\not\Leftarrow} \;\; [\,\varphi \in Y \iff (\forall K \subseteq Mod_t)\,((K \models \varphi) \Rightarrow (CK \models \varphi))\,] \;.$$

THEOREM 9 (axiomatizability theorems)

1) Uf Up = *Mod Th*

2) $H_w\, S^+_w\, P = Mod\ Eq$

3) $H_w\, S^+_s\, P = Mod\ Af$

4) $S^+_s\, P\, Up = Mod\ Qaf$

5) $S^+_w\, P\, Up = Mod\ Qeq$

6) $S^+_s\, P^+\, Up = Mod\ Uhf$

7) $H_w\, S^+_w\, Up = Mod\ Ude$

8) $H_w\, S^+_s\, Up = Mod\ Uda$

9) $S^+_s\, Up = Mod\ Unv$

PROOF

The proof follows from Theorem 1 and 3 in Németi–Sain [30].

COROLLARY 9.1

Each proposition (1-9) of Theorem 9 implies a preservation theorem. For example, proposition 2 ($H_w\, S^+_w\, P = Mod\ Eq$) implies the following theorem:

Operator $H_w\, S^+\, P$ preserves every equality. More precisely, let $\varphi \in Eq_t$ be an equality and let $K \subseteq Mod_t$ a class of models. Then

$$K \models \varphi \implies H_w\, S^+_w\, P\, K \models \varphi .$$

3.8 FREE MODEL

DEFINITION 34 (many-sorted K-free model over X)

Let $K \subseteq Mod_t^{\circ}$ be a t-type class of models and let $\mathfrak{A} \in Mod_t^{\circ}$ be a model such that $\mathfrak{A} = \langle\langle A_s\rangle_{s\in S}, \langle r^{\mathfrak{A}}\rangle_{r\in Dom(t_1)}\rangle$.

Let $X = \langle X_s\rangle_{s\in S}$ be a sequence of sets such that $(\forall s \in S)\, X_s \subseteq A_s$. Then $\mathfrak{A}$ is a K-free model over X or the model $\mathfrak{A}$ is K-freely generated by the set X, iff

(i) $\mathfrak{A} \in K$.

(ii) for every model $\mathfrak{B} \in K$ and for every sequence of functions $f = \langle f_s\rangle_{s\in S}$ such that $(\forall s \in S)\, f_s : X_s \to B_s$, there exists a unique homomorphism $g : \mathfrak{A} \to \mathfrak{B}$ such that $(\forall s \in S)\, g_s \supseteq f_s$.

■

DEFINITION 35 (many-sorted K-free model)

Let $K \subseteq Mod_t^{\circ}$ and let $\mathfrak{A} \in Mod_t^{\circ}$. $\mathfrak{A}$ is said to be a K-free model iff there exists a set X such that $\mathfrak{A}$ is a K-free model over X.

■

PROPOSITION 10

Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are K-free models over X. Then $\mathfrak{A} \cong \mathfrak{B}$.

PROOF

The proof is easy by Definition 34.

QED .

NOTATION

1) $\mathrm{Fr}_X K$ denotes the K-free model over X. The notation is correct because of Proposition 10, since a free model is unique up to isomorphism. More precisely, $\mathrm{Fr}_X K$ denotes an arbitrary element of an isomorphism class.

2) Let $K \subseteq Mod_t^{\circ}$.
$\mathbf{P}^+ K \stackrel{d}{=} \mathbf{I} \; \{ \mathbf{P}_{i \in I} \mathfrak{A}_i : \mathfrak{A} \in {}^I K$ and $I \neq 0$ and I is a set $\}$, where $\mathbf{I}$ is said to be the isomorphism operator

$$\mathbf{I} : Sb\, Mod_t^{\circ} \rightarrow Sb\, Mod_t^{\circ} \; .$$

$$\mathbf{I}\, K \stackrel{d}{=} \{ \mathfrak{A} \in Mod_t^{\circ} : (\exists \mathfrak{B} \in K)\, (\mathfrak{A} \cong \mathfrak{B}) \} \; .$$

THEOREM 11

Let t be a fixed similarity type and let $K \subseteq Mod_t^{\circ}$ be a class of models such that $\mathbf{S}_s \mathbf{P}^+ K = K$.
Suppose $(\forall s \in S)\, (\exists$ normal $\mathfrak{A} \in K)\; |A_s| > 1$.

