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DISCRETE-TIME IDENTIFICATION OF LINEAR DYNAMIC PROCESS

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ABSTRACT

In this report the discrete-time identification methods of linear dynamic systems are considered. The report is formulated on the basis ot the authors' candidate dissertations. This work gives a survey on the related topics, on the one hand, and contains the own results of the authors, on the other. The report is recommended for experts dealing with algorithmical problems of computer aided process identification.

The purpose of the work, the applied different models and estimation methods are formulated in the Introduction. Section 2 deals with the off-line identification methods for SISO and MISO systems. Off-line identification methods for MIMO systems are considered in Section 3. On-line identification algorithms of MISO or MIMO systems are discussed in Section 4 or Section 5, respectively. Section 6 is devoted to the appendices. Section 7 gives a very detailed bibliography of the related topic.

SYMBOLS

$=$	equals; is equivalent to
$\neq$	does not equal
$\equiv$	identically equals to
$>$	greater than
$<$	less than
$\geqq$	greater than or equal to
$\leqq$	less than or equal to
$\hat{Y}$	estimated value of variable y
$E\{\ldots\}$	operator stands for the mathematical expectation
$\underline{\underline{A}}$, $\underline{\underline{B}}$, $\underline{\underline{C}}$	matrices
$\underline{\underline{A}}^T$, $\underline{\underline{B}}^T$, $\underline{\underline{C}}^T$	transposed matrices
$\underline{\underline{A}}^A$, $\underline{\underline{A}}^A$, $\underline{\underline{A}}^A$	adjoint matrices
$\underline{x}$, $\underline{u}$, $\underline{v}$	column vectors
$\underline{x}^T$, $\underline{u}^T$, $\underline{v}^T$	row vectors
$\underline{\underline{0}}$	zero matrix
$\underline{0}$	zero vector
$\underline{\underline{I}}$	unit matrix
diag $<\ldots>$	diagonal matrix
$\lvert\underline{\underline{A}}\rvert$	determinant of matrix $\underline{\underline{A}}$
$\mathrm{tr}(\underline{\underline{A}})$	trace of matrix $\underline{\underline{A}}$
$\underline{\underline{A}}^{-1}$	inverse of matrix $\underline{\underline{A}}$
$\frac{d}{d\underline{x}}$	ordinary derivative column operator
$\frac{df(\underline{x})}{d\underline{x}}$	derivate or gradient of function $f(\underline{x})$
$\frac{d\underline{f}(\underline{x})}{d\underline{x}^T}$	Jacobian matrix, vector derivative of function $f(\underline{x})$

$\frac{d^2 f(\underline{x})}{d\underline{x} d\underline{x}^T}$	Hessian matrix of the second derivatives of function $f(\underline{x})$
$\sup_{\underline{u}}$	supremum over all $\underline{u}$ (least upper bound)
$\inf_{\underline{u}}$	infinum over all $\underline{u}$ (greatest lower bound)
$\lim_{N \to \infty}$	limit of series
$\operatorname{plim}_{N \to \infty}$	probability limits of series
$\otimes$	symbol of the KRONECKER matrix product
LS	Least Squares Method
GLS	Generalized Least Squares Method
ML	Maximum Likelihood Method
IV	Instrumental Variable Method
PKF	Priori Knowledge Fitting Method
K-L	KOOPMANS-LEVIN Method
STA	STochastic Approximation
EXM	EXtended Matrix Method
FEXM	First EXtended Matrix Method
SEXM	Second EXtended Matrix Method
TEXM	Third EXtended Matrix Method
SISO	Single Input Single Output
MISO	Multiple Input Single Output
MIMO	Multiple Input Multiple Output
MLG	Maximum Likelihood Method with noise model of Generalized structure
SGLS	STEIGLITZ' Generalized Least Squares Method
OI	Observability Index
AR	Auto-Regressive
MA	Moving-Average
ARMA	Auto-Regressive and Moving Average

I. INTRODUCTION

1.1 Formulation of the problem

This report - according to its title - is devoted to the identification of linear dynamic processes using sampled data.

The process identification beginning with the classical grapho-analytical methods up to the modern, computer-aided procedures, has been very popular among the researchers. Now the issues of identification terminology, the classification of the methods are not dealt with. These questions have been discussed on a very high level by famous authors on several IFAC Symposia and Congresses on Identification [12, 15, 46] and many books have been published in this field [47, 110, 111] . In this report the terminology of the international literature dealing with identification is used, the notations are the same as in the papers of ÅSTRÖM' school and for the teaching of control theory at the Technical University, Budapest. Since the terminology of the international literature is widely used, definitions are elaborated on only in the most important cases.

The object of the identification is assumed to be a linear, dynamic system with lumped parameters, or this latter restriction is eased sometimes by allowing serial dead-time (transportation delay). It is also assumed that the input and output signals of the process can be recorded in every sampling time ΔT. (The measurements are supposed to be coherent.) The above measuring situation is shown on Fig. 1.1-1 for a single input, single output system where $u(t)$ and $y(t)$ mean the sampled values of the input and output signals, respectively. The argument t means (in the whole report) the discrete-time and taking the sampling time ΔT for a unit, the values of t can only be integers. The situation is the same

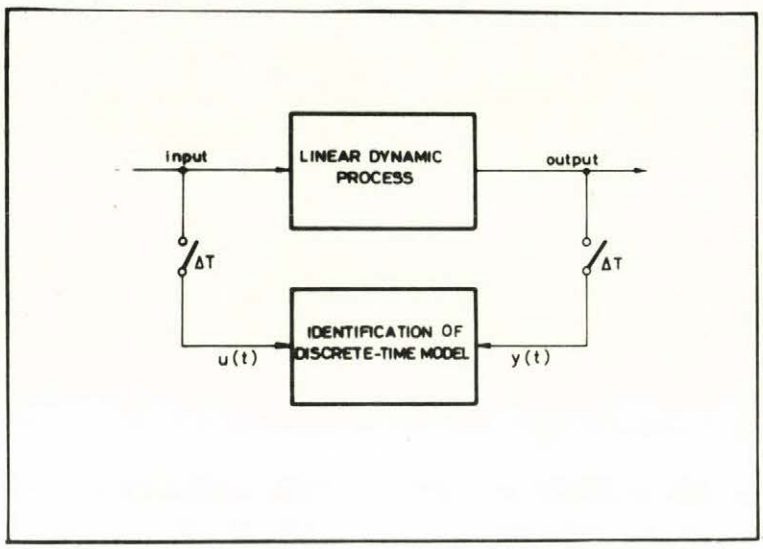


Fig. 1.1-1

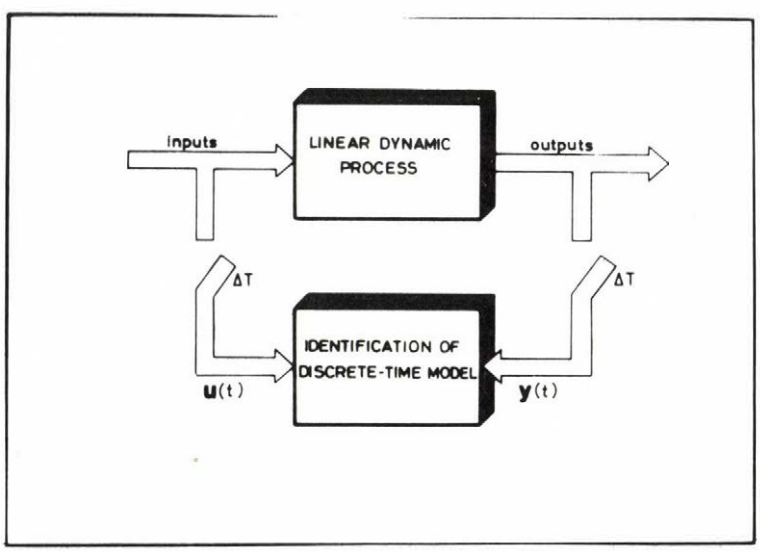


Fig. 1.1-2

for multiple input multiple output systems, too, but everything is valid only for vector-variables.

The measuring situation is suitable for both passive and active identification. So it includes e.g. the identification based on the classical step responses and contains at the same time the application possibilities of other, more up-to-date methods. This approach is called "discrete identification" in the literature, and the term refers to the discreteness of the measuring circumstances and also to the character of the applicable models.

At the beginning of the 60s, the attention has been directed towards the discrete system description and discrete process models. Its reason is partly the availability of data in discrete forms and, perhaps mainly, the possibility of (often on-line) data processing and evaluating by digital computers. In consequence of this, there was a quick upsurge in the development of the theory and techniques of the discrete identification. Perhaps KALMAN [77] was the first to publish in this field, but the basically pilot activity was due to the group at IBM led by ÅSTRÖM [8]. Even up to now, it is their off-line method that gives the best result and affords the basis for comparison with other methods. Since then, several other approaches have been suggested which might have been advantageous under their special conditions, but they actually correspond to the special cases of the original idea [12, 47]. The aim of our report is - among others - to clarify the relations between the several types of approaches and also to show the restrictions of the applications.

Thus numerous methods have been worked out for the off-line identification of single input single output systems, although the comprehensive analysis of their interconnections has figured until now either not at all or only with regard to very few methods in the literature. On-line versions of

methods have been elaborated, their comparative analysis has been carried out until now only within the frame of simulation tests [72]. The situation is not so clear in the field of identification methods of multiple input - multiple output systems. Though the generalization of several methods elaborated for a single output system has already been completed [57, 91, 108, 109, 101, 136] but easy to use methods can not be mentioned neither in off-line, nor in on-line relation. Neither has any author made the comparison.

Considering the practical application, we can state without fear that the off-line discrete identification methods of single input single output systems already mean in many places of research, in many countries the everyday identification technique and the literature reported already of the accomplishment of a great number of industrial modelling tasks. The demand for on-line application is considerably less which accounts for the rather scarce references to practical results. There are likewise very few reports on practical application in the international literature for the case of multiple-output systems whether we consider off-line or on-line approach.

According to the above survey, this report has set the following objectives:

1. Comparison of off-line identification algorithms of single output systems and consideration - where it is possible - of new ideas and perspectives.
2. Comparison of the on-line algorithms of the single input single output systems on the basis of the canonic algorithm of the stochastic approximation.
3. Generalization of the methods worked out for single output systems to multiple output systems and comparison of algorithms.

rational spectrum can be generated by a linear filter C/D). Let the variance of e(t) be λ^2 and to ensure unambiguousness, the zero-order term of the polynomial $C(z^{-1})$ be a unit. We can follow the connection of the particular signals in Fig. 1.2-1. Quantities a_i, b_i, c_i, d_i are the system parameters, d is the deadtime (integer) measured in the unit of the sampling interval, so that its magnitude is $d\Delta t$. Applying the various parameter estimation procedures it is usually required that u(t) be measured without error and its value be uncorrelated with the source noise. Maximum likelihood (hereinafter ML) type estimation can be used if the condition of normal distribution of e(t) holds.

In control theory the direct use of the formal description in Fig. 1.2-1, i.e.

$$y(t)= z^{-d}\,\frac{B(z^{-1})}{A(z^{-1})}\,u(t)+\frac{C(z^{-1})}{D(z^{-1})}\,e(t)=v(t)+w(t) \qquad (1.2\text{-}3)$$

is more useful. Although $B(z^{-1})/A(z^{-1})$ and $C(z^{-1})/D(z^{-1})$ correspond here formally to the pulse transfer functions, in the reality, however, we can restore the original input and output signal only with the help of appropriate holding elements so that we never have to imagine a pulse on the input of the discrete model. Therefore, in many cases one speaks in the literature simply of a discrete transfer function.

In the expression (1.2-3) $A(z^{-1})$, $B(z^{-1})$, $C(z^{-1})$, $D(z^{-1})$ are the polynomials of z^{-1} and z^{-1} means the backward shift operator which practically coincides with the inverse of the variable of the Z-transformation. So that for a signal $x(t)=z^{-1}x(t)=x(t-1)$, which facilitates very much the handling of difference equations of the type (1.2-1). Let the structure of the polynomials have the following form:

1.2 Structure of the applied models

Let us consider first Single Input Single Output systems (called hereinafter by their abbreviation used in the international literature SISO systems).

As already mentioned in the preceding section, we are engaged in linear dynamic systems where the system is assumed to be as a series connection of at most one member with concentrated parameter and one with dead-time. On the basis of the sampled values of the input and output signal the discrete--time model equivalent to the original system has to be determined in the sampling instants. The task is therefore to perform the experimental identification with the knowledge of the primary structure in order to establish the secondary structure, resp. the estimation of parameters [47].

ÅSTRÖM and his research colleagues have recommended in their paper listed as sources, the application of the model in Fig. 1.2-1. [8, 9, 12, 13]. This model describes the process with the following linear stochastic difference equation

$$y(t)= \sum_{i=0}^{n} b_i u(t-d-i) - \sum_{i=1}^{n} a_i y(t-i) + \sum_{i=1}^{n} a_i w(t-i) + w(t), \qquad (1.2\text{-}1)$$

where

$$w(t)= e(t)+ \sum_{i=1}^{n} c_i e(t-i) - \sum_{i=1}^{n} d_i w(t-i). \qquad (1.2\text{-}2)$$

Here u(t) is the input signal, v(t) the output signal of the process. y(t) denotes the measured output signal, corrupted by the measurement and environmental noise w(t) reduced to the output. e(t) is the source of the output noise, the so-called source noise assumed practically as a zero--mean white noise (from which arbitrary output noise with

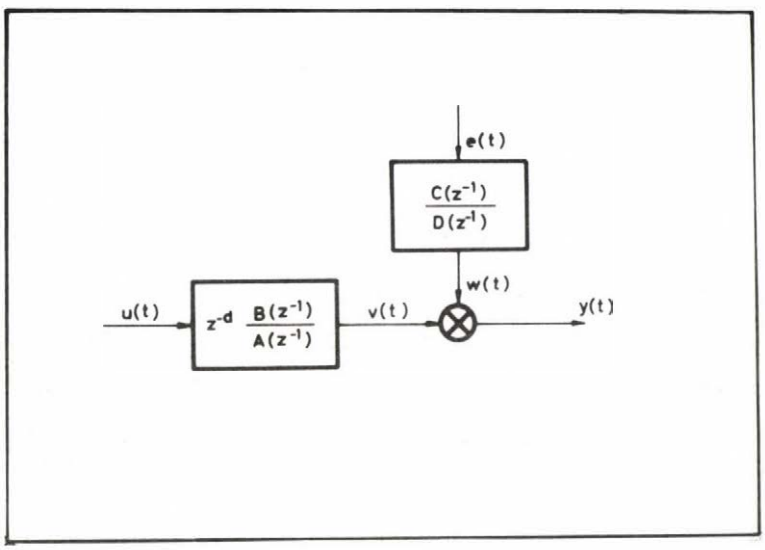


Fig. 1.2-1

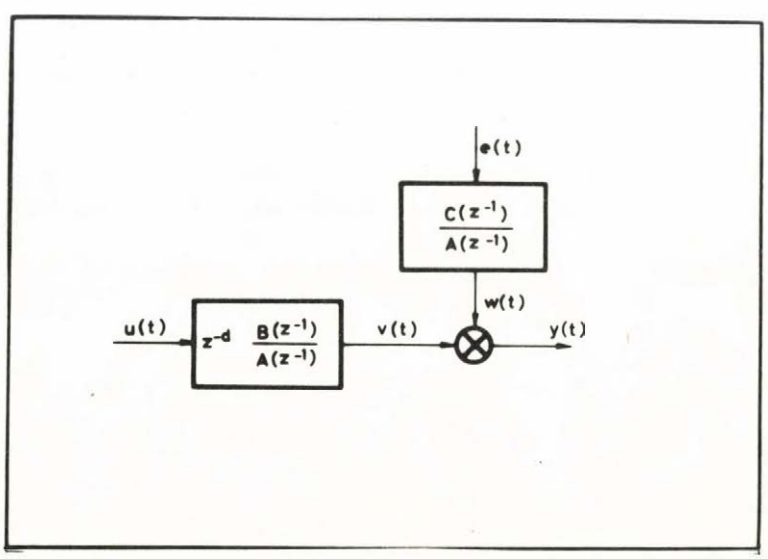


Fig. 1.2-2

$$\left.\begin{aligned} A(z^{-1}) &= 1+a_1 z^{-1}+ \dots +a_n z^{-n} = 1+\tilde{A}(z^{-1}) \\ B(z^{-1}) &= b_o+b_1 z^{-1}+ \dots +b_n z^{-n} \\ C(z^{-1}) &= 1+c_1 z^{-1}+ \dots +c_n z^{-n} \\ D(z^{-1}) &= 1+d_1 z^{-1}+ \dots +d_n z^{-n} \end{aligned}\right\} \qquad (1.2\text{-}4)$$

where the n order is common in the equations, allowing for the particular coefficients being zeros within the frame of physical realizability. This latter condition means that the order of denominators must be greater than or equal to those of the numerators. (Hereinafter reference a_o, c_o means according to (1.2-4) $a_o \equiv c_o \equiv 1$).

ÅSTRÖM and his co-workers have afterwards pointed out that because of practical considerations it is not desirable to distinguish the polynomials $A(z^{-1})$ and $D(z^{-1})$ for the reduction to a common denominator can always be achieved and the shape of our model takes the simplest form as shown in Fig. 1.2-2. On the other hand, redundant poles and zeros introduced by reduction to a common denominator can be eliminated by relatively simple procedure. Thus in this case $D(z^{-1}) = A(z^{-1})$ and the system equation will take the form

$$y(t) = z^{-d}\,\frac{B(z^{-1})}{A(z^{-1})}\,u(t) + \frac{C(z^{-1})}{A(z^{-1})}\,e(t) = v(t)+w(t) \qquad (1.2\text{-}5)$$

It is easy to see on the right side of the relations (1.2-3) and (1.2-5) the separation of the process model and the so-called noise model.

It is to be noted that the forms of models presented until now are partly or totally linear in parameters. The relations (1.2-1) and (1.2-2) can be written e.g. in the form

$$y(t)= \underline{f}^T(u,y,w,t)\,\underline{p}_{baa}+w(t)=\underline{f}^T(u,y,t)\,\underline{p}_{ba}+A(z^{-1})\,w(t), \quad (1.2\text{-}6)$$

or

$$w(t)= \underline{f}^T(e,w,t)\,\underline{p}_{cd}+e(t) \quad (1.2\text{-}7)$$

where

$$\underline{f}^T(u,y,w,t)=[u(t-d),\ldots,u(t-d-n);-y(t-1),\ldots,-y(t-n);$$
$$w(t-1),\ \ldots\ ,w(t-n)], \quad (1.2\text{-}8)$$

$$\underline{f}^T(u,y,t)=[u(t-d),\ldots,u(t-d-n);-y(t-1),\ldots,-y(t-n)], \quad (1.2\text{-}9)$$

and

$$\underline{f}^T(e,w,t)=[e(t-1),\ldots,e(t-n);-w(t-1),\ldots,-w(t-n)]. \quad (1.2\text{-}10)$$

Further

$$\underline{p}_{baa}=[b_o,b_1,\ldots,b_n;a_1,\ldots,a_n;a_1,\ldots,a_n]^T, \quad (1.2\text{-}11)$$

$$\underline{p}_{ba}=[b_o,b_1,\ldots,b_n;a_1,\ldots,a_n]^T \quad (1.2\text{-}12)$$

and

$$\underline{p}_{cd}=[c_1,\ldots,c_n;d_1,\ldots,d_n]^T. \quad (1.2\text{-}13)$$

Likewise the noiseless output $v(t)$ of the system can be expressed in the form linear in parameters (apply the substitution $w(t) \equiv 0$):

$$v(t)=\underline{f}^T(u,v,t)\,\underline{p}_{ba} \quad (1.2\text{-}14)$$

where

$$\underline{f}^T(u,v,t)=[u(t-d),\ldots,u(t-d-n);-v(t-1),\ldots,-v(t-n)]. \quad (1.2\text{-}15)$$

Similarly to what has gone before the form linear in parameters of (1.2-5) is e.g.

$$y(t)=\underline{f}^{T}(u,y,t)\,\underline{p}_{ba}+C(z^{-1})\,e(t). \qquad (1.2\text{-}16)$$

Here the so-called equation error $C(z^{-1})e(t)$ is now a stochastic process with a moving average.

The discrete-time systems can also be given by state equations. The state equations of system described by the Eqs. (1.2-1), (1.2-2) are [6, 7, 8]:

$$\underline{x}(t+1)=\underline{\underline{F}}\underline{x}(t)+\underline{g}u(t)+\underline{q}e(t) \qquad (1.2\text{-}17)$$

$$y(t)=\underline{h}^{T}\underline{x}(t)+b_{o}u(t)+e(t) \qquad (1.2\text{-}18)$$

where $\underline{\underline{F}}$, $\underline{g}$, $\underline{q}$ and $\underline{h}$ can be constructed in infinite many ways according to the various resolution of the pulse transfer functions. We present now for the system equation (1.2-5) a possible mode of resolution

$$\underline{x}(t+1)=\begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \\ -a_{n-1} & 0 & 0 & \dots & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{bmatrix}\underline{x}(t)+\begin{bmatrix} b_1-a_1b_o \\ b_2-a_2b_o \\ \vdots \\ b_{n-1}-a_{n-1}b_o \\ b_n-a_nb_o \end{bmatrix}u(t)+\begin{bmatrix} c_1-a_1 \\ c_2-a_2 \\ \vdots \\ c_{n-1}-a_{n-1} \\ c_n-a_n \end{bmatrix}e(t) \qquad (1.2\text{-}19)$$

and

$$y(t)=[1,0,\dots,0]\,\underline{x}(t)+b_{o}u(t)+e(t), \qquad (1.2\text{-}20)$$

which practically corresponds to the cross-resolution [8, 42, 60].

The system equations given by state equations usually do not fit directly for identification purposes for they contain too many parameters. However, special forms containing minimum number of parameters, so e.g. equations (1.2-19) and (1.2-20), can be well used, also suitable models can be ob-

tained by other methods of resolution too [42], (phase variable form, etc.).

We have presented above the model types used for discrete identification methods. The methods recommended by various authors can be considered actually as sub-cases of (1.2-3). Out of these, we give now only the most important ones in the following table:

Table 1.2-I.

Symbol	$C(z^{-1})/D(z^{-1})$	Author	Output noise	Equation error
MLG	$C(z^{-1})/D(z^{-1})$	ÅSTRÖM	ARMA	ARMA
ML	$C(z^{-1})/A(z^{-1})$	ÅSTRÖM	ARMA	MA
LS	$1/A(z^{-1})$	KALMAN	AR	white
SGLS	1	STEIGLITZ	white	MA
GLS	$1/A(z^{-1})H(z^{-1})$	CLARKE	AR	AR
IV	$C(z^{-1})/A(z^{-1})H(z^{-1})$	TALMON	ARMA	ARMA

Notations in the Symbol column of the Table can be found in the list of abbreviations at the beginning of the report on the one hand, and will be explained in detail in the course of the discussion of the corresponding methods on the other.

One of the most important considerations in constructing the discrete-time identification models of the multiple input multiple output systems is to obtain the identification methods of the MIMO systems by the formal extension of the methods worked out for SISO systems and already generally used, to multiple output models. (Hereinafter we are going to denote by the abbreviation MIMO the multiple input multiple output systems.) We would like to emphasize that in the case of the MIMO systems the multiple output is stressed, since the discrete model of the multiple input single output

systems does not differ with regard to parameter estimation from the SISO models (multiple input single output systems are denoted by the abbreviation MISO). The MISO version of the system model (1.2-5) can be constructed in a simple way.

$$y(t)=\sum_{j=1}^{m} \frac{B_j(z^{-1})}{A(z^{-1})} u_j(t-d_j)+\frac{C(z^{-1})}{A(z^{-1})} e(t) \qquad (1.2\text{-}21)$$

Here $u_j(t)$ is the j-th input signal from the m inputs and

$$B_j(z^{-1})=b_{j0}+b_{j1}z^{-1}+\ldots+b_{jn}z^{-n};\quad j=1,2,\ldots,m. \qquad (1.2\text{-}22)$$

The block diagram of the process model is shown in Fig.1.2-3. The quantities d_j denote the dead-time by input variables (channels). This system is also linear in parameters, i.e.

$$y(t)=\underline{f}^T(u_1,\ldots,u_m,y,t)\underline{p}_{b_1\ldots b_m a}+C(z^{-1})e(t), \qquad (1.2\text{-}23)$$

where

$$\underline{f}^T(u_1,\ldots,u_m,y,t)=[u_1(t-d_1),\ldots,u_1(t-d_1-n);\ldots;u_m(t-d_m),$$

$$\ldots,u_m(t-d_m-n);-y(t-1),\ldots,-y(t-n)] \qquad (1.2\text{-}24)$$

and

$$\underline{p}_{b_1\ldots b_m a}=[b_{10},\ldots,b_{1n};\ldots;b_{m0},\ldots,b_{mn};a_1,\ldots,a_n]^T. \qquad (1.2\text{-}25)$$

If the system has several outputs, then the system model needs essential change. Let $\underline{u}(t)$ be the m dimensional vector of the input signals, $\underline{y}(t)$ and $\underline{e}(t)$ the q dimensional vectors of the output signal and source noise, respectively. The discrete time MIMO process model has to construct relation between these vector variables so that the "basic philosophy" of the SISO models remains unchanged resp. we can use the

corresponding multivariate generalizations (e.g. parameter matrix) of the various notions (e.g. parameter vector). Without going into details regarding the analysis of the inner structure, we simply consider the system model as according to Fig. 1.2-4.

We obtain the MIMO extension of the SISO process model (1.2-5), i.e. a linear stochastic vector difference equation if the coefficient matrices $\underline{\underline{A}}_i$, $\underline{\underline{B}}_i$, $\underline{\underline{C}}_i$ are used instead of coefficients a_i, b_i, c_i and the already defined vectors $\underline{u}(t)$, $\underline{y}(t)$, $\underline{e}(t)$ instead of $u(t)$, $y(t)$, $e(t)$.
Thus the MIMO system model:

$$\underline{y}(t)=\underline{\underline{A}}^{-1}(z^{-1})\,\underline{\underline{B}}(z^{-1})\,\underline{u}(t-d)+\underline{\underline{A}}^{-1}(z^{-1})\,\underline{\underline{C}}(z^{-1})\,\underline{e}(t) \qquad (1.2\text{-}26)$$

where

$$\underline{\underline{A}}(z^{-1})=\underline{\underline{I}}+\underline{\underline{A}}_1 z^{-1}+\ldots+\underline{\underline{A}}_n z^{-n} = \underline{\underline{I}} + \tilde{\underline{\underline{A}}}(z^{-1}) \qquad (1.2\text{-}27)$$

$$\underline{\underline{B}}(z^{-1})=\underline{\underline{B}}_0+\underline{\underline{B}}_1 z^{-1} +\ldots+ \underline{\underline{B}}_n z^{-n} \qquad (1.2\text{-}28)$$

$$\underline{\underline{C}}(z^{-1})=\underline{\underline{I}}+\underline{\underline{C}}_1 z^{-1} +\ldots+ \underline{\underline{C}}_n z^{-n} \qquad (1.2\text{-}29)$$

are matrix polynomials [108, 109, 145]. Here $\underline{\underline{I}}$ denotes the unit matrix (therefore $\underline{\underline{A}}_0$ and $\underline{\underline{C}}_0$ denote hereinafter unit matrices if they are to be distinguished), further it is assumed that $\underline{e}(t)$ is an element of the independent stochastic vector series with zero mean and covariance matrix $\underline{\underline{\Lambda}}$. Fig. 1.2-5 shows the block diagram of the model. By multiplying both sides of (1.2-26) by $A(z^{-1})$ and considering the meaning of the shift operator z^{-1}, we obtain the vector difference equation of the MIMO system:

$$\underline{y}(t)= \sum_{i=0}^{n} \underline{\underline{B}}_i\underline{u}(t-d-i) - \sum_{i=1}^{n} \underline{\underline{A}}_i\underline{y}(t-i)+ \sum_{i=1}^{n} \underline{\underline{C}}_i\underline{e}(t-i)+\underline{e}(t). \qquad (1.2\text{-}30)$$

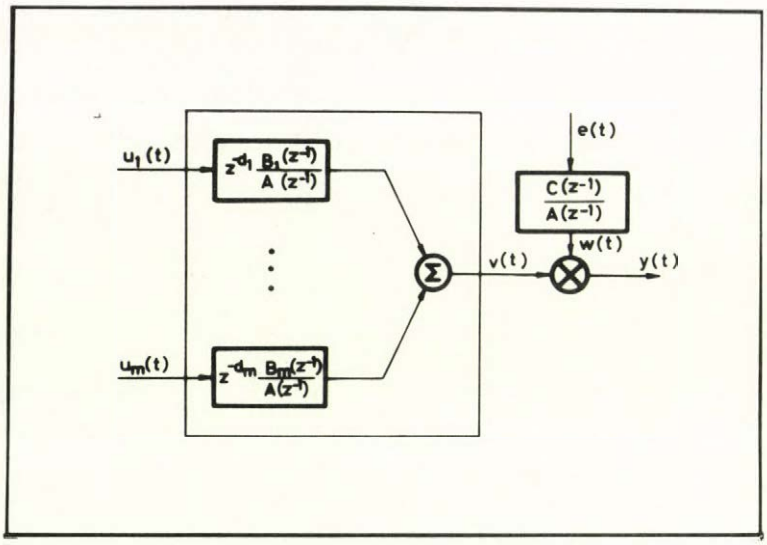


Fig. 1.2-3

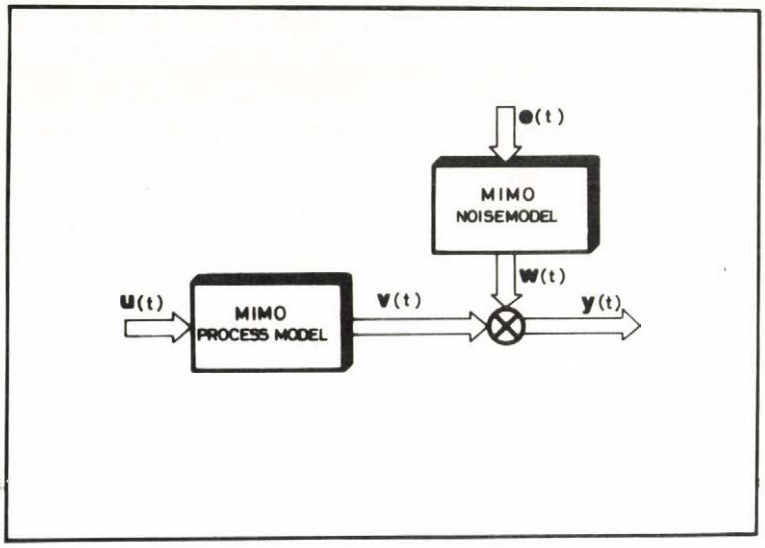


Fig. 1.2-4

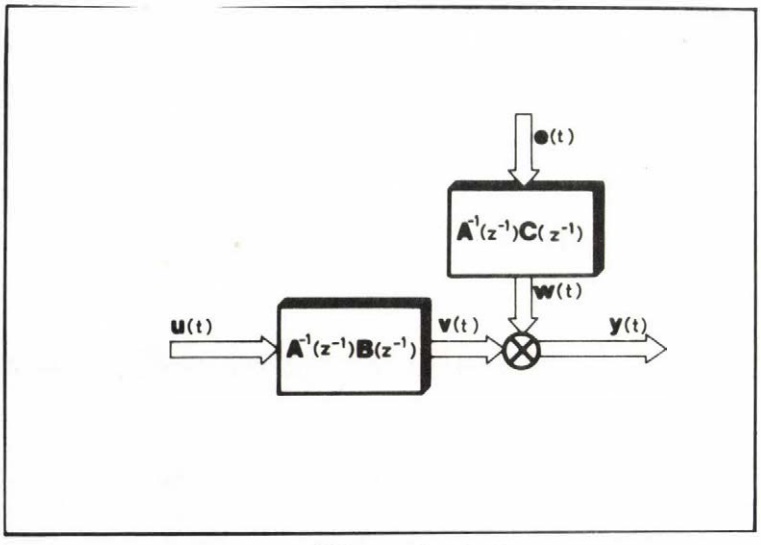


Fig. 1.2-5

This form is like (1.2-16) linear in the parameter matrices of the process, i.e. can be written also in the form [108]

$$\underline{y}(t)=\underline{\underline{P}}_{BA}\ \underline{f}(\underline{u},\underline{y},t)+\underline{\underline{C}}(z^{-1})\,\underline{e}(t) \qquad (1.2\text{-}31)$$

where

$$\underline{\underline{P}}_{BA}=[\underline{\underline{B}}_o,\underline{\underline{B}}_1,\ldots,\underline{\underline{B}}_n,\underline{\underline{A}}_1,\ldots,\underline{\underline{A}}_n] \qquad (1.2\text{-}32)$$

and

$$\underline{f}^T(\underline{u},\underline{y},t)=[\underline{u}^T(t-d),\ldots,\underline{u}^T(t-d-n);-\underline{y}^T(t-1),\ldots,-\underline{y}^T(t-n)]. \qquad (1.2\text{-}33)$$

In equation (1.2-30) we are bound to have same dead-time for every input variable because of the demand of formal description. We can approximate better the real cases by choosing the minimum dead-time value for d and so certain elements of the matrices $\underline{\underline{B}}_i$ become zeros at small i values. This can result from estimation too, but can be considered in advance by prohibition, masking of the corresponding coefficients (cf. section 1.3).

The MIMO version of the state equations (1.2-17), (1.2-18) is obtained simply by increasing the dimension of $\underline{g}$, $\underline{q}$, $\underline{h}$:

$$\underline{x}(t+1)=\underline{\underline{F}}\underline{x}(t)+\underline{\underline{G}}\underline{u}(t)+\underline{\underline{Q}}\underline{e}(t) \qquad (1.2\text{-}34)$$

$$\underline{y}(t)=\underline{\underline{H}}\underline{x}(t)+\underline{\underline{B}}_o\underline{u}(t)+\underline{e}(t). \qquad (1.2\text{-}35)$$

It holds also for MIMO systems that there exist many types of state-space description forms. A form which can simply be constructed is the following:

$$\underline{x}(t+1)=\begin{bmatrix} -\underline{\underline{A}}_1 & \underline{\underline{I}} & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ -\underline{\underline{A}}_2 & \underline{\underline{0}} & \underline{\underline{I}} & \cdots & \underline{\underline{0}} \\ \vdots & \vdots & \vdots & & \vdots \\ -\underline{\underline{A}}_{n-1} & \underline{\underline{0}} & \underline{\underline{0}} & & \underline{\underline{I}} \\ -\underline{\underline{A}}_n & \underline{\underline{0}} & \underline{\underline{0}} & & \underline{\underline{0}} \end{bmatrix}\underline{x}(t)+\begin{bmatrix} \underline{\underline{B}}_1-\underline{\underline{A}}_1\underline{\underline{B}}_0 \\ \underline{\underline{B}}_2-\underline{\underline{A}}_2\underline{\underline{B}}_0 \\ \vdots \\ \underline{\underline{B}}_{n-1}-\underline{\underline{A}}_{n-1}\underline{\underline{B}}_0 \\ \underline{\underline{B}}_n-\underline{\underline{A}}_n\underline{\underline{B}}_0 \end{bmatrix}\underline{u}(t)+\begin{bmatrix} \underline{\underline{C}}_1-\underline{\underline{A}}_1 \\ \underline{\underline{C}}_2-\underline{\underline{A}}_2 \\ \vdots \\ \underline{\underline{C}}_{n-1}-\underline{\underline{A}}_{n-1} \\ \underline{\underline{C}}_n-\underline{\underline{A}}_n \end{bmatrix}\underline{e}(t) \qquad (1.2\text{-}36)$$

and

$$\underline{y}(t)=[\underline{\underline{I}},\underline{\underline{0}},\ldots,\underline{\underline{0}}]\underline{x}(t)+\underline{\underline{B}}_0\underline{u}(t)+\underline{e}(t), \qquad (1.2\text{-}37)$$

which can be constructed in a quite similar way to the cross--resolution. The problem of the determination of the minimum dimension of the state vector is a very essential problem from both identification and other considerations, these questions, however, will not be considered in this report. In App.1., on the other hand, we will point to some very important relationships which will promote the analysis of the structural problems of the MIMO systems.

All noise models used for SISO systems can formally be generalized for MIMO systems. The most important of these are summed up in Table 1.2-II. The notions here, of course, have to be related to vector variables.

Table 1.2-II.