Let $X \in {}^S Sets$ be an arbitrary function. Then $\mathrm{Fr}_X K$ exists.

PROOF

The proof follows from Proposition 1 in Andréka–Németi [4] .

QED .

THEOREM 12

(i) Let t be a fixed similarity type, and suppose $\mathbf{S}_s^+ \mathbf{P}^+ K = K$ and $(\forall s \in S)(\exists \mathfrak{A} \in K)\ |A_s| > 1$.

Let $X \in {}^S(Sets \sim 1)$ an arbitrary function. Then there exists $\mathbf{Fr}_X K$.

(ii) There exists a similarity type t and a function $X \in {}^S Sets$ such that $\mathbf{Fr}_X K$ does not exist.

PROOF

(i) The proof follows from Theorem 11.

(ii) Let t be a similarity type such that there is no constant term of the sort s for every $s \in S$.

Let $X = \langle 0 : s \in S \rangle$. Then $\mathbf{Fr}_X Mod_t$ does not exist.

QED .

EXAMPLE 8 (many-sorted free model)

Let us fix the similarity type t of the class of models Mod_t as follows:

$$t = \langle S, t_1, H \rangle$$

where

$$S = \{0, 1\},$$
$$t_1 = \{\langle f, \langle 0, 1 \rangle\rangle, \langle g, \langle 1, 1 \rangle\rangle\},$$
$$H = Dom\ t_1 = \{f, g\}.$$

Let $K \subseteq Mod_t$ be a class of t-type models such that

$$K \overset{d}{=} Mod\ \{g(f(v_0^0)) = f(v_0^0)\}\ .$$

Let us consider the models $\mathcal{B}$, $\mathcal{C}$ $\in Mod_t$ (see Figure 14) .

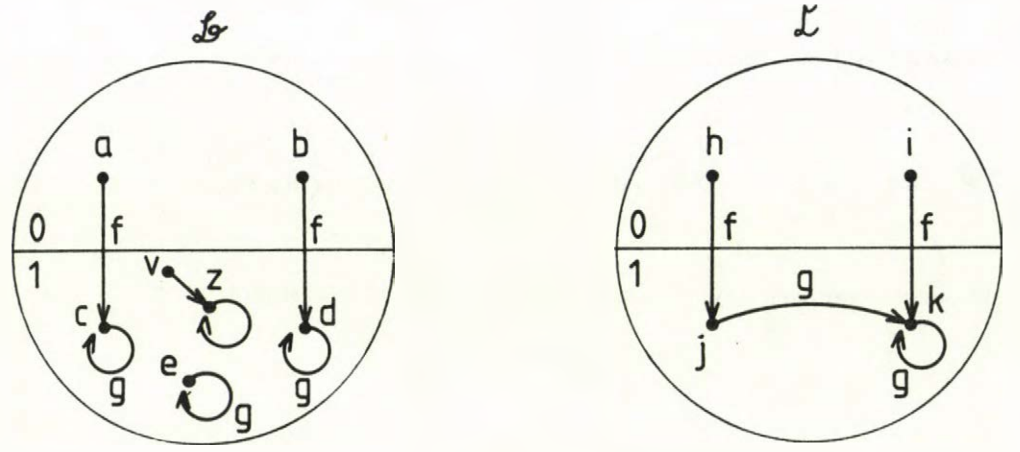

Figure 14.

$\mathcal{B} = \ll B_0 , B_1 > , < f^{\mathcal{B}}, g^{\mathcal{B}} \gg$ **where**

$B_0 = \{ a , b \}$,

$B_1 = \{ c , d , e , v , z \}$.

$f^{\mathcal{B}} = \{ < a, c > , < b, d > \}$.

$g^{\mathcal{B}} = \{ < c, c > , < d, d > , < e, e > , < v, z > , < z, z > \}$.

$\mathcal{C} = \ll C_0 , C_1 > , < f^{\mathcal{C}} , g^{\mathcal{C}} \gg$ **where**

$C_0 = \{ h , i \}$,

$C_1 = \{ j , k \}$.