Symbol	$\underline{\underline{D}}^{-1}(z^{-1})\underline{\underline{C}}(z^{-1})$	Author	Output noise	Equation error
MLG	$\underline{\underline{D}}^{-1}(z^{-1})\underline{\underline{C}}(z^{-1})$	EATON	ARMA	ARMA
ML	$\underline{\underline{A}}^{-1}(z^{-1})\underline{\underline{C}}(z^{-1})$	ROWE	ARMA	MA
LS	$\underline{\underline{A}}^{-1}(z^{-1})$	ROWE	AR	white
GLS	$\underline{\underline{A}}^{-1}(\underline{z}^{-1})\underline{\underline{H}}^{-1}(z^{-1})$	BÁNYÁSZ	AR	AR

The use of a system model describing the behaviour of the process simultaneously in all N time instants has a particular importance for off-line identification methods when N related input - output signal data pairs are processed at the same time. Such system description forms can be obtained by the so-called TOEPLITZ matrices [6, 7, 145]. Assume at a SISO system that the related data pairs of the input and output signals { u(t), y(t); t=1,2,...,N } are available in vectors $\underline{u}$ and $\underline{y}$. On the basis of the system equation (1.2-3) we can write the equation of the process for the vectors containing N measurements, too:

$$\underline{y} = \underline{\underline{S}}^{d}\ \underline{\underline{A}}^{-1}\underline{\underline{B}}\underline{u} + \underline{\underline{D}}^{-1}\ \underline{\underline{C}}\underline{e} + \underline{\underline{A}}^{-1}\ \underline{\underline{E}}\underline{z}_{o}. \qquad (1.2\text{-}38)$$

Here $\underline{e}$ is, alike $\underline{u}$ and $\underline{y}$, vector with N elements, containing the values of the source noise. $\underline{\underline{E}}$ is a (N x n) matrix in whose upper part there is an n x n unit matrix, its other parts contain zeros:

$$\underline{\underline{E}} = \begin{bmatrix} \underline{\underline{I}} \\ \underline{\underline{0}} \end{bmatrix}. \qquad (1.2\text{-}39)$$

$\underline{z}_o$ is an (n x 1) vector containing the initial conditions (the system's past preceding the observations). As the parameter estimations become asymptotically independent from the initial conditions [6, 8], we can use hereinafter the assumption $\underline{z}_o = \underline{0}$ without loss of generality. The rest of the matrices in (1.2-38) have a (N x N) size, whose structures are:

$$\begin{aligned}
\underline{\underline{A}} &= \underline{\underline{I}} + a_1\underline{\underline{S}} + \ldots + a_n\underline{\underline{S}}^n = \underline{\underline{I}} + \tilde{\underline{\underline{A}}}, \\
\underline{\underline{B}} &= b_o\underline{\underline{I}} + b_1\underline{\underline{S}} + \ldots + b_n\underline{\underline{S}}^n, \\
\underline{\underline{C}} &= \underline{\underline{I}} + c_1\underline{\underline{S}} + \ldots + c_n\underline{\underline{S}}^n, \\
\underline{\underline{D}} &= \underline{\underline{I}} + d_1\underline{\underline{S}} + \ldots + d_n\underline{\underline{S}}^n,
\end{aligned} \qquad (1.2.40)$$

where $\underline{\underline{S}}^g$ is an (N x N) TOEPLITZ matrix (a special lower triangle matrix), where 1-s are only in the g-th row under the main diagonal, the rest of its elements are zeros [6, 145]. I.e. for the ij-th element of $\underline{\underline{S}}^g$ the equality $S^g_{ij} = \delta_{(i-g),j}$ holds, where $\delta_{p,q}$ is the KRONECKER symbol. Some essential features of the so-called shift matrices $\underline{\underline{S}}^g$ are analyzed in detial in Appendix 2.

(As can be seen from the foregoing, it is often necessary to use the same letter for the notation of different notions as the alphabet proves to be insufficient. Therefore, we do draw your attention to the meaning of indices and arguments!)

The joint equation of the MIMO systems related to N measurements can be written is a concise form with the help of the so-called KRONECKER matrix products [97]. The operational identities referring to the KRONECKER matrix product are summed up in Appendix 3. Assume that with a MIMO system the related data pairs of input and output vectors $\{\underline{u}(t), \underline{y}(t);\ t=1,2,\ldots,N\}$ are available in the form of a

vector

$$\underline{u}_M = [\underline{u}^T(1), \ldots, \underline{u}^T(N)]^T \qquad (1.2\text{-}41)$$

and

$$\underline{y}_M = [\underline{y}^T(1), \ldots, \underline{y}^T(N)]^T. \qquad (1.2\text{-}42)$$

Let the vector

$$\underline{e}_M = [\underline{e}^T(1), \ldots, \underline{e}^T(N)]^T \qquad (1.2\text{-}43)$$

be also of the same construction. The analogue form of the MIMO system (1.2-26) with (1.2-38) for N observations is

$$\underline{y}_M = \Big[\sum_{i=0}^{n} \underline{\underline{S}}_N^i(1) \otimes \underline{\underline{A}}_i\Big]^{-1} \Big[\sum_{i=d}^{n+d} \underline{\underline{S}}_N^i(1) \otimes \underline{\underline{B}}_i\Big] \underline{u}_M +$$

$$+ \Big[\sum_{i=0}^{n} \underline{\underline{S}}_N^i(1) \otimes \underline{\underline{A}}_i\Big]^{-1} \Big[\sum_{i=0}^{n} \underline{\underline{S}}_N^i(1) \otimes \underline{\underline{C}}_i\Big] \underline{e}_M. \qquad (1.2\text{-}44)$$

Here $\otimes$ denotes the KRONECKER matrix product and $\underline{\underline{C}}_0$ must be taken, obviously, for $\underline{\underline{I}} \cdot \underline{\underline{S}}_N^i(1)$ is an (N x N) shift matrix. For the simplicity in (1.2-44) we have disregarded the initial conditions.

1.3 Parameter estimation methods

We shall use the system equation of the form

$$y(t)= f(u,y,\underline{p},t) + e(t) \qquad (1.3\text{-}1)$$

in general for the parametric identification of the SISO models, where $\underline{p}$ denotes the parameter vector. Assume that the estimation of the structure was exact and the expected value of $e(t)$ is zero so that in the difference $y(t)-f(u,y,\underline{p},t)$ only random (and not systematic) error components can arise. The classical parameter estimation problem is the determination of the estimated value $\hat{\underline{p}}$ of the parameter vector $\underline{p}$ so that $f(u,y,\underline{p},t)$ fits best the observed $y(t)$ values according to certain optimality criterion. The various statistical estimation methods are usually classified on the basis of the nature of this quality criterion on the one hand, and on that of the previous knowledge of the residual error (measurement noise) on the other [12, 15, 47].

The system equation (1.3-1) enables us to calculate the residual errors $e(1),\ldots,e(N)$ for every $\underline{p}$ parameter vector on basis of the samples containing the value of number N of the input and output signal. Let us arrange these in a vector $\underline{e}=[e(1),\ldots,e(N)]^T$. According to the maximum likelihood (hereinafter ML) method, realizing the principle of maximum probability, the estimation of the parameter vector $\underline{p}$ is determined so that it maximizes the joint probability density of N elements of the error vector $\underline{e}$ that is the probability of coincidence of the real random error and $\underline{e}$ calculated from the related $u(t)$ and $y(t)$ values, as well as $\hat{\underline{p}}$. Thus in the ML method the conditional probability density function $d_N(\underline{e}|\underline{p})$ has to be maximized. Instead of the conditional probability density function $d_N(\underline{e}|\underline{p})$ in most cases its natural logarithm, the so-called likelihood funciton $L(\underline{p})=\ln d_N(\underline{e}|\underline{p})$ is maximized. In the ML method, consequently, the likelihood function

can be constructed only on basis of the apriori knowledge (or assumption) of the density function. In the discrete identification methods e(t) is assumed to be of normal distribution in applying the ML method and this assumption is valid for the majority of practical cases. (Otherwise the normality examination of the residuals, - that is the e(t)-s calculated after the parameter estimation - is a routine statistical investigation [8, 47, 57, 80].)

Assuming $\underline{e}$ to be a vector variable of normal distribution, the its probability density function is [80]

$$d_N(\underline{e}|\underline{p}) = (2\pi)^{-\frac{N}{2}} \left|\underline{\underline{Z}}\right|^{-\frac{1}{2}} \exp\{-\frac{1}{2}\,\underline{e}^T\,\underline{\underline{Z}}^{-1}\,\underline{e}\}, \qquad (1.3\text{-}2)$$

where

$$\underline{\underline{Z}} = E\{\underline{e}\,\underline{e}^T\}, \qquad (1.3\text{-}3)$$

i.e. the covariance matrix of the noise. Here $|\ldots|$ denotes the determinant. Therefore the likelihood function for normal distribution is

$$L(\underline{p}) = -\frac{N}{2}\ln 2\pi - \frac{1}{2}\ln\left|\underline{\underline{Z}}\right| - \frac{1}{2}\,\underline{e}^T\,\underline{\underline{Z}}^{-1}\,\underline{e}. \qquad (1.3\text{-}4)$$

In case of given $\underline{\underline{Z}}$ we can get the maximum of $L(\underline{p})$ from the equvalent condition [57]:

$$Q(\hat{\underline{p}}) = \min_{\underline{p}} Q(\underline{p}) = \min_{\underline{p}} \frac{1}{2}\,\underline{e}^T\,\underline{\underline{Z}}^{-1}\underline{e}. \qquad (1.3\text{-}5)$$

The solution obviously requires the knowledge of the covariance matrix $\underline{\underline{Z}}$ of the noise. By assuming furthermore also the elements of $\underline{\underline{Z}}$ to be unknown, the minimization has to be extended for these variables, too.

For a stationary white noise e(t) having standard deviation λ

$$\underline{\underline{Z}} = \lambda^2 \underline{\underline{I}}. \qquad (1.3\text{-}6)$$

In this case the ML estimation is obtained by solving the extremum problem

$$Q(\hat{\underline{p}}) = \min_{\underline{p}} \frac{1}{2} \underline{e}^T \underline{e}. \qquad (1.3\text{-}7)$$

Both tasks (1.3-5) and (1.3-7) are nonlinear extremum problems often to be solved in a general case by taking into account, in addition, some restrictions related to the parameters. Assume now that the system equation (1.3-1) can be written in the form linear in parameters

$$y(t) = \underline{f}^T(u,y,t)\underline{p} + e(t). \qquad (1.3\text{-}8)$$

(Thus $\underline{f}(u,y,t)$ does not depend on $\underline{p}$, it contains only the function components characterizing the structure). If N related values of the u(t) and y(t) signals are available in vectors $\underline{u}$ and $\underline{y}$, then the minimum of the cost or loss function (1.3-7) is supplied by the estimated parameter vector.

$$\hat{\underline{p}} = \left[\underline{\underline{F}}^T(\underline{u},\underline{y}) \underline{\underline{F}}(\underline{u},\underline{y}) \right]^{-1} \underline{\underline{F}}^T(\underline{u},\underline{y}) \underline{y} \equiv$$

$$\equiv (\underline{\underline{F}}_{uy}^T \underline{\underline{F}}_{uy})^{-1} \underline{\underline{F}}_{uy}^T \underline{y}. \qquad (1.3\text{-}9)$$

(cf. Appendix 5.)

Here

$$\underline{\underline{F}}(\underline{u},\underline{y}) \equiv \underline{\underline{F}}_{uy} = \begin{bmatrix} \underline{f}^T(u,y,1) \\ \cdot \\ \cdot \\ \cdot \\ \underline{f}^T(u,y,N) \end{bmatrix}. \qquad (1.3\text{-}10)$$

Note that the loss function of form (1.3-7) can be minimized not only in case of normal distribution error, so that the expression (1.3-9) has a general validity and is called the least squares method, hereinafter denoted by LS [77]. In case of normal distribution error, the LS estimation yields simultaneously ML estimation, too. The most important statements related to the LS estimation are summed up in Appendix 5.

Let us consider matrix $\underline{\underline{Z}}$ to be given at the minimization of the loss function (1.3-5). By proper choice of $\underline{\underline{Z}}$, we can influence objectively or subjectively the form of the fitting. By choosing e.g. for $\underline{\underline{Z}}$ a diagonal matrix, where the elements are large in the point where the uncertainty of the measurement is great and small there, where the accuracy is great, the loss function (1.3-5) makes, in a fairly apparent way, measurement points with small measurement error dominant. The estimated parameter vector assuring the minimum in case of system equation (1.3-8) is (cf. Appendix 5.):

$$\hat{\underline{p}} = \left[\underline{\underline{F}}^T(\underline{u},\underline{y}) \underline{\underline{Z}}^{-1} \underline{\underline{F}}(\underline{u},\underline{y}) \right]^{-1} \underline{\underline{F}}^T(\underline{u},\underline{y}) \underline{\underline{Z}}^{-1} \underline{y} \equiv$$

$$\equiv (\underline{\underline{F}}_{uy}^T \underline{\underline{Z}}^{-1} \underline{\underline{F}}_{uy})^{-1} \underline{\underline{F}}_{uy}^T \underline{\underline{Z}}^{-1} \underline{y}. \qquad (1.3\text{-}11)$$

This method is called Generalized Least Squares and denoted by GLS. The method GLS yields at the same time, by the choice of $\underline{\underline{Z}}$ according to (1.3-3), an ML estimation. The method is dealt with more in detail in Appendix 5.

It can be realized (App.5.) that the parameter estimation

$$\hat{\underline{p}} = \left[\underline{\underline{G}}^T(\underline{u},\underline{v}) \underline{\underline{F}}(\underline{u},\underline{y}) \right]^{-1} \underline{\underline{G}}^T(\underline{u},\underline{v}) \underline{y} \equiv$$

$$\equiv (\underline{\underline{G}}_{uv}^T \underline{\underline{F}}_{uy})^{-1} \underline{\underline{G}}_{uv}^T \underline{y} \qquad (1.3\text{-}12)$$

has also satisfactory statistical properties. Here $\underline{v}$ denotes the vector of N elements constructed from the noiseless output $v(t)$ of the system, furthermore $\underline{\underline{G}}(\underline{u},\underline{v}) \equiv \underline{\underline{G}}_{uv}$. As this cannot be measured in reality, it is usually estimated by the assumed parameter values. The construction of $\underline{\underline{G}}(\underline{u},\underline{v}) \equiv \underline{\underline{G}}_{uv}$ can otherwise correspond with (1.3-10) but now v is used instead of y. This procedure is called method of Instrumental Variables and denoted by IV [57, 143].

All of the above surveyed methods are applied for the determination of parameter estimation among the discrete time identification methods. These estimation methods yield, in case of the fulfilment of the necessary conditions, an asymptotically unbiased, consistent estimation. In case of a finite number of samples, the bias is much smaller than the deviation of the parameters which makes the methods completely suitable for practical purposes.

For the general SISO system equations (1.2-1) and (1.2-2) we have assumed that the source noise $e(t)$ is a white noise with zero mean and has, in stationary case, a time-independent λ deviation and a covariance matrix according to (1.3-6). In an analogue way in the MIMO system equation (1.2-26) the source noise vector $\underline{e}(t)$ with zero mean is uncorrelated, and has constant covariance matrix $\underline{\underline{\Lambda}}$. (Note that $\underline{\underline{\Lambda}}$ actually corresponds to the generalization of λ^2). If there are N numbers of measurement vector values available, then the joint density function of the N numbers of $\underline{e}(t)$ values calculated from the measured ones is [4, 57, 108]:

$$d(\underline{e}_{\underline{M}} \,|\, \underline{\underline{P}}) = \prod_{t=1}^{N} d(\underline{e}(t) \,|\, \underline{\underline{P}}) = (2\pi)^{\frac{-qN}{2}} \left|\underline{\underline{\Lambda}}\right|^{-\frac{N}{2}} \exp\left[-\frac{1}{2} \sum_{t=1}^{N} \underline{e}^{T}(t) \underline{\underline{\Lambda}}^{-1} \underline{e}(t)\right], \tag{1.3-13}$$

where

$$d(\underline{e}(t)\left|\underline{\underline{P}}\right.)=(2\pi)^{\frac{q}{2}}\left|\underline{\underline{\Lambda}}\right|^{-\frac{1}{2}}\exp(-\frac{1}{2}\,\underline{e}^{T}(t)\,\underline{\underline{\Lambda}}^{-1}\,\underline{e}(t)) \qquad (1.3\text{-}14)$$

is the conditional probability density function of $\underline{e}(t)$. Here $\underline{\underline{P}}$ is the parameter matrix of the MIMO system and q is the dimension of $\underline{e}(t)$, i.e. the number of outputs. On the basis of (1.3-13) the likelihood function for the MIMO systems is:

$$L = -\frac{qN}{2}\ln 2\pi - \frac{N}{2}\ln\left|\underline{\underline{\Lambda}}\right| - \frac{1}{2}\sum_{t=1}^{N}\underline{e}^{T}(t)\,\underline{\underline{\Lambda}}^{-1}\,\underline{e}(t). \qquad (1.3\text{-}15)$$

According to the deduction presented in App.5. the maximization of the likelihood function (1.3-15) is an equivalent task with the minimization of the loss function (i.e. determinant of a matrix) [108]

$$Q(\hat{\underline{\underline{P}}}) = \min_{\underline{\underline{P}}} Q(\underline{\underline{P}}) = \min_{\underline{\underline{P}}} \left|\frac{1}{N}\sum_{t=1}^{N}\underline{e}(t)\underline{e}^{T}(t)\right|. \qquad (1.3\text{-}16)$$

1.4 On the continuous-time correspondence of discrete-time system models

In the sub-section 1.1 it was already mentioned that the linear, dynamic, discrete-time process models investigated in this report are - according to the sampling principle - equivalent with the original continuous system in respect of the sampling times.

The discrete-time models obtained by identification methods can be directly utilized in the design of DDC circuits and in general in computer process control systems. In practice, however, it occurs more frequently that the identification is used as a preparatory step of the optimal design of classical control circuits and in this case even today the continuous-time system model resp. the transfer function description mode connected therewith is the most widespread. It is therefore necessary to transform the discrete transfer function obtained by parametric identification (we shall revert to this terminology) to the conventional Laplace transfer function. In this subsection we would like to draw attention in short to that the discrete-continuous transformation can be considered unambiguous for the approximation of a given type of the input signal.

Consider now the description of a continuous SISO system by state equations in the form

$$\dot{\underline{x}}(\tau) = \underline{\underline{G}}\, \underline{x}(\tau) + \underline{b}\, u(\tau) \qquad (1.4\text{-}1)$$

$$y(\tau) = \underline{h}^T\, \underline{x}(\tau) + b_o\, u(\tau) \qquad (1.4\text{-}2)$$

The solution of continuous state equations is [42]:

$$\underline{x}(\tau)= \underline{\underline{e}}^{\underline{\underline{G}}\tau}\ \underline{x}(0)+ \int_0^{\tau} \underline{\underline{e}}^{\underline{\underline{G}}(\tau-\nu)}\ \underline{b}u(\nu)\ \ d\nu\ , \qquad (1.4\text{-}3)$$

where τ is the continuous time of the system. Let $\tau = \Delta T$, then

$$\underline{x}\ (\Delta T)= \underline{\underline{e}}^{\underline{\underline{G}}\,\Delta T}\ \underline{x}(0)+ \int_0^{\Delta T} \underline{\underline{e}}^{\underline{\underline{G}}(\Delta T-\nu)}\ u(\nu)\ d\nu\ \underline{b} =$$

$$= \underline{\underline{e}}^{\underline{\underline{G}}\,\Delta T}\ \underline{x}(0)+\underline{a}\left[u(0),\underline{\underline{G}},\underline{b},\ \Delta T\right]. \qquad (1.4\text{-}4)$$

By comparing this latter result with equation (1.2-17), taking the noiseless case $\underline{q} = \underline{0}$ and evaluating the so-called input integral for the approximation of a given type of the input signal $u(\nu)$ between $\tau = 0$ and $\tau = \Delta T$, i.e. determining $\underline{a}\left[u(0),\underline{\underline{G}},\underline{b},\Delta t\right]$, we obtain unambiguous relations between the continuous and discrete state equations, by the comparison of the coefficient matrices. On the basis of the equation (1.4-4), by taking advantage of the time-independence of the system, we can write for any two states following each other by ΔT that

$$\underline{x}(t+1)= \underline{\underline{e}}^{\underline{\underline{G}}\Delta T}\ \underline{x}(t)+\underline{a}\left[u(t),\underline{\underline{G}},\underline{b},\ \Delta T\right]. \qquad (1.4\text{-}6)$$

If suitable canonical equations are chosen for comparison, then the utilization of common notations applied in (1.4-2), resp. (1.2-18) is justified. Otherwise, the change-over to such a form can be carried out by simple transformations [42].

By assuming $u(\nu)= u(0) = \text{const.}$ in the whole sampling period, it can be deduced that the coefficient matrices of the continuous state equation equivalent for the given approximation can be obtained from the $\underline{\underline{F}}$ and $\underline{g}$ in the following way:

$$\underline{\underline{G}} = \frac{1}{\Delta T} \ln(\underline{\underline{F}}) \qquad (1.4\text{-}7)$$

and

$$\underline{b} = \frac{1}{\Delta T} \ln(\underline{\underline{F}}).(\underline{\underline{F}}-\underline{\underline{I}})^{-1}\, \underline{g}. \qquad (1.4\text{-}8)$$

If we need the transfer function of the SISO system, then we transform the state equations on the basis of the obtained $\underline{\underline{G}}$ and $\underline{b}$ to a canonical form where the coefficients of the numerator and denominator can be directly read out [42].

Note that some authors prefer to carry out the retransformation by decomposition to subsystems (separately for every eigenvalue of the system), however the numerical technique required by the relations (1.4-7) and (1.4-8) - logarithm and inversion of the matrix, etc. - can be considered nowadays as elaborated.

In case of another type of approximation of the input signal the formula (1.4-7) remains unchanged, on the other hand (1.4-8) changes. We wish to emphasize again that in case of a given approximation the discrete-continuous change-over, often called "z → s" transformation is unambiguous.

These considerations make understandable why we prefer to speak, maybe with more exactness, of discrete transfer functions. The pulse transfer function assumes namely an approximation $u(\nu) \equiv u(0)\,\delta(0)$, i.e. a pulse of magnitude $u(0)$ occurring in the moment $\nu = 0$, but this approximation is only a subcase of the possible approximation of the input signal. The higher-order quadrature is used for the approximation of the input signal, the more exact transformation will be obtained for the coefficients in the numerator, the conversion of the denominator is performed in an exact way. In practice, due to the appropriate choice of the sampling

period it is usually sufficient to consider the input signal to be constant for every sampling (i.e. to imagine a zero--order hold element at the transformation).

II. OFF-LINE IDENTIFICATION METHODS FOR SINGLE OUTPUT SYSTEMS

In the introduction we have indicated that in the description forms the basic differences are caused by the number of outputs. Therefore, we classify also the off-line identification methods according to the number of the outputs of the process or the model. This section deals with the identification methods of single output systems. The methods will be discussed mainly for SISO systems but the way of direct generalization for multiple input case will be given, too. Similarly the issue of deadtime will also be dealt with. Detailed investigations will be made with respect to the case d=0 and the possibility of generalization will also be pointed out. Issues related to deadtime will be discussed in the greatest detail with the LS method which is otherwise the simplest with regard to identification.

At the discussion of off-line methods, simultaneous presence of the related values of N number of the input signals and the output signal is assumed. The task is to minimize a criterion constructed from these signals and corresponding to a loss function of subsection 1.3 in the space of parameters. For this reason the off-line identification methods mean <u>deterministic</u> extremum seeking problems which can be solved by one of the methods of mathematical programming. They are called deterministic because the loss function yields always the same value for a given measurement situation at a given parameter vector.

The most of the off-line identification methods require matrix inversion, furthermore the intermediate iteration steps or the explicit solution have usually the form

$$\underline{p} = \underline{\underline{M}}^{-1}\underline{v} \qquad (2.1)$$

where $\underline{\underline{M}}$ is a square matrix, $\underline{v}$ a vector of appropriate dimension. If we now do not wish to estimate the i-th p_i element of parameter vector $\underline{p}$ or in another formulation we wish to estimate under restriction $p_i \equiv 0$, then it can be achieved the following way. The i-th column and row of matrix $\underline{\underline{M}}$ will be made equal to zero and e.g. 1. is chosen for the element M_{ii} at their intersection. Thereafter the i-th element of vector $\underline{v}$ is made equal to zero. So we obtain definitely zero for p_i

$$\underline{p} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ p_i \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdots\cdots\ 0\ \cdots\cdots \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0\ldots0\ 1\ 0\ldots0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdots\cdots\ 0\ \cdots\cdots \end{bmatrix}^{-1} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \begin{bmatrix} \cdots\cdots\ 0\ \cdots\cdots \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0\ldots0\ 1\ 0\ldots0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdots\cdots\ 0\ \cdots\cdots \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \tag{2.2}$$

This technique is called masking. In practical applications we used to give a mask vector whose elements have value 1 if we wish to estimate the coefficient concerned and value zero if we wish to exclude it from the estimation.

2.1 Least squares method

General information about the least square method for SISO systems is given in App.5. The condition of the asymtotically unbiased parameter estimation (App.5-6) is fulfilled by the assumption $C(z^{-1}) \equiv 1$ and $D(z^{-1}) \equiv A(z^{-1})$ from the general system equation (1.2-3), in addition it should be required from the input signal that it should be uncorrelated with the source noise and fulfils the CESARO condition of persistently exciting (cf.App.6) [12, 88].

The system equation can now be written in the form of the following difference equation:

$$A(z^{-1})y(t) = z^{-d} B(z^{-1}) u(t) + e(t). \qquad (2.1\text{-}1)$$

By assuming d=0 and utilizing the meaning of $A(z^{-1})$ and $B(z^{-1})$:

$$y(t) = \sum_{i=0}^{n} b_i u(t-i) - \sum_{i=1}^{n} a_i y(t-i) + e(t) = \underline{f}^T(u,y,t)\underline{p}_{ba} + e(t), \qquad (2.1\text{-}2)$$

where

$$\underline{f}(u,y,t) = \left[u(t),\ u(t-1),\ldots,u(t-n);\ -y(t-1),\ldots,-y(t-n)\right]^T \qquad (2.1\text{-}3)$$

and

$$\underline{p}_{ba} = \left[b_0, b_1, \ldots, b_n;\ a_1, \ldots, a_n\right]^T, \qquad (2.1\text{-}4)$$

as it follows also from (1.2-16).

In case of zero initial condition (which does not reduce

substantially the generality) [7, 8], the vector equation simultaneously valid for N samples can be obatined from relations (1.2-38) and (1.2-40):

$$\underline{y} = b_o \underline{\underline{I}}\underline{u} + b_1 \underline{\underline{S}}\underline{u} + \ldots + b_n \underline{\underline{S}}^n \underline{u} - a_1 \underline{\underline{S}}\underline{y} - \ldots - a_n \underline{\underline{S}}^n \underline{y} + \underline{e} =$$

$$= \underline{\underline{F}}(\underline{u},\underline{y})\underline{p}_{ba} + \underline{e} \equiv \underline{\underline{F}}_{uy}\, \underline{p}_{ba} + \underline{e} \qquad (2.1\text{-}5)$$

where

$$\underline{\underline{F}}_{uy} \equiv \underline{\underline{F}}(\underline{u},\underline{y}) = \left[\underline{u}, \underline{\underline{S}}\underline{u}, \ldots, \underline{\underline{S}}^n \underline{u}; \; -\underline{\underline{S}}\underline{y}, \ldots, -\underline{\underline{S}}^n \underline{y}\right] =$$

$$= \begin{bmatrix} \underline{f}^T(u,y,1) \\ \cdot \\ \cdot \\ \cdot \\ \underline{f}^T(u,y,N) \end{bmatrix} . \qquad (2.1\text{-}6)$$

We have deduced in App. 5 that the LS estimation of the parameters of the above system equation is given by

$$\hat{\underline{p}}_{ba} = (\underline{\underline{F}}_{uy}^T \underline{\underline{F}}_{uy})^{-1} \underline{\underline{F}}_{uy}^T \underline{y} = \left[\underline{\underline{F}}^T(\underline{u},\underline{y})\underline{\underline{F}}(\underline{u},\underline{y})\right]^{-1} \underline{\underline{F}}^T(\underline{u},\underline{y},) \underline{y} . \qquad (2.1\text{-}7)$$

This relation is also called - after the classical regression analysis - the solution of the GAUSSIAN normal equation system, and in case of the discrete-time identification the KALMAN estimation. [57, 77, 80].

By taking into account the relations

$$\underline{\underline{F}}_{uy}^T \underline{\underline{F}}_{uy} = \sum_{t=1}^{N} \underline{f}(u,y,t)\underline{f}^T(u,y,t) \qquad (2.1\text{-}8)$$

and

$$\underline{\underline{F}}_{uy}^{T}\,\underline{Y} = \sum_{t=1}^{N} \underline{f}(u,y,t)\,y(t) \qquad (2.1\text{-}9)$$

the solution (2.1-7) can be made easier with respect to the computations.

Estimation properties are discussed in App.5. more in detail, here only some remarks are made on the conditions (App. 5-6) of asymptotic unbiasedness. Of the two conditions the second is more important. This means in case of an estimation linear in parameters that in the expression on the right side of (2.1-2) the vector of the function components in the scalar product (now f(u,y,t)) and the additive disturbing term, the so-called equation error (now e(t)) can not be in correlation. This uncorrelatedness can be easily understood in our case. The input signal has been assumed to be uncorrelated with e(t); on the other hand, $\underline{f}(u,y,t)$ contains only the values of the output signal of a preceeding time- and through them - source noise values of a preceding time. Hence the uncorrelatedness of f(u,y,t) and e(t) already follows. The more exact proof is given in App. 7.

With respect to equation (2.1-6) we note that although the description mode applied there with the shift matrices requires the zero initial conditions (that is the start with the zero $\underline{f}$ vector in the sums (2.1-8) and(2.1-9)), we have no problem in changing over to the non-zero initial conditions on the basis of the right side of the equation. Here it is easy to be seen that this change-over means only that at t=1 (at the beginning of the investigations) the measurements u(o), y(o),..., u(-n+1), y(-n+1) have also to be available in order to construct $\underline{f}(u,y,1)$ which needs N+n samples altogether. This problem can be of interest for small N, as the effect of the initial conditions on the estimation disappears asymptotically [6, 7].

Our further statements refer to the coefficient b_o and deadtime d. In case d=0 the coefficient b_o has an increased role. Then the non-zero value of b_o means that either a proportional "channel" without dynamics is also represented in the process or the process has a differentiating character. Those systems have b_o in whose transient function there is a jump at zero moment. In the identification programs written by the various research workshops, two kinds of conceptions can be discovered [8, 60, 117]. According to one, the estimation procedure is suitable fitted, as a matter of course, also for the estimation of b_o and if one does not wish to estimate it, then it is excluded from the approximation model by the masking technique described above. According to the other, the estimation of b_o is not built in the program (the parameter vector begins by b_1) and the effect corresponding to b_o is obtained by the choice d=-1. Then b_1 will be b_o and so on b_n will be b_{n-1}. If the program has been originally written for the estimation of polynomials of the order n, then in this case $B(z^{-1})$ will always be of one degree less than $A(z^{-1})$. (By special masking it can be achieved to make the order of $B(z^{-1})$, $A(z^{-1})$ equal.)

The consideration of the given deadtime d can be attained on the one hand by the adequate contruction of the function component vectors (special programming demand), on the other by the suitable shift of the input and output signals with respect to each other. The latter possibility covers a far more general possibility than the former. For it is not difficult to write a computer program which performs the shifts corresponding to the deadtimes by inputs even for a multiple input system so that it determines at the same time the number of sampling elements available for identification, too. (The shifts caused by the deadtimes reduce namely the number of the usable related values). On the basis of the shifted samples we can already perform the identification with a program written for the case d=0 and after all, in the final

result we take the deadtime again into account according to the operation z^d applied.

In connection with the special issues, also not that opinions differ also with regard to the order of the polynomials $B(z^{-1})$ and $A(z^{-1})$. According to one approach, both orders can be considered identical (programming is simpler because of cycles of same effect) and use, if necessary, masking. With the second approach different orders are also allowed (more complicated cycles). The first solution proved more reasonable in the practice.

We would like also to advert to the extension of the LS method to multiple input, i.e. MISO systems. For these systems the LS method can be formally used according to the same equations (2.1-6) and (2.1-7), only $\underline{f}(u,y,t)$ will be modified according to (1.2-24) and obviously $\underline{p}_{ba}$ to (1.2-25). The system equation will now have the form:

$$y(t) = \sum_{j=1}^{m} \frac{B_j(z^{-1})}{A(z^{-1})} u_j(t-d_j) + \frac{1}{A(z^{-1})} e(t) =$$

$$= \underline{f}^T(u_1,\ldots,u_m,y,t)\underline{p}_{b_1\ldots b_m a} + e(t) \qquad (2.1\text{-}10)$$

where

$$\underline{f}^T(u_1,\ldots,u_m,y,t) = \left[u_1(t-d_1),\ldots,u_1(t-d_1-n);\ldots;u_m(t-d_m),\ldots,\right.$$

$$\left.,u_m(t-d_m-n);-y(t-1),\ldots,-y(t-n)\right] \qquad (2.1\text{-}11)$$

and

$$\underline{p} = \left[b_{10},\ldots,b_{1n};\ldots;b_{mo},\ldots,b_{mn};\ a_1,\ldots,a_n\right]^T. \qquad (2.1\text{-}12)$$

2.2 Maximum likelihood method

ÅSTRÖM and his co-workers worked out the method of maximum likelihood first for the general SISO system equation (1.2-3) and prepared programs for this task for the firm IBM [8]. Later, in the course of practical applications, it became completely clear that the use of the model (1.2-5) is sufficient for identification tasks [9]. Meanwhile the ÅSTRÖM or ÅSTRÖM-BOHLIN and the maximum likelihood method resp. ML structure have practically merged.

The ML method is discussed by assuming $d \equiv 0$ and $b_o \equiv 0$ as already mentioned in the introduction of this chapter and their estimation is going to be performed via shifting of the input and output signals with respect to each other.

In case of zero initial conditions the system equation for N samples is now

$$\underline{y} = \underline{\underline{A}}^{-1}\ \underline{\underline{B}}\ \underline{u} + \underline{\underline{A}}^{-1}\ \underline{\underline{C}}\ \underline{e}, \qquad (2.2\text{-}1)$$

can be obtained from (1.2-38) by the assumption $\underline{z}_o \equiv \underline{0}$ and $\underline{\underline{D}} = \underline{\underline{A}}$. As $e(t)$ is uncorrelated and

$$E\{\underline{e}\ \underline{e}^T\} = \lambda^2\ \underline{\underline{I}}, \qquad (2.2\text{-}2)$$

we obtain the maximum likelihood estimation on basis of (1.3-5) resp. (1.3-7) by solving the extremum problem

$$Q(\hat{\underline{p}}) = \min_{\underline{p}}\ \frac{1}{2}\underline{e}^T\ \underline{e}\ . \qquad (2.2\text{-}3)$$

The parameters of the SISO system are contained now by the parameter vector

$$\underline{p}_{bac} = [\ b_1,\ldots,b_n;\ a_1,\ldots,a_n,\ c_1,\ldots,c_n\ ]^T \qquad (2.2\text{-}4)$$

as we can write on basis of (1.2-16) that

$$y(t)= \underline{f}^T(u,y,t)\underline{p}_{ba} + C(z^{-1})e(t)= \underline{f}^T(u,y,e,t)\underline{p}_{bac} + e(t), \qquad (2.2\text{-}5)$$

where $\underline{f}(u,y,e,t)$ is according to (2.2-14). Here $e(t)$ is not measurable but can be calculated at a given $\underline{p}_{bac}$. Both the equation

$$e(t)= \frac{A(z^{-1})y(t)- B(z^{-1})u(t)}{C(z^{-1})}= y(t)- \underline{f}^T(u,y,e,t)\underline{p}_{bac} \qquad (2.2\text{-}6)$$

or the vector equation

$$\underline{e} = \underline{\underline{C}}^{-1}(\underline{\underline{A}}\underline{y} - \underline{\underline{B}}\underline{u}) \qquad (2.2\text{-}7)$$

are equally suitable for its calculation. From the scalar equation (2.2-6) by the cross-resolution we can get a state equation much simpler from the point of view of computational aspect for the calculation of $e(t)$ which is the following:

$$\underline{x}(t)= \begin{bmatrix} -c_1 & 1 & 0\ldots0 \\ -c_2 & 0 & 1\ldots0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ -c_{n-1} & 0 & 0\ldots1 \\ -c_n & 0 & 0\ldots0 \end{bmatrix} \underline{x}(t-1) + \begin{bmatrix} 1 & a_1 & -b_1 \\ 0 & a_2 & -b_2 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & a_{n-1} & -b_{n-1} \\ 0 & a_n & -b_n \end{bmatrix} \begin{bmatrix} y(t) \\ y(t-1) \\ u(t-1) \end{bmatrix} \qquad (2.2\text{-}8)$$

$$\underline{e}(t)= [\,1,0,\ldots,0,0\,]\,\underline{x}(t); \quad t=1,2,\ldots,N \qquad (2.2\text{-}9)$$

(These equations do not mean that the coefficient matrices have to be constructed in the computer, they only refer to a simple computational algorithm.)

As neither the parameters are known, it would be more exact to use the symbol ^ to denote the estimated value both for e(t) and the coefficients, for the sake of simplicity, however disregard from this.

The ML estimation requires essentially the solution of a nonlinear extremum problem. The nonlinearity is caused by the fact that in $\underline{f}(u,y,e,t)$ in (2.2-5) e(t) can be generated as a function of the parameters.

For the minimum seeking of $Q(\underline{p})$ one can choose between many methods of mathematical programming. Pure seeking procedures are generally not applied as the first and second derivatives of Q are relatively easy to be calculated. This latter fact encourages us to have recourse to methods using derivatives. Although already several authors have reported the use of a first-order method or of what asymptotically converges to the second order ones (POWELL, FLETSCHER-POWELL, gradient, etc.) but in the practice mostly the NEWTON-RAPHSON second-order seeking method is used, with certain modifications [8, 15, 60, 14]. The canonical form of this latter algorithm:

$$\hat{\underline{p}}_{bac}[j+1] = \hat{\underline{p}}_{bac}[j] - \alpha[j]\underline{\underline{R}}[j]\,\frac{dQ(\hat{\underline{p}}_{bac}[j])}{d\hat{\underline{p}}_{bac}[j]} \qquad (2.2\text{-}10)$$

Here j refers to the iteration cycle and ^ denotes the estimated value. The optimum choice of $\underline{\underline{R}}[j]$

$$\underline{\underline{R}}[j] = \underline{\underline{T}}(\hat{\underline{p}}_{bac}[j])^{-1}, \qquad (2.2\text{-}11)$$

where $\underline{\underline{T}}(\hat{\underline{p}}_{bac}[j])$ is the Hessian matrix of the second derivatives of Q calculated by $\hat{\underline{p}}_{bac}[j]$. Modifications differing from the original algorithm are implied in factor $\alpha[j]$. The computer realization of the method is somewhat different [60, 145]. After the presentation of the computational pos-

sibility of the derivatives, we are going to describe the applied seeking strategy.