$f^{\mathcal{C}} = \{ < h, j > , < i, k > \}$.

$g^{\mathcal{C}} = \{ < j, k > , < k, k > \}$.

CLAIM

$\mathcal{B} \in K$ and $\mathcal{C} \notin K$.

PROOF

Let us denote formula $g(f(v_0^o)) = f(v_0^o)$ by φ , i.e.

$$\varphi \overset{d}{=} g(f(v_0^o)) = f(v_0^o) \ .$$

(i) $\mathfrak{B} \in K$

It is obvious that formula φ is valid in model $\mathfrak{B}$, since all the valuations of variable v_0^o ($k(v_0^o) = a$ and $k'(v_0^o) = b$) satisfy formula φ in model $\mathfrak{B}$. In more details:

If $k(v_0^o) = a$ then $g(f(a)) = g(c) = c$ and $f(a) = c$.

If $k'(v_0^o) = b$ then $g(f(b)) = g(d) = d$ and $f(b) = d$.

Therefore $\mathfrak{B} \in K$.

(ii) $\mathfrak{C} \notin K$

Formula φ is not valid in model $\mathfrak{C}$, since there exists a valuation k'' into model $\mathfrak{C}$, which do not satisfy formula φ , namely :

If $k''(v_0^o) = h$ then $g(f(h)) = g(j) = k$ and $f(h) = j \neq k$.

Thus $\mathfrak{C} \notin K$, therefore

$$K \subset Mod_t \ .$$

QED .

Now we show that there exists a K-free model for class of models

$$K = Mod \{ g(f(v_0^o)) = f(v_0^o) \} \ .$$

Let $\mathfrak{A}$ be a t-type model, where $A_0 = \{x\}$, $A_1 = \{y\}$, $f^{\mathfrak{A}} = \{<x, y>\}$, $g^{\mathfrak{A}} = \{<y, y>\}$ (see Figure 15) .

Let $X = <X_0, X_1> = <\{x\}, 0>$.

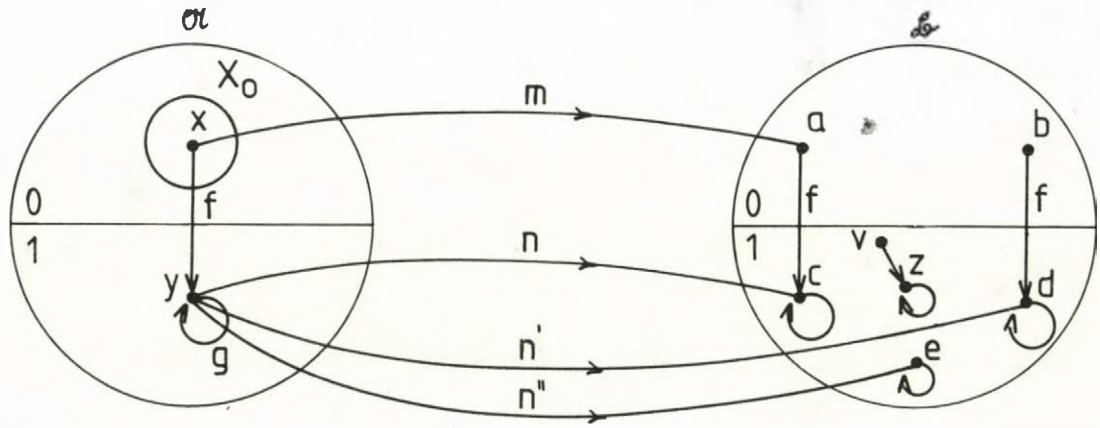

Figure 15.

CLAIM

$\mathfrak{A}$ is a K-free model over X.

PROOF

(i) Obviously, $\mathfrak{A} \in K$, since $\mathfrak{A} \in Mod_t$ and the unique valuation k of variable v_0^0 $(k(v_0^0) = x)$ satisfies formula φ in model $\mathfrak{A}$.

(ii) Let $m = < m_s >_{s \in S}$ be an arbitrary function $m : X \to B$, where $\mathfrak{B} \in K$ is an arbitrary but fixed model (see Figure 15). For example, let $m = < m_0, 0 >$ such that $m_0(x) = a$.