Consider the following for the construction of the gradient:

$$\frac{dQ(\underline{p}_{bac})}{d\underline{p}_{bac}} = \frac{d\underline{e}^T}{d\underline{p}_{bac}}\,\underline{e} = -\underline{\underline{F}}_{uye}^T(\underline{\underline{C}}^{-1})^T\,\underline{e} = -\sum_{t=1}^{N}\underline{f}(u^F,y^F,e^F,t)\,\underline{e}(t), \tag{2.2-12}$$

where

$$\underline{\underline{F}}_{uye} \equiv \underline{\underline{F}}(\underline{u},\underline{y},\underline{e})=\left[\underline{\underline{S}}\underline{u},\ldots,\underline{\underline{S}}^n\underline{u};\ -\underline{\underline{S}}\underline{y},\ldots,-\underline{\underline{S}}^n\underline{y};\underline{\underline{S}}\underline{e},\ldots,\underline{\underline{S}}^n\underline{e}\right]=$$

$$=\begin{bmatrix}\underline{f}^T(u,y,e,1)\\ \cdot\\ \cdot\\ \cdot\\ \underline{f}^T(u,y,e,N)\end{bmatrix}, \tag{2.2-13}$$

further

$$\underline{f}(u,y,e,t)=\left[u(t-1),\ldots,u(t-n);-y(t-1),\ldots,-y(t-n);\right.$$
$$\left.e(t-1),\ldots,e(t-n)\right]^T \tag{2.2-14}$$

and

$$\underline{f}(u^F,y^F,e^F,t)=\left[u^F(t-1),\ldots,u^F(t-n);-y^F(t-1),\ldots,-y^F(t-n);\right.$$
$$\left.e^F(t-1),\ldots,e^F(t-n)\right]^T. \tag{2.2-15}$$

Here

$$u^F(t)=\frac{1}{C(z^{-1})}\,u(t), \tag{2.2-16}$$

$$y^F(t) = \frac{1}{C(z^{-1})}\, y(t), \qquad (2.2\text{-}17)$$

$$e^F(t) = \frac{1}{C(z^{-1})}\, e(t) \qquad (2.2\text{-}18)$$

i.e. auxiliary parameters obtained by autoregressive filtering (indicated by the upper script F: "filtered"). $\underline{\underline{S}}$ denotes the shift matrix.

To justify (2.2-12) we show the calculation of the derivatives of $\underline{e}$ by coefficients b_i, a_i and c_i. By utilizing the equation (2.2-7), as well as (1.2-40),

$$\frac{\partial \underline{e}}{\partial b_i} = \frac{\partial}{\partial b_i}\, \underline{\underline{C}}^{-1}(\underline{\underline{A}}\underline{y} - \underline{\underline{B}}\underline{u}) = -\,\underline{\underline{C}}^{-1}\, \frac{\partial \underline{\underline{B}}_i}{\partial b_i}\, \underline{u} = -\underline{\underline{C}}^{-1}\, \underline{\underline{S}}^i\, \underline{u}, \qquad (2.2\text{-}19)$$

$$\frac{\partial \underline{e}}{\partial a_i} = \frac{\partial}{\partial a_i}\, \underline{\underline{C}}^{-1}(\underline{\underline{A}}\underline{y} - \underline{\underline{B}}\underline{u}) = \underline{\underline{C}}^{-1}\, \frac{\partial \underline{\underline{A}}}{\partial a_i}\, \underline{y} = \underline{\underline{C}}^{-1}\, \underline{\underline{S}}^i\, \underline{y}, \qquad (2.2\text{-}20)$$

$$\frac{\partial \underline{e}}{\partial c_i} = \frac{\partial}{\partial c_i}\, \underline{\underline{C}}^{-1}(\underline{\underline{A}}\underline{y} - \underline{\underline{B}}\underline{u}) = -\underline{\underline{C}}^{-1}\, \frac{\partial \underline{\underline{C}}}{\partial c_i}\, \underline{\underline{C}}^{-1}(\underline{\underline{A}}\underline{y} - \underline{\underline{B}}\underline{u}) = -\underline{\underline{C}}^{-1}\, \underline{\underline{S}}^i \underline{e}. \qquad (2.2\text{-}21)$$

In the last equation the identity $\partial \underline{\underline{C}}^{-1} / \partial c_i \equiv -\underline{\underline{C}}^{-1}\, \partial \underline{\underline{C}} / \partial c_i \underline{\underline{C}}^{-1}$ is used. These equations enable us to construct the complete gradient vector according to (2.2-12).

The partial derivatives constructed for one element e(t) of $\underline{e}$, are in complete analogy with Eqs. (2.2-19) - (2.2-21):

$$\frac{\partial e(t)}{\partial b_i} = -z^{-i}\,\frac{u(t)}{C(z^{-1})} = -\,\frac{u(t-i)}{C(z^{-1})} \qquad (2.2\text{-}22)$$

$$\frac{\partial e(t)}{\partial a_i} = z^{-i}\,\frac{y(t)}{C(z^{-1})} = \frac{y(t-i)}{C(z^{-1})} \qquad (2.2\text{-}23)$$

$$\frac{\partial e(t)}{\partial c_i} = -z^{-i}\,\frac{e(t)}{C(z^{-1})} = -\,\frac{e(t-i)}{C(z^{-1})}\,. \qquad (2.2\text{-}24)$$

At the end of the '60s, the calculations were still based on these relations, considering that (2.2-12) can be written also in the form of

$$\frac{dQ(\underline{p}_{bac})}{d\underline{p}_{bac}} = \sum_{t=1}^{N} \frac{\partial e(t)}{\partial \underline{p}_{bac}}\; e(t)\,. \qquad (2.2\text{-}25)$$

The programming of these calculations required, however, a large-size memory, while $\underline{f}(u^F, y^F, e^F, t) = \underline{f}_1^F(t)$ in (2.2-12) can be got by a very simple calculation. This can be achieved by the following state equation:

$$\underline{f}_1^F(t) = \underline{\underline{\Omega}}_1\, \underline{f}_1^F(t-1) + \underline{\omega}_1(t) =$$

$$
\begin{bmatrix} u^F(t-1) \\ u^F(t-2) \\ \vdots \\ u^F(t-n) \\ \hline -y^F(t-1) \\ -y^F(t-2) \\ \vdots \\ -y^F(t-n) \\ \hline e^F(t-1) \\ e^F(t-2) \\ \vdots \\ e^F(t-n) \end{bmatrix} = \left[\begin{array}{cccc:cccc:cccc} -c_1 & -c_2 & \cdots & -c_n & & & & & & & & \\ 1 & 0 & \cdots & 0 & & & & & & & & \\ \vdots & & & & & & & & & & & \\ 0 & 0 & \cdots & 1 & & \underline{0} & & & & \underline{0} & & \\ \hline & & & & -c_1 & -c_2 & \cdots & -c_n & & & & \\ & & & & 1 & 0 & \cdots & 0 & & & & \\ & \underline{0} & & & \vdots & & & & & \underline{0} & & \\ & & & & 0 & 0 & \cdots & 1 & & & & \\ \hline & & & & & & & & -c_1 & -c_2 & \cdots & -c_n \\ & & & & & \underline{0} & & & 1 & 0 & \cdots & 0 \\ & \underline{0} & & & & & & & \vdots & & & \\ & & & & & & & & 0 & 0 & \cdots & 1 \end{array}\right] \begin{bmatrix} u^F(t-2) \\ u^F(t-3) \\ \vdots \\ u^F(t-n-1) \\ \hline -y^F(t-2) \\ -y^F(t-3) \\ \vdots \\ -y^F(t-n-1) \\ \hline e(t-2) \\ e(t-3) \\ \vdots \\ e(t-n-1) \end{bmatrix} + \begin{bmatrix} u(t-1) \\ 0 \\ \vdots \\ 0 \\ \hline -y(t-1) \\ 0 \\ \vdots \\ 0 \\ \hline e(t-1) \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

(2.2-26)

The Hessian matrix consists of two parts:

$$\underline{\underline{T}} = \frac{d^2 Q(\underline{p}_{bac})}{d\underline{p}_{bac}\, d\underline{p}_{bac}^T} = \underline{\underline{T}}_1 + \underline{\underline{T}}_2 = \frac{d\underline{e}^T}{d\underline{p}_{bac}} \frac{d\underline{e}}{d\underline{p}^T_{bac}} +$$

$$+\left[\frac{d}{d\underline{p}^T_{bac}}\left(\frac{d\underline{e}^T}{d\underline{p}_{bac}}\right)\right]\underline{e}. \qquad (2.2\text{-}27)$$

Here

$$\underline{\underline{T}}_1 = \underline{\underline{F}}^T_{uye}(\underline{\underline{C}}^{-1})^T\, \underline{\underline{C}}^{-1}\underline{\underline{F}}_{uye} = \sum_{t=1}^{N} \underline{f}(u^F, y^F, e^F, t)\underline{f}(u^F, y^F, e^F, t)^T, \qquad (2.2\text{-}28)$$

i.e. can be formed from quantities already known as we have already given the recursive calculation of $\underline{f}_1^F(t)$.

The construction of $\underline{\underline{T}}_2$ deserves more attention

$$\underline{\underline{T}}_2 = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{T}}_{bc} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{T}}_{ac} \\ \underline{\underline{T}}_{cb} & \underline{\underline{T}}_{ca} & \underline{\underline{T}}_{cc} \end{bmatrix} \qquad (2.2\text{-}29)$$

In this matrix $\underline{\underline{T}}_{cb} = \underline{\underline{T}}^T_{bc}$ and $\underline{\underline{T}}_{ca} = \underline{\underline{T}}^T_{ac}$, further $\underline{\underline{T}}_{cc}$ is symmetric ($\underline{\underline{T}}$ itself is also symmetric!). Only the first row and last column of the matrices $\underline{\underline{T}}_{bc}$, $\underline{\underline{T}}_{ac}$ and $\underline{\underline{T}}_{cc}$ contain informative new element. For the other elements it is valid that $t_{ij}=t_{(i-1),(j+1)}$. Thus a $(3\text{-}2n\text{-}1)$ dimensional vector is sufficient to construct $\underline{\underline{T}}_2$. The construction of this vector

$$\underline{g}^{FF}(t)=\left[u^{FF}(t-1),\ldots,u^{FF}(t-2n+1);-y^{FF}(t-1),\ldots,-y^{FF}(t-2n+1);\right.$$

$$\left.2e^{FF}(t-1),\ldots,2e^{FF}(t-2n+1)\right]^T, \qquad (2.2\text{-}30)$$

where

$$u^{FF}(t)=\frac{u^F(t)}{C(z^{-1})} \qquad (2.2\text{-}31)$$

$$y^{FF}(t)=\frac{y^F(t)}{C(z^{-1})} \qquad (2.2\text{-}32)$$

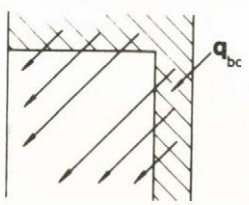
$$e^{FF}(t)=\frac{e^F(t)}{C(z^{-1})} \; . \qquad (2.2\text{-}33)$$

In order to form recursively $g^{FF}(t)$ we could also now construct a state equation similar to (2.2-26), but as it would differ only in size, we do not go now into details of its construction. $\underline{\underline{T}}_2$ can be formed from the vector

$$\underline{q}^{FF} = \sum_{t=1}^{N} \underline{g}^{FF}(t)e(t)=\left[\, \underline{q}_{bc}^T, \underline{q}_{ac}^T, \underline{q}_{cc}^T \,\right]^T \qquad (2.2\text{-}34)$$

since this contains the elements of $\underline{\underline{T}}_2$ different from 0. $\underline{\underline{T}}_{bc}$ can be constructed from $\underline{q}_{bc}$, containing the first 2n-1 elements of $\underline{q}^{FF}$, in the following way:

$$\underline{\underline{T}}_{bc} = \qquad (2.2\text{-}35)$$



where the repetition of identical elements was denoted by arrows. A similar regularity is valid also for the other submatrices.

The above special structural relations follow from the scalar form

$$t_{ij} = \sum_{t=1}^{N} \frac{\partial^2 e(t)}{\partial p_i \, \partial p_j} e(t) \qquad (2.2\text{-}36)$$

deduced from the elements of matrix $\underline{\underline{T}}_2$ where the value of the second derivatives of $e(t)$:

$$\left.\begin{aligned}
&\frac{\partial^2 e(t)}{\partial b_i \partial b_j} = 0; \quad \frac{\partial^2 e(t)}{\partial b_i \partial a_j} = 0; \quad \frac{\partial^2 e(t)}{\partial a_i \partial a_j} = 0; \\
&\frac{\partial^2 e(t)}{\partial b_i \partial c_j} = z^{-(i+j)} \frac{u(t)}{C^2(z^{-1})} = \frac{u^F(t-i-j)}{C(z^{-1})}; \\
&\frac{\partial^2 e(t)}{\partial a_i \partial c_j} = -z^{-(i+j)} \frac{y(t)}{C^2(z^{-1})} = -\frac{y^F(t-i-j)}{C(z^{-1})}; \\
&\frac{\partial^2 e(t)}{\partial c_i \partial c_j} = 2z^{-(i+j)} \frac{e(t)}{C^2(z^{-1})} = 2\frac{e^F(t-i-j)}{C(z^{-1})}
\end{aligned}\right\} \qquad (2.2\text{-}37)$$

Note that in case of $N \to \infty$ $\underline{\underline{T}}_2$ becomes a zero matrix. This is a simple consequence of the fact that of $\underline{\underline{T}}_2$ is formed from vector $\underline{q}$ according to (2.2-34), which is, however, asymptotically equal with the zero vector, as it contains the correlations of $e(t)$ formed by quantities uncorrelated with it.

Alike the gradient, we use today already for the calculation of the Hessian matrix, too, the recursive relations presented above, and not the difference equations represented by the scalar equations according to (2.2-37) . We acknowledge, however, that without these we would be unable to perform the detailed analysis for the deduction of the simpler algorithms. As in the Newton-Raphson technique, the calculation of inverse matrix, the gradient and the second derivatives require the greater part of the calculation time by so the described relations can form a basis for fast computer programs.

The algorithm (2.2-10) ensures, of course, only local extremum values.The result of the LS estimation is considered - in the absence of other information - as the starting point of the seeking under condition $c_i \equiv 0$. The computer programs can usually start also from other starting points given in advance. By this operation mode we can check also the globality of the local minimum obtained. The extremum seeking is further complicated by the fact that the roots of the polynomial $z^n\, C(z^{-1})$ must fall inside of the unit circle (the noise model is now of "minimum phase") because of the autoregressive filtering by $1/C(z^{-1})$ required for the calculation of the auxiliary quantities.

Our seeking strategy is as follows. The search can start from the LS estimation or external starting point. We use the $\underline{\underline{T}} \cong \underline{\underline{T}}_1$ approximation in some of the first steps. The calculation of $\underline{\underline{T}}_1$ is much more simple and the speed of the search fairly great. But in the vicinity of the optimum $\underline{\underline{T}}_2$ has already to be taken into consideration. During the search, the positive definiteness of $\underline{\underline{T}}^{-1}$ has to be checked. If this is not fulfilled, then the convergence of the search is no more ensured (in this case the step would have been made in the direction of the gradient, although the right direction in searching for the minimum is the other way round). Therefore, in every step of the search we investi-

gate the non-negativeness of the main diagonal elements of $\underline{\underline{T}}$ and $\underline{\underline{T}}^{-1}$. If any of them is negative, then we use the $\underline{\underline{T}} \cong \underline{\underline{T}}_1$ approximation, as $\underline{\underline{T}}_1$ is by definition a positive definite matrix. (The investigation of the main diagonal yields a not sufficient only necessary condition, but as computation this is the simplest and in practice completely satisfactory solution). The stability of $C(z^{-1})$ is ensured by a simple strategy.In every iteration step we check for $z^n\, C(z^{-1})$ given by the new parameter estimation by a numerical method whether the roots fall into the permitted domain. If yes, then we accept the new point, if not, then we consider the arithmetic mean of the old and new points as a new point. The halving was, of course, meant only for the part of the parameter vector containing the c_i-s. (With this method we get again in a permissibly domain, if the old point was also there). In consequence of the numerical errors of inversion, on the one hand, and complexity of the surface, on the other hand, a situation can arise in which the value of the loss function Q is greater in the new point than in the old one. In this case we change over to the suboptimal minimum search in an opposite direction to the gradient. This can be ensured by the choice of

$$\alpha[j]\,\underline{\underline{R}}[j] = \frac{\dfrac{dQ(\underline{p}_{bac}[j])}{d\underline{p}^T_{bac}[j]}\;\dfrac{dQ(\underline{p}_{bac}[j])}{d\underline{p}_{bac}[j]}}{\dfrac{dQ(\underline{p}_{bac}[j])}{d\underline{p}^T_{bac}[j]}\;\underline{\underline{T}}(\underline{p}_{bac}[j])\;\dfrac{dQ(\underline{p}_{bac}[j])}{d\underline{p}_{bac}[j]}} \cdot \underline{\underline{I}} \qquad (2.2\text{-}38)$$

(See rf. [81].)

If here, too, the value of the loss function appears to be greater, then we can again pass over to the halving strategy (now for the complete $\underline{p}_{bac}$ vector) for it is sure that after some steps Q will again be smaller than its value in

the preceding step. According to our experiences this is the combined strategy which has proved the best with both our simulation and industrial identification tasks. We always had various problems with other solutions suggested in the literature.

The extension of the algorithm discussed just now to the MISO system (1.2-21) is very simple as the role of the polynomials $B_i(z^{-1})$ regarding the particular input signals $u_i(t)$ completely coincides. So the consideration of m inputs can be simply realized for the calculation of $e(t)$, the gradient and the Hessian matrix. Thanks to this the computer program working according to the ML method can at present consider simultaneously several inputs, too.

For the calculation of both the gradient vector and the Hessian matrix the output signal of the autoregressive filter $1/C(z^{-1})$ has to be calculated for various input signals. On the other hand, the "memory" of these filters has to be known at the start of the calculations. As we, in general, do not know this, we assume the output signal of the filter to be zero for the moments before the beginning of the calculations. Experience shows that this approximation has no relevant influence on the exactness of the calculation and its effect can be neglected asymptotically.

2.3 The generalized least squares method

For the case of the system equation containing $C(z^{-1}) \equiv 1$ and $D(z^{-1}) \equiv A(z^{-1}) * H(z^{-1})$, CLARKE suggested an estimation technique [33]. This structure is called the generalized least squares structure (GLS) because the estimation method obtained this way agrees asymptotically with the GLS method of AITKEN [57].

So the equation of the SISO system is now (by assuming d=0):

$$y(t) = \frac{B(z^{-1})}{A(z^{-1})} u(t) + \frac{1}{A(z^{-1})H(z^{-1})} e(t) \qquad (2.3\text{-}1)$$

where

$$H(z^{-1}) = 1 + h_1 z^{-1} + \ldots + h_s z^{-s}. \qquad (2.3\text{-}2)$$

By multiplying the Eq. (2.3-1) by $H(z^{-1})$, we obtain the form

$$y^F(t) = \frac{B(z^{-1})}{A(z^{-1})} u^F(t) + \frac{1}{A(z^{-1})} e(t) \qquad (2.3\text{-}3)$$

which exactly corresponds to the LS structure. Here

$$y^F(t) = H(z^{-1}) y(t) \text{ and } u^F(t) = H(z^{-1}) u(t). \qquad (2.3\text{-}4)$$

The analogous version of the eq. (2.1-2) is now

$$y^F(t) = \sum_{i=1}^{n} b_i u^F(t-i) - \sum_{i=1}^{n} a_i y^F(t-i) + e(t) =$$

$$= \underline{f}^T(u^F, y^F, t)\, \underline{p}_{ba} + e(t), \qquad (2.3\text{-}5)$$

where

$$\underline{f}(u^F,y^F,t)=\left[u^F(t-1),\ldots,u^F(t-n);\ -y^F(t-1),\ldots,-y^F(t-n)\right]^T. \tag{2.3-6}$$

The vector equation corresponding to (2.1-5) is now

$$\begin{aligned}\underline{y}^F &= b_1\underline{\underline{S}}\underline{u}^F + \ldots + b_n\underline{\underline{S}}^n\underline{u}^F - a_1\underline{\underline{S}}\underline{y}^F - \ldots - a_n\underline{\underline{S}}^n\underline{y}^F + \underline{e} = \\ &= \underline{\underline{F}}(\underline{u}^F,\underline{y}^F)\underline{p}_{ba} + \underline{e} = \underline{\underline{H}}\ \underline{\underline{F}}(\underline{u},\underline{y})\underline{p}_{ba} + \underline{e} = \\ &= \underline{\underline{H}}\ \underline{\underline{F}}_{uy}\ \underline{p}_{ba} + \underline{e}\ ,\end{aligned} \tag{2.3-7}$$

where the commutativity of the TOEPLITZ matrices have been utilized. Here in consequence of (2.3-4)

$$\underline{y}^F = \underline{\underline{H}}\underline{y} \qquad \text{and} \qquad \underline{u}^F = \underline{\underline{H}}\ \underline{u} \tag{2.3-8}$$

where

$$\underline{\underline{H}} = \underline{\underline{I}} + h_1\underline{\underline{S}} + \ldots + h_s\underline{\underline{S}}^s = \underline{\underline{I}} + \tilde{\underline{\underline{H}}}\ . \tag{2.3-9}$$

Further

$$\begin{aligned}\underline{\underline{F}}(\underline{u}^F,\underline{y}^F)&=\left[\underline{\underline{S}}\ \underline{u}^F,\ldots,\ \underline{\underline{S}}^n\ \underline{u}^F;\ -\underline{\underline{S}}\ \underline{y}^F,\ldots,-\underline{\underline{S}}^n\ \underline{y}^F\right]= \\ &=\begin{bmatrix}\underline{f}^T(u^F,y^F,1)\\ \vdots \\ \underline{f}^T(u^F,y^F,N)\end{bmatrix}.\end{aligned} \tag{2.3-10}$$

As the equation error in Eq. (2.3-7) is also now white noise, by minimizing the loss function

$$Q_1(\underline{p}_{ba},\underline{p}_h)=\left[\underline{y}^F-\underline{\underline{F}}(\underline{u}^F,\underline{y}^F)\underline{p}_{ba}\right]^T\left[\underline{y}^F-\underline{\underline{F}}(\underline{u}^F,\underline{y}^F)\underline{p}_{ba}\right] \tag{2.3-11}$$

an asymptotically unbiased parameter estimation can be obtained.

In accordance with (2.1-7) this will now be

$$\hat{\underline{p}}_{ba} = \left[\underline{\underline{F}}^T(\underline{u}^F,\underline{y}^F)\underline{\underline{F}}(\underline{u}^F,\underline{y}^F)\right]^{-1} \underline{\underline{F}}^T(\underline{u}^F,\underline{y}^F)\,\underline{y}^F =$$

$$= \left[\underline{\underline{F}}_{uy}^T \underline{\underline{H}}^T \underline{\underline{H}}\,\underline{\underline{F}}_{uy}\right]^{-1} \underline{\underline{F}}_{uy}^T \underline{\underline{H}}^T \underline{\underline{H}}\,\underline{y} = \hat{\underline{p}}_{ba}(\underline{p}_h). \qquad (2.3\text{-}12)$$

But here we have to consider that the solution is obtained as a function of

$$\underline{p}_h = \left[h_1, h_2, \ldots, h_s\right]^T. \qquad (2.3\text{-}13)$$

As we do not know $\underline{p}_h$, this also has to be estimated.

We obtain from Eq. (2.3-1) after some arrangement that

$$\left[y(t) - \underline{f}^T(u,y,t)\underline{p}_{ba}\right] = r(t) = -\underline{g}^T(r,t)\underline{p}_h + e(t), \qquad (2.3\text{-}14)$$

where

$$\underline{g}(r,t) = \left[r(t-1), \ldots, r(t-s)\right]^T. \qquad (2.3\text{-}15)$$

Here $r(t)$ is the so-called residual

$$r(t) = y(t) - \underline{f}^T(u,y,t)\underline{p}_{ba}. \qquad (2.3\text{-}16)$$

As (2.3-14) is linear in the parameters h_i and the equation error is white noise, we can use again the LS method. The vector equation for N samples is now

$$\underline{y} - \underline{\underline{F}}_{uy}\underline{p}_{ba} = \underline{r} = -h_1\underline{\underline{S}}\underline{r} - \ldots - h_s\,\underline{\underline{S}}^s\,\underline{r} + \underline{e} =$$

$$= -\underline{\underline{G}}(\underline{r})\underline{p}_h + \underline{e} = -\underline{\underline{G}}_r\,\underline{p}_h + \underline{e}. \qquad (2.3\text{-}17)$$

Here $\underline{r}$ contains the N values of r(t) in a form similar to $\underline{u}$, $\underline{y}$ and $\underline{e}$. The loss function to be minimized is

$$Q_2(\underline{p}_h,\underline{p}_{ba})=(\underline{r} + \underline{\underline{G}}_r\underline{p}_h)^T(\underline{r} + \underline{\underline{G}}_r \underline{p}_h) \qquad (2.3\text{-}18)$$

and the parameter estimation ensuring the minimum will be

$$\hat{\underline{p}}_h = -(\underline{\underline{G}}_r^T \underline{\underline{G}}_r)^{-1} \quad \underline{\underline{G}}_r^T \underline{r} = \hat{\underline{p}}_h(\underline{p}_{ba}). \qquad (2.3\text{-}19)$$

Thus the parameter estimation $\hat{\underline{p}}_h$ is the function of $\underline{p}_{ba}$. So in order to carry out a complete parameter estimation, we can use a successive approximation method, recommended in the literature first by CLARKE [33]. This estimation technique consists of the following steps:

1. By assuming $\underline{p}_h = \underline{0}$ a simple LS estimation is carried out according to (2.1-7) for $\underline{p}_{ba}$
2. Thereafter on the basis of (2.3-16) the residuals r(t), i.e. $\underline{r}$ is calculated.
3. In the next step an estimation for $\underline{p}_h$ is made according to (2.3-19).
4. With the help of the obtained $\hat{\underline{p}}_h$, $\underline{y}^F$ and $\underline{u}^F$, i.e. the values of the input and output signal filtered by $H(z^{-1})$ are calculated.
5. The procedure is continued by an estimation according to (2.3-12), thereafter the particular steps of the iteration will be repeated beginning with point 2. until the achievement of the required accuracy.

The CLARKE's iteration method means essentially the minimization of the loss function Q_1 according to the relaxation (or GAUSS-SEIDELL) method to be used in the space of the joint parameter vector $\underline{p}_{bah} = [\underline{p}_{ba}^T, \underline{p}_h^T]^T$. I.e. the minimization proceeds in the subspace corresponding to parameters $\underline{p}_{ba}$, resp. $\underline{p}_h$ step by step.

There is a nonlinear estimation with respect to the parameter vector $\underline{p}_{bah}$ of the process model (2.3-1) and the CLARKE method decomposes this task to two estimation problems linear in parameters by iteration steps. It is also possible, of course, to minimize $Q_1(\underline{p}_{ba},\underline{p}_h)= Q_1(\underline{p}_{bah})$ directly according to $\underline{p}_{bah}$ with the NEWTON-RAPHSON technique (2.2-10). Alike to those discussed with the ML structure, here the relations to be used for the calculation of the gradient vector and the Hessian matrix for the system equation (2.3-1) are also given.

From the vector equation (2.3-7)

$$\underline{e} = \underline{\underline{H}}(\underline{\underline{A}}\ \underline{y} - \underline{\underline{B}}\ \underline{u}) = \underline{\underline{A}}\ \underline{y}^F - \underline{\underline{B}}\ \underline{u}^F = \underline{\underline{H}}\ \underline{r} \qquad (2.3\text{-}20)$$

hence the derivatives of $\underline{e}$ by parameters are

$$\frac{\partial \underline{e}}{\partial b_i} = -\frac{\partial \underline{\underline{B}}}{\partial b_i}\ \underline{u}^F = -\underline{\underline{S}}^i\ \underline{u}^F, \qquad (2.3\text{-}21)$$

$$\frac{\partial \underline{e}}{\partial a_i} = \frac{\partial \underline{\underline{A}}}{\partial a_i}\ \underline{y}^F = \underline{\underline{S}}^i\ \underline{y}^F, \qquad (2.3\text{-}22)$$

$$\frac{\partial \underline{e}}{\partial h_i} = \frac{\partial \underline{\underline{H}}}{\partial h_i}\ (\underline{\underline{A}}\underline{y} - \underline{\underline{B}}\underline{u}) = \underline{\underline{S}}^i\ \underline{r} \qquad (2.3\text{-}23)$$

On basis of the above equations as well as (2.2-12) now

$$\frac{dQ(\underline{p}_{bah})}{d\underline{p}_{bah}} = \frac{d\underline{e}^T}{d\underline{p}_{bah}}\ \underline{e} = -\underline{\underline{F}}^{*T}_{uye}\ \underline{\underline{H}}^T\ \underline{e} =$$

$$= -\sum_{t=1}^{N} \underline{f}(u^F,y^F,r,t)e(t), \qquad (2.3\text{-}24)$$

where

$$\underline{\underline{F}}^{*}_{uye} \equiv \underline{\underline{F}}^{*}(\underline{u},\underline{y},\underline{e}^{F})=\left[\underline{\underline{S}}\underline{u},\ldots,\underline{\underline{S}}^{n}\underline{u},-\underline{\underline{S}}\underline{y},\ldots,-\underline{\underline{S}}^{n}\underline{y},\ -\underline{\underline{S}}\underline{\underline{H}}^{-2}\underline{e},\ldots,-\underline{\underline{S}}^{n}\underline{\underline{H}}^{-2}\underline{e}\right]=$$

$$=\left[\underline{\underline{S}}\underline{u},\ldots,\underline{\underline{S}}^{n}\underline{u},-\underline{\underline{S}}\underline{y},\ldots,-\underline{\underline{S}}^{n}\underline{y},-\underline{\underline{S}}\ \underline{\underline{H}}^{-1}\ \underline{r},\ldots,-\underline{\underline{S}}^{n}\ \underline{\underline{H}}^{-1}\ \underline{r}\right] \qquad (2.3\text{-}25)$$

further

$$\underline{f}(u^{F},y^{F},r,t)=\left[u^{F}(t-1),\ldots,u^{F}(t-n);-y^{F}(t-1),\ldots,-y^{F}(t-n);\right.$$

$$\left.-r(t-1),\ldots,-r(t-n)\right]^{T}, \qquad (2.3\text{-}26)$$

where $u^{F}(t)$ and $y^{F}(t)$ are according to (2.3-4), and $r(t)$ can be calculated from (2.3-16). $e(t)$ is still necessary to (2.3-24) which can be obtained from (2.3-5). (It is important to note that $r(t) = y(t) - \underline{f}^{T}(u,y,t)\underline{p}_{ba}$ and $e(t) = y^{F}(t) - \underline{f}^{T}(u^{F},y^{F},t)\underline{p}_{ba}$.) The $\underline{f}(u^{F},y^{F},r,t)= \underline{f}_{2}^{F}(t)$ required for the gradient vector can, of course, be formed even now in a simple recursive way, as with the ML structure:

$$\underline{f}_{2}^{F}(t)= \underline{\underline{\Omega}}_{2}\ \underline{f}_{2}^{F}(t-1)+ \underline{\underline{S}}(t-1)\underline{p}_{h} + \underline{\omega}_{2}(t)=$$

$$
\begin{bmatrix} u^F(t-1) \\ u^F(t-2) \\ \vdots \\ u^F(t-n) \\ \hline -y^F(t-1) \\ -y^F(t-2) \\ \vdots \\ -y^F(t-n) \\ \hline -r(t-1) \\ -r(t-2) \\ \vdots \\ -r(t-n) \end{bmatrix} = \left[\begin{array}{c|c|c} \begin{matrix} 0 & 0 \dots 0 & 0 \\ 1 & 0 \dots 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \dots 1 & 0 \end{matrix} & \underline{0} & \underline{0} \\ \hline \underline{0} & \begin{matrix} 0 & 0 \dots 0 & 0 \\ 1 & 0 \dots 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \dots 1 & 0 \end{matrix} & \underline{0} \\ \hline \underline{0} & \underline{0} & \begin{matrix} 0 & 0 \dots 0 & 0 \\ 1 & 0 \dots 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \dots 1 & 0 \end{matrix} \end{array}\right] \begin{bmatrix} u^F(t-2) \\ u^F(t-3) \\ \vdots \\ u^F(t-n-1) \\ \hline -y^F(t-2) \\ -y^F(t-3) \\ \vdots \\ -y^F(t-n-1) \\ \hline -r(t-2) \\ -r(t-3) \\ \vdots \\ -r(t-n-1) \end{bmatrix} + \begin{bmatrix} u(t-2) \dots u(t-s-1) \\ 0 \quad \dots \quad 0 \\ \vdots \quad\quad \vdots \\ 0 \quad \dots \quad 0 \\ \hline -y(t-2) \dots -y(t-s-1) \\ 0 \quad \dots \quad 0 \\ \vdots \quad\quad \vdots \\ 0 \quad \dots \quad 0 \\ \hline 0 \quad \dots \quad 0 \\ \vdots \quad\quad \vdots \\ 0 \quad \dots \quad 0 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_s \end{bmatrix} + \begin{bmatrix} u(t-1) \\ 0 \\ \vdots \\ 0 \\ \hline -y(t-1) \\ 0 \\ \vdots \\ 0 \\ \hline -r(t-1) \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

(2.3-27)

Alike (2.2-27) the Hessian matrix consists here, too, of two parts. Now

$$\underline{\underline{T}}_1 = \underline{\underline{F}}^{*T}_{uye}\ \underline{\underline{H}}^{T}\ \underline{\underline{H}}\ \underline{\underline{F}}^{*}_{uye} = \sum_{t=1}^{N} \underline{f}(u^F, y^F, r, t)\underline{f}^T(u^F, y^F, r, t), \tag{2.3-28}$$

the construction of $\underline{\underline{T}}_2$ is the following:

$$\underline{\underline{T}}_2 = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{T}}_{bh} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{T}}_{ah} \\ \underline{\underline{T}}_{hb} & \underline{\underline{T}}_{ha} & \underline{\underline{0}} \end{bmatrix}. \tag{2.3-29}$$

Here $\underline{\underline{T}}_{bh} = \underline{\underline{T}}^{T}_{hb}$ and $\underline{\underline{T}}_{ah} = \underline{\underline{T}}^{T}_{ha}$. Only the first row and last column of the matrices $\underline{\underline{T}}_{bh}$ and $\underline{\underline{T}}_{ab}$ contain informative new elements. For the other elements it is valid that $t_{ij} = t_{(i-1),(j+1)}$. So a $2(2n-1)$ dimensional vector is sufficient for storing the elements of $\underline{\underline{T}}_2$. The construction of $\underline{\underline{T}}_2$ can be understood on the basis of the second derivatives of $\underline{e}$ by parameters, as

$$\frac{\partial^2 \underline{e}}{\partial b_i\, b_j} = \underline{0};\quad \frac{\partial^2 \underline{e}}{\partial b_i \partial a_j} = \underline{0};\quad \frac{\partial^2 \underline{e}}{\partial a_i \partial a_j} = \underline{0};\quad \frac{\partial^2 \underline{e}}{\partial h_i \partial h_j} = \underline{0}$$

$$\frac{\partial^2 \underline{e}}{\partial h_j \partial b_i} = -\underline{\underline{S}}^{i}\ \frac{\partial \underline{\underline{H}}}{\partial h_j}\ \underline{u} = -\underline{\underline{S}}^{i+j}\ \underline{u} \tag{2.3-30}$$

$$\frac{\partial^2 \underline{e}}{\partial h_j \partial a_i} = \underline{\underline{S}}^{i}\ \frac{\partial \underline{\underline{H}}}{\partial h_j}\ \underline{y} = \underline{\underline{S}}^{i+j}\ \underline{y}$$

and t_{ij} is calculated according to (2.2-36). On the basis of the latter equations the construction of the $2(2n-1)$ dimensional vector facilitating the calculation of $\underline{\underline{T}}_2$ is:

$$\underline{g}(t)=\left[u(t-2),\ldots,u(t-2n);-y(t-2),\ldots,-y(t-2n)\right]^T \qquad (2.3\text{-}31)$$

and the filling of $\underline{\underline{T}}_2$ is carried out on basis of the vector

$$\underline{q} = -\sum_{t=1}^{N} \underline{g}(t)e(t)=\left[\underline{q}_{bh}^T,\ \underline{q}_{ah}^T\right]^T \qquad (2.3\text{-}32)$$

according to the procedure presented with (2.2-35). With the help of the above relations also in case of the GLS structure the NEWTON-RAPHSON extremum seeking method can be realized. This yields faster convergence than the original CLARKE procedure, but at the price of much more work of computation. It may be reasonable to reduce the calculations by the approximation $\underline{\underline{T}} \cong \underline{\underline{T}}_1$ as $\underline{\underline{T}}_2$ disappears asymptotically, thus in case of sufficiently large number of samples our approximation will already be justified.

If we do not need separately the parameters $\underline{p}_h$ and our aim is only the determination of an equivalent model, then we can proceeed in a much simpler way. The system equation (2.3-1) can be reduced by simple transformation to the form

$$y(t)= \frac{B(z^{-1})H(z^{-1})}{A(z^{-1})H(z^{-1})}\,u(t) + \frac{1}{A(z^{-1})H(z^{-1})}\,e(t)=$$

$$= \frac{P(z^{-1})}{Q(z^{-1})}\,u(t) + \frac{1}{Q(z^{-1})}\,e(t) \qquad (2.3\text{-}33)$$

i.e. to a system equation corresponding to the LS structure. Therefore, in case of increased order, also the simple LS estimation yields an equivalent system model with the GLS

structure. The $P(z^{-1})/Q(z^{-1})$ - which contains now common factors in the numerator and denominator - can be reduced by appropriate numerical methods to the minimum number of parameters.

Thus the GLS structure uses instead of the autoregressive noise model $1/A(z^{-1})$ of the LS structure a more complex, but likewise autoregressive noise model $1/A(z^{-1}).H(z^{-1})$. The only advantage of this expansion can be to achieve with this latter noise model, by choosing the order of $H(z^{-1})$ for sufficiently large, a good approximation of the general noise model $C(z^{-1})/D(z^{-1})$.

We have to mention the second CLARKE's method by which he recommends in the particular iteration steps not the filtering of the original data according to (2.3-4), but always the renewed filtering of the previously already filtered values. This method corresponds to the autoregressive noise model of the type

$$\frac{C(z^{-1})}{D(z^{-1})} = \frac{1}{A(z^{-1}) \prod_{i=1}^{\infty} H_i(z^{-1})} \quad . \qquad (2.3\text{-}34)$$

In many cases this model approximates better the general noise model and its application is more practical.