Then there exists a unique function $n \supseteq m$ such that $n : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism. Namely $n = < n_0, n_1 >$ where $n_0(x) = m_0(x) = a$ and $n_1(y) = c$.

It is clear that function n is a homomorphism, and this function is unique, since the other possible extentions of function m, e.g.

$$n' = < \{ < x, a >, < y, d > \} >$$

$$n'' = < \{ < x, a >, < y, e > \} >$$

are not homomorphisms.

QED.

REMARK 16

We construct a free model $\mathrm{Fr}_{<1,1>} K$ for class of models K in Example 8. That is, we need a K-free model generated by a sequence of sets $X = < \{0\} , \{0\} >$, i.e. both X_o and X_1 have only one element.

Let us consider model $\mathcal{D}$ such that

$$\mathcal{D} = << D_o , D_1 > , < r^{\mathcal{D}} >_{r \in Dom(t_1)} > \quad \text{where}$$

$$D_o = \{x\}, \quad |D_1| = \omega$$

$$f^{\mathcal{D}} = \{ < x, y > \} \quad \text{where} \quad y \in D_1 \quad \text{and}$$

$$g^{\mathcal{D}} = \{ < y, y > , < p_i , p_{i+1} > : i \in \omega \} \text{ . (See Fig. 16.)}$$

Let $X = < X_o , X_1 > = < \{x\} , \{p_o\} >$.

CLAIM

$\mathcal{D}$ is a K-free model over X .

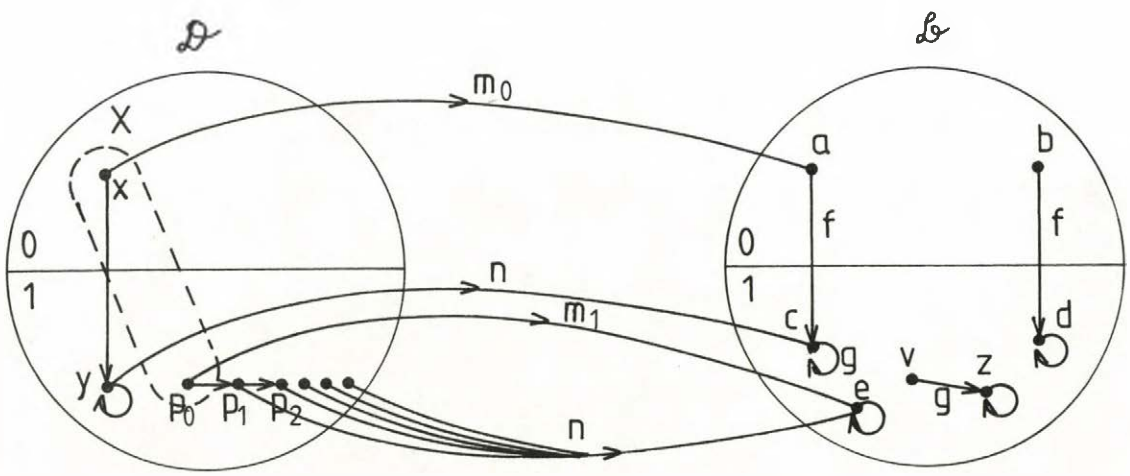

Figure 16.

PROOF

Let $\mathfrak{B} \in K$ be an arbitrary model. Easy to see that any sequence of functions $m : X \to B$ (i.e. $< m_0 , m_1 > : < \{x\} , \{p_0\} > \longrightarrow < B_0 , B_1 >$) is extendable to a unique homomorphism $n : \mathfrak{F} \to \mathfrak{B}$.

QED .

REMARK 17

Let $X = < X_0 , X_1 >$ be an arbitrary sequence of sets such that $X_0 \neq 0$. Then there exists $\mathrm{Fr}_X K$ for class of models K in Example 8. The proof is easy.

4. ACKNOWLEDGEMENTS

I would like to express my appreciation to Hajnal Andréka and István Németi for their encouragement at the time of writing this paper and for the numerous helpful and inspiring discussions we had about the topic. Special thanks are due to Ildikó Sain for many helpful remarks and suggestions. I wish to thank Zoltán Pásztor and Gábor Márkus for careful reading.

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