The conditions of the convergence of the GLS method were dealt in detail by SÖDERSTRÖM [123].

2.4 The extended matrix methods. Quasilinearization

In subsection 1.2 the linearity of the models in parameter was underlined several times. SMETS [120] and later TALMON [130, 131] have utilized this advantage in developing the extended matrix methods.

SMETS elaborated the so-called "First Extended Matrix Method" (hereinafter FEXM) for the system equation (2.3-1) of the GLS structure. On the basis of the vector equation (2.3-7) it can be written that

$$\underline{y} = \underline{\underline{H}}\, \underline{\underline{F}}_{uy}\, \underline{p}_{ba} - \underline{\underline{\tilde{H}}}\, \underline{y} + \underline{e} = \underline{\underline{F}}_{uy}\, \underline{p}_{ba} - \underline{\underline{\tilde{H}}}(\underline{y} - \underline{\underline{F}}_{uy}\, \underline{p}_{ba}) + \underline{e} =$$

$$= \underline{\underline{F}}_{uy}\, \underline{p}_{ba} - \underline{\underline{\tilde{H}}}\, \underline{r} + \underline{e} = [\underline{\underline{F}}_{uy}, -\underline{\underline{G}}_{r}]\, \underline{p}_{bah} + \underline{e} =$$

$$= \underline{\underline{F}}_{uyr}\, \underline{p}_{bah} + \underline{e} = \underline{\underline{F}}(\underline{u}, \underline{y}, \underline{r})\, \underline{p}_{bah} + \underline{e} \qquad (2.4\text{-}1)$$

where

$$\underline{\underline{F}}(\underline{u}, \underline{y}, \underline{r}) \equiv \underline{\underline{F}}_{uyr} = [\underline{\underline{F}}_{uy}, -\underline{\underline{G}}_{r}]. \qquad (2.4\text{-}2)$$

The definition of $\underline{\underline{\tilde{H}}}$, $\underline{\underline{G}}_{r}$, as well as $\underline{r}$ and $\underline{p}_{bah}$ can be found in subsection 2.3, the other notations have occurred already several times.

The system equation linear in parameters (2.4-1) formally exactly corresponds to the form applicable for LS estimation, so that $\underline{p}_{bah}$ is the LS estimation of the joint parameter vector.

$$\hat{\underline{p}}_{bah} = (\underline{\underline{F}}_{uyr}^{T}\, \underline{\underline{F}}_{uyr})^{-1}\, \underline{\underline{F}}_{uyr}^{T}\, \underline{y}. \qquad (2.4\text{-}3)$$

In $\underline{\underline{F}}_{uyr}$ the value $\underline{r}$ is required which, of course, is not known in advance, therefore $\underline{r}$ can only be calculated, i.e. estimated. This leads necessarily to an iterative solution. A possible construction of the iterative solution is the following:

1. For vector $\underline{p}_{ba}$ a LS estimation is performed on the basis of the relation (2.1-7)
2. On the basis of (2.3-14) by taking the obtained $\hat{\underline{p}}_{ba}$, the values $\hat{r}(t)$ and thus $\hat{\underline{r}}$, too are calculated. By using $\hat{\underline{r}}$ the auxiliary quantities necessary for (2.4-3) are also computed.
3. On the basis of (2.4-3) the estimation $\hat{\underline{p}}_{bah}$ is determined thereafter by utilizing the subvector $\hat{\underline{p}}_{ba}$ of $\hat{\underline{p}}_{bah}$, the iteration is continued from 2. until the achievement of the required accuracy.

SMETS has compared in details the methods of CLARKE's GLS and his FEXM. The basic difference between the two methods is that CLARKE's method asymptotically converges to the classical GLS method of AITKEN, while SMETS' method tends towards a non optimal instrumental variables method hereinafter (IV).

The principle of the FEXM method was used by TALMON for the system equation (1.2-5) applied with the ML method. He called this method "Second EXtended Matrix" method (hereinafter SEXM).

The system equation (2.2-1) can be arranged into the following form:

$$\underline{y} = \underline{\underline{B}}\underline{u} - \underline{\underline{\tilde{A}}}\underline{y} + \underline{\underline{\tilde{C}}}\underline{e} + \underline{e} = \underline{\underline{F}}_{uye}\underline{p}_{bac} + \underline{e} =$$

$$= \underline{\underline{F}}(\underline{u},\underline{y},\underline{e})\underline{p}_{bac} + \underline{e} \equiv \underline{\underline{F}}_{uye}\,\underline{p}_{bac} + \underline{e} \,. \qquad (2.4\text{-}4)$$

As this system equation is also linear in parameters with a white noise equation error, we can again use the LS estimation according to which

$$\hat{\underline{p}}_{bac} = (\underline{\underline{F}}_{uye}^{T} \underline{\underline{F}}_{uye})^{-1} \underline{\underline{F}}_{uye}^{T} \underline{y}. \qquad (2.4\text{-}5)$$

Alike FEXM, here also an iterative solution is only possible, as $\underline{e}$ had to be known to form $\underline{\underline{F}}(\underline{u},\underline{y},\underline{e})$. A possible iteration scheme is the following.

1. LS estimation is made for the vector $\hat{\underline{p}}_{ba}$ according to (2.1-7). $\hat{\underline{p}}_{ba}$ is put in the corresponding part of $\hat{\underline{p}}_{bac}$, the values of the c_i-s are still zeros.
2. The e(t) i.e. $\underline{e}$ values belonging to the given $\hat{\underline{p}}_{bac}$, are calculated by Eq. (2.2-6). Let us form the auxiliary quantities required for (2.4-5).
3. The estimation $\hat{\underline{p}}_{bac}$ is calculated according to (2.4-5), thereafter the iteration is continued from point 2. until the attainment of a given accuracy.

The detailed analysis of the method can be found in [131].

TALMON has applied the principle of matrix extension, i.e. the consideration of linearity in parameters also for the general SISO system equations (1.2-1) - (1.2-2). This method is called "Third EXtended Matrix" method (hereinafter TEXM). This report will not advert to it, as the ML estimation of the general form (1.2-1) has not been discussed either.

In the following we intend to represent another method connected with those discussed in this subsection in the sense that it takes advantage of the linearity in parameters, resp. carries out in a certain sense quasilinearization. Already ÅSTRÖM [12, 13] pointed at the possibility of quasilinearization, but the following method was suggested by FURHT [49].

It is well known that while we perform the ML identification of the system (1.2-5), we minimize a prediction error relating to a fictitious output signal

$$v_f(t) = \frac{B(z^{-1})u(t) - [A(z^{-1}) - C(z^{-1})]\,y(t)}{C(z^{-1})} \qquad (2.4\text{-}6)$$

(See rf. [49].)

Although FURHT carried out the deduction of his method, in another way, we are applying now a train of thoughts enabling us to further generalizations. Let us write the system equation (2.2-1) in the form

$$\underline{e} = \underline{\underline{C}}^{-1}(\underline{\underline{A}}\underline{y} - \underline{\underline{B}}\underline{u}) = \underline{\underline{A}}\,\underline{\underline{C}}^{-1}\,\underline{y} - \underline{\underline{B}}\,\underline{\underline{C}}^{-1}\underline{u} = \underline{\underline{A}}\underline{y}^F - \underline{\underline{B}}\underline{u}^F \qquad (2.4\text{-}7)$$

where the fact has been considered that the multiplication by $\underline{\underline{C}}^{-1}$ consisting of the sum of shift matrices is interchangeable (cf. App. 2-20), further that now

$$\underline{y}^F = \underline{\underline{C}}^{-1}\underline{y} \quad \text{and} \quad \underline{u}^F = \underline{\underline{C}}^{-1}\underline{u}. \qquad (2.4\text{-}8)$$

The scalar autoregressive filtering corresponding to these latter equations means the signal formation according to

$$y^F(t) = \frac{1}{C(z^{-1})}\,y(t) \quad \text{and} \quad u^F(t) = \frac{1}{C(z^{-1})}\,u(t). \qquad (2.4\text{-}9)$$

By an additional formation of (2.4-7), we obtain

$$\begin{aligned}
\underline{e} &= \underline{\underline{A}}\underline{y}^F - \underline{\underline{B}}\underline{u}^F + \underline{y} - \underline{v} - \underline{\underline{C}}\underline{\underline{C}}^{-1}(\underline{y}-\underline{v}) = \\
&= \underline{\underline{A}}\underline{y}^F - \underline{\underline{B}}\underline{u}^F + \underline{y} - \underline{v} - \underline{\underline{C}}\underline{y}^F + \underline{\underline{C}}\underline{v}^F = \\
&= \underline{y}^F + \underline{\underline{\tilde{A}}}\underline{y}^F - \underline{\underline{B}}\underline{u}^F + \underline{y} - \underline{v} - \underline{y}^F - \underline{\underline{\tilde{C}}}(\underline{y}^F - \underline{v}^F) + \underline{v}^F = \\
&= (\underline{y} - \underline{v} + \underline{v}^F) - [\underline{\underline{B}}\underline{u}^F - \underline{\underline{\tilde{A}}}\underline{y}^F + \underline{\underline{\tilde{C}}}(\underline{y}^F - \underline{v}^F)],
\end{aligned} \qquad (2.4\text{-}10)$$

where $\underline{v}$ is an arbitrary (Nx1) vector.

From (2.4-10) we obtain the equation

$$\underline{y}_f = \underline{y} - \underline{v} + \underline{v}^F = \underline{\underline{B}}\underline{u}^F - \underline{\underline{\tilde{A}}}\underline{y}^F + \underline{\underline{\tilde{C}}}(\underline{y}^F - \underline{v}^F) + \underline{e} =$$

$$= \underline{\underline{F}}(\underline{u}^F, \underline{y}^F, \underline{y}^F - \underline{v}^F)\, \underline{p}_{bac} + \underline{e} \qquad (2.4\text{-}11)$$

where $\underline{y}_f$ is a (Nx1) vector containing values of not a really measured output signal. The construction of the matrix $\underline{\underline{F}}(\underline{u}^F, \underline{y}^F, \underline{y}^F - \underline{v}^F)$ is

$$\underline{\underline{F}}(\underline{u}^F, \underline{y}^F, \underline{y}^F - \underline{v}^F) = \left[\underline{\underline{S}}\underline{u}^F, \ldots, \underline{\underline{S}}^n \underline{u}^F;\ -\underline{\underline{S}}\underline{y}^F, \ldots, -\underline{\underline{S}}^n \underline{y}^F;\ \underline{\underline{S}}(\underline{y}^F - \underline{v}^F), \right.$$

$$\left. \ldots,\ \underline{\underline{S}}^n (\underline{y}^F - \underline{v}^F) \right] =$$

$$= \begin{bmatrix} \underline{f}^T(u^F, y^F, y^F - v^F, 1) \\ \vdots \\ \underline{f}^T(u^F, y^F, y^F - v^F, N) \end{bmatrix} \qquad (2.4\text{-}12)$$

where

$$\underline{f}(u^F, y^F, y^F - v^F, t) = \left[u^F(t-1), \ldots, u^F(t-n); -y^F(t-1), \ldots, -y^F(t-n); \right.$$

$$\left. y^F(t-1) - v^F(t-1), \ldots, y^F(t-n) - v^F(t-n) \right]^T. \qquad (2.4\text{-}13)$$

As in (2.4-11) the equation error is white noise, we can again apply the LS estimation, whose result is

$$\hat{\underline{p}}_{bac} = \left[\underline{\underline{F}}^T(\underline{u}^F, \underline{y}^F, \underline{y}^F - \underline{v}^F) \underline{\underline{F}}(\underline{u}^F, \underline{y}^F, \underline{y}^F - \underline{v}^F) \right]^{-1} *$$

$$* \underline{\underline{F}}^T(\underline{u}^F, \underline{y}^F, \underline{y}^F - \underline{v}^F) \underline{y}_f. \qquad (2.4\text{-}14)$$

In (2.4-14) FURHT has chosen for $\underline{v}$ the fictive model output

$$\underline{v}_f = \underline{\underline{C}}^{-1}[\underline{\underline{B}}\underline{u} - (\underline{\underline{A}}-\underline{\underline{C}})\underline{y}] \qquad (2.4\text{-}15)$$

which can be calsulated on the basis of (2.4-6). Starting from the LS estimation, we can - alike the extended matrix methods - construct here also an iterative estimation scheme, where the calculated values of $\underline{v}$ have to be updated step by step.

We have chosen an arbitrary $\underline{v}$ in the deduction in order to indicate that the solution proposed by FURHT is not the only one, but a lot of other approaches can also be formed. Another possible approach is e.g. to take the noiseless output of the process for $\underline{v}$, for it can be obtained in a simpler form than $\underline{v}_f$, in the form of

$$\underline{v} = \underline{\underline{A}}^{-1}\,\underline{\underline{B}}\underline{u} \qquad (2.4\text{-}16)$$

i.e.

$$v(t) = \frac{B(z^{-1})}{A(z^{-1})}\,u(t) \qquad (2.4\text{-}17)$$

We obtain an even more simple possibility of quasilinearization on the basis of Eq. (2.4-7):

$$\underline{y}^F = \underline{\underline{B}}\underline{u}^F - \underline{\underline{\tilde{A}}}\underline{y}^F + \underline{e} = \underline{\underline{F}}(\underline{u}^F,\underline{y}^F)\underline{p}_{ba} + \underline{e}, \qquad (2.4\text{-}18)$$

where now $\underline{y}^F$ and $\underline{u}^F$ are according to (2.4-8). As the equation error is also here white noise, the LS estimation can be applied:

$$\hat{\underline{p}}_{ba} = [\underline{\underline{F}}^T(\underline{u}^F,\underline{y}^F)\underline{\underline{F}}(\underline{u}^F,\underline{y}^F)]^{-1}\,\underline{\underline{F}}^T(\underline{u}^F,\underline{y}^F)\underline{y}^F. \qquad (2.4\text{-}19)$$

On the other hand, the calculation of the $\underline{u}^F$ and $\underline{y}^F$ containing the filtered values, requires also the c_i parameters. We are presenting now a simple method for the estimation of the coefficients of the polynomial $\underline{\underline{C}}(z^{-1})$. We can write the equation (2.4-7) also in the form of

$$(\underline{\underline{A}}\underline{y} - \underline{\underline{B}}\underline{u}) = \underline{r} = \underline{\underline{C}}\underline{e} = \tilde{\underline{\underline{C}}}\underline{e} + \underline{e} = \underline{\underline{G}}(\underline{e})\underline{p}_c + \underline{e} \equiv \underline{\underline{G}}_e\underline{p}_c + \underline{e} \tag{2.4-20}$$

where

$$\underline{p}_c = [c_1, \ldots, c_n]^T \tag{2.4-21}$$

and

$$\underline{\underline{G}}(\underline{e}) = [\underline{\underline{S}}\underline{e}, \ldots, \underline{\underline{S}}^n\underline{e}] = \begin{bmatrix} \underline{g}^T(e,1) \\ \vdots \\ \underline{g}^T(e,N) \end{bmatrix}. \tag{2.4-22}$$

Here

$$\underline{g}(e,t) = [e(t-1), \ldots, e(t-n)]^T. \tag{2.4-23}$$

The calculation of the signals $r(t)$ and $e(t)$ with a given $\hat{p}_{ba}$, resp. $\hat{\underline{p}}_{bac}$ (known from a previous iterative step) is performed on the basis of Eqs. (2.3-14) and (2.2-6). The iterative procedure agrees, according to the meaning, with what was discussed in respect of the extended matrix methods. (In the practice the values a_i proved to be fairly good initial values for the coefficients c_i as in this case $C(z^{-1}) \equiv A(z^{-1})$, which corresponds to a white noise output error. The $C(z^{-1})$ corresponding to the real noise transfer function can be easier derived, hence than from the much more gross $C(z^{-1}) \equiv 1$ approximation, since this latter supposes a special noise

spectrum as a start).

Among the methods of quasilinearization have to be mentioned those of STIEGLITZ and McBRIDE [129] which also take advantage of the linearity in parameters of the system equation and perform LS estimation by iterations. Their method was elaborated for the white noise output error, i.e. when the system equation has the form

$$y(t) = \frac{B(z^{-1})}{A(z^{-1})}\, u(t) + e(t). \qquad (2.4\text{-}24)$$

For N samples

$$\underline{y} = \underline{\underline{A}}^{-1}\, \underline{\underline{B}}\underline{u} + \underline{e} = \underline{\underline{B}}\underline{u} - \tilde{\underline{\underline{A}}}\underline{y} + \underline{\underline{A}}\underline{e} =$$

$$= \underline{\underline{F}}(u,y)\, \underline{p}_{ba} + \underline{\underline{A}}\underline{e} \equiv \underline{\underline{F}}_{uy}\, \underline{p}_{ba} + \underline{\underline{A}}\underline{e}. \qquad (2.4\text{-}25)$$

This latter equation is linear in parameters, with a moving average equation error. By a simple transformationwe get the equation

$$\underline{\underline{A}}^{-1}\underline{y} \equiv \underline{y}^F = \underline{\underline{B}}\underline{\underline{A}}^{-1}\underline{u} - \tilde{\underline{\underline{A}}}\underline{\underline{A}}^{-1}\, \underline{y} + \underline{e} = \underline{\underline{B}}\underline{u}^F - \tilde{\underline{\underline{A}}}\underline{y}^F + \underline{e} =$$

$$= \underline{\underline{F}}(\underline{u}^F, \underline{y}^F)\underline{p}_{ba} + \underline{e} \qquad (2.4\text{-}26)$$

where the equation error is already white noise. Here the equations

$$\underline{y}^F = \underline{\underline{A}}^{-1}\underline{y} \quad \text{and} \quad \underline{u}^F = \underline{\underline{A}}^{-1}\, \underline{u} \qquad (2.4\text{-}27)$$

mean the filtering, and the corresponding scalar equations

$$y^F(t) = \frac{1}{A(z^{-1})}\, y(t) \quad \text{and} \quad u^F(t) = \frac{1}{A(z^{-1})}\, u(t). \qquad (2.4\text{-}28)$$

On the basis of (2.4-26) the LS equation can be got in the form

$$\hat{\underline{p}}_{ba} = [\underline{\underline{F}}^T(\underline{u}^F,\underline{y}^F)\ \underline{\underline{F}}(\underline{u}^F,\underline{y}^F)]^{-1}\ \ \underline{\underline{F}}^T(\underline{u}^F,\underline{y}^F)\underline{y}^F \qquad (2.4\text{-}29)$$

for the parameters of the system. It is now very simple to obtain an iterative scheme:

1. An LS estimation is made for $\underline{p}_{ba}$ with the assumption $\underline{y}^F=\underline{y}$ and $\underline{u}^F=\underline{u}$, i.e. $A(z^{-1})=1$ in the filtering equations.
2. The filtered values or other auxiliary quantities are calculated.
3. The estimation according to (2.4-29) is performed, thereafter from point 2. the procedure is contained until the attainment of the given accuracy.

The authors proposed two more procedures which accelerate the convergence near the minimum point. The necessity of such a correction can be understood in the following way. Let us write the derivatives of

$$\underline{e} = \underline{y} - \underline{\underline{A}}^{-1}\ \underline{\underline{B}}\underline{u} \qquad (2.4\text{-}30)$$

by the particular parameters:

$$\frac{\partial \underline{e}}{\partial b_i} = -\underline{\underline{A}}^{-1}\ \ \frac{\partial \underline{\underline{B}}}{\partial b_i}\ \underline{u} = -\underline{\underline{A}}^{-1}\ \underline{\underline{S}}^i\underline{u} = -\underline{\underline{S}}^i\ \underline{\underline{A}}^{-1}\underline{u} = -\underline{\underline{S}}^i\underline{u}^F \qquad (2.4\text{-}31)$$

$$\frac{\partial \underline{e}}{\partial a_i} = -\ \frac{\partial \underline{\underline{A}}^{-1}}{\partial a_i}\ \underline{\underline{B}}\underline{u} = \underline{\underline{A}}^{-1}\ \frac{\partial \underline{\underline{A}}}{\partial a_i}\ \underline{\underline{A}}^{-1}\ \underline{\underline{B}}\underline{u} = \underline{\underline{A}}^{-1}\ \underline{\underline{S}}^i\ \underline{v} = \underline{\underline{S}}^i\ \underline{v}^F. \qquad (2.4\text{-}32)$$

Here

$$\underline{v} = \underline{\underline{A}}^{-1}\ \underline{\underline{B}}\ \underline{u} \qquad (2.4\text{-}33)$$

contains the values of the noiseless output signal and

$$\underline{v}^F = \underline{\underline{A}}^{-1}\,\underline{v}. \qquad (2.4\text{-}34)$$

The derivative of the loss function of the ML method is therefore

$$\frac{dQ(\underline{p}_{ba})}{d\underline{p}_{ba}} = \frac{d\underline{e}^T}{d\underline{p}_{ba}}\,\underline{e} = -\,\underline{\underline{F}}^T(\underline{u}^F,\underline{v}^F)\underline{e} =$$

$$= -\underline{\underline{F}}^T(\underline{u}^F,\underline{v}^F)\left[\underline{y}^F - \underline{\underline{F}}(\underline{u}^F,\underline{y}^F)\underline{p}_{ba}\right] \qquad (2.4\text{-}35)$$

where we have considered the equation (2.4-26) further

$$\underline{\underline{F}}(\underline{u}^F,\underline{y}^F) = \left[\underline{\underline{S}}\,\underline{u}^F,\ldots,\underline{\underline{S}}^n\underline{u}^F;\ -\underline{\underline{S}}\underline{v}^F,\ldots,-\underline{\underline{S}}^n\underline{v}^F\right] \qquad (2.4\text{-}36)$$

and

$$\underline{\underline{F}}(\underline{u}^F,\underline{y}^F) = \left[\underline{\underline{S}}\,\underline{u}^F,\ldots,\underline{\underline{S}}^n\underline{u}^F;\ -\underline{\underline{S}}\underline{y}^F,\ldots,-\underline{\underline{S}}^n\underline{y}^F\right]. \qquad (2.4\text{-}37)$$

(This latter has already been mentioned several times above.)

By making the gradient (2.4-35) equal to zero vector, we get

$$\hat{\underline{p}}_{ba} = \left[\underline{\underline{F}}^T(\underline{u}^F,\underline{v}^F)\underline{\underline{F}}(\underline{u}^F,\underline{y}^F)\right]^{-1}\ \underline{\underline{F}}^T(\underline{u}^F,\underline{v}^F)\underline{y}^F, \qquad (2.4\text{-}38)$$

which corresponds formally to an IV estimation, as it is easy to see. The estimation (2.4-38) makes the gradient (under given $A(z^{-1})$) equal to zero in every step, but it can be applied only if we have got a fairly good estimation for $\underline{v}$, i.e. in the vicinity of the optimum.

As we can express $\underline{e}$ on the basis of (2.4-25), also in the following way:

$$\underline{e} = \underline{y} - \underline{\underline{A}}^{-1}\ \underline{\underline{B}}\ \underline{u} = \underline{y} - \underline{v} = \underline{y} - (\underline{\underline{B}}\ \underline{u} - \tilde{\underline{\underline{A}}}\ \underline{v}) =$$

$$= -\underline{y} - \underline{\underline{F}}(\underline{u},\underline{v})\ \underline{p}_{ba}, \qquad (2.4\text{-}39)$$

so substituting this into (2.4-35), we get the LS estimation

$$\hat{\underline{p}}_{ba} = \left[\underline{\underline{F}}^{T}(\underline{u}^{F},\underline{v}^{F})\ \underline{\underline{F}}(\underline{u},\underline{v})\right]^{-1}\ \underline{\underline{F}}^{T}(\underline{u}^{F},\underline{v}^{F})\underline{y}\ . \qquad (2.4\text{-}40)$$

While the solution (2.4-29) is an asymptotically GLS estimation, the relations (2.4-38) and (2-4.40) mean an IV estimation each.

Experiences shows that these procedures are preferably to be applied sequentially, in order to achieve more an more exact results. For changes in order heuristic strategies can be designed.

2.5 The method of instrumental variables

According to the method of "Instrumental Variables" (hereinafter IV) in case of a system equation linear in parameters the equation

$$\hat{\underline{p}} = \left[\underline{\underline{G}}^T(\underline{u},\underline{v})\ \underline{\underline{F}}(\underline{u},\underline{y})\right]^{-1}\ \underline{\underline{G}}^T(\underline{u},\underline{v})\underline{y} =$$

$$= (\underline{\underline{G}}_{uv}^T\ \underline{\underline{F}}_{uy})^{-1}\ \underline{\underline{G}}_{uv}^T\ \underline{y} \qquad (2.5\text{-}1)$$

yields an asymptotically unbiased parameter estimation if the conditions

$$\operatorname*{plim}_{N\to\infty}\left(\frac{1}{N}\ \underline{\underline{G}}_N^T\ \underline{\underline{F}}_N\right) = \underline{\underline{S}}_3 < \infty \qquad \text{and} \qquad \operatorname*{plim}_{N\to\infty}\left(\frac{1}{N}\ \underline{\underline{G}}_N^T\ \underline{r}_N\right) = \underline{0} \qquad (2.5\text{-}2)$$

are fulfilled where N subscript indicates the number of the samples. In (2.5-2) the notation $\underline{r}_N$ whose value is

$$\underline{r}_N = \underline{y}_N - \underline{\underline{F}}_N\underline{p} = \underline{y} - \underline{\underline{F}}(\underline{u},\underline{y})\underline{p}, \qquad (2.5\text{-}3)$$

emphasizes still more that the asymptotic unbiasedness depends on the fact whether the elements of $\underline{\underline{G}}$ are uncorrelated with the equation error or not.

The IV method was dealt with in detail by POLLAK and WONG, who deduced a great number of theorems on estimation properties. Equations for the identification of discrete-time SISO systems were transformed by YOUNG [143] to a form easy to be used.

The general SISO system equation (1.2-1) can be also written in the following form linear in parameters

$$y(t) = \underline{f}^T(u,y,t)\underline{p}_{ba} + A(z^{-1})w(t) \qquad (2.5\text{-}4)$$

or for N samples

$$\underline{y} = \underline{\underline{F}}(\underline{u},\underline{y})\ \underline{p}_{ba} + \underline{\underline{A}}^{-1}\ \underline{w}. \qquad (2.5\text{-}5)$$

The IV estimation of $\underline{p}_{ba}$ is yielded by

$$\hat{\underline{p}}_{ba} = \left[\underline{\underline{G}}^{T}(\underline{u},\underline{v})\ \underline{\underline{F}}(\underline{u},\underline{y})\right]^{-1}\ \underline{\underline{G}}^{T}(\underline{u},\underline{v})\ \underline{y} \qquad (2.5\text{-}6)$$

where

$$\underline{\underline{G}}(\underline{u},\underline{v})=\left[\underline{\underline{S}}\underline{u},\ldots,\underline{\underline{S}}^{n}\underline{u};\ -\underline{\underline{S}}\underline{v},\ldots,-\underline{\underline{S}}^{n}\ \underline{v}\right]. \qquad (2.5\text{-}7)$$

(cf. Appendix F.5.)

Here

$$\underline{v} = \underline{\underline{A}}^{-1}\ \underline{\underline{B}}\underline{u}\ = \underline{\underline{B}}\underline{u} - \tilde{\underline{\underline{A}}}\underline{v} = \underline{\underline{G}}(\underline{u},\underline{v})\underline{p}_{ba}, \qquad (2.5\text{-}8)$$

i.e. contains the values of the noiseless output signal of the system. As the equation error $A(z^{-1})w(t)$ depends only on the uncorrelated $e(t)$ source noise, the elements of $\underline{\underline{G}}(\underline{u},\underline{v},) = \underline{\underline{G}}(\underline{u},\underline{\underline{A}}^{-1}\underline{\underline{B}}\underline{u})$ will be uncorrelated with the equation error because they are only the functions or $u(t)$ uncorrelated with $e(t)$. Note that the described IV estimation yields the asymptotically unbiased estimation of the process parameters for every noise model of Table 1.2-I.

The quasilinearization of the equation (1.2-2) of the general noise model enables an iterative scheme to be designed also for the estimation of the noise parameters. This noise model type, however, - alike the third extended matrix method - will not be discussed in this paper.

The IV estimation (2.5-6) can, of course, be realized only by an iteration technique as $\underline{v}$ can not be produced without knowing the parameters. A possible iterative solution is the following:

1. Simple LS estimation for $\underline{p}_{ba}$
2. Calculate $\underline{v}$ with the help of $\hat{\underline{p}}_{ba}$ and for the auxiliary quantities required for (2.5-6)
3. Perform the IV estimation according to (2.5-6), thereupon return to 2 and continue the iteration until the achievement of the given accuracy.

Note that also with the IV algorithm such a convergence accelerating process can be designed, as with the STIEGLITZ - McBRIDE method, when we make the gradient zero in every step.

2.6 The priori knowledge fitting method

Apart from the LS method, all other ones until now contained iterative procedures. By using only those assumptions had been applied heretofore, PETERKA and his co-workers elaborated an explicit method [102, 103, 104] which yields the estimation of the process parameters in a single step. The process model applied is

$$y(t) = \frac{B(z^{-1})}{A(z^{-1})}\, u(t) + w(t) \qquad (2.6\text{-}1)$$

so that it formally agrees with the previous models. But it is sufficient to assume that $w(t)$ is a stationary random signal series, having zero mean and uncorrelated with the input signal. On the other hand, it yields a substantial ease in the assumptions regarding the noise that the type of the distribution is indifferent and there is no need to assume a given structure for the noise model.

The method elaborated by PETERKA uses the uncorrelatedness of the input signal and the output noise. The name of the method comes from the fact that the fitting of the model strongly depends upon the prior knowledge. ("Priori Knowledge Fitting", hereinafter PKF). Thus we used to speak of a method using prior knowledge, after its English abbreviation PKF method, or a method based on the principle of tallying the uncorrelatedness of the input signal and output noise. ("Tally Principle" = TP) [103].

The vector form of Eq. (2.6-1) for N samples:

$$\underline{y} = \underline{\underline{A}}^{-1}\, \underline{\underline{B}}\, \underline{u} + \underline{w} = \underline{\underline{B}}\, \underline{u} - \tilde{\underline{\underline{A}}}\, \underline{y} + \underline{\underline{A}}\, \underline{w} = \underline{\underline{F}}(\underline{u},\underline{y})\underline{p}_{ba} + \underline{r}, \qquad (2.6\text{-}2)$$

when the notations used heretofore were applied and $\underline{r}$, the vector of the residuals

$$\underline{r} = \underline{\underline{A}}\,\underline{w} = \underline{y} - \underline{\underline{F}}(\underline{u},\underline{y})\underline{p}_{ba} = \underline{y} - \underline{\underline{F}}_{uy}\,\underline{p}_{ba} \,. \qquad (2.6\text{-}3)$$

As w(t) and u(t) are assumed to be uncorrelated, it follows from (2.6-3) that r(t) and u(t) are uncorrelated, too. Assume the opposite i.e. that in the difference $\underline{y}-\underline{\underline{F}}_{uy}\underline{p}_{ba}$ yielding the residuals terms dependent on $\underline{u}$ or yielding correlation with $\underline{u}$ occur. This dependence is assumed to be linear, then according to our latter statement the relations

$$\underline{y} - \underline{\underline{F}}_{uy}\,\hat{\underline{p}}_{ba} = \underline{r} + \underline{\underline{K}}\underline{u} = \underline{r} + \underline{\underline{M}}(\underline{u})\underline{k} \equiv \underline{r}+\underline{\underline{M}}_{u}\,\underline{k} \qquad (2.6\text{-}4)$$

should have to be fulfilled. Let here be

$$\underline{\underline{K}} = \sum_{i=1}^{\nu} \underline{\underline{S}}^{i}\, k_i \qquad (2.6\text{-}5)$$

$$\underline{k} = \left[k_1,\ldots,k_\nu\right]^T, \qquad (2.6\text{-}6)$$

and

$$\underline{\underline{M}}(\underline{u})\equiv\underline{\underline{M}}_{u} = \begin{bmatrix} \underline{m}^T(\underline{u},1) \\ \vdots \\ \underline{m}^T(\underline{u},N) \end{bmatrix} = \left[\underline{\underline{S}}\underline{u},\ldots,\underline{\underline{S}}^{\nu}\underline{u}\right], \qquad (2.6\text{-}7)$$

where

$$\underline{m}(\underline{u},t)=\left[u(t-1),\ldots,u(t-\nu)\right]^T. \qquad (2.6\text{-}8)$$

If, on the other hand, the uncorrelatedness exists in fact, then we must get such a $\hat{\underline{p}}_{ba}$ estimation, for which $\underline{k} = \underline{0}$ or at least

$$Q(\underline{p}_{ba})= \frac{1}{2}\,\underline{k}^T\,\underline{k} \qquad (2.6\text{-}9)$$

is minimum.

On the basis of (2.6-4) we can write formally in a simple way the LS estimation of $\underline{k}$

$$\hat{\underline{k}} = (\underline{\underline{M}}_u^T \underline{\underline{M}}_u)^{-1} \underline{\underline{M}}_u^T (\underline{y} - \underline{\underline{F}}_{uy} \underline{p}_{ba}). \qquad (2.6\text{-}10)$$

By substituting the obtained $\hat{\underline{k}}$ into (2.6-9) and minimizing it according to $\underline{p}_{ba}$ (cf. Appendix 8) we get

$$\hat{\underline{p}}_{ba} = \left[\underline{\underline{F}}_{uy}^T \underline{\underline{M}}_u (\underline{\underline{M}}_u^T \underline{\underline{M}}_u)^{-1} (\underline{\underline{M}}_u^T \underline{\underline{M}}_u)^{-1} \underline{\underline{M}}_u^T \underline{\underline{F}}_{uy} \right]^{-1} *$$

$$* \; \underline{\underline{F}}_{uy}^T \underline{\underline{M}}_u (\underline{\underline{M}}_u^T \underline{\underline{M}}_u)^{-1} (\underline{\underline{M}}_u^T \underline{\underline{M}}_u)^{-1} \underline{\underline{M}}_u^T \underline{y}. \qquad (2.6\text{-}11)$$

The obtained solution gives an explicit expression for the estimation of $\hat{\underline{p}}_{ba}$ without any iterative process. The asymptotic properties of the estimation were studied in detail in [103] (otherwise the estimation is asymptotically unbiased which is not proved here because of lack of space.)

The formula (2.6-11) can be considered as a least squares estimation weighted by the weighting matrix

$$\underline{\underline{M}}_u (\underline{\underline{M}}_u^T \underline{\underline{M}}_u)^{-1} (\underline{\underline{M}}_u^T \underline{\underline{M}}_u)^{-1} \underline{\underline{M}}_u^T$$

but also as an IV estimation where the auxiliary matrix is

$$\underline{\underline{F}}_{uy}^T \underline{\underline{M}}_u (\underline{\underline{M}}_u^T \underline{\underline{M}}_u)^{-1} (\underline{\underline{M}}_u^T \underline{\underline{M}}_u)^{-1} \underline{\underline{M}}_u^T .$$

Many authors are keen on using the PKF method although its accuracy is somewhat behind that of the ML method [62]. However, the method is credited with the extraordinary advantage of yielding an explicit solution. As one of its heaviset drawbacks has been considered, its failure to give an estimation for the parameters of the noise model. Herebelow an iteration method enabling also an ÅSTRÖM noise

model to be determined on the basis of the $\hat{\underline{p}}_{ba}$ parameter estimation obtained by the explicit PKF method will be presented. Accordingly assume for w(t) the existence of the relation

$$w(t) = \frac{C(z^{-1})}{A(z^{-1})}\, e(t) \qquad (2.6\text{-}12)$$

where the notations on the right side of the equation are already known. This means that on the basis of (2.6-2) the equation error will be according to

$$r(t) = C(z^{-1})e(t) = \tilde{C}(z^{-1})e(t) + e(t) \qquad (2.6\text{-}13)$$

where

$$C(z^{-1}) = 1 + c_1 z^{-1} + \ldots + c_n z^{-n} = 1 + \tilde{C}(z^{-1}). \qquad (2.6\text{-}14)$$

On the basis of $\underline{u}$ and $\underline{y}$ containing the measurements, as well as on the basis of the $\hat{\underline{p}}_{ba}$ estimation according to (2.6-11) the estimation of $\underline{r}$ can be determined from (2.6-3):

$$\hat{\underline{r}} = \underline{y} - \underline{\underline{F}}_{uy}\, \hat{\underline{p}}_{ba} \,. \qquad (2.6\text{-}15)$$

By taking the values r(t) on the basis of the equation (2.6-13) an iteration procedure consisting of repeated LS estimations can be formed for the estimation of the noise parameters

$$\underline{p}_c = [\, c_1, c_2, \ldots, c_n \,]^T \qquad (2.6\text{-}16)$$

in the form

$$\hat{\underline{p}}_c[j] = \left[\, \underline{\underline{G}}^T(\hat{\underline{e}}_{j-1})\, \underline{\underline{G}}(\hat{\underline{e}}_{j-1}) \,\right]^{-1} \underline{\underline{G}}^T(\hat{\underline{e}}_{j-1})\, \hat{\underline{r}} \,. \qquad (2.6\text{-}17)$$

Here

$$\hat{\underline{e}}_j = \hat{\underline{r}} - \underline{\underline{G}}(\hat{\underline{e}}_{j-1})\hat{\underline{p}}_c[j] \tag{2.6-18}$$

and

$$\underline{\underline{G}}(\hat{\underline{e}}_j) = \begin{bmatrix} \underline{g}_j^T(1) \\ \cdot \\ \cdot \\ \cdot \\ \underline{g}_j^T(N) \end{bmatrix} \tag{2.6-19}$$

where

$$\underline{g}_j(t) = [\hat{e}_j(t-1),\ldots,\hat{e}_j(t-n)]^T. \tag{2.6-20}$$

2.7 The KOOPMANS-LEVIN method

With the methods discussed heretofore, the condition of getting asymptotically unbiased estimation was to assume the input signal to be measured without error. There are fairly divergent opinions in the literature, as to the identifiability in the case of input noise. Several authors assert that the elimination of input noises by measurement techniques is much more efficient than the construction of such a statistical (sometimes very artificial) estimation which is usually inefficient because of the "a priori" information required for its proper application.

The methods applied in the regression analysis for the elimination of the input noise are essentially various trends of the FRISCH confluent analysis. From the identification methods of linear dynamic systems applicable in case of noisy inputs, the method of KOOPMANS and LEVIN can be considered as best founded in theory. In the following this method is presented and its modified version developed by us is also given [26].

For the study of this method consider Fig. 2.7-1. Denote here $u_o(t)$ and $y_o(t)$ the noiseless input and output of the process, $w_u(t)$ and $w_y(t)$ the measurement noises at the input, and the output, respectively. (As this method has a logic totally different from those mentioned heretofore, the noiseless output will be denoted here by y_o instead of v). Let further be $B(z^{-1})$ and $A(z^{-1})$ according to the relation (2.1-4). Considering that the differece equation of the noiseless system is determined by the polynomials $B(z^{-1})$ and $A(z^{-1})$ the system equation can also be written in a form linear in parameters (arranged for 0) not discussed until now, viz.

$$\underline{g}_o^T(u_o,y_o,t)\ \underline{p}_{bla} = 0, \qquad (2.7\text{-}1)$$

where

$$\underline{g}_o(u_o,y_o,t)=[u_o(t),u_o(t-1),\ldots,u_o(t-n);-y_o(t),-y_o(t-1),\ldots,-y_o(t-n)]^T \qquad (2.7\text{-}2)$$

and

$$\underline{p}_{bla}=[b_o,b_1,\ldots,b_n,1,a_1,a_2,\ldots,a_n]^T. \qquad (2.7\text{-}3)$$

By writing (2.7-1) in detail:

$$\sum_{i=0}^{n} b_i u_o(t-i)-y_o(t)-\sum_{i=0}^{n} a_i y_o(t-i)=0. \qquad (2.7\text{-}4)$$

If we introduce the vector

$$\underline{w}(t)=[w_u(t),w_u(t-1),\ldots,w_u(t-n);-w_y(t),-w_y(t-1),\ldots,-w_y(t-n)]^T \qquad (2.7\text{-}5)$$

representing the input and output noises, then for the vector containing measurements corresponding to the noisy measurement situation

$$\underline{g}(u,y,t)=[u(t),u(t-1),\ldots,u(t-n);-y(t),-y(t-1),\ldots,-y(t-n)]^T \qquad (2.7\text{-}6)$$

the equation

$$\underline{w}(t)=\underline{g}(u,y,t)-\underline{g}_o(u_o,y_o,t) \qquad (2.7\text{-}7)$$

holds.

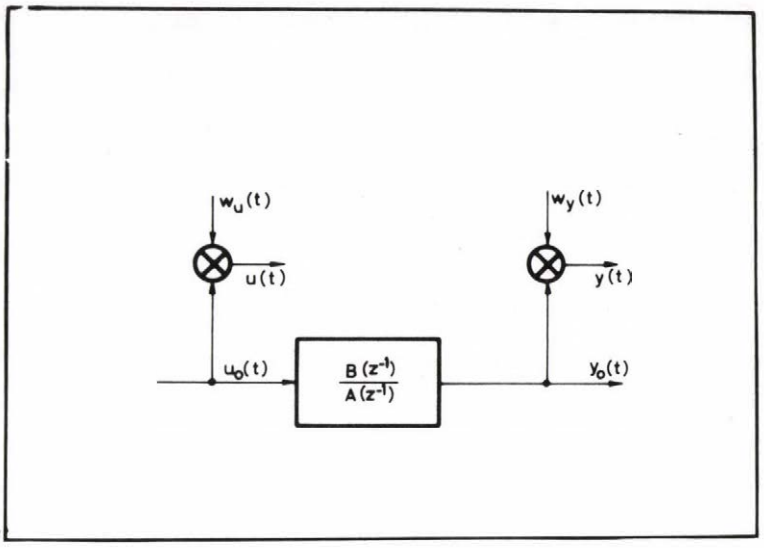


Fig. 2.7-1

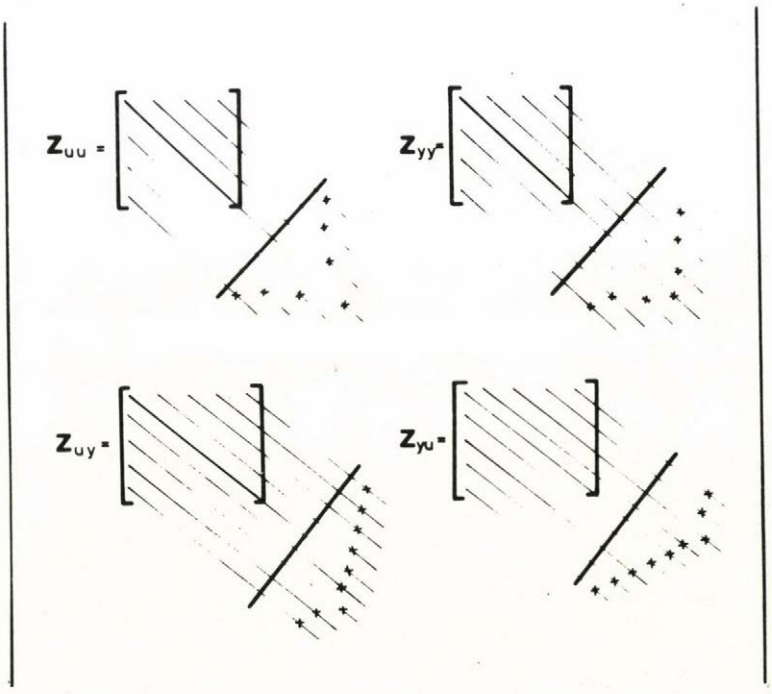


Fig.2.7-2

It is assumed that the disturbances in w(t) have normal distribution, zero mean and their covariance matrix

$$E\{\underline{w}(t)\underline{w}^T(t)\} = \underline{\underline{Z}} \qquad (2.7\text{-}8)$$

is known. Here the stationarity of the disturbances has also been assumed as $\underline{\underline{Z}}$ is a time-independent constant. The covariance matrix $\underline{\underline{Z}}$ can be logically decomposed to the following four submatrices:

$$\underline{\underline{Z}} = \begin{bmatrix} \underline{\underline{Z}}_{uu} & \underline{\underline{Z}}_{uy} \\ \underline{\underline{Z}}_{yu} & \underline{\underline{Z}}_{yy} \end{bmatrix}. \qquad (2.7\text{-}9)$$

Here $\underline{\underline{Z}}_{uu}$ contains the values of the autocorrelation series of the input noise $w_u(t)$, $\underline{\underline{Z}}_{yy}$ those of the output noise, in a way shown in Fig. 2.7-2. Likewise $\underline{\underline{Z}}_{uy}$ and $\underline{\underline{Z}}_{yu}$ contain the values of the cross-correlation products. From the special cases the assumptions

1. $\underline{\underline{Z}}_{uu} = \delta^2\underline{\underline{I}}$, $\underline{\underline{Z}}_{uy} = \underline{\underline{Z}}_{yu} = \underline{\underline{0}}$, $\underline{\underline{Z}}_{yy} = \lambda^2\underline{\underline{I}}$;

2. $\underline{\underline{Z}}_{uu} = \delta^2\underline{\underline{I}}$, $\underline{\underline{Z}}_{uy} = \underline{\underline{Z}}_{yu} = \underline{\underline{0}}$

hold several times in the practice and make the calculations as well as the preliminary determination (estimation) of $\underline{\underline{Z}}$ easier.

Generally the probability density function of the vector w(t) of dimension q=2(n+1)

$$d(\underline{w}|\underline{g}_o) = (2\pi)^{-\frac{q}{2}} |\underline{\underline{Z}}|^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}\,\underline{w}^T(t)\underline{\underline{Z}}^{-1}\,\underline{w}(t)\}, \qquad (2.7\text{-}10)$$

as we have seen e.g. in (1.3-2).

The joint density function for N independent samplings

$$d_N(\underline{w}|\underline{g}_o) = (2\pi)^{-\frac{qN}{2}} \left|\underline{\underline{Z}}\right|^{-\frac{N}{2}} \exp\{-\frac{1}{2}\sum_{t=1}^{N} \underline{w}^T(t)\underline{\underline{Z}}^{-1}\underline{w}(t)\} \quad (2.7\text{-}11)$$

Let us write the likelihood function for the application of the ML estimation method, i.e. the natural logarithm of (2.7-11)

$$L(\underline{p}_{bla}) = -\frac{qN}{2}\ln(2\pi) - \frac{N}{2}\ln\left|\underline{\underline{Z}}\right| - \frac{1}{2}\sum_{t=1}^{N} \underline{w}^T(t)\underline{\underline{Z}}^{-1}\underline{w}(t) \quad (2.7\text{-}12)$$

The ML estimation of the $\underline{p}_{bla}$ parameter vector is obtained by the maximization of this term in according to $\underline{p}_{bla}$. The task is further complicated by the fact that the maximization has to be performed under constraint (2.7-1), so that

$$\underline{g}_o^T(u_o,y_o,t)\underline{p}_{bla} = 0, \text{ for } t=1,2,\ldots,N \quad (2.7\text{-}13)$$

should be fulfilled. This constraint means that simultaneously with the estimation of the parameters $\underline{p}_{bla}$ the noiseless inputs and outputs $\underline{g}_o(u_o,y_o,t)$ have also to be estimated from the noisy measurement data.

The maximization task with this constraint can be solved in a well-known way with the method of Lagrange multipliers [41] tracing back it to the minimization of the following term:

$$Q(\underline{p}_{bla}) = \frac{1}{2}\sum_{t=1}^{N}\left[\underline{w}^T(t)\underline{\underline{Z}}^{-1}\underline{w}(t) + 2\Psi(t)\underline{g}_o^T(u_o,y_o,t)\underline{p}_{bla}\right]. \quad (2.7\text{-}14)$$

In point A.9 of the Appendix it is deduced that the minimum according to $\underline{g}_o(u_o,y_o,t)$ of the term (2.7-14) can be achieved by the minimization of the quantity

$$Q^{*}(\underline{p}_{bla}) = \frac{1}{2} \frac{\underline{p}_{bla}^{T} \underline{\underline{G}} \underline{p}_{bla}}{\underline{p}_{bla}^{T} \underline{\underline{Z}} \underline{p}_{bla}} \qquad (2.7\text{-}15)$$

by $\underline{p}_{bla}$, where

$$\underline{\underline{G}} = \sum_{t=1}^{N} \underline{g}(u,y,t)\underline{g}^{T}(u,y,t). \qquad (2.7\text{-}16)$$

It is known from linear algebra that the greatest lower bound of the generalized quadratic form (2.7-15) is yielded by the smallest eigenvalue of the generalized eigenvalue problem

$$\underline{\underline{G}}\ \underline{p}_{bla} = \mu \underline{\underline{Z}}\ \underline{p}_{bla} \qquad (2.7\text{-}17)$$

and the $\underline{p}_{bla}$ yielding the minimum is the eigenvector belonging to the smallest eigenvalue.

Although there are known numerical methods [94] to solve the generalized eigenvalue problems, because of their complexity we reduce the problem (2.7-17) in Appendix 10 to a simple eigenvalue problem, viz. to a solution requiring only the determination of the eigenvalue of the symmetric matrix. The finding of the real eigenvalues of a symmetric matrix is already considered nowadays a routine task in numerical analysis.

Assume that $\underline{p}^{*}$ is obtained for the eigenvalue belonging to the least eigenvalue of the eigenvalue problem (2.7-17). Since $\underline{p}_{bla}$ is of a special construction in the sense that the value of its (n+2)nd component has to be 1 under any circumstances. This can be simple ensured by dividing the elements of vector $\underline{p}^{*}$ by the (n+2)nd element of the vector, p^{*}_{n+2}

$$\hat{\underline{p}}_{bla} = \frac{1}{p^{*}_{n+2}} \underline{p}^{*} \quad . \qquad (2.7\text{-}18)$$

It can be demonstrated [107] that the estimation (2.7-18) is consistent and generally biased for finite N. If the standard deviation of the input noises are substantially less than that of the input signals, this bias is less than the standard derivation of the parameters.

The basic principle of the method can be demonstrated as the deduction of the covariance matrix representing the statistical property of the noises from the covariance matrix formed from observations. This latter one is approximately the sum of the covariance matrices of the useful signals and the perturbing noises, the deduction will therefore be in case of infinitely large samples asymptotically perfect. If our knowledges about the covariance matrix of the noises are not sure, the estimation is generally biased, because the deduction of the covariance matrix of the noises does not occur even asymptotically. (The above explanation means, of course, only a very rough demonstration of the correct statistical investigations).

From the point of view of computational techniques the KOOPMANS-LEVIN method is a relatively simple procedure, on the other hand, the demand of priori information, the requirement of the knowledge of $\underline{\underline{Z}}$ seems to be quite considerable.

In practical cases the submatrix $\underline{\underline{Z}}_{uu}$ of $\underline{\underline{Z}}$ is estimated but for the other part of $\underline{\underline{Z}}$ only assumptions can be taken. Because of these difficulties, we have elaborated a method for the case $\underline{\underline{Z}}_{uu} = \underline{\underline{0}}$, $\underline{\underline{Z}}_{uy} = \underline{\underline{0}}$ in order to update $\underline{\underline{Z}}_{yy}$ and got the iterative version of the KOOPMANS-LEVIN method.

In accordance with our previous assumptions now

$$\underline{w}(t) = [0,0,\ldots,0;\ -w_y(t),-w_y(t-1),\ldots,-w_y(t-n)]^T \qquad (2.7\text{-}19)$$

and also further on

$$w_y(t) = y(t) - y_o(t). \qquad (2.7\text{-}20)$$

Here $y(t)$ is the measured value, and $y_o(t)$ can be calculated by the given $\underline{\hat{p}}_{bla}$ (previously estimated) i.e.

$$y_o(t) = \sum_{i=0}^{n} \hat{b}_i\, u(t-i) - \sum_{i=1}^{n} \hat{a}_i y_o(t-i). \qquad (2.7\text{-}21)$$

By calculating the values of $y_o(t)$, $w_y(t)$ for $t=1,2,\ldots,N$ and constructing $\underline{w}(t)$, an estimation of $\underline{\underline{Z}}$ can be obtained by the expression

$$\hat{\underline{\underline{Z}}} = \frac{1}{N} \sum_{t=1}^{N} \underline{w}(t)\underline{w}^T(t). \qquad (2.7\text{-}22)$$

Of course, it is enough to estimate only the part $\underline{\underline{Z}}_{yy}$ of $\underline{\underline{Z}}$, for the rest is zero.

On the basis of the above, we can from an iterative process for the estimation of the process parameters by using the modified KOOPMANS-LEVIN method. The iterative method can be constructed in the following way:

1. Determine the preliminary estimation of the process parameters $\underline{p}_{ba}$ by the simple LS estimation.
2. Calculate $w_y(t)$, thereafter $\underline{\underline{Z}}$ through $\underline{w}(t)$.
3. Determine the parameter estimation $\underline{\hat{p}}_{bla}$ yielding the minimum of (2.7-15).
4. Return to point 2. and continue the iteration until the attainment of a given accuracy.

The application of the iterative method is made easier by $\underline{\underline{D}}$ being independent from $\underline{p}_{bla}$ with the decomposition of $\underline{\underline{G}}$, so that it has to be calculated only once. Compared with the iteration methods published in other chapters of this report, here therefore not inversion, but eigenvalue search has to be performed by iterations.

2.8 Identification in closed loop

As already indicated in the Introduction, this report deals with the identification methods applicable in open loop, when the input signal of the process is independent from the output signal and the output noise. The discussion of identification methods applicable in closed loop, would demand an another way of approach, on the one hand, the estimation techniques applicable there are also very different from those applied in this report on the other hand. Methods, computer procedures elaborated for open loop can be applied, however, directly or only with slight modification for some particular cases of the identification in closed loops. These possibilities will be illustrated in this subsection [65, 73, 145].

Let us investigate the identification in the closed loop system shown in Fig. 2.8-1. Beside the notations used heretofore, here $u_a(t)$ is the reference value and $u_z(t)$ the artifical disturbance independent from $e(t)$, further the $P(z^{-1})/Q(z^{-1})$ is the transfer function, where

$$P(z^{-1}) = p_o + p_1 z^{-1} + \ldots + p_k z^{-k} \qquad (2.8\text{-}1)$$

$$Q(z^{-1}) = 1 + q_1 z^{-1} + \ldots + q_k z^{-k} . \qquad (2.8\text{-}2)$$

By taking advantage of the rules of determination of the resultant transfer functions for the closed loop, we obtain that

$$y(t) = \frac{P(z^{-1})\,B(z^{-1})}{A(z^{-1})\,Q(z^{-1}) + B(z^{-1})\,P(z^{-1})}\,u_a(t) +$$

$$= \frac{Q(z^{-1})\,B(z^{-1})}{A(z^{-1})\,Q(z^{-1}) + B(z^{-1})\,P(z^{-1})}\,u_z(t) +$$

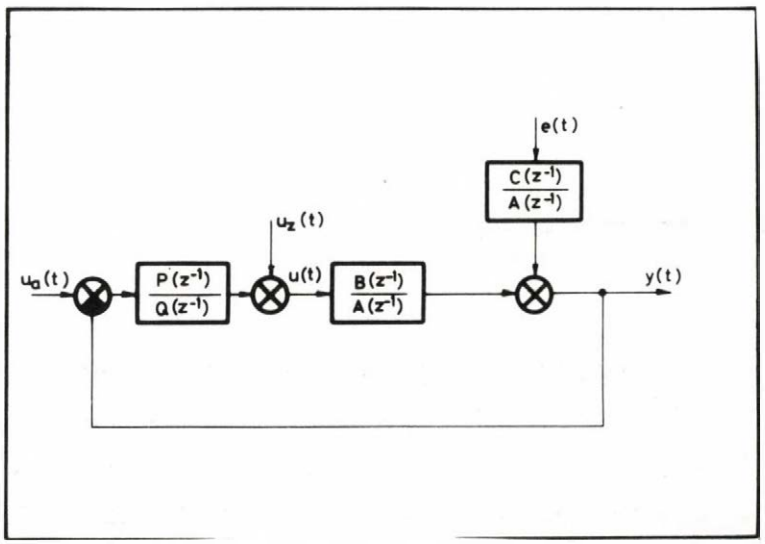


Fig.2.8-1

$$+ \frac{Q(z^{-1})C(z^{-1})}{A(z^{-1})Q(z^{-1})+P(z^{-1})B(z^{-1})} e(t). \qquad (2.8\text{-}3)$$

If any of $u_a(t)$ or $u_z(t)$ is stationary, then by considering either the

$$\frac{P.B}{AQ+B.P} u_a(t) + \frac{QC}{AQ+PB} e(t) \qquad (2.8\text{-}4)$$

or

$$\frac{QB}{AQ+B.P} u_z(t) + \frac{QC}{AQ+PB} e(t) \qquad (2.8\text{-}5)$$

error terms as "output" noise, we can apply one of the methods worked out for the SISO systems, by taking the $u_z(t)$ or $u_a(t)$ variables as input signals. Regarding the structure, the ML method seems to be here the best fitting for the parameter estimation. The estimation of the parameters of the denominator AQ+BP can be an intermediary objective, hence in the knowledge of P and Q, A and B can already be determined (see below).

Assume that during the investigations the reference value $u_a(t)$ is zero (or it is constant and then we work with a model valid for changes). In this case the closed system is described by the following two equations:

$$y(t) = \frac{B(z^{-1})}{A(z^{-1})} u(t) + \frac{C(z^{-1})}{A(z^{-1})} e(t) \qquad (2.8\text{-}6)$$

and

$$u(t) = - \frac{P(z^{-1})}{Q(z^{-1})} y(t). \qquad (2.8\text{-}7)$$

For this latter equation, we have assumed that the artifical distrubance $u_z(t)$ was also zero. By a simple substitution the two equations can be reduced to a single one

$$y(t) = -\frac{B(z^{-1})P(z^{-1})}{A(z^{-1})Q(z^{-1})}y(t) + \frac{C(z^{-1})}{A(z^{-1})}e(t) \qquad (2.8\text{-}8)$$

i.e.

$$\left[A(z^{-1})Q(z^{-1}) + P(z^{-1})B(z^{-1})\right]y(t) = Q(z^{-1})C(z^{-1})e(t). \qquad (2.8\text{-}9)$$

This latter equation corresponds to a mixed autoregressive - moving average (ARMA) stochastic process, i.e.

$$R(z^{-1})y(t) = S(z^{-1})e(t), \qquad (2.8\text{-}10)$$

where by assuming $b_o \equiv 0$ the equations have the form

$$R(z^{-1}) = 1 + r_1 z^{-1} + \ldots + r_{k+n} z^{-(k+n)} \qquad (2.8\text{-}11)$$

and

$$S(z^{-1}) = 1 + s_1 z^{-1} + \ldots + s_{k+n} z^{-(k+n)}. \qquad (2.8\text{-}12)$$

Assume that the coefficients $R(z^{-1})$ could have been determined by a certain identification method. If the parameters of the regulator are known, i.e. $P(z^{-1})$ and $Q(z^{-1})$, then from the equation

$$A(z^{-1})Q(z^{-1}) + B(z^{-1})P(z^{-1}) = R(z^{-1}) \qquad (2.8\text{-}13)$$

$A(z^{-1})$ and $B(z^{-1})$ can be obtained. By performing the polynomial multiplications in (2.8-13), by comparing the cofficients we obtain that

$$\underline{\underline{M}}\,\underline{p}_{ba} = \left[\begin{array}{cccc|cccc} p_o & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ p_1 & p_o & \dots & 0 & q_1 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ p_{k-1} & p_{k-2} & \dots & p_o & q_{k-1} & q_{k-2} & \dots & 1 \\ p_k & p_{k-1} & \dots & p_1 & q_k & q_{k-1} & \dots & q_1 \\ 0 & p_k & \dots & p_2 & \dots & q_k & \dots & q_2 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & p_k & & 0 & \dots & q_n \end{array}\right] \left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{array}\right] = \left[\begin{array}{c} r_1-q_1 \\ r_2-q_2 \\ \vdots \\ r_k-q_k \\ r_{k+1} \\ \vdots \\ r_{k+n} \end{array}\right] = \underline{m}. \qquad (2.8\text{-}14)$$

The condition of the determination of $\underline{p}_{ba}$ is that either $Q(z^{-1})$ or $P(z^{-1})$ is of the n-th degree as we have 2n unknown parameters and thus $R(z^{-1})$ has to be of 2n-th degree which is fulfilled only under condition k=n. In this case we obtain the estimation of the process parameters in the form

$$\hat{\underline{p}}_{ba} = \underline{\underline{M}}^{-1}\,\underline{m} \; . \qquad (2.8\text{-}15)$$

(Note that similar considerations can be made for the estimation of $C(z^{-1})$, too.)

Several methods have already been elaborated for the estimation of the parameters of the ARMA stochastic process (2.8-10)[2, 5, 33]. Here we would like to point out that by prohibiting the estimation of the coefficients relating to the input signals $u(t)$, i.e. by masking, the estimation of the coefficients of $R(z^{-1})$ can be obtained either by the ML, or the SEXM method.

If $u_a(t)$ is zero but $u_z(t)$ not, then the equation (2.8-7) will have the form

$$u(t) = - \frac{P(z^{-1})}{Q(z^{-1})} y(t) + u_z(t). \qquad (2.8\text{-}16)$$

By substituting this into (2.8-6), we obtain the system equation

$$\left[A(z^{-1})Q(z^{-1}) + B(z^{-1})P(z^{-1})\right] y(t) =$$

$$= A(z^{-1})Q(z^{-1})B(z^{-1})u_z(t) + Q(z^{-1})C(z^{-1})e(t). \qquad (2.8\text{-}17)$$

It is obvious that the ML structure according to (1.2-5) was formally achieved, so that either the ML, or the SEXM or any other identification method valid for similar models can be applied. Now, of course, here is no need of masking and $u_z(t)$ has to be considered as input signal. The decomposition of the obtained denominator can be performed also according to (2.8-15).

If the $u_a(t)$ and $u_z(t)$ signals are neither of zero value, nor are they stationary, then the closed loop or the denominator AQ+BP can be identified on the basis of the system equation (2.8-31) as a two-input (u_a, u_z) and single output (y) system.Thus, e.g. the application of the ML procedure worked out for the MISO systems can be thought of.

It is not our intention to examine now thoroughly the various possible cases (which transfer is known, which signal can be measured, which preliminary conditions are true, etc.), we have pointed out that the identification methods elaborated for open loops can be applied to closed loops, too, if the system is analyzed in detail and certain supplementary calculations are made in a given case.

III. OFF-LINE IDENTIFICATION METHODS FOR MULTIPLE OUTPUT SYSTEMS

As seen in the preceding chapter, the number of the inputs of a system does not influence substantially the identification technique. Despite the multiple inputs a MISO system equation does not require, compared with the SISO system equation, any form of description different in kind, only the sizes of the featuring memory resp. the parameter vectors, increase. Likewise the loss functions to be minimized retain their character unchanged.

On the other hand, turning to the examination of the MIMO systems by the increase of the number of outputs there is a change in dimension: the memory vector becomes memory matrix, the parameter vector parameter matrix, etc. This could be recognized in subsection 1.2, where also the MIMO system equations were discussed. Accordingly, the loss functions also depend in a substantially more complex way from the parameters.

Methods used for single output systems were discussed for the case of d=0 as a given d value can always be set by shifting the series of input and output values with respect to each other. This can be achieved, of course, also in the case of a MISO system by considering the deadtimes by inputs in their mutual dependence, only the algorithm will be somewhat more sophisticated. With MIMO systems we are unable to form a deadtime given for all inputs by shifting input and output signals with regard to each other. Therefore, it will now be required to denote the deadtimes by zero symbols of certain coefficients of the matrix polynomial $\underline{\underline{B}}(z^{-1})$. (For that matter this is possible also with single output systems but there the shifting technique has proved to be more simple.)

Here, among the introductory remarks referring to the MIMO systems, we will present the technique for the masking (prohibition) of the parameters not included in the estimation and not wanted. (The same was shown for the SISO systems at the beginning of Chapter 2.) Most of the off-line identification methods of the multiple output systems require matrix inversion. The place of the inversion, either an explicit solution or an iterative method is considered, can be demonstrated as follows:

$$\underline{\underline{P}} = \underline{\underline{V}}\ \underline{\underline{M}}^{-1} \qquad (3\text{-}1)$$

where $\underline{\underline{M}}$ is a symmetric square matrix, $\underline{\underline{V}}$ a matrix of suitable dimension. With MIMO systems only one column of the parameter matrix $\underline{\underline{P}}$ can be prohibited at a time, the single elements not. This, however, is completely suitable from the theoretical point of view, since the conditions of uniqueness [108] refer to the columns of $\underline{\underline{P}}$. If we now do not wish to estimate the i-th column of the parameter matrix or, with other words, we wish to prescribe a zero constraint for the coefficients in the i-th column, then we can achieve this in the following way. We make the i-th row and column of matrix $\underline{\underline{M}}$ equal to zero, thereafter we choose e.g. 1 for the M_{ii} element at their intersection. Thereupon we make the i-th column of matrix $\underline{\underline{V}}$ equal to zero. Thus we obtain absolutely zero for the i-th column of $\underline{\underline{P}}$:

$$\underline{\underline{P}} = \begin{bmatrix} & 0 & \\ \cdots & \vdots & \cdots \\ & 0 & \end{bmatrix} \begin{bmatrix} & & 0 & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & 0 & & \end{bmatrix}^{-1} = \begin{bmatrix} & 0 & \\ \cdots & \vdots & \cdots \\ & 0 & \end{bmatrix} \qquad (3\text{-}2)$$

For the practical use (programming) a mask matrix used to be given whose dimensions are identical with $\underline{\underline{P}}$, and it contains value 1 where we wish to estimate the element and 0 where we do not. The expounding of simple examples can convince us that this technique can in fact be applied only to the pro-

hibition of columns and not to elements occurring in any places.

Also for the off-line identification methods of the MIMO system is valid the statement that if the parameter matrix $\underline{\underline{P}}$ has to be estimated on the basis of N conjugate $\underline{u}(t)$ and $\underline{y}(t)$ vectors by simultaneous processing, then we face even now deterministic extremum seeking problem where the corresponding loss function has to be minimized as a criterion in the space of the numerous parameters of the MIMO system.

Hereinafter we will give a survey of the parametric identification methods of the MIMO systems, all those which can be considered as direct generalizations of procedures well-known for SISO systems.

3.1 Least squares method

We get an uncorrelated equation error vector term from the general equation (1.2-26) of the discrete-time MIMO system models under the condition $\underline{\underline{C}}(z^{-1})\equiv\underline{\underline{I}}$, i.e. possibility to use the simple LS method. In this case the system equation is

$$\underline{y}(t) = \sum_{i=0}^{n} \underline{\underline{B}}_i\, \underline{u}(t-i) - \sum_{i=1}^{n} \underline{\underline{A}}_i\, \underline{y}(t-i) + \underline{e}(t) \qquad (3.1\text{-}1)$$

where the deadtime was not stated separately, but implied in the parameter matrices $\underline{\underline{B}}_i$. This form is linear in the parameter matrices, i.e. it can be written also in the form

$$\underline{y}(t) = \underline{\underline{P}}_{BA}\, \underline{f}(\underline{u},\underline{y},t) + \underline{e}(t) \qquad (3.1\text{-}2)$$

where

$$\underline{\underline{P}}_{BA} = \left[\underline{\underline{B}}_o, \underline{\underline{B}}_1, \ldots, \underline{\underline{B}}_n, \underline{\underline{A}}_1, \ldots, \underline{\underline{A}}_n\right] \qquad (3.1\text{-}3)$$

and

$$\underline{f}^T(\underline{u},\underline{y},t) = \left[\underline{u}^T(t), \ldots, \underline{u}^T(t-n); -\underline{y}^T(t-1), \ldots, -y^T(t-n)\right] = \underline{f}^T(t). \qquad (3.1\text{-}4)$$

If we wish to estimate the parameters on the basis of N conjugate $\underline{u}(t)$, $\underline{y}(t)$ values, then it is reasonable to use also here the system equation relating to N samples, similarly to the SISO systems.

Let us introduce the notations

$$\underline{\underline{Y}} = \left[\underline{y}(1), \ldots, \underline{y}(N)\right], \qquad (3.1\text{-}5)$$

$$\underline{\underline{F}} = [\underline{f}(1),\ldots,\underline{f}(N)] \qquad (3.1\text{-}6)$$

and

$$\underline{\underline{E}} = [\underline{e}(1),\ldots,\underline{e}(N)]\,. \qquad (3.1\text{-}7)$$

The joint system matrix equation

$$\underline{\underline{Y}} = \underline{\underline{P}}_{BA}\,\underline{\underline{F}} + \underline{\underline{E}}\,. \qquad (3.1\text{-}8)$$

With the help of the operation vec(...) (see Appendix 3), we obtain from (3.1-8) that

$$\mathrm{vec}(\underline{\underline{Y}}) = \mathrm{vec}(\underline{\underline{P}}_{BA}\,\underline{\underline{F}} + \underline{\underline{E}}) = \mathrm{vec}(\underline{\underline{P}}_{BA}\,\underline{\underline{F}}) + \mathrm{vec}(\underline{\underline{E}}) =$$

$$= (\underline{\underline{F}}^{T} \otimes \underline{\underline{I}}_{q})\ \mathrm{vec}(\underline{\underline{P}}_{BA}) + \mathrm{vec}(\underline{\underline{E}}), \qquad (3.1\text{-}9)$$

where we have used the advantage of the identity (A.4-3) and $\underline{\underline{I}}_{q}$ is a (qxq) unit matrix, where q is the number of outputs. By applying the notations of (1.2-42) and (1.2-43), according to which

$$\underline{y}_{M} = \mathrm{vec}(\underline{\underline{Y}}) \qquad (3.1\text{-}10)$$

and

$$\underline{e}_{M} = \mathrm{vec}(\underline{\underline{E}}) \qquad (3.1\text{-}11)$$

further $\underline{u}_{M}$, too, has a similar meaning, the equation (3.1-9) will take the form

$$\underline{y}_{M} = \underline{\underline{G}}\ \underline{p}_{BA} + \underline{e}_{M}. \qquad (3.1\text{-}12)$$

Here we have used the notations

$$\underline{\underline{G}} = \underline{\underline{F}}^{T} \otimes \underline{\underline{I}}_{q} = \underline{\underline{G}}(\underline{u}_{M}, \underline{y}_{M}) \qquad (3.1\text{-}13)$$

and

$$\underline{p}_{BA} = \mathrm{vec}(\underline{\underline{P}}_{BA}) \cdot \qquad (3.1\text{-}14)$$

Note that the detailed analysis of (1.2-44) leads to the system equation (3.1-12) also under the assumption of $\underline{\underline{C}}(z^{-1}) = \underline{\underline{I}}$, i.e. in consequence of the above

$$\underline{y}_M = \Big[\sum_{i=0}^{n} \underline{\underline{S}}_N^i(1) \otimes \underline{\underline{B}}_i\Big]\underline{u}_M - \Big[\sum_{i=1}^{n} \underline{\underline{S}}_N^i(1) \otimes \underline{\underline{A}}_i\Big]\underline{y}_M + \underline{e}_M =$$

$$= \underline{\underline{G}}(\underline{u}_M, \underline{y}_M)\underline{p}_{BA} + \underline{e}_M \cdot \qquad (3.1\text{-}15)$$

It is deduced in Appendix 11 that even starting from various modes of approaches the LS estimation of the system parameters can be obtained in the forms

$$\hat{\underline{\underline{P}}}_{BA} = \underline{\underline{Y}}\ \underline{\underline{F}}^T(\underline{\underline{F}}\ \underline{\underline{F}}^T)^{-1} \qquad (3.1\text{-}16)$$

or

$$\hat{\underline{p}}_{BA} = \Big[(\underline{\underline{F}}\ \underline{\underline{F}}^T)^{-1}\ \underline{\underline{F}} \otimes \underline{\underline{I}}_q\Big]\underline{y}_M \cdot \qquad (3.1\text{-}17)$$

3.2 Maximum likelihood method

Alike the SISO systems the ML parameter estimation of the MIMO systems has conceptually been incorporated to a certain structure and this corresponds to the generalization of the single output ÅSTRÖM model, thus according to (1.2-30) [108]

$$\underline{y}(t) = \sum_{i=0}^{n} \underline{\underline{B}}_i \, \underline{u}(t-i) - \sum_{i=1}^{n} \underline{\underline{A}}_i \, \underline{y}(t-i) + \sum_{i=1}^{n} \underline{\underline{C}}_i \, \underline{e}(t-i) + \underline{e}(t). \tag{3.2-1}$$

This system equation can be written also in the form

$$\underline{y}(t) = \underline{\underline{P}}_{BAC} \, \underline{f}(\underline{u},\underline{y},\underline{e},t) + \underline{e}(t) \tag{3.2-2}$$

where

$$\underline{\underline{P}}_{BAC} = [\underline{\underline{B}}_0, \underline{\underline{B}}_1, \ldots, \underline{\underline{B}}_n; \underline{\underline{A}}_1, \ldots, \underline{\underline{A}}_n; \underline{\underline{C}}_1, \ldots, \underline{\underline{C}}_n] = [\underline{\underline{P}}_B; \underline{\underline{P}}_A; \underline{\underline{P}}_C] \tag{3.2-3}$$

and

$$\underline{f}(\underline{u},\underline{y},\underline{e},t) = [\underline{u}^T(t), \ldots, \underline{u}^T(t-n); -\underline{y}^T(t-1), \ldots, -\underline{y}^T(t-n); \underline{e}^T(t-1), \ldots, \underline{e}^T(t-n)]^T. \tag{3.2-4}$$

It is deduced in App.5 that for MIMO systems the maximization of the likelihood function is equivalent with the solution of the minimization problem

$$Q(\hat{\underline{\underline{P}}}_{BAC}) = \min_{\underline{\underline{P}}_{BAC}} Q(\underline{\underline{P}}_{BAC}) = \min_{\underline{\underline{P}}_{BAC}} \left| \hat{\underline{\underline{\Lambda}}}(\underline{\underline{P}}_{BAC}) \right| =$$

$$= \min_{\underline{\underline{P}}_{BAC}} \left| \frac{1}{N} \sum_{t=1}^{N} \underline{e}(t) \underline{e}^T(t) \right|. \tag{3.2-5}$$

Here $\hat{\underline{\underline{\Lambda}}}$ is the ML estimation of the $\underline{\underline{\Lambda}}$ covarience matrix of the source noise vector $\underline{e}(t)$. The calculation of $\hat{\underline{\underline{\Lambda}}}$ is given by (App.5-25):

$$\hat{\underline{\underline{\Lambda}}} = \frac{1}{N} \sum_{t=1}^{N} \underline{e}(t)\underline{e}^{T}(t). \qquad (3.2\text{-}6)$$

Herebelow we shall use the forms ordered in vectors of $\underline{\underline{P}}_{BA}$ and its submatrices:

$$\mathrm{vec}(\underline{\underline{P}}_{BAC}) = \underline{p}_{BAC}, \qquad (3.2\text{-}7)$$

$$\mathrm{vec}(\underline{\underline{P}}_{B}) = \underline{p}_{B}, \qquad (3.2\text{-}8)$$

$$\mathrm{vec}(\underline{\underline{P}}_{A}) = \underline{p}_{A}, \qquad (3.2\text{-}9)$$

$$\mathrm{vec}(\underline{\underline{P}}_{C}) = \underline{p}_{C}. \qquad (3.2\text{-}10)$$

The minimization of the loss function (3.2-5) is a complicated nonlinear extremum seeking task. As the cost function is a more complex formation than a quadratic one in the parameters, so explicit solution can not be given for the minimum, only iterative processes can be chosen. As it yields the fastest convergence speed in the quadratic sense, being at the same time the most widespread method for the ML estimation of the SISO systems, we suggest now again the application of the NEWTON-RAPHSON procedure for the extremum seeking. The canonical form of the algorithm is:

$$\underline{p}(j+1) = \underline{p}(j) - \underline{\underline{T}}^{-1}[\underline{p}(j)]\frac{dQ[\underline{p}(j)]}{d\,\underline{p}(j)}, \qquad (3.2\text{-}11)$$

i.e. the gradient of Q and the Hessian matrix $\underline{\underline{T}}$ are required in every iterative step. Although the loss function (3.2-5) has a physical meaning which can be well interpreted, it is more practical to consider for minimization the cost function

$$\tilde{Q}(\underline{\underline{P}}_{BAC}) = \frac{N}{2} \ln \left| \hat{\underline{\underline{\Lambda}}}(\underline{\underline{P}}_{BAC}) \right| \qquad (3.2\text{-}12)$$

whose handling for numerical calculations and deductions is more convenient, at the same time, its extremum is equal to the minimum of $Q(\underline{\underline{P}}_{BAC})$ as the logarithm function is a monotonous function. Thus our iterative algorithm will finally take the form

$$\underline{p}(j+1) = \underline{p}(j) - \underline{\underline{T}}^{-1}[\underline{p}(j)] \frac{d\tilde{Q}[\underline{p}(j)]}{d\,\underline{p}(j)} \quad . \qquad (3.2\text{-}13)$$

(Here, of course, $\underline{p} = \underline{p}_{BAC} = \mathrm{vec}(\underline{\underline{P}}_{BAC})$)

Let us examine the generation of the first and second derivatives. The deduction is given in App.12 that the gradient of the loss function can be calculated according to the following relation:

$$\frac{d\tilde{Q}(\underline{\underline{P}}_{BAC})}{d\underline{p}_{BAC}} = \sum_{t=1}^{N} \frac{\partial \underline{e}^T(t)}{\partial \underline{p}_{BAC}} \hat{\underline{\underline{\Lambda}}}^{-1} \underline{e}(t). \qquad (3.2\text{-}14)$$

By introducing the notations

$$\underline{f}_{\underline{u}}(t) = [\underline{u}^T(t), \underline{u}^T(t-1), \ldots, \underline{u}^T(t-n)]^T, \qquad (3.2\text{-}15)$$

$$\underline{f}_{\underline{y}}(t) = [\underline{y}^T(t-1), \underline{y}^T(t-2), \ldots, \underline{y}^T(t-n)]^T, \qquad (3.2\text{-}16)$$

$$\underline{f}_{\underline{e}}(t) = [\underline{e}^T(t-1), \underline{e}^T(t-2), \ldots, \underline{e}^T(t-n)]^T \qquad (3.2\text{-}17)$$

the system equation can be written also in the form of

$$\underline{\underline{C}}(z^{-1})\underline{e}(t) = \underline{y}(t) + \underline{\underline{P}}_A \underline{f}_{\underline{y}}(t) - \underline{\underline{P}}_B \underline{f}_{\underline{u}}(t). \qquad (3.2\text{-}18)$$

Hence a formal derivation yields that

$$\underline{\underline{C}}(z^{-1})\frac{\partial \underline{e}(t)}{\partial \underline{p}_B^T} = -\underline{f}_{\underline{u}}^T(t)\otimes\underline{\underline{I}}_q, \qquad (3.2\text{-}19)$$

$$\underline{\underline{C}}(z^{-1})\frac{\partial \underline{e}(t)}{\partial \underline{p}_A^T} = \underline{f}_{\underline{y}}^T(t)\otimes\underline{\underline{I}}_q, \qquad (3.2\text{-}20)$$

$$\underline{\underline{C}}(z^{-1})\frac{\partial \underline{e}(t)}{\partial \underline{p}_C^T} = -\underline{f}_{\underline{e}}^T(t)\otimes\underline{\underline{I}}_q. \qquad (3.2\text{-}21)$$

These moving-average filter equations already yield the mode of calculation of the partial derivatives of the $\underline{e}(t)$. By taking into account the relation (3.2-4), the three equations can be reduced to solve one equation:

$$\underline{\underline{C}}(z^{-1})\frac{\partial \underline{e}(t)}{\partial \underline{p}_{BAC}^T} = -\underline{f}^T(\underline{u},\underline{y},\underline{e},t)\otimes\underline{\underline{I}}_q. \qquad (3.2\text{-}22)$$

In order to interprete the filter equations, further to facilitate computer programming, it is worth expanding the relations by the particular elements of parameter matrices. By expanding the particular components we get that

$$\underline{\underline{C}}(z^{-1})\frac{\partial \underline{e}(t)}{\partial B_{kij}} = \begin{bmatrix} 0 \\ \vdots \\ -z^{-k}\,u_j(t) \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{i-th row}, \qquad (3.2\text{-}23)$$

$$\underline{\underline{C}}(z^{-1}) \frac{\partial \underline{e}(t)}{\partial A_{kij}} = \begin{bmatrix} 0 \\ \vdots \\ z^{-k}\, y_j(t) \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{i-th row}, \qquad (3.2\text{-}24)$$

$$\underline{\underline{C}}(z^{-1}) \frac{\partial \underline{e}(t)}{\partial C_{kij}} = \begin{bmatrix} 0 \\ \vdots \\ -z^{-k}\, e_j(t) \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{i-th row}. \qquad (3.2\text{-}25)$$

Here B_{kij}, A_{kij}, C_{kij} are the i, j-th elements of the corresponding $\underline{\underline{B}}$, $\underline{\underline{A}}$, $\underline{\underline{C}}$ matrices. It is obvious from the relations that

$$\frac{\partial \underline{e}(t)}{\partial B_{kij}} = \frac{\partial \underline{e}(t-k+1)}{\partial B_{lij}} \qquad (3.2\text{-}26)$$

which saves substantial calculation. The same holds also for the derivatives according to A_{kij} and C_{kij}. This means that alike with the single output systems, it is also here possible to form recursively partial derivatives. The relations in this respect are also presented in App. 12.

As the calculation of both $\hat{\underline{\underline{A}}}$ and the gradient requires the error vector $\underline{e}(t)$, it is reasonable to calculate it by a recursive model with state vectors:

$$\underline{x}(t) = \begin{bmatrix} -\underline{\underline{C}}_1 & \underline{\underline{I}} & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ -\underline{\underline{C}}_2 & \underline{\underline{0}} & \underline{\underline{I}} & \cdots & \underline{\underline{0}} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ -\underline{\underline{C}}_{n-1} & \underline{\underline{0}} & \underline{\underline{0}} & \cdots & \underline{\underline{I}} \\ -\underline{\underline{C}}_n & \underline{\underline{0}} & \underline{\underline{0}} & & \underline{\underline{0}} \end{bmatrix} \underline{x}(t-1) + \begin{bmatrix} \underline{\underline{A}}_1 & \underline{\underline{I}} - \underline{\underline{B}}_1 & \underline{\underline{B}}_0 \\ \underline{\underline{A}}_2 & \underline{\underline{0}} - \underline{\underline{B}}_2 & \underline{\underline{0}} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \underline{\underline{A}}_{n-1} & \underline{\underline{0}} - \underline{\underline{B}}_{n-1} & \underline{\underline{0}} \\ \underline{\underline{A}}_n & \underline{\underline{0}} - \underline{\underline{B}}_n & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} \underline{y}(t-1) \\ \underline{y}(t) \\ \underline{u}(t-1) \\ \underline{u}(t) \end{bmatrix}$$

(3.2-27)

and

$$\underline{e}(t) = \left[\underline{\underline{I}}, \underline{\underline{0}}, \ldots, \underline{\underline{0}}, \underline{\underline{0}}\right] \underline{x}(t). \qquad (3.2\text{-}28)$$

This latter equation pair can be considered as the generalization of the equations (2.2-8) and (2.2-9).

An i, j-th element of the matrix of the second order derivatives is obtained by the derivation of the i-th element of the gradient vector according to the j-th coefficient

$$\frac{\partial^2 \tilde{Q}(\underline{\underline{P}}_{BAC})}{\partial p_i \partial p_j} = \sum_{t=1}^{N} \frac{\partial \underline{e}^T(t)}{\partial p_i} \hat{\underline{\underline{\Lambda}}}^{-1} \frac{\partial \underline{e}(t)}{\partial p_j} +$$

$$+ \sum_{t=1}^{N} \underline{e}^T(t) \hat{\underline{\underline{\Lambda}}}^{-1} \frac{\partial^2 \underline{e}(t)}{\partial p_i \partial p_j} . \qquad (3.2\text{-}29)$$

Here the first term makes the always positive $\underline{\underline{T}}_1$ part of the Hessian matrix, while the second one, the $\underline{\underline{T}}_2$ part which sometimes changes even its definiteness. (Here the assumption $\underline{\underline{T}} = \underline{\underline{T}}_1 + \underline{\underline{T}}_2$ was used.) The $\underline{\underline{T}}_1$ matrix can be expressed also in a concise form:

$$\underline{\underline{T}}_1 = \sum_{t=1}^{N} \frac{d\underline{e}^T(t)}{d\underline{p}_{BAC}} \hat{\underline{\underline{\Lambda}}}^{-1} \frac{d\underline{e}(t)}{d\underline{p}^T_{BAC}} \quad . \qquad (3.2\text{-}30)$$

As the construction of $\underline{\underline{T}}_2$ is very complicated with the MIMO systems, it is reasonable to use the approximation $\underline{\underline{T}} \cong \underline{\underline{T}}_1$ which yields in the majority of the cases a satisfactory result. Although the relations of the calculation of $\underline{\underline{T}}_2$, are given in App.12., we did not use them in the practice because of their complexity. This means that our minimization process is an approximative NEWTON-RAPHSON technique. This version ensures, far from the minimum, practically the same convergence speed slows down with small samples, on the other hand the requirement of the investigation of the definiteness does not arise. For a large number of samples, the two versions of the method are practically equivalent with each other, because by increasing N, $\underline{\underline{T}}_2$ becomes asymptotically zero. (This has been already pointed out with the SISO systems.) Note furthermore that with the computer program written for the method, the NEWTON-RAPHSON searching technique is also combined with the determination of a suboptimal step as with the ML method of the SISO systems but, this part of the algorithm, will not be discussed here in detail.

3.3 The generalized least squares method

The GLS method of CLARKE worked out for the SISO systems [33] can be generalized also for MIMO systems, although this cannot be performed by simple formal tricks, e.g. by the appropriate extensions of the dimensions. The main reason for this is that the "whitening" filtration for the MIMO systems is more complicated than for the SISO systems.

In the system equation (1.2-26) we obtain the GLS structure of the MIMO system by introducing the matrix polynomial $\underline{\underline{H}}(z^{-1})$ which formally satisfies the equality $\underline{\underline{C}}(z^{-1}) = \underline{\underline{H}}^{-1}(z^{-1})$

$$\underline{y}(t) = \underline{\underline{A}}^{-1}(z^{-1})\,\underline{\underline{B}}(z^{-1})\,\underline{u}(t) + \underline{\underline{A}}^{-1}(z^{-1})\,\underline{\underline{H}}^{-1}(z^{-1})\,\underline{e}(t). \qquad (3.3\text{-}1)$$

The system equation can be written also in the rearranged form

$$\underline{\underline{H}}(z^{-1})\,\underline{y}(t) = \underline{\underline{H}}(z^{-1})\,\underline{\underline{B}}(z^{-1})\,\underline{u}(t) - \underline{\underline{H}}(z^{-1})\,\tilde{\underline{\underline{A}}}(z^{-1})\,\underline{y}(t) + \underline{e}(t). \qquad (3.3\text{-}2)$$

As the multiplication by $\underline{\underline{H}}(z^{-1})$ cannot be transposed in a general case either by $\underline{\underline{B}}(z^{-1})$, or by $\underline{\underline{A}}(z^{-1})$, thus we cannot construct the filtered values of $\underline{u}(t)$ and $\underline{y}(t)$ in the way usual and permissible with the SISO systems.

Let us write the equation (3.3-2) more in detail:

$$\underline{\underline{H}}(z^{-1})\,\underline{y}(t) = \underline{\underline{H}}(z^{-1})\,\underline{\underline{B}}_0\,\underline{u}(t) + \ldots + \underline{\underline{H}}(z^{-1})\,\underline{\underline{B}}_n\,\underline{u}(t-n) -$$

$$- \underline{\underline{H}}(z^{-1})\,\underline{\underline{A}}_1\,\underline{y}(t-1) - \ldots - \underline{\underline{H}}(z^{-1})\,\underline{\underline{A}}_n\,\underline{y}(t-n) + \underline{e}(t). \qquad (3.3\text{-}3)$$

By utilizing $\mathrm{vec}(\underline{\underline{A}}\ \underline{\underline{B}}\ \underline{\underline{C}}) = (\underline{\underline{C}}^{T} \otimes \underline{\underline{A}})\,\mathrm{vec}(\underline{\underline{B}})$, the terms in (3.3-3) can be written also in the following way:

$$\underline{\underline{H}}(z^{-1})\,\underline{\underline{A}}_i\,\underline{y}(t-i) = \left[\,\underline{y}^T(t-i) \otimes \underline{\underline{H}}(z^{-1})\right] \mathrm{vec}(\underline{\underline{A}}_i)$$

$$i=1,2,\ldots,n \qquad (3.3\text{-}4)$$

namely

$$\mathrm{vec}\left[\underline{\underline{H}}(z^{-1})\underline{\underline{A}}_i\ \underline{y}(t-i)\right] = \underline{\underline{H}}(z^{-1})\underline{\underline{A}}_i\ \underline{y}(t-i), \qquad (3.3\text{-}5)$$

as the vec(...) of a vector is itself.

In a similar way we obtain that

$$H(z^{-1})\underline{\underline{B}}_i\ \underline{u}(t-i) = \left[\,\underline{u}^T(t-i) \otimes \underline{\underline{H}}(z^{-1})\right] \mathrm{vec}(\underline{\underline{B}}_i) \qquad (3.3\text{-}6)$$

and

$$\underline{\underline{H}}(z^{-1})\,\underline{y}(t) = \left[\,\underline{y}^T(t) \otimes \underline{\underline{H}}(z^{-1})\right] \mathrm{vec}(\underline{\underline{I}}_q). \qquad (3.3\text{-}7)$$

Let $\underline{\underline{H}}(z^{-1})$ be of the following structure:

$$\underline{\underline{H}}(z^{-1}) = \underline{\underline{I}}_q + \underline{\underline{H}}_1 z^{-1} + \ldots + \underline{\underline{H}}_k z^{-k}, \qquad (3.3\text{-}8)$$

then, e.g.

$$\underline{y}^T(t-i) \otimes \underline{\underline{H}}(z^{-1}) = \left[y_1(t-i)\underline{\underline{I}}_q + \ldots + y_1(t-i-k)\underline{\underline{H}}_k,\right.$$

$$\left.\ldots, y_q(t-i)\underline{\underline{I}}_q + \ldots + y_q(t-i-k)\underline{\underline{H}}_k\right]. \qquad (3.3\text{-}9)$$

It is easy to see that the matrix elements can be calculated without difficulty. Let us apply the following notations for the matrices of the filtered values

$$\underline{\underline{Y}}_F(t-i) = \underline{y}^T(t-i) \otimes \underline{\underline{H}}(z^{-1}), \tag{3.3-10}$$

$$\underline{\underline{U}}_F(t-i) = \underline{u}^T(t-i) \otimes \underline{\underline{H}}(z^{-1}), \tag{3.3-11}$$

$$\underline{y}_F(t-i) = \left[\underline{y}^T(t) \otimes \underline{\underline{H}}(z^{-1})\right] \mathrm{vec}(\underline{\underline{I}}_q). \tag{3.3-12}$$

The system equation can be reduced by these notations to the form

$$\underline{y}_F(t) = \underline{\underline{U}}_F(t)\,\mathrm{vec}(\underline{\underline{B}}_0) + \ldots + \underline{\underline{U}}_F(t-n)\mathrm{vec}(\underline{\underline{B}}_n) -$$

$$- \underline{\underline{Y}}_F(t-1)\mathrm{vec}(\underline{\underline{A}}_1) - \ldots - \underline{\underline{Y}}_F(t-n)\,\mathrm{vec}(\underline{\underline{A}}_n) + \underline{e}(t) \tag{3.3-13}$$

or still more briefly

$$\underline{y}_F(t) = \underline{\underline{X}}_F(t)\underline{p}_{BA} + \underline{e}(t). \tag{3.3-14}$$

Here $\underline{p}_{BA} = \mathrm{vec}(\underline{\underline{P}}_{BA})$ holds also hereinafter, as well as

$$\underline{\underline{X}}_F(t) = \left[\underline{\underline{U}}_F(t), \ldots, \underline{\underline{U}}_F(t-n); -\underline{\underline{Y}}_F(t-1), \ldots, -\underline{\underline{Y}}_F(t-n)\right]. \tag{3.3-15}$$

If N measurements are available, then by using the notations

$$\underline{w}_F = \left[\underline{y}_F^T(1), \underline{y}_F^T(2), \ldots, \underline{y}_F^T(N)\right]^T \tag{3.3-16}$$

$$\underline{\underline{F}}_F = \left[\underline{\underline{X}}_F^T(1), \underline{\underline{X}}_F^T(2), \ldots, \underline{\underline{X}}_F^T(N)\right]^T \tag{3.3-17}$$

$$\underline{d} = \left[\underline{e}^T(1), \underline{e}^T(2), \ldots, \underline{e}^T(N)\right]^T \tag{3.3-18}$$

the joint system equation will be

$$\underline{w}_F = \underline{\underline{F}}_F \underline{p}_{BA} + \underline{d}. \tag{3.3-19}$$

As in (3.3-14) - in consequence of the filtering - the equation error is white noise, we can use the LS method which leads on the basis of the equation (3.3-19) to the well-known explicit solution

$$\hat{\underline{P}}_{BA} = (\underline{\underline{F}}_F^T \underline{\underline{F}}_F)^{-1} \underline{\underline{F}}_F^T \underline{w}_F \cdot \qquad (3.3\text{-}20)$$

The filtering equations (3.3-10) - (3.3-12) require by all means the knowledge of $\underline{\underline{H}}(z^{-1})$, therefore let us examine the estimation possibility of the parameter matrices $\underline{\underline{H}}_i$. Assume the knowledge of $\hat{\underline{\underline{P}}}_{BA}$.

Rewrite the system equation (3.3-1) into the form

$$\underline{\underline{H}}(z^{-1})\left[\underline{\underline{A}}(z^{-1})\underline{y}(t)-\underline{\underline{B}}(z^{-1})\underline{u}(t)\right]= \underline{\underline{H}}(z^{-1})\underline{r}(t) = \underline{e}(t) \qquad (3.3\text{-}21)$$

where

$$\underline{r}(t) = \underline{\underline{A}}(z^{-1})\underline{y}(t)-\underline{\underline{B}}(z^{-1})\underline{u}(t) = \underline{y}(t)- \hat{\underline{\underline{P}}}_{BA}\, \underline{f}(\underline{u},\underline{y},t) \qquad (3.3\text{-}22)$$

denotes the equation error vector (cf. also the notations in (3.1-2)!). Considering the construction of $\underline{\underline{H}}(z^{-1})$, the autoregressive $\underline{r}(t)$ will be as follows:

$$\underline{r}(t) = -\underline{\underline{H}}_1\, \underline{r}(t-1)-\ldots-\underline{\underline{H}}_k\, \underline{r}(t-k)+\underline{e}(t). \qquad (3.3\text{-}23)$$

This is a system equation linear in parameter matrices with white noise equation error, thus the LS estimation of the form (3.1-16) can be employed here too, i.e.:

$$\hat{\underline{\underline{P}}}_H = \underline{\underline{R}}\, \underline{\underline{G}}^T(\underline{\underline{G}}\, \underline{\underline{G}}^T)^{-1} \qquad (3.3\text{-}24)$$

where

$$\underline{\underline{P}}_H = \left[\underline{\underline{H}}_1, \underline{\underline{H}}_2, \ldots, \underline{\underline{H}}_k \right] , \tag{3.3-25}$$

$$\underline{\underline{R}} = \left[\underline{r}(1), \underline{r}(2), \ldots, \underline{r}(N) \right], \tag{3.3-26}$$

$$\underline{\underline{G}} = \left[\underline{g}(1), \underline{g}(2), \ldots, \underline{g}(N) \right], \tag{3.3-27}$$

and

$$\underline{g}(t) = \left[-\underline{r}^T(t-1), \ldots, -\underline{r}^T(t-k) \right] . \tag{3.3-28}$$

As according to (3.3-22) $\underline{r}(t)$ depends directly only on $\hat{\underline{\underline{P}}}_{BA}$ and not on $\hat{\underline{\underline{P}}}_H$, we can also now develop the well-known iteration (successive approximation) technique of the GLS method elaborated for the SISO systems. This means that first the equality $\underline{\underline{H}}(z^{-1}) = \underline{\underline{I}}_q$ is assumed and thus $\hat{\underline{p}}_{BA}$ (resp. $\hat{\underline{\underline{P}}}_{BA}$) is estimated. That is a simple LS estimation. Thereupon the $\underline{r}(t)$ vectors are calculated, the matrices $\underline{\underline{R}}$ and $\underline{\underline{G}}$ are constructed, thereafter the estimation $\underline{\underline{P}}_H$ is determined. With the obtained $\underline{\underline{H}}(z^{-1})$ the filtering is carried out and the estimation $\hat{\underline{p}}_{BA}$ is updated.

It should now be stressed again that the basic difference from the GLS method worked out for SISO systems consisted now in that the multiplication by $\underline{\underline{H}}(z^{-1})$, that is the filtering, could not be replaced by other matrix polynomials. This accounted for the need of the introduction of the special filter equations (3.3-10) - (3.3-12).

3.4 Extended matrix methods

The extended matrix methods are virtually based on LS technique, so that their generalization for MIMO systems can be simply achieved. As the method to be presented now can be employed for every extended matrix method elaborated for the SISO systems, therefore the generalization for MIMO systems of only the SEXM method, as entitled to the most widespread application, will be shown in detail.

The equation (1.2-30) of the MIMO system can be written also in the form of

$$\underline{y}(t) = \underline{\underline{P}}_{BAC}\, \underline{f}(\underline{u},\underline{y},\underline{e},t) + \underline{e}(t) \qquad (3.4\text{-}1)$$

where

$$\underline{\underline{P}}_{BAC} = [\underline{\underline{B}}_o, \underline{\underline{B}}_1, \ldots, \underline{\underline{B}}_n, \underline{\underline{A}}_1, \ldots, \underline{\underline{A}}_n, \underline{\underline{C}}_1, \ldots, \underline{\underline{C}}_n] \qquad (3.4\text{-}2)$$

and

$$\underline{f}(\underline{u},\underline{y},\underline{e},t) = [\underline{u}^T(t), \ldots, \underline{u}^T(t-n); -\underline{y}^T(t-1), \ldots, -\underline{y}^T(t-n); \underline{e}^T(t-1), \ldots, \underline{e}^T(t-n)]^T = \underline{f}(t). \qquad (3.4\text{-}3)$$

The joint matrix equation relating to the N samples is

$$\underline{\underline{Y}} = \underline{\underline{P}}_{BAC}\, \underline{\underline{F}}_{UYE} + \underline{\underline{E}}\,. \qquad (3.4\text{-}4)$$

Here

$$\underline{\underline{F}}_{UYE} = [\underline{f}(1), \ldots, \underline{f}(N)] \qquad (3.4\text{-}5)$$

furthermore the definitions of $\underline{\underline{Y}}$ and $\underline{\underline{E}}$ are according to subsection 3.1.

Formally (3.4-4) is equivalent with (3.1-8), so that we can employ also now the LS estimation of the form (3.1-16) according to which

$$\hat{\underline{\underline{P}}}_{BAC} = \underline{\underline{Y}}\ \underline{\underline{F}}_{UYE}^{T}\ (\underline{\underline{F}}_{UYE}\ \underline{\underline{F}}_{UYE}^{T})^{-1} \qquad (3.4\text{-}6)$$

As $\underline{f}(\underline{u},\underline{y},\underline{e},t)$ and so $\underline{\underline{F}}_{UYE}$ also contain the values $\underline{e}(t)$, which we do not know in advance, only an iterative solution can be applied. In order to use (3.4-6) an $\underline{e}(t)$ series (e.g. zeros corresponding to the LS estimate) have to be assumed at the start, thereafter with the help of the estimated parameter matrix $\hat{\underline{\underline{P}}}_{BAC}$, a new estimation can be made for the $\underline{e}(t)$-s in the form of

$$\hat{\underline{e}}(t) = \underline{y}(t) - \hat{\underline{\underline{P}}}_{BAC}\ \underline{f}(\underline{u},\underline{y},\underline{e},t). \qquad (3.4\text{-}7)$$

The iterative process will be carried until the attainment of a given accuracy as with the SISO systems.

3.5 The priori knowledge fitting method

As the PKF method worked out for SISO systems actually uses the LS technique with certain constraints, it is much easier to be extended to MIMO systems than the GLS or ML methods, where the generalization of the autoregressive or moving average filtering caused difficulties [136].

The MIMO generalization of the SISO system equation (2.6-1) is:

$$\underline{y}(t) = \underline{\underline{A}}^{-1}(z^{-1})\underline{\underline{B}}(z^{-1})\underline{u}(t) + \underline{w}(t) \qquad (3.5\text{-}1)$$

i.e. coincides with the MIMO models presented until now, only it does not assume a separate noise model for the output noise $\underline{w}(t)$. This system equation can be written also in the form

$$\underline{y}(t) = \sum_{i=0}^{n} \underline{\underline{B}}_i \, \underline{u}(t-i) - \sum_{i=1}^{n} \underline{\underline{A}}_i \, \underline{y}(t-i) + \underline{r}(t) =$$

$$= \underline{\underline{P}}_{BA} \, \underline{f}(\underline{u},\underline{y},t) + \underline{r}(t) \qquad (3.5\text{-}2)$$

where

$$\underline{r}(t) = \underline{w}(t) + \sum_{i=1}^{n} \underline{\underline{A}}_i \, \underline{w}(t-i). \qquad (3.5\text{-}3)$$

According to the PKF method, the parameters of the model have to be fitted to the measured values, so as to fulfil at the same time (as a constraint) the uncorrelatedness of the output noise vector (as a calculated value) and the input signal vector. Assume of the $\underline{w}(t)$ output noise, similarly to the SISO systems, that it is ergodic, has zero mean and is a random stochastic vector variable uncorrelated with the input vector [103, 136].

It is easy to see that the uncorrelatedness of $\underline{r}(t)$ and $\underline{u}(t)$ follows from the uncorrelatedness of $\underline{w}(t)$ and $\underline{u}(t)$ as $\underline{r}(t)$ is the linear combination of the values $\underline{w}(t)$ of preceding instants. Assume the contrary, i.e. the occurrence of terms dependent from or correlated with $\underline{u}$ in the difference $\underline{y}(t) - \underline{P}_{BA}\ \underline{f}(\underline{u},\underline{y},t)$ yielding the residuals. Assume this dependence to be linear, then according to our latter statement the relation

$$\underline{y}(t)-\underline{\underline{P}}_{BA}\ \underline{f}(\underline{u},\underline{y},t) = \underline{r}(t)+ \underline{\underline{P}}_{k}\ \underline{g}(\underline{u},t)=$$

$$= \underline{r}(t) + \sum_{i=0}^{s} \underline{\underline{K}}_{i}\ \underline{u}(t-i) \qquad (3.5\text{-}4)$$

had to be fulfilled. Here

$$\underline{\underline{P}}_{k} =\left[\underline{\underline{K}}_{0}, \underline{\underline{K}}_{1},\ldots,\underline{\underline{K}}_{s}\right] \qquad (3.5\text{-}5)$$

and

$$\underline{g}(\underline{u},t) =\left[\underline{u}^{T}(t),\underline{u}^{T}(t-1),\ldots,\underline{u}^{T}(t-s)\right]^{T}. \qquad (3.5\text{-}6)$$

As a result of the identification we have to get a $\hat{\underline{\underline{P}}}_{BA}$ estimation for which $\underline{\underline{P}}_{k} = \underline{\underline{0}}$ or at least

$$Q(\underline{\underline{P}}_{BA}) = \frac{1}{2}\left[\mathrm{vec}(\underline{\underline{P}}_{k})\right]^{T}\left[\mathrm{vec}(\underline{\underline{P}}_{k})\right]= \frac{1}{2}\ \mathrm{tr}(\underline{\underline{P}}_{k}^{T}\ \underline{\underline{P}}_{k}) \qquad (3.5\text{-}7)$$

is minimal.

On the basis of (3.5-4), the joint matrix equation for N samples

$$\underline{\underline{Y}} - \underline{\underline{P}}_{BA}\ \underline{\underline{F}} = \underline{\underline{P}}_{k}\ \underline{\underline{G}} + \underline{\underline{R}} \qquad (3.5\text{-}8)$$

where

$$\underline{\underline{Y}} = [\underline{y}(1),\ldots,\underline{y}(N)], \tag{3.5-9}$$

$$\underline{\underline{R}} = [\underline{r}(1),\ldots,\underline{r}(N)], \tag{3.5-10}$$

$$\underline{\underline{F}} = [\underline{f}(1),\ldots,\underline{f}(N)], \tag{3.5-11}$$

and

$$\underline{\underline{G}} = [\underline{g}(1),\ldots,\underline{g}(N)]. \tag{3.5-12}$$

By formal comparison of the equations (3.1-8) and (3.5-8), we can simply set the LS estimation according to (3.1-16) for the auxiliary parameter matrix $\underline{\underline{P}}_k$

$$\hat{\underline{\underline{P}}}_k = (\underline{\underline{Y}} - \underline{\underline{P}}_{BA}\ \underline{\underline{F}})\ \underline{\underline{G}}^T(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}. \tag{3.5-13}$$

Substitute now $\hat{\underline{\underline{P}}}_k$ into (3.5-7) and perform the minimization according to $\underline{\underline{P}}_{BA}$. We get according to App. 13. that

$$\hat{\underline{\underline{P}}}_{BA} = \underline{\underline{Y}}\ \underline{\underline{G}}^T(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}\ \underline{\underline{G}}\ \underline{\underline{F}}^T\ *$$

$$*[\underline{\underline{F}}\ \underline{\underline{G}}^T(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}\ \underline{\underline{G}}\ \underline{\underline{F}}^T]^{-1}, \tag{3.5-14}$$

which is an explicit expression for the estimated parameter matrix [136].

IV. ON-LINE IDENTIFICATION METHODS FOR SINGLE OUTPUT SYSTEMS

As already mentioned above, off-line and on-line identifications have to be distinguished with respect to data processing. We speak of off-line identification if the method is based on the simultaneous presence of data. From the point of view of loss function minimization this is a deterministic problem (for the same parameter vector the loss function has always the same value for a given measurement data set). On the other hand, the on-line method, by means of direct contact with the process, updates the previous results of the estimation according to the new measurements. With the on-line strategies beside the estimated parameters belonging to the preceding samples, there are also required such auxiliary quantities, concentrating in themselves the information of the preceding measurements and renewable by new measurements, by means of which the parameter estimates can be corrected. From the point of view of the loss function minimization the on-line methods are stochastic processes as in the case of the same parameter vector the loss function can have different values depending on the new data. The on-line method is called recursive if a procedure is obtained which yields, after processing N related input-output data pairs, the same result as the off-line evaluation of N samples.

As the on-line methods minimize loss functions, where the value of the function is disturbed by stochastic disturbances, these procedures can be discussed uniformly by the stochastic approximation. From the various approaches [12, 37, 38, 47, 113, 119, 126] the TZYPKIN approach will be used as being the best applicable for identification tasks and nearest to the engineering concept [37, 38]. One of the possible general forms of the step-by-step solution formula of the stochastic approximation is the following:

$$\underline{p}(t) = \underline{p}(t-1) + \underline{k}_t[\underline{x}(t), \underline{p}(t-1)]. \qquad (4\text{-}1)$$

Here t, the subscript of the steps of stochastic approximation continues to denote discrete time with the on-line methods. $\underline{p}$ denotes the parameters to be determined, $\underline{x}$ the vector of the stochastic observations. $\underline{k}_t$ means the correction function.

According to TZYPKIN [37], if the general step-by-step formula (4-1) can be transformed to the canonical relation

$$\underline{p}(t) = \underline{p}(t-1) - \underline{\underline{R}}(t)\underline{g}[\underline{x}(t), \underline{p}(t-1)], \qquad (4\text{-}2)$$

this latter can be attributed with an expressive meaning. If namely we are facing the task

$$J(\underline{p}^*) = \min_{\underline{p}} E_{\underline{x}} \{Q[\underline{x}(t), \underline{p}(t)]\} \qquad (4\text{-}3)$$

then the algorithm (4-2) ensures to find the minimum if Q is unimodal [90] and the convergence or weighting matrix $\underline{\underline{R}}(t)$ meets certain conditions. This requires the coincidence of $\underline{g}$ with the gradient of Q, as a realization, by $\underline{p}$. Here $E_{\underline{x}}\{...\}$ denotes the expectation according to $\underline{x}$. The train of thought can be followed also if it is converted the solution of the stochastic extremum seeking problem (4-3) can be obtained by iteration algorithms of form (4-2).

In this report the analysis of the necessary and sufficient conditions of the convergence is not discussed, we refer only to [37, 38, 90], according to which the maximum eigenvalue of $\underline{\underline{R}}(t)$ has to tend to zero in a prescribed extent. Although the stochastic approximation and its mathematical background are covered by an extensive literature, only very few authors deal with its identification applications, with special regard to the convergence behaviours [90, 102, 113].

In this subsection the applicability of the canonical algorithm (4-2) for identification methods will be presented on the one hand, and in some cases new algorithms also elaborated by the use of formal analogies, on the other.

With the methods of the stochastic approximation (hereinafter STA) the loss function to be minimized has the form

$$J[\underline{p}(t)] = E_{\underline{x}}\{Q[\underline{x}(t),\underline{p}(t)]\} \tag{4-4}$$

i.e. it is the expected value of the stochastic function $Q[\underline{x}(t).\underline{p}(t)]$. In the majority of the practical cases, by assuming the stationary $\underline{x}(t)$ random process rather the loss function

$$J[\underline{p}(t)] = M_{\underline{x}}\{Q[\underline{x}(t),\underline{p}(t)]\} \tag{4-5}$$

is applied. Here $M_{\underline{x}}\{...\}$ denotes the average value in time, according to $\underline{x}$.

The canonical form of the algorithm yielding the solution [37]

$$\underline{p}(t) = \underline{p}(t-1) - \underline{\underline{R}}(t)\,\frac{dQ[\underline{x}(t),\ \underline{p}(t-1)]}{d\underline{p}(t-1)} \tag{4-6}$$

or in another way

$$\Delta\underline{p}(t-1) = \underline{p}(t)-\underline{p}(t-1) = -\underline{\underline{R}}(t)\,\frac{dQ[\underline{x}(t),\underline{p}(t-1)]}{d\underline{p}(t-1)}\,. \tag{4-7}$$

It is easy to see that the preceding equations are formally completely identical with those of the methods employing the gradient applicable for deterministic minimization tasks. As

to their content, the meaning of the equations is, on the other hand, of much greater importance. The equation (4-6) means that although the loss function to be minimized is stochastic and Q is now a stochastic variable, it is sufficient, to utilize the gradient of a realization of Q (with a given $\underline{x}(t)$) for the minimum seeking. The conditions of convergence have already been mentioned [37, 90].

In App.14 it is deduced that the seeking method optimal in quadratic sense can be ensured by the convergence matrix

$$\underline{\underline{R}}(t)=\Big[\sum_{j=1}^{t} \underline{\underline{H}}\{Q[\underline{x}(j),\underline{p}(t-1)]\}\Big]^{-1} . \qquad (4-8)$$

Here $\underline{\underline{H}}$ means the Hessian matrix formed from the second derivatives of Q by $\underline{p}(t-1)$ so that

$$\underline{\underline{H}}\{Q[\underline{x}(j),\underline{p}(t-1)]\} = \frac{d^2Q[\underline{x}(j),\underline{p}(t-1)]}{d\underline{p}(t-1)\, d\,\underline{p}^{T}(t-1)} . \qquad (4-9)$$

Note that in $\underline{\underline{R}}(t)$ the argument of $\underline{p}$ does not depend on the running subscript j.

In the following the on-line algorithms relating to the various models and basic identification methods will be discussed on the basis of the algorithm (4-6). This approach can, in our opinion, be considered more general than the transformation of the off-line methods into recursive form by using various identities. If namely there exists a recursive solution, then STA with the optimal $\underline{\underline{R}}(t)$ yields likewise the recursive on-line algorithm (cf.e.g. the recursive on-line LS method).

It is important to remark that for the purpose of identification not only the optimal STA algorithm, but also a convergence matrix or scalar convergence coefficient of another construction can be used. In these cases the convergence can also be assured but with different convergence speed. [37, 90]

Note further that a procedure completely identical with the STA algorithm can be used also for solving off-line identification problems, but the task is here basically deterministic and therefore the method, too, can be considered as a version of the gradient methods.

4.1 The on-line least squares method

We have seen in subsection 2.1 that the system equation for the LS method has the form

$$y(t)= \underline{f}^T(u,y,t)\ \underline{p}_{ba} + e(t)= \underline{f}^T(t)\underline{p}_{ba} + e(t). \qquad (4.1\text{-}1)$$

Thus choose for Q the quantity

$$Q = Q[\underline{x}(t),\underline{p}(t)]= \frac{1}{2}\, e^2(t)= \frac{1}{2}[y(t)- \underline{f}^T(t)\underline{p}_{ba}]^2 \qquad (4.1\text{-}2)$$

which corresponds to the least squares principle. Now

$$\underline{x}(t)=[u(t),y(t)]^T, \qquad (4.1\text{-}3)$$

which is of common construction for the SISO system with the on-line algorithms [77].

Determine the quantities required for the algorithm (4-6). Form first the gradient vector:

$$\frac{dQ[\underline{x}(t),\underline{p}_{ba}(t-1)]}{d\underline{p}_{ba}(t-1)} = -\underline{f}(t)[y(t)- \underline{f}^T(t)\underline{p}_{ba}(t-1)]\ . \qquad (4.1\text{-}4)$$

By deriving again the expression (4.1-4), we obtain that

$$\frac{d^2Q[\underline{x}(j),\underline{p}_{ba}(t-1)]}{d\underline{p}_{ba}(t-1)\, d\underline{p}^T_{ba}(t-1)} = \underline{f}(j)\underline{f}^T(j)= \underline{f}(u,y,j)\underline{f}^T(u,y,j). \qquad (4.1\text{-}5)$$

Thus on the basis of (4-8) the optimal convergence matrix is

$$\underline{\underline{R}}(t) =\Big[\sum_{j=1}^{t} \underline{f}(j)\underline{f}^T(j)\Big]^{-1}. \qquad (4.1\text{-}6)$$

We deduce in App.15 that $\underline{\underline{R}}(t)$ can in this case be recursively formed

$$\underline{\underline{R}}(t) = \underline{\underline{R}}(t-1) - \frac{\underline{\underline{R}}(t-1)\underline{f}(t)\underline{f}^T(t)\underline{\underline{R}}(t-1)}{1+\underline{f}^T(t)\underline{\underline{R}}(t-1)\underline{f}(t)} \qquad (4.1\text{-}7)$$

where it must not be forgotten that $\underline{\underline{R}}(t)$ is, on the basis of (4.1-6), a symmetric matrix.

The on-line least squares method requires therefore the step-by-step updating of $\underline{\underline{R}}(t)$ according to (4.1-7), furthermore the form of the relation (4-6) is now by considering (4.1-4)

$$\underline{p}_{ba}(t) = \underline{p}_{ba}(t-1) + \underline{\underline{R}}(t)\underline{f}(t)\left[y(t)-\underline{f}^T(t)\underline{p}_{ba}(t-1)\right]. \qquad (4.1\text{-}8)$$

The off-line LS estimation for N data pairs is yielded by the relation (2.1-7) in the form

$$\underline{p}_{ba}(N) = \left[\underline{\underline{F}}^T_{uy}(N)\underline{\underline{F}}_{uy}(N)\right]^{-1} \underline{\underline{F}}^T_{uy}(N)\underline{y}_N \,. \qquad (4.1\text{-}9)$$

The number of the processed samples was denoted here by N. Assume now that the solution according to (4.1-9) is available. The equation yielding the LS estimation for the (N+1)-th measurements u(N+1), y(N+1) can be given by using $\underline{f}(N+1)$ formed according to (2.1-3),

$$\underline{p}_{ba}(N+1) = \left[\underline{\underline{F}}^T_{uy}(N+1)\underline{\underline{F}}_{uy}(N+1)\right]^{-1} \underline{\underline{F}}^T_{uy}(N+1)\underline{y}_{N+1} =$$

$$= \left\{\begin{bmatrix}\underline{\underline{F}}_{uy}(N)\\ \underline{f}^T(N+1)\end{bmatrix}^T \begin{bmatrix}\underline{\underline{F}}_{uy}(N)\\ \underline{f}^T(N+1)\end{bmatrix}\right\}^{-1} \begin{bmatrix}\underline{\underline{F}}_{uy}(N)\\ \underline{f}^T(N+1)\end{bmatrix}^T \begin{bmatrix}\underline{y}_N\\ y(N+1)\end{bmatrix} =$$

$$= \left[\underline{\underline{F}}^T_{uy}(N)\underline{\underline{F}}_{uy}(N) + \underline{f}(N+1)\,\underline{f}^T(N+1)\right]^{-1}\left[\underline{\underline{F}}^T_{uy}(N)\underline{y}_N + \underline{f}(N+1)\,y(N+1)\right].$$

(4.1-10)

On the basis of the Eqs. (2.1-8), (4.1-6), (A.15-1) and (A.15-6) we can write that

$$\left[\underline{\underline{F}}_{uy}^{T}(N+1)\,\underline{\underline{F}}_{uy}(N+1)\right]^{-1}$$

$$=\left[\underline{\underline{F}}_{uy}^{T}(N)\underline{\underline{F}}_{uy}(N)\right]^{-1}-\frac{\left[\underline{\underline{F}}_{uy}^{T}(N)\underline{\underline{F}}_{uy}(N)\right]^{-1}\,\underline{f}(N+1)\,\underline{f}^{T}(N+1)\left[\underline{\underline{F}}_{uy}^{T}(N)\underline{\underline{F}}_{uy}(N)\right]^{-}}{1+\underline{f}^{T}(N+1)\left[\underline{\underline{F}}_{uy}^{T}(N)\underline{\underline{F}}_{uy}(N)\right]^{-1}\,\underline{f}(N+1)} \qquad (4.1\text{-}11)$$

Hence we obtain by the notation

$$\underline{\underline{R}}(N)=\left[\underline{\underline{F}}_{uy}^{T}(N)\underline{\underline{F}}_{uy}(N)\right]^{-1} \qquad (4.1\text{-}12)$$

the direct relation between the recursive least squares method and the stochastic approximation.

It is deduced in App.16 that the recursive form of (4.1-10)

$$\underline{p}_{ba}(N+1)=\underline{p}_{ba}(N)+\left[\underline{\underline{F}}_{uy}^{T}(N+1)\,\underline{\underline{F}}_{uy}(N+1)\right]^{-1}\,\underline{f}(N+1)\left[y(N+1)-\underline{f}^{T}(N+1)\,\underline{p}_{ba}(N\right. \qquad (4.1\text{-}13)$$

formally exactly corresponds to the Eq. (4.1-8) obtained by STA using the notation (4.1-12) if the $\underline{\underline{R}}(t)$ convergence matrix is optimized.

The following should be noted of the choice of the initial values $\underline{\underline{R}}(0)$ and $\underline{p}_{ba}(0)$. The numerical examination of many authors [37, 81] has found that the convergence of the on--line LS estimation is practically independent from the starting point $\underline{p}_{ba}(0)$. On the one hand we can choose a matrix calculated from a preliminary off-line estimate or stored data according to the relation (4.1-12), on the other hand, we achieve a satisfactory result also with a suffi-

ciently large matrix $\underline{\underline{R}}(0) = \text{const.}\underline{\underline{I}}$.

The above presented equivalence of the recursive LS method and the optimal STA algorithm means that by starting from identical initial conditions $\underline{p}_{ba}(0)$ and $\underline{\underline{R}}(0)$ the two methods yield the same result.

In discussing the on-line LS estimation algorithm, we have to mention the application of the forgetting factor, resp. the various forgetting strategies [15, 47]. If, namely instead of the matrix

$$\underline{\underline{F}}_{uy}(N+1) = \begin{bmatrix} \underline{\underline{F}}_{uy}(N) \\ \underline{f}^{T}(N+1) \end{bmatrix} \qquad (4.1\text{-}14)$$

the expansion of the matrix $\underline{\underline{F}}_{uy}(t)$ is formed according to the relation

$$\underline{\underline{F}}^{*}_{uy}(N+1) = \begin{bmatrix} \rho(N+1)\,\underline{\underline{F}}_{uy}(N) \\ \underline{f}^{T}(N+1) \end{bmatrix} \qquad (4.1\text{-}15)$$

then it will be possible to take the previous measurement results into account with a weight less than 1 by applying the forgetting factor $0 \leq \rho(t) \leq 1$. If $\rho(t)$ is independent of time ($\rho = \text{const}$), then we have the so-called exponential forgetting [130]. The appropriate choice of the time function $\rho(t)$ can yield, of course, manifold forgetting strategies. The application of a constant ρ corresponds to a weighted LS estimation achieved by the weight matrix

$$\underline{\underline{Z}}_{t}^{-1} = \text{diag} < \rho^{t-1}, \rho^{t-2}, \ldots, \rho, 1> \; . \qquad (4.1\text{-}16)$$

(Here subscript t refers to the size of $\underline{\underline{Z}}$.)

In case of ρ = const it is easy to see [130] that in on-line LS algorithms only the relation with the purpose to update $\underline{\underline{R}}(t)$ is modified

$$\underline{\underline{R}}(t)= \frac{1}{\rho 2}\left[\underline{\underline{R}}(t-1)- \frac{\underline{\underline{R}}(t-1)\,\underline{f}(t)\underline{f}^{T}(t)\underline{\underline{R}}(t-1)}{\rho^{2}+ \underline{f}^{T}(t)\underline{\underline{R}}(t-1)\,\underline{f}(t)} \right]. \qquad (4.1\text{-}17)$$

The application of the forgetting factor - by forgetting the previous measurements - makes the on-line LS method in fact adaptive, i.e. enables the changes having occurred in the process parameters to be followed up.

4.2 The on-line maximum likelihood method

On the basis of the loss function (2.2-3) of the off-line ML method the momentary loss function to be applied for the on--line STA algorithms is

$$Q(t)= \frac{1}{2}\, e^2(t). \tag{4.2-1}$$

The STA algorithm optimal in quadratic sense requires the gradient of $Q(t)$ and the Hessian matrix. On the basis of the relation (2.2-12) and the respective equations

$$\frac{de(t)}{dp_{bac}} = -\underline{f}(u^F,y^F,e^F,t) \tag{4.2-2}$$

where

$$\begin{aligned}\underline{f}(u^F,y^F,e^F,t)= \\ =\big[&u^F(t-1),\ldots,u^F(t-n);-y^F(t-1),\ldots,-y^F(t-n); \\ &e^F(t-1),\ldots,e^F(t-n)\big]^T.\end{aligned} \tag{4.2-3}$$

Here

$$u^F(t)= \frac{1}{C(z^{-1})}\, u(t);y^F(t)= \frac{1}{C(z^{-1})}\, y(t);e^F(t)= \frac{1}{C(z^{-1})}\, e(t). \tag{4.2-4}$$

Thus

$$\frac{dQ(t)}{dp_{bac}} = \frac{de(t)}{dp_{bac}}\, e(t) = -\underline{f}(u^F,y^F,e^F,t)\, e(t), \tag{4.2-5}$$

where

$$e(t) = y(t) - \underline{f}^T(u,y,e,t)\underline{p}_{bac} \qquad (4.2\text{-}6)$$

and the structure of $\underline{f}(u,y,e,t)$ is according to (2.2-14).

According to the canonical algorithm (4-6) of the STA the quantities according to the relations (4.2-5) and (4.2-6) are to be computed on the basis of the newest (at the moment t) observations and the preceding (i.e. known heretofore as best) parameter estimates $\underline{p}_{bac}(t-1)$. The on-line verison of the above discussed off-line methods of the discrete-time models can - apart from the LS method - be elaborated only by certain approximations just by reason of this prior fact. The problem is caused by the occurrence of the parameter-dependent time function in the momentary gradient (and the same way, of course, with the Hessian matrix). Then namely beside the current parameter vector, the complete realization of the given variable had to be calculated by starting from the initial conditions of the system in order to obtain the theoretically exact values. In the present case the values $u^F(t), y^F(t), e^F(t)$ and $e(t)$ had also to be calculated by the parameter vector $\underline{p}_{bac}(t-1)$. This, on the other hand, would require the storing of all previous measurements which is obviously impossible in the on-line procedure. As an approximate solution a method can be chosen according to which the above-mentioned auxiliary quantities are generated recursively (this can always be done, for they can be written with linear difference equations) and in every step the parameter values $\underline{p}_{bac}(t-1)$ will be considered only in such parts of the recursive equations which relates to the updating. If the on-line procedure is really convergent, then this type of approach will be asymptotically perfect and the algorithm tends toward the canonical algorithm of STA.

The algorithm of the on-line ML method takes the following form according to our assumptions up till now

$$\underline{p}_{bac}(t)=\underline{p}_{bac}(t-1)+\underline{\underline{R}}(t)\underline{f}(u^F,y^F,e^F,t)\left[y(t)-\underline{f}^T(u,y,e,t)\underline{p}_{bac}(t-1)\right]. \tag{4.2-7}$$

Here the updating of the auxiliary quantities is made as follows (cf. the equations in subsection 2.2):

$$\underline{f}(u^F,y^F,e^F,t)=\underline{\underline{\Omega}}_1\left[\underline{p}_{bac}(t-1)\right]\underline{f}(u^F,y^F,e^F,t-1)+\underline{\omega}_1(t) \tag{4.2-8}$$

which can be detailed after (2.2-26). The e(t) can be calculated recursively on the basis of (2.2-8) and (2.2-9). Also in these relations $\underline{p}_{bac}(t-1)$ has to be used.

Thereafter let us investigate the formation of the matrix of the second derivatives. It can be established from (2.2-27) that here the generation of the parameter-dependent auxiliary quantities is also needed. Therefore we have to use the same approximations discussed with the formation of the gradient, i.e. only the updating of the auxiliary quantities is made by the last estimated parameter values.

New difficulties are met when forming $\underline{\underline{R}}(t)$ on the basis of the second derivatives. The Hessian matrix according to (2.2-27) cannot be formed recursively, only its first part, the $\underline{\underline{T}}_1$ determined by a diadic sum. When calculating the momentary second derivatives, it is therefore reasonable to take advantage of the approximation

$$\underline{\underline{H}}\{Q(t)\}=\frac{d^2Q(t)}{d\underline{p}_{bac}\,d\underline{p}_{bac}^T}\cong\frac{de(t)}{d\underline{p}_{bac}}\cdot\frac{de(t)}{d\underline{p}_{bac}^T}=$$

$$=\underline{f}(u^F,y^F,e^F,t)\underline{f}^T(u^F,y^F,e^F,t). \tag{4.2-9}$$

Thus

$$\underline{\underline{R}}(t)=\Big[\sum_{i=1}^{t}\underline{f}(u^F,y^F,e^F,i)\underline{f}^T(u^F,y^F,e^F,i)\Big]^{-1}, \qquad (4.2\text{-}10)$$

which can already be formed recursively if the vector $\underline{p}_{bac}(t-1)$ is considered only for the updating of the recursive equation. Thus, on the basis of the formal analogy with (4.1-7)

$$\underline{\underline{R}}(t)=\underline{\underline{R}}(t-1)-\frac{\underline{\underline{R}}(t-1)\underline{f}(u^F,y^F,e^F,t)\underline{f}^T(u^F,y^F,e^F,t)\underline{\underline{R}}(t-1)}{1+\underline{f}^T(u^F,y^F,e^F,t)\underline{\underline{R}}(t-1)\underline{f}(u^F,y^F,e^F,t)}. \qquad (4.2\text{-}11)$$

As in this relation only the dependence of $f(u^F,y^F,e^F,t)$ on $\underline{p}_{bac}(t-1)$ is assumed, here also the equation (4.2-8) is used.

Our approximate assumption relating to the Hessian matrix holds also asymptotically as for infinitely large samples $\underline{\underline{T}}_2$ tends with the real parameters toward zero, as it was already mentioned. In spite of this it is possible, to form the convergence matrix (4.2-11) according to the principle described in 4.3 in connection with on-line GLS method in order to approximate better the exact Hessian matrix (cf. equation (4.3-20)), the required calculations are, however, much more complex here. The main difference comes, of course, from the calculation of $\underline{\underline{W}}(t)$. With the on-line GLS method only $e(t)$ is needed to form $\underline{\underline{W}}(t)$, since $u(t)$ and $y(t)$ are available. On the other hand, with the ML method in consequence of the construction of $\underline{\underline{T}}_2$, values $u^{FF}(t)$, $y^{FF}(t)$ and $e^{FF}(t)$ according to (2.2-31) -(2.2-33) have also to be generated.

To our best knowledge, we have been the first to publish the detailed discussion of the approximations needed for the realization of the on-line ML method in [17] in Hungarian.

4.3 The on-line version of the generalized least squares method

The GLS structure assuming the special autoregressive noise model and the related CLARKE' off-line estimation method were discussed in subsection 2.3. The on-line identification algorithm for the GLS structure was elaborated by HASTINGS-JAMES and SAGE [68, 123]. The system equation (2.3-1) can be written in two equations linear in parameters:

$$y^F(t) = \underline{f}^T(u^F, y^F, t)\underline{p}_{ba} + e(t) \qquad (4.3\text{-}1)$$

and

$$r(t) = -g^T(r,t)\underline{p}_h + e(t) \qquad (4.3\text{-}2)$$

(cf. the relations (2.3-5) and (2.3-14) with the related notations and definitions.)
Here

$$r(t) = y(t) - \underline{f}^T(u,y,t)\underline{p}_{ba}\,. \qquad (4.3\text{-}3)$$

The equations (4.3-1) and (4.3-2) are linear in parameters and correspond formally to the system equation (4.1-1) applied with the LS method. Thus, considering the loss functions

$$Q_1(t) = \frac{1}{2}\left[y^F(t) - \underline{f}^T(u^F, y^F, t)\underline{p}_{ba}\right]^2 \qquad (4.3\text{-}4)$$

and

$$Q_2(t) = \frac{1}{2}\left[r(t) + \underline{g}^T(r,t)\underline{p}_h\right]^2 \qquad (4.3\text{-}5)$$

which can be formed now also according to the LS principle, the optimal canonical algorithm of the STA can be applied, namely

$$\underline{p}_{ba}(t) = \underline{p}_{ba}(t-1) + \underline{\underline{R}}_1(t)\underline{f}(u^F, y^F, t)\left[y^F(t) - \underline{f}^T(u^F, y^F t)\underline{p}_{ba}(t-1)\right], \tag{4.3-6}$$

$$\underline{\underline{R}}_1(t) = R_1(t-1) - \frac{\underline{\underline{R}}_1(t-1)\underline{f}(u^F, y^F, t)\underline{f}^T(u^F, y^F, t)\underline{\underline{R}}_1(t-1)}{1+\underline{f}^T(u^F, y^F, t)\underline{\underline{R}}_1(t-1)\underline{f}(u^F, y^F, t)}, \tag{4.3-7}$$

as well as

$$\underline{p}_h(t) = \underline{p}_h(t-1) - \underline{\underline{R}}_2(t)\underline{g}(r,t)\left[r(t) + \underline{g}^T(r,t)\underline{p}_h(t-1)\right], \tag{4.3-8}$$

$$\underline{\underline{R}}_2(t) = \underline{\underline{R}}_2(t-1) - \frac{\underline{\underline{R}}_2(t-1)\underline{g}(r,t)\underline{g}^T(r,t)\underline{\underline{R}}_2(t-1)}{1+\underline{g}^T(r,t)\underline{\underline{R}}_2(t-1)\underline{g}(r,t)}. \tag{4.3-9}$$

These equations are to be used together with the relations (2.3-4) resp. (4.3-3) applied for the formation of $y^F(t)$, $u^F(t)$ and $r(t)$. The sequence of the application of the equations follows accordingly.

The on-line GLS method uses the outputs of a moving-average filter which is more advantageous than the autoregressive filter of the ML structure in the sense that after the collection of data in a required quantity these filter equations can be started without the uncertainty of the initial state as the memories of the filters can be filled up. On the other hand, with the autoregressive filter the initial state is usually not known. The moving-average filter is more advantageous also from the consideration that in case of missing data the on-line parameter estimation can be continued by the filling up of memory vectors and restarting the filtering.

For the on-line GLS method - as in general for methods where additional auxiliary quantities have to be calculated - it is especially important by which time the estimated parameters get in an acceptably close neighbourhood of their real values. In itself the on-line LS algorithm is practically convergent independently from the initial value. On the other hand, the on-line GLS method given by the equations (4.3-6) - (4.3-9) is very sensitive for the start of the algorithms, viz. in the following sense. The relation between the on-line updating of the estimation $\underline{p}_{ba}$ and $\underline{p}_h$ is actually given by the filter equations (2.3-4) and the relation (2.3-14). Hence it follows that if the estimated values of $\underline{p}_{ba}$ and $\underline{p}_h$ are not sufficiently exact then the error can accumulate easily through $r(t)$, the filtered $y^F(t)$ and $u^F(t)$, i.e. we get a divergent procedure. In the practice this problem can usually be eliminated only by using - after the start of the algorithms for a given time - the simple on-line LS method instead of (3.4-6), i.e. the values $y(t)$ and $u(t)$ instead of $y^F(t)$ and $u^F(t)$. During this, of course, $r(t)$ and the estimation of $\underline{p}_h$ is calculated. At a given moment, on the other hand, the LS method is already replaced by the GLS method, by taking advantage of the values of the filtered variables. In the majority of cases a convergent algorithm is obtained this way, but for the time of the LS-GLS replacement any statement can be made only on the basis of experience. Another useful advice is to delay the start of the estimation of $\underline{p}_h$ with respect to the LS estimation of $\underline{p}_{ba}$ in order to obtain $r(t)$-s with sufficient exactness. Therefore the on-line estimation of $\underline{p}_h$ is practically situated between the LS start and the GLS start.

In particular very noisy systems it can be useful if for the calculation of $r(t)$ at any time not directly $\underline{p}_{ba}(t)$ but its averaged value, i.e. filtered by a first-class filter is used. By this, the disturbing fluctuations in the parameter estimation can be reduced to a great extent. The equation of such a filter for the input $\underline{p}_{ba}$ and output $\underline{p}^F_{ba}$ of the filter is

$$\underline{p}^{F}_{ba}(t)=(1-\alpha)\underline{p}_{ba}(t)+\alpha\underline{p}^{F}_{ba}(t-1). \qquad (4.3\text{-}10)$$

$(0 \leq \alpha \geq 1)$

Here the "time constant" of the smoothing can be set by α.

The Eqs. (4.3-6) - (4.3-9) minimizing the average value of the two functionals in the separated subspaces of $\underline{p}_{ba}$ and $\underline{p}_{h}$, could have been set on the basis of the properties of $Q_1(t)$ and $Q_2(t)$ that for a fixed $\underline{p}_{h}$, $Q_1(t)$ is linear in parameters $\underline{p}_{ba}$, and $Q_2(t)$ in $\underline{p}_{h}$ for a fixed $\underline{p}_{ba}$. Otherwise, $Q_1(t)$ and $Q_2(t)$ represent actually the same loss, only with respect to the two subspaces.

It has to be mentioned that there is possibility for minimizing the loss function $Q_1(t)$ in the space of the joint parameter vector, too:

$$\underline{p}_{bah} = \left[\underline{p}^{T}_{ba},\underline{p}^{T}_{h}\right]^{T}. \qquad (4.3\text{-}11)$$

Since the on-line solution of HASTINGS-JAMES performed the minimization with optimal steps realized by subspaces [123], it is obvious that the optimal steps feasible in the whole parameter space yield greater convergence speed.

In discussing the on-line ML method, it was already explained that because of practical considerations it is impossible to update the auxiliary quantities in the gradient and Hessian matrices beginning from the starting point of the observations, accordingly both now and later on the parameters estimated at last will be taken into account for the updating of the recursive relations. This circumstance makes no great problem with the GLS structure as for the values $u^{F}(t)$, $y^{F}(t)$ formed by moving-average filtering, there is no need of recalculation, the above bound has to be observed only with the calculation of $r(t)$, resp. $e(t)$.

On the basis of the Eqs. (2.3-21) - (2.3-23), resp. (2.3-24)

$$\frac{dQ_1(t)}{d\underline{p}_{bah}} = -\underline{f}(u^F,y^F,r,t)e(t), \qquad (4.3-12)$$

where the construction of $\underline{f}(u^F,y^F,r,t)$ is according to (2.3-26) and $e(t)$ can be obtained by the following equivalent calculation methods:

$$e(t) = y^F(t) - \underline{f}^T(u^F,y^F,t)\underline{p}_{ba} =$$

$$= r(t) + \underline{g}^T(r,t)\underline{p}_h = y(t) - \underline{f}^T(u,y,r)\underline{p}_{bah} \qquad (4.3-13)$$

(cf. relations (2.3-5), (2.3-14) and (2.4-1)). The on-line estimation of $\underline{p}_{bah}$ can therefore be carried out on the basis of (4-6) according to the equation:

$$\underline{p}_{bah}(t) = \underline{p}_{bah}(t-1) + \underline{\underline{R}}(t)\underline{f}(u^F,y^F,r,t)e(t). \qquad (4.3-14)$$

In connection with the Eqs. (2.3-28), (2.3-29) and (2.3-30), on the basis of which the second order derivatives can be calculated, we get the following result (by using the notation (4-1)), i.e. the Hessian matrix belonging to the t-th sample can be formed according to [53]

$$\underline{\underline{H}}\{Q_1(t)\} = \underline{f}(u^F,y^F,r,t)\underline{f}^T(u^F,y^F,r,t) + \underline{\underline{W}}(t). \qquad (4.3-15)$$

Here $\underline{\underline{W}}(t)$ is of structure of $\underline{\underline{T}}_2$ in (2.3-30) and consists of elements of types $-u(t-i-j)e(t)$ and $y(t-i-j)e(t)$ in the non-zero submatrix places according to the Eqs. (2.3-3o) - (2.3-32). The construction of $\underline{\underline{W}}(t)$ or $\underline{\underline{T}}_2$ means that the seeking in the subspaces $\underline{p}_{ba}$ and $\underline{p}_h$ independently from each other is performed optimally according to the Eqs. (4.3-6) - (4.3-9), while in the joint $\underline{p}_{bah}$ parameter space the steps done in the two subspacesare already in interaction.

Unfortunately, the formula (4.3-15) of the instantaneous Hessian matrix has the consequence that the convergence matrix $\underline{\underline{R}}(t)$ can no more be formed recursively. Following (4-8) let us think of that now

$$\underline{\underline{R}}^{-1}(t)= \underline{\underline{R}}^{-1}(t-1)+\underline{f}(u^F,y^F,r,t)\underline{f}^T(u^F,y^F,r,t)+ \underline{\underline{W}}(t), \quad (4.3-16)$$

which is already not the customary diadic sum.

The problem is caused by the presence of $\underline{\underline{W}}(t)$ and the recursivity can be achieved only by certain approximation. (Although it must not be forgotten that the particular construction of $\underline{\underline{W}}(t)$ reduces the calculations through its diadic resolution compared with a matrix of general construction.)

Let

$$\underline{\underline{R}}^{-1}(t)= \underline{\underline{R}}_{\ast}^{-1}(t)+ \underline{\underline{W}}(t), \quad (4.3-17)$$

then $\underline{\underline{R}}_{\ast}(t)$ can be updated according to the relation already known:

$$\underline{\underline{R}}_{\ast}(t)= \underline{\underline{R}}_{\ast}(t-1)- \frac{\underline{\underline{R}}_{\ast}(t-1)\,\underline{f}(u^F,y^F,r,t)\underline{f}^T(u^F,y^F,r,t)\underline{\underline{R}}_{\ast}(t-1)}{1+\underline{f}^T(u^F,y^F,r,t)\underline{\underline{R}}_{\ast}(t-1)\,\underline{f}(u^F,y^F,r,t)} . \quad (4.3-18)$$

Then we obtain from (4.3-17) that

$$\underline{\underline{R}}(t)=\left[\underline{\underline{R}}_{\ast}^{-1}(t)+ \underline{\underline{W}}(t)\right]^{-1} = \underline{\underline{R}}_{\ast}(t)\left[\underline{\underline{I}} + \underline{\underline{W}}(t)\underline{\underline{R}}_{\ast}(t)\right]^{-1} . \quad (4.3-19)$$

Approximate now the second term of the right side by its first order matrix serie:

$$\underline{\underline{R}}(t)\tilde{=} \underline{\underline{R}}_{\ast}(t)\left[\underline{\underline{I}} - \underline{\underline{W}}(t)\underline{\underline{R}}_{\ast}(t)\right] = \underline{\underline{R}}_{\ast}(t)-\underline{\underline{R}}_{\ast}(t)\underline{\underline{W}}(t)\underline{\underline{R}}_{\ast}(t). \quad (4.3-20)$$

The applied approximation enabled us to obtain not very complicated relations, while the recursivity still holds.

With respect to the application of the equation (4.3-20) it has to be remarked that this relation - because of its strongly approximating character - can change the positive definiteness of $\underline{\underline{R}}(t)$ which on its turn entails the loss of convergence. Therefore the relevant investigations (e.g. the signs of the elements in the main diagonal) have to be made in every step and if a negative definiteness arises, then only $\underline{\underline{R}}_{*}(t)$ has to be used as $\underline{\underline{R}}(t)$. This is feasible also because - as already pointed out in the preceding chapters, while discussing the off-line methods - as the number of the processed samples grows the effect of $\underline{\underline{T}}_2$ on the identification diminishes. This follows also from (4.3-20), since simultaneously with the transition $\underline{\underline{R}}_{*}(t) \to \underline{\underline{0}}$, $\underline{\underline{R}}_{*}(t) \to \underline{\underline{R}}(t)$ also comes about. This is the fact why the approximate first part of the Hessian matrix is used - in particular with the extended matrix methods - for the formation of the convergence matrix, the error due to the approximation is decreasing asymptotically, on the other hand the part corresponding to $\underline{\underline{T}}_1$ can always be constructed recursively, being a diadic sum.

4.4 On-line extended matrix methods

The extended matrix methods discussed in subsection 2.4 are based on the observation that the system equations assuming various noise models can be written in a form linear in parameters. This fact enabled the off-line LS method to be used almost exclusively although because of the application of variables which cannot be directly measured (only estimated or calculated) only iterative solutions could be chosen. Over and above the preceding remarks the on-line versions of the extended matrix methods also utilize the fact that the Hessian matrix of the loss function asymptotically agrees with its first part, the matrix $\underline{\underline{T}}_1$. This matrix is, on the other hand, a diadic sum making the recursivity, discussed with the LS method, possible. This, of course, does not exclude that we count also here, alike the solution investigated at the on-line GLS method, with the exact second derivatives, but in this case we will lose the advantages of the recursivity.

The first extended matrix method (FEXM) related to the GLS structure was elaborated for the system equation

$$y(t)= \underline{f}^T(u,y,r,t)\,\underline{p}_{bah} + e(t) \qquad (4.4\text{-}1)$$

obtainable from the Eqs. (2.3-1), (2.3-3), (2.3-5) and (2.3-14) [120]. Here

$$f(u,y,r,t)=[u(t-1),\ldots,u(t-n);-y(t-1),\ldots,-y(t-n);$$

$$-r(t-1),\ldots,-r(t-n)]^T, \qquad (4.4\text{-}2)$$

where $r(t)$ is according to (2.3-14). The algorithm of the on-line FEXM method (following the on-line LS method) is

$$\underline{p}_{bah}(t) = \underline{p}_{bah}(t-1) + \underline{\underline{R}}(t)\underline{f}(u,y,r,t)\left[y(t) - \underline{f}^{T}(u,y,r,t)\underline{p}_{bah}(t-1)\right]. \qquad (4.4\text{-}3)$$

$\underline{\underline{R}}(t)$ is updated by $\underline{f}(u,y,r,t)$ using the relation (4.1-7). By comparing (4.3-14) and (4.4-3), it is easy to see that the two methods lead to not identical results. The difference comes from the fact that there is $\underline{f}(u^F,y^F,\ldots)$ instead of $f(u,y,\ldots)$ in the real gradient of the loss function

$$Q = \frac{1}{2}\left[y(t) - \underline{f}^{T}(u,y,r,t)\underline{p}_{bah}\right]^2 \qquad (4.4\text{-}4)$$

as we have already seen with (4.3-12). Beside the approximation of the second derivatives the FEXM method applies also an approximation of the first derivative making the calculation easier as we can get the Eq. (4.4-3) only by neglecting the dependence of r on $\underline{p}_{bah}$ in Q according to (4.4-4).
The above idea has to be followed essentially also in presenting the on-line version of the SEXM method [130, 131]. The SEXM method refers to the Eq. (2.2-5) which gives the ML structure in a form linear in parameters:

$$y(t) = \underline{f}^{T}(u,y,e,t)\underline{p}_{bac} + e(t). \qquad (4.4\text{-}5)$$

Thus the equation of the on-line algorithm is now:

$$\underline{p}_{bac}(t) = \underline{p}_{bac}(t-1) + \underline{\underline{R}}(t)\underline{f}(u,y,e,t)\left[y(t) - \underline{f}^{T}(u,y,e,t)\underline{p}_{bac}(t-1)\right], \qquad (4.4\text{-}6)$$

where $\underline{\underline{R}}(t)$ is updated by $\underline{f}(u,y,e,t)$ according to the relation (4.1-7). $e(t)$ is, of course, not measured but it is now only an estimated resp. calculated value:

$$e(t) = y(t) - \underline{f}^{T}(u,y,e,t)\underline{p}_{bac}(t). \qquad (4.4\text{-}7)$$

Here $\underline{p}_{bac}$ is taken at the t-th moment because it is the newest estimate and gives also for e(t) the most accurate result. The same approximations are valid for the on-line version of the SEXM method, as refer to the on-line FEXM method. Thus on the one hand the relation (4.4-6) does not use the real gradient because we disregard the dependence of e(t) on $\underline{p}_{bac}$, on the other hand, only the $\underline{\underline{T}}_1$ part of the Hessian matrix is taken to form $\underline{\underline{R}}(t)$.

On the basis of completely similar considerations, the on-line estimation algorithm can be given also for other, quasilinearized structures discussed in subsection 2.4, since every method could be formally reduced to the LS estimation by the linearization in parameters. The approximations hold, of course, also hereinafter.

Although this report does not deal with a more detailed analysis of the above methods, we note that the preceding approximations relate to the algorithm of STA optimal in second order sense as a successive approximative solution to the gradient technique.

4.5 The on-line version of the instrumental variable method

The on-line version of the instrumental variable (IV) method can be practically derived from the off-line solution (2.5-6). By following the deduction of the recursive version of the LS method [143], we get the algorithm:

$$\underline{p}_{ba}(t)= \underline{p}_{ba}(t-1)+ \underline{\underline{R}}(t)\underline{f}(u,v,t)[y(t)- \underline{f}^{T}(u,y,t)\underline{p}_{ba}(t-1)], \tag{4.5-1}$$

where

$$\underline{f}(u,y,t)=[u(t-1),\ldots,u(t-n);-y(t-1),\ldots,-y(t-n)]^{T} \tag{4.5-2}$$

has the usual form and

$$\underline{f}(u,v,t)=[u(t-1),\ldots,u(t-n);-v(t-1),\ldots,-v(t-n)]^{T}. \tag{4.5-3}$$

Here $v(t)$ is the calculated (estimated) value of the output of the noiseless system:

$$v(t)= \underline{f}^{T}(u,v,t)\underline{p}_{ba}(t). \tag{4.5-4}$$

As following from (2.5-6) now:

$$\underline{\underline{R}}(t)=\Big[\sum_{i=1}^{t} \underline{f}(u,v,i)\underline{f}^{T}(u,y,i)\Big]^{-1} \tag{4.5-5}$$

therefore the recursive relation updating $\underline{\underline{R}}(t)$ will have the form:

$$\underline{\underline{R}}(t)= \underline{\underline{R}}(t-1) - \frac{\underline{\underline{R}}(t-1)\underline{f}(u,v,t)\underline{f}^{T}(u,y,t)\underline{\underline{R}}(t-1)}{1+\underline{f}^{T}(u,v,t)\underline{\underline{R}}(t-1)\underline{f}(u,y,t)}. \tag{4.5-6}$$

In consequence, $\underline{\underline{R}}(t)$ here is not symmetric!

V. ON-LINE IDENTIFICATION METHODS FOR MULTIPLE OUTPUT SYSTEMS

The proportions of the preceding sections indicate our stand point already mentioned according to which in the Introduction under the present technical conditions, we attribute a much greater practical importance to the off-line identification methods than to the on-line ones. All this holds in particular for the multiple output systems.

As also with the estimation methods of multiple output systems, a scalar loss function has to be minimized, in discussing their on-line versions we could apply also now the canonical algorithm of the STA, but as the well applicable on-line identification algorithms of the MIMO systems correspond to the recursive solutions of the LS-type methods, hereinafter we, too, will follow this approach.

5.1 The on-line least squares method

The generalization of the LS method valid for the MIMO systems has been discussed in detail in subsection 3.1. The explicit estimation obtained for the parameter matrix $\underline{\underline{P}}_{BA}$ of the system equation (3-2) is according to (3.1-16)

$$\hat{\underline{\underline{P}}}_{BA} = \underline{\underline{Y}}\ \underline{\underline{F}}^T\ (\underline{\underline{F}}\ \underline{\underline{F}}^T)^{-1}, \tag{5.1-1}$$

where

$$\underline{\underline{Y}} = [\underline{y}(1), \ldots, \underline{y}(N)] \tag{5.1-2}$$

and

$$\underline{\underline{F}} = [\underline{f}(1), \ldots, \underline{f}(N)]. \tag{5.1-3}$$

The definition of $\underline{f}(t)$ corresponds to (3.1-4).

By analyzing the relation (5.1-1) in detail, we can obtain a recursive solution. Let us write the off-line LS estimation for the case of containing N and N+1 samples:

$$\hat{\underline{\underline{P}}}_{BA}(N) = \underline{\underline{Y}}_N\ \underline{\underline{F}}_N^T\ (\underline{\underline{F}}_N\ \underline{\underline{F}}_N^T)^{-1} \tag{5.1-4}$$

$$\hat{\underline{\underline{P}}}_{BA}(N+1) = \underline{\underline{Y}}_{N+1}\ \underline{\underline{F}}_{N+1}^T\ (\underline{\underline{F}}_{N+1}\ F_{N+1}^T)^{-1}. \tag{5.1-5}$$

Consider further that

$$\underline{\underline{F}}_{N+1} = [\underline{\underline{F}}_N,\ \underline{f}(N+1)], \tag{5.1-6}$$

as well as

$$\underline{\underline{Y}}_{N+1} = [\underline{\underline{Y}}_N,\ \underline{y}(N+1)], \tag{5.1-7}$$

thus

$$\underline{\underline{F}}_{N+1}\ \underline{\underline{F}}_{N+1}^{T} = \underline{\underline{F}}_{N}\ \underline{\underline{F}}_{N}^{T} + \underline{f}(N+1)\underline{f}^{T}(N+1), \qquad (5.1\text{-}8)$$

and

$$\underline{\underline{Y}}_{N+1}\ \underline{\underline{F}}_{N+1}^{T} = \underline{\underline{Y}}_{N}\ \underline{\underline{F}}_{N}^{T} + \underline{y}(N+1)\,\underline{f}^{T}(N+1). \qquad (5.1\text{-}9)$$

Under the diadic extension according to (5.1-8) the recursive equation is

$$(\underline{\underline{F}}_{N+1}\ \underline{\underline{F}}_{N+1}^{T})^{-1} = (\underline{\underline{F}}_{N}\ \underline{\underline{F}}_{N}^{T})^{-1} - \frac{(\underline{\underline{F}}_{N}\ \underline{\underline{F}}_{N}^{T})^{-1}\ \underline{f}(N+1)\,\underline{f}^{T}(N+1)\,(\underline{\underline{F}}_{N}\ \underline{\underline{F}}_{N}^{T})^{-1}}{1+\underline{f}^{T}(N+1)\,(\underline{\underline{F}}_{N}\ \underline{\underline{F}}_{N}^{T})^{-1}\ \underline{f}(N+1)} \qquad (5.1\text{-}10)$$

as it was deduced in App. 15.

On the basis of all these, the recursive form of $\hat{\underline{\underline{P}}}_{BA}(N+1)$ is

$$\hat{\underline{\underline{P}}}_{BA}(N+1) = \underline{\underline{Y}}_{N+1}\ \underline{\underline{F}}_{N+1}^{T}\ (\underline{\underline{F}}_{N+1}\ \underline{\underline{F}}_{N+1}^{T})^{-1} =$$

$$= \left[\underline{\underline{Y}}_{N}\ \underline{\underline{F}}_{N}^{T} + \underline{y}(N+1)\,\underline{f}^{T}(N+1)\right]\left[(\underline{\underline{F}}_{N}\ \underline{\underline{F}}_{N}^{T})\right]^{-1} -$$

$$- \frac{(\underline{\underline{F}}_{N}\ \underline{\underline{F}}_{N}^{T})^{-1}\ \underline{f}(N+1)\underline{f}^{T}(N+1)\,(\underline{\underline{F}}_{N}\ \underline{\underline{F}}_{N}^{T})^{-1}}{1+\underline{f}^{T}(N+1)\,(\underline{\underline{F}}_{N}\ \underline{\underline{F}}_{N}^{T})^{-1}\ \underline{f}(N+1)}\ . \qquad (5.1\text{-}11)$$

Hence with the notation

$$\underline{\underline{R}}_{N+1} = (\underline{\underline{F}}_{N+1}\ \underline{\underline{F}}_{N+1}^{T})^{-1} \qquad (5.1\text{-}12)$$

we obtain

$$\hat{\underline{\underline{P}}}_{BA}(N+1) = \hat{\underline{\underline{P}}}_{BA}(N) + \underline{y}(N+1)\,\underline{f}^{T}(N+1)\,\underline{\underline{R}}_{N+1} - \hat{\underline{\underline{P}}}_{BA}(N)\,\underline{f}(N+1)\,\underline{f}^{T}(N+1)\,\underline{\underline{R}}_{N+1} =$$

$$= \hat{\underline{\underline{P}}}_{BA}(N) + \left[\underline{y}(N+1) - \hat{\underline{\underline{P}}}_{BA}(N)\,\underline{f}(N+1)\right]\underline{f}^{T}(N+1)\,\underline{\underline{R}}_{N+1}. \qquad (5.1\text{-}13)$$

The great formal similarity with the algorithm of the on-line LS method applied for the single output systems can be recognized easily. The updating of the convergence matrix $\underline{\underline{R}}_{N+1} = \underline{\underline{R}}(N+1)$ can be performed by the recursive relation (5.1-10) in the same way as for the on-line methods of the single output systems. [109].

The above presented on-line LS algorithm of the multiple output systems can be applied without any particular change to the realization of the on-line version of the GLS or extended matrix methods. As also these methods refer to the system equations linear in parameter matrices, the formal generalization is very simple to be achieved, so that we will not deal with their detailed discussion.

A P P E N D I X

Appendix 1.

In this section our statements concerning the structural problems of the MIMO system equations are summarized from identification point of views.

The inner construction and structural problems of the state--space description of multiple input multiple output discrete-time, linear dynamic systems have already been dealt whith a great number of authors who achieved valuable, basic theoretical results [42]. Thus the most important canonical forms of the state equations of the MIMO systems, the conditions of description in state-space of minimal dimension, the requirements of controllability and observability are already well-known.

While with the methods of control or state estimation (forecasting and filtering) the authors usually prefer the state space description form, with the identification methods, the models given by ordinary or vector difference equations are much more frequent. This certainly can be explained by the simpler form of the dependence of the output signals from the parameter vector (or matrix). The procedures which transform a state space description to a vector difference equation and vice versa are well-known in the literature. [42, 108].

From the aspect of identification, the rules of observability and identifiability of the MIMO systems are the most important regarding the theory. From the point of view of the practical identification technique, however, the construction of the OI observability index and the theorems relating to them, are considered as most essential. Here we refer to the theorem - most clearly presented by Rowe - according to which necessary and sufficient condition of

the unambiguous identification of the coefficient matrices of a vector difference equation is to have at least OI number of nonzero columns among (in the) matrices $\underline{\underline{A}}_j$ [108]. This involves from the point of view of the identification technique that in the case of matrix elements estimated as non-significant, the omission of the whole columns is practically or theoretically justified.

As the above can be considered known from the literature, hereinafter we make such independently formed statements about MIMO systems' structures suitable for identification which have a direct relevance to the computer processing of measurements.

Consider in Fig. A.1-1 the discrete-time, linear, dynamic, MIMO system with an m-dimensional input vector $\underline{u}$ and a q dimensional output vector $\underline{v}$. Disregard first the interaction of the outputs, i.e. every input signal should have an effect upon every output, but the latter ones not upon each other. In this case the dynamic property of the process can be described by a discrete transfer function matrix $\underline{\underline{W}}(z^{-1})$ whose $W_{ij}(z^{-1})$-th element represents the effect of the j-th input signal upon the i-th output signal. In a general case also the numerators and denominators of the W_{ij}-s are different:

$$\underline{\underline{W}}(z^{-1}) = \begin{bmatrix} \dfrac{F_{11}(z^{-1})}{G_{11}(z^{-1})} & \cdots & \dfrac{F_{1m}(z^{-1})}{G_{1m}(z^{-1})} \\ \vdots & & \\ \dfrac{F_{q1}(z^{-1})}{G_{q1}(z^{-1})} & \cdots & \dfrac{F_{qm}(z^{-1})}{G_{qm}(z^{-1})} \end{bmatrix} \qquad \text{(A.1-1)}$$

By allowing certain redundancy in the particular elements, after reduction to a common denominator, we obtain that

$$\underline{\underline{W}}(z^{-1}) = \frac{1}{A(z^{-1})} \begin{bmatrix} B_{11}(z^{-1}) & \dots & B_{1m}(z^{-1}) \\ \vdots & & \vdots \\ B_{q}(z^{-1}) & \dots & B_{qm}(z^{-1}) \end{bmatrix} = \frac{1}{A(z^{-1})} \underline{\underline{B}}(z^{-1}), \qquad (A.1\text{-}2)$$

where $B_{ij}(z^{-1})$-s are polynomials (without denominator). On the basis of this latter expression the vector difference equation of the system can be written as:

$$\underline{v}(t) = \sum_{i=0}^{n} \underline{\underline{B}}_i \, \underline{u}(t-i) - \sum_{i=1}^{n} \underline{\underline{A}}_j \, \underline{v}(t-i) =$$

$$= \underline{\underline{B}}(z^{-1})\underline{u}(t) - \tilde{\underline{\underline{A}}}(z^{-1})\underline{v}(t). \qquad (A.1\text{-}3)$$

Here

$$\underline{\underline{A}}(z^{-1}) = \underline{\underline{I}} + \underline{\underline{A}}_1 z^{-1} + \dots + \underline{\underline{A}}_n z^{-n} = \underline{\underline{I}} + \tilde{\underline{\underline{A}}}(z^{-1}), \qquad (A.1\text{-}4)$$

$$\underline{\underline{B}}(z^{-1}) = \underline{\underline{B}}_o + \underline{\underline{B}}_1 z^{-1} + \dots + \underline{\underline{B}}_n z^{-n}, \qquad (A.1\text{-}5)$$

where

$$\underline{\underline{A}}_i = a_i \underline{\underline{I}} \qquad (A.1\text{-}6)$$

namely

$$A(z^{-1}) = 1 + a_1 z^{-1} + \dots + a_n z^{-n}. \qquad (A.1\text{-}7)$$

The above train of thought serves to demonstrate that if we can neglect in advance the interaction of the outputs, the vector difference equation of the system can always be sought in the form of (A.1-3) by assuming special diagonal $\underline{\underline{A}}_j$ coefficient matrices. In this case the transfer functions referring to the dependence result directly in the form $B_{ij}(z^{-1})/A(z^{-1})$. As this structure allows certain redundancy in the elements of the transition matrix, their further examination for possible reducibility is advisable. This form is particularly advantageous for the estimation of structure (order of the system) because practically only n has to be changed. Under the preceding assumption, these same assumptions refer to the noise model, too, of which we can easily be convinced, for also there $A(z^{-1})$ is the denominator. This is similarly true for the whole noisy system, as according to the Eq. (1.2-26) the measured noisy output signal vector results from the superposition of the process and the noise model.

Consider now Fig. A.1-2, where the interaction of the output signals is taken into consideration by a $\underline{\underline{Q}}(z^{-1})$ transfer function matrix. On the basis of the Figure

$$\underline{v}(t) = \underline{\underline{P}}(z^{-1})\underline{u}(t) + \underline{\underline{Q}}(z^{-1})\underline{v}(t), \qquad (A.1\text{-}8)$$

whence

$$\underline{v}(t) = \left[\underline{\underline{I}} - \underline{\underline{Q}}(z^{-1})\right]^{-1} \underline{\underline{P}}(z^{-1})\underline{u}(t), \qquad (A.1\text{-}9)$$

i.e. by comparing with Fig. A.1-1 the system can be substituted by the equivalent function matrix

$$\underline{\underline{W}}(z^{-1}) = \left[\underline{\underline{I}} - \underline{\underline{Q}}(z^{-1})\right]^{-1} \underline{\underline{P}}(z^{-1}). \qquad (A.1\text{-}10)$$

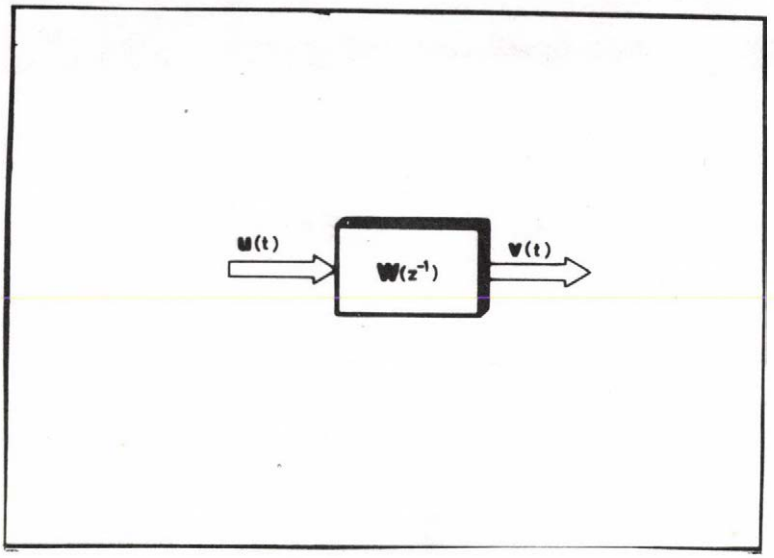


Fig. A.1-1

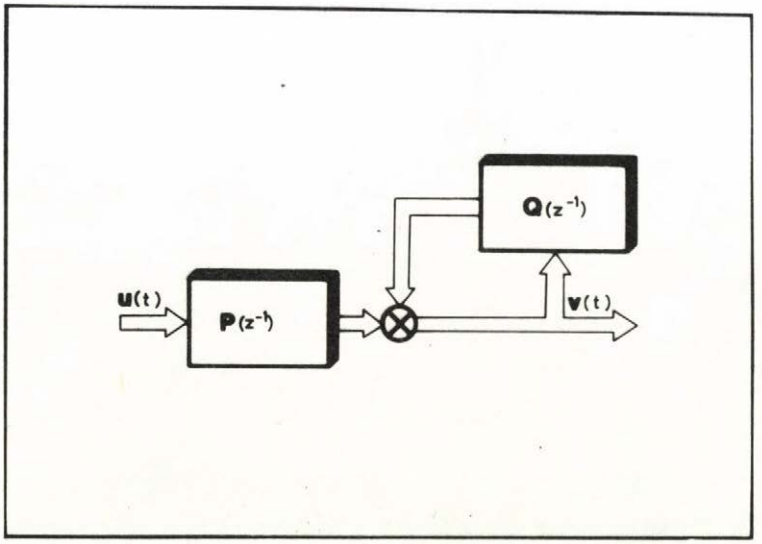


Fig. A.1-2

Let be

$$\underline{\underline{I}} - \underline{\underline{Q}}(z^{-1}) = \frac{1}{F(z^{-1})} \underline{\underline{G}}(z^{-1}) \qquad \text{(A.1-11)}$$

and

$$\underline{\underline{P}}(z^{-1}) = \frac{1}{H(z^{-1})} \underline{\underline{K}}(z^{-1}). \qquad \text{(A.1-12)}$$

The transformations according to these two assumpitons, can always be carried out. By these notations we obtain from the Eq. (A.1-9) that

$$\frac{1}{F(z^{-1})} \underline{\underline{G}}(z^{-1}) \underline{v}(t) = \frac{1}{H(z^{-1})} \underline{\underline{K}}(z^{-1}) \underline{u}(t). \qquad \text{(A.1-13)}$$

By introducing the notations

$$H(z^{-1})\underline{\underline{G}}(z^{-1}) = \underline{\underline{L}}(z^{-1}) = \underline{\underline{L}}_o + \underline{\underline{L}}_1 z^{-1} + \ldots = \underline{\underline{L}}_o\left[\underline{\underline{I}} + \tilde{\underline{\underline{L}}}(z^{-1})\right] \qquad \text{(A.1-14)}$$

and

$$F(z^{-1})\underline{\underline{K}}(z^{-1}) = \underline{\underline{M}}(z^{-1}) = \underline{\underline{M}}_o + \underline{\underline{M}}_1 z^{-1} + \ldots \qquad \text{(A.1-15)}$$

and assuming that $\underline{\underline{L}}_o$ is regular, we get the form

$$\underline{v}(t) = \underline{\underline{L}}_o^{-1} \underline{\underline{M}}(z^{-1}) \underline{u}(t) - \tilde{\underline{\underline{L}}}(z^{-1}) \underline{v}(t) = \underline{\underline{B}}'(z^{-1}) \underline{u}(t) - \tilde{\underline{\underline{A}}}'(z^{-1}) \underline{v}(t) \qquad \text{(A.1-16)}$$

for the system equation. It is easy to see that this equation shows a complete formal identity with the Eq. (A.1-3). The deduction shows that in this case the matrices $\underline{\underline{A}}_i$ have no special construction (no form $a_i \underline{\underline{I}}$), further that the

$\underline{\underline{P}}(z^{-1})$ and $\underline{\underline{Q}}(z^{-1})$ matrices can not be reset from the vector difference equation (A.1-16) and the equivalent (A.1-10) corresponds formally to the general equation

$$\underline{\underline{W}}(z^{-1}) = \underline{\underline{A}}'^{-1}(z^{-1})\underline{\underline{B}}'(z^{-1}) \qquad \text{(A.1-17)}$$

and not to (A.1-2).

The reduction to a common denominator can be, of course, achieved also with the (A.1-16), resp. (A.1-17) transfer function matrices and then again we get $\underline{\underline{A}}_i$ matrices of special construction, but the memory represented by $\underline{\underline{B}}'(z^{-1})$ increases. Our statements can be extended to the noise model, too, and as a result of similar considerations we get there a general form $\underline{\underline{A}}'^{-1}(z^{-1})\underline{\underline{C}}'(z^{-1})$.

The above can be summarized as follows. Either, the outputs have an interaction or not, a MIMO system in a general case can be written by a vector difference equation of general construction from whose coefficient matrices the original (real) structure cannot be unambiguously reset. (Let us think of that in the i-th output signal a given dynamic change can be ensured in a direct channel from the j-th input signal, moreover via the interaction of the output signals there are a lot of other ways, too. This question is closely related to the fact that the state space description of a system and thus its inner construction also is determined only to the extent of a linear transformation). In both cases there exists also a vector difference equation with special construction (with diagonal $\underline{\underline{A}}_i$ matrices), (perhaps with a greater memory, i.e. model order), which, on the other hand, does not contain the interaction of the outputs.

The identification therefore ensures essentially an input--output equivalent model without clearing the relations of the outputs. On the other hand, we can mention as its advan-

tage that the prescription concerning the observability index can be simply respected by special $\underline{\underline{A}}_i$ matrices. This involves that the search of structure in n contains the search in observability index, too.

Appendix 2. [6, 7, 145]

Definition:

The following matrix operator is called a generalized TOEPLITZ matrix:

$$\underline{\underline{S}}_N^i(\underline{\underline{A}}) = \begin{bmatrix} \underline{\underline{0}} & & & & & & & & \\ \cdot & \cdot & & & & & & & \\ \cdot & \cdot & \cdot & & & & & & \\ \cdot & & \cdot & \cdot & & & & & \\ \underline{\underline{0}} & & & \cdot & \cdot & & & & \\ \underline{\underline{A}} & \underline{\underline{0}} & & & \cdot & & & & \\ \underline{\underline{0}} & \underline{\underline{A}} & \cdot & & & \cdot & & & \\ \cdot & & \cdot & \cdot & & & \cdot & & \\ \cdot & & & \cdot & \cdot & & & \cdot & \\ \cdot & & & & \cdot & \cdot & & & \cdot \\ \underline{\underline{0}} & \underline{\underline{0}} & \cdots & \underline{\underline{0}} & \underline{\underline{A}} & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \end{bmatrix} \qquad \text{(A.2-1)}$$

(i blocks under the last block row; N blocks under the first block column)

and $\underline{\underline{S}}_N^i(\underline{\underline{A}}) = \underline{\underline{0}}$, if $i \geq N$.

If the size of matrix $\underline{\underline{A}}$ is mxn, then the size of $\underline{\underline{S}}_N^i(\underline{\underline{A}})$ is N.m×N.n. The generalized TOEPLITZ matrix is easy to de defined also by the KRONECKER matrix product (cf. A.3).

(A.2-1) can be rewritten in the form as:

$$\underline{\underline{S}}_N^i(\underline{\underline{A}}) = \underline{\underline{S}}_N^i(1) \otimes \underline{\underline{A}} . \qquad \text{(A.2-2)}$$

Here $\underline{\underline{S}}_N^i(1)$ has a size of N*N and is defined also by (A.2-1), so that units are only in its i-th row under the main diagonal, its other elements are zeros. $\underline{\underline{S}}_N^i(1)$ is called on the basis of the following simple example a shift matrix and denoted only by $\underline{\underline{S}}^i$. Let namely be

$$\underline{\underline{S}}_3^1(1) = \underline{\underline{S}}^1 = \underline{\underline{S}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{A.2-3}$$

and

$$\underline{y} = [y_1 \; y_2 \; y_3]^T = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} . \tag{A.2-4}$$

Then

$$\underline{\underline{S}}_3^1(1)\underline{y} = \underline{\underline{S}}\,\underline{y} = \begin{bmatrix} 0 \\ y_1 \\ y_2 \end{bmatrix} , \tag{A.2-5}$$

so that the meaning of "shifting" is easy to be followed.

The most important identities valid for the generalized TOEPLITZ matrices will be presented in brief. Let

$$\underline{\underline{T}}_{NL} = \sum_{i=0}^{n_L} \underline{\underline{S}}_N^i(\underline{\underline{L}}_i) \tag{A.2-6}$$

and

$$\underline{\underline{T}}_{NM} = \sum_{i=0}^{n_M} \underline{\underline{S}}_N^i(\underline{\underline{M}}_i) , \tag{A.2-7}$$

where

$$\underline{\underline{L}}(z) = \sum_{i=0}^{n_L} z^i \, \underline{\underline{L}}_i \qquad \text{(A.2-8)}$$

as well as

$$\underline{\underline{M}}(z) = \sum_{i=0}^{n_M} z^i \, \underline{\underline{M}}_i \, . \qquad \text{(A.2-9)}$$

Then we can declare the following rules:

1. If

$$\underline{\underline{L}}(z) \pm \underline{\underline{M}}(z) = \underline{\underline{K}}(z) = \sum_{i=0}^{n_K} z^i \, \underline{\underline{K}}_i \, , \qquad \text{(A.2-10)}$$

where

$$\underline{\underline{K}}_i = \underline{\underline{L}}_i \pm \underline{\underline{M}}_i \quad \text{and} \quad n_K = \max \{n_L, n_M\} \, , \qquad \text{(A.2-11)}$$

then the relation

$$\underline{\underline{T}}_{NL} \pm \underline{\underline{T}}_{NM} = \sum_{i=0}^{n_K} \underline{\underline{S}}_N^i \, (\underline{\underline{K}}_i) \qquad \text{(A.2-12)}$$

holds.

2. If

$$\underline{\underline{L}}(z) \underline{\underline{M}}(z) = \sum_{i=0}^{n_K} z^i \, \underline{\underline{K}}_i \, , \qquad \text{(A.2-13)}$$

where

$$\underline{\underline{K}}_i = \sum_{i=0}^{i} \underline{\underline{L}}_j \, \underline{\underline{M}}_{i-j} \quad \text{and} \quad n_K = n_L + n_M \, , \qquad \text{(A.2-14)}$$

then

$$\underline{\underline{T}}_{NL} \cdot \underline{\underline{T}}_{NM} = \sum_{i=0}^{n_K} \underline{\underline{S}}_N^i (\underline{\underline{K}}_i) \qquad \text{(A.2-15)}$$

holds.

3. Let

$$\underline{\underline{L}}(z) = \underline{\underline{L}}_o + \sum_{i=1}^{n_L} z^i \underline{\underline{L}}_i \,. \qquad \text{(A.2-16)}$$

If $\underline{\underline{L}}_i$ is quadratic and $\underline{\underline{L}}_o$ not singular, then

$$\underline{\underline{T}}_{NL}^{-1} = \sum_{i=0}^{N-1} \underline{\underline{S}}_N^i(\underline{\underline{K}}_i) \,, \qquad \text{(A.2-17)}$$

where

$$\underline{\underline{K}}(z) = \underline{\underline{L}}^{-1}(z) = \sum_{i=0}^{\infty} z^i \underline{\underline{K}}_i \qquad \text{(A.2-18)}$$

and

$$\left.\begin{aligned}
&\underline{\underline{K}}_o = \underline{\underline{L}}_o^{-1} \\
&\underline{\underline{K}}_1 = -\underline{\underline{L}}_o^{-1} \underline{\underline{L}}_1 \underline{\underline{K}}_o \\
&\underline{\underline{K}}_2 = -\underline{\underline{L}}_o^{-1}(\underline{\underline{L}}_1 \underline{\underline{K}}_1 + \underline{\underline{L}}_2 \underline{\underline{K}}_o) \\
&\vdots \\
&\underline{\underline{K}}_{N-1} = -\underline{\underline{L}}_o^{-1}(\underline{\underline{L}}_1 \underline{\underline{K}}_{N-2} + \underline{\underline{L}}_2 \underline{\underline{K}}_{N-3} + \ldots + \underline{\underline{L}}_{N-1} \underline{\underline{K}}_o)
\end{aligned}\right\} \text{(A.2-19)}$$

$\underline{\underline{L}}_i = \underline{\underline{0}}$, of course, if $i > n_L$.

A very important property of the shift matrices $\underline{\underline{S}}_N^i(1) = \underline{\underline{S}}^i$ is the interchangeability which means the following:

$$\underline{\underline{S}}^i \cdot \underline{\underline{S}}^j = \underline{\underline{S}}^j \cdot \underline{\underline{S}}^i = \underline{\underline{S}}^{i+j}. \qquad \text{(A.2-20)}$$

Appendix 3. [97]

Definition:

Let $\underline{\underline{A}}$ be an (mxn) matrix and $\underline{\underline{B}}$ an (rxs) matrix. The KRONECKER product of the $\underline{\underline{A}}$ an $\underline{\underline{B}}$ matrices means the following expression:

$$\underline{\underline{A}} \otimes \underline{\underline{B}} = \begin{bmatrix} a_{11}\underline{\underline{B}} & a_{12}\underline{\underline{B}} & \cdots & a_{1n}\underline{\underline{B}} \\ a_{21}\underline{\underline{B}} & a_{22}\underline{\underline{B}} & \cdots & a_{2n}\underline{\underline{B}} \\ \vdots & & & \\ a_{m1}\underline{\underline{B}} & a_{m2}\underline{\underline{B}} & \cdots & a_{mn}\underline{\underline{B}} \end{bmatrix} \qquad \text{(A.3-1)}$$

where the size of matrix $\underline{\underline{A}} \otimes \underline{\underline{B}}$ is $(m.r)x(n.s)$.

The following operational identities are the most important for the KRONECKER matrix product:

1. If $\underline{\underline{A}}$ and $\underline{\underline{C}}$, as well as $\underline{\underline{B}}$ and $\underline{\underline{D}}$ matrices are multipliable, then

$$(\underline{\underline{A}} \otimes \underline{\underline{B}})(\underline{\underline{C}} \otimes \underline{\underline{D}}) = (\underline{\underline{A}}\underline{\underline{C}} \otimes \underline{\underline{B}}\underline{\underline{D}}). \qquad \text{(A.3-2)}$$

2. If the inverses of the matrices $\underline{\underline{A}}$ and $\underline{\underline{B}}$ exist, then

$$(\underline{\underline{A}} \otimes \underline{\underline{B}})^{-1} = \underline{\underline{A}}^{-1} \otimes \underline{\underline{B}}^{-1}. \qquad \text{(A.3-3)}$$

3. The transposition rule of the KRONECKER products can be formulated as follows:

$$(\underline{\underline{A}} \otimes \underline{\underline{B}})^{T} = \underline{\underline{A}}^{T} \otimes \underline{\underline{B}}^{T}. \qquad \text{(A.3-4)}$$

4. If the condition of the addition of matrices is fulfilled, then:

$$(\underline{\underline{A}}+\underline{\underline{B}})\otimes(\underline{\underline{C}}+\underline{\underline{D}}) = \underline{\underline{A}}\otimes\underline{\underline{C}} + \underline{\underline{A}}\otimes\underline{\underline{D}} + \underline{\underline{B}}\otimes\underline{\underline{C}} + \underline{\underline{B}}\otimes\underline{\underline{D}}\,. \qquad (A.3\text{-}5)$$

5. The order of operations can be prescribed:

$$\underline{\underline{A}}\otimes(\underline{\underline{B}}\otimes\underline{\underline{C}}) = (\underline{\underline{A}}\otimes\underline{\underline{B}})\otimes\underline{\underline{C}}\,. \qquad (A.3\text{-}6)$$

6. The trace of the KRONECKER products can be calculated as follows:

$$\mathrm{tr}(\underline{\underline{A}}\otimes\underline{\underline{B}}) = \mathrm{tr}(\underline{\underline{A}})\,.\,\mathrm{tr}(\underline{\underline{B}})\,. \qquad (A.3\text{-}7)$$

The identities can be understood in the simplest way by the expansion of simple examples.

Appendix 4. [97]

The generation rules of vectors from matrices can be defined by KRONECKER products.

Definition:

Let $\underline{\underline{A}}$ be (mxn) matrix, then denote by vec($\underline{\underline{A}}$) the following arrangement (arrangement into vectors) of the elements of the matrix:

$$\text{vec}(\underline{\underline{A}}) = \begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \vdots \\ \underline{a}_n \end{bmatrix} = \left[\underline{a}_1^T, \underline{a}_2^T, \ldots, \underline{a}_n^T\right]^T , \qquad (A.4\text{-}1)$$

where $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$ are the column vectors of the matrix $\underline{\underline{A}}$.

The most important identities related to the arrangement into vectors are the following:

1. If $\underline{y}$ is a column vector, then

$$\text{vec}(\underline{y}) = \text{vec}(\underline{y}^T) = \underline{y} . \qquad (A.4\text{-}2)$$

2. The arrangement into vectors of the product matrices can be defined by KRONECKER products as:

$$\text{vec}(\underline{\underline{A}}\ \underline{\underline{B}}\ \underline{\underline{C}}) = (\underline{\underline{C}}^T \otimes \underline{\underline{A}})\,\text{vec}(\underline{\underline{B}}) . \qquad (A.4\text{-}3)$$

3. If $\underline{\underline{A}}$ is a (mxn) and $\underline{\underline{B}}$ a (nxr) matrix, then from (A.4-3) it is obvious that

$$\mathrm{vec}(\underline{\underline{A}}\ \underline{\underline{B}}) = (\underline{\underline{B}}^T \otimes \underline{\underline{I}}_m)\mathrm{vec}(\underline{\underline{A}}) = (\underline{\underline{B}}^T \otimes \underline{\underline{A}})\,\mathrm{vec}(\underline{\underline{I}}_n) = (\underline{\underline{I}}_r \otimes \underline{\underline{A}})\mathrm{vec}(\underline{\underline{B}}), \tag{A.4-4}$$

where the index of the unit matrices refers to their size. Here we have taken advantage of the trivial identities

$$\mathrm{vec}(\underline{\underline{A}}\ \underline{\underline{B}}) = \mathrm{vec}(\underline{\underline{I}}_m\ \underline{\underline{A}}\ \underline{\underline{B}}) = \mathrm{vec}(\underline{\underline{A}}\ \underline{\underline{I}}_n\ \underline{\underline{B}}) = \mathrm{vec}(\underline{\underline{A}}\ \underline{\underline{B}}\ \underline{\underline{I}}_r). \tag{A.4-5}$$

4. The following identities refer to the calculation of the trace of the product matrices:

$$\mathrm{tr}(\underline{\underline{A}}\ \underline{\underline{B}}\ \underline{\underline{C}}) = \left[\mathrm{vec}(\underline{\underline{A}}^T)\right]^T (\underline{\underline{I}} \otimes \underline{\underline{B}})\mathrm{vec}(\underline{\underline{C}}) \tag{A.4-6}$$

$$\mathrm{tr}(\underline{\underline{A}}\ \underline{\underline{B}}) = \left[\mathrm{vec}(\underline{\underline{A}}^T)\right]^T \mathrm{vec}(\underline{\underline{B}}) \tag{A.4-7}$$

$$\mathrm{tr}(\underline{\underline{A}}\ \underline{\underline{Z}}^T\ \underline{\underline{B}}\ \underline{\underline{Z}}\ \underline{\underline{C}}) = \left[\mathrm{vec}(\underline{\underline{Z}})\right]^T (\underline{\underline{A}}^T\ \underline{\underline{C}}^T \otimes \underline{\underline{B}})\mathrm{vec}(\underline{\underline{Z}}) =$$

$$= \left[\mathrm{vec}(\underline{\underline{Z}})\right]^T (\underline{\underline{C}}\ \underline{\underline{A}} \otimes \underline{\underline{B}}^T)\mathrm{vez}(\underline{\underline{Z}}). \tag{A.4-8}$$

The identities can be understood in the most simple way by expanding simple examples.

Appendix 5.

In this section the estimation methods summarized in subsection 1.3 and to be applied in this report are discussed in detail.

Consider first the least squares method. With the LS method the loss function of the form (1.3-7) has to be minimized, i.e. the extreme value problem

$$Q(\hat{\underline{p}}) = \min_{\underline{p}} Q(\underline{p}) = \min_{\underline{p}} \frac{1}{2} \sum_{t=1}^{N} [y(t) - \underline{f}^T(u,y,t)\underline{p}]^2 =$$

$$= \min_{\underline{p}} \frac{1}{2} (\underline{y} - \underline{\underline{F}}_{uy}\, \underline{p})^T (\underline{y} - \underline{\underline{F}}_{uy}\, \underline{p}) \qquad (A.5\text{-}1)$$

has to be solved. Here $Q(\underline{p})$ is

$$Q(\underline{p}) = \frac{1}{2} (\underline{y}^T \underline{y} - 2\underline{y}^T \underline{\underline{F}}_{uy}\, \underline{p} + \underline{p}^T \underline{\underline{F}}_{uy}^T \underline{\underline{F}}_{uy}\, \underline{p}), \qquad (A.5\text{-}2)$$

where the notations according to subsections 1.3 and 2.1 have been used and the joint vector equation valid for N samples of the equation (1.3-8) is applied in the form

$$\underline{y} = \underline{\underline{F}}(\underline{u},\underline{y})\underline{p} + \underline{e} = \underline{\underline{F}}_{uy}\, \underline{p} + \underline{e}\,. \qquad (A.5\text{-}3)$$

The loss function $Q(\underline{p})$ is therefore quadratic in $\underline{p}$ with the LS method for the (1.3-8) system equation linear in parameters. This means at the same time that by making the gradient

$$\frac{d}{d\underline{p}} Q(\underline{p}) = -\underline{\underline{F}}_{uy}^T \underline{y} + \underline{\underline{F}}_{uy}^T \underline{\underline{F}}_{uy}\, \underline{p} \qquad (A.5\text{-}4)$$

of the loss function by $\underline{p}$ equal to zero vector an explicit solution is obtained for the parameter estimation $\hat{\underline{p}}$ [46, 57].

$$\hat{\underline{p}} = (\underline{\underline{F}}_{uy}^T \underline{\underline{F}}_{uy})^{-1} \underline{\underline{F}}_{uy}^T \underline{y} = \left[\underline{\underline{F}}^T(\underline{u},\underline{y}) \underline{\underline{F}}(\underline{u},\underline{y}) \right]^{-1} \underline{\underline{F}}^T(\underline{u},\underline{y}) \underline{y} . \quad (A.5\text{-}5)$$

The Hessian matrix of the second derivatives $\underline{\underline{F}}_{uy}^T \underline{\underline{F}}_{uy}$ is a Grammian matrix that is non-negative definite. Thus the solution (A.5-5) (the solution of the Gauss normal equation system) gives a minimum point if $\underline{\underline{F}}_{uy}^T \underline{\underline{F}}_{uy}$ is not singular (i.e. has a full rank). The necessary condition of the regularity is that the number of the rows of $\underline{\underline{F}}_{uy}$ (i.e. the measurement points) be greater than or equal with the number of the columns (i.e. parameters), further that the elements of the vector of the function components $\underline{f}(u,y,t)$ have a linear independence.

$\underline{\underline{F}}_{uy}$ contains also a stochastic variable with the models generally dynamic used for discrete identification. In these cases only asymptotically unbiased estimation can be obtained for $\underline{p}$ but it has to be assumed that [15, 46, 57]

$$\operatorname*{plim}_{N\to\infty} \frac{1}{N} (\underline{\underline{F}}_N^T \underline{\underline{F}}_N) = \underline{\underline{S}}_1 < \infty \quad \text{and} \quad \operatorname*{plim}_{N\to\infty} \frac{1}{N} (\underline{\underline{F}}_N^T \underline{e}_N) = \underline{0} , \quad (A.5\text{-}6)$$

where the N subscript denotes now the number of the processed samples. Then namely on the basis of (A.5-5) and (A.5-3)

$$\operatorname*{plim}_{N\to\infty} \hat{\underline{p}}_N = \operatorname*{plim}_{N\to\infty} (\underline{\underline{F}}_N^T \underline{\underline{F}}_N)^{-1} \underline{\underline{F}}_N \underline{y}_N = \hat{\underline{p}}_N +$$

$$+ \operatorname*{plim}_{N\to\infty} (\frac{1}{N} \underline{\underline{F}}_N^T \underline{\underline{F}}_N)^{-1} \operatorname*{plim}_{N\to\infty} (\frac{1}{N} \underline{\underline{F}}_N^T \underline{e}_N) = \underline{p} . \quad (A.5\text{-}7)$$

Here the identity has been used according to which the probability limit of the product agrees with the product of the

probability limits of the factors. Note that just now because of the statistical dependence of the two factors the formation of the expected value cannot be applied (with a given finite N), only the probability limit can. This accounts for the estimation being only asymptotically unbiased.

Under the conditions of (A.5-6) the LS estimation is asymptotically unbiased and, as proven by several authors [80,125], of all linear estimates it has the least standard deviation, is consistent and asymptotically efficient one. May we also remark at this point that the LS estimation can be applied not only in case of an error with normal distribution but agrees with the ML estimation in case of normal distribution.

With the generalized least squares method the detailed form of the loss function (1.3-5) to be minimized is by considering (A.5-3) and an arbitrary symmetrical $\underline{\underline{W}}$ weight matrix:

$$Q(\underline{p}) = \frac{1}{2} (\underline{y} - \underline{\underline{F}}_{uy}\, \underline{p})^T\, \underline{\underline{W}} (\underline{y} - \underline{\underline{F}}_{uy}\, \underline{p}) =$$

$$= \frac{1}{2} (\underline{y}^T\, \underline{\underline{W}}\, \underline{y} - 2\underline{y}^T\, \underline{\underline{W}}\, \underline{\underline{F}}_{uy}\, \underline{p} + \underline{p}^T\, \underline{\underline{F}}_{uy}^T\, \underline{\underline{W}}\, \underline{\underline{F}}_{uy}\, \underline{p}). \qquad (A.5\text{-}8)$$

By forming the gradient and making it equal to zero vector for the necessary condition required by the minimum, we obtain that [15, 57]

$$\hat{\underline{p}} = [\underline{\underline{F}}^T(\underline{u},\underline{y})\, \underline{\underline{W}}\, \underline{\underline{F}}(\underline{u},\underline{y})]^{-1}\, \underline{\underline{F}}^T(\underline{u},\underline{y})\, \underline{\underline{W}}\, \underline{y} =$$

$$= (\underline{\underline{F}}_{uy}^T\, \underline{\underline{W}}\, \underline{\underline{F}}_{uy})^{-1}\, \underline{\underline{F}}_{uy}^T\, \underline{\underline{W}}\, \underline{y}. \qquad (A.5\text{-}9)$$

This equation is the general solution of the GLS method and with the choice of $\underline{\underline{W}} = \underline{\underline{Z}}^{-1}$ it agrees with the ML estimate in case of normal distribution. For the asymptotically unbiased estimate now the conditions

$$\operatorname*{plim}_{N\to\infty} \frac{1}{N} (\underline{\underline{F}}_N^T \underline{\underline{W}}_N \underline{\underline{F}}_N) = \underline{\underline{S}}_2 < \infty \quad \text{and} \quad \operatorname*{plim}_{N\to\infty} \frac{1}{N} (\underline{\underline{F}}_N^T \underline{\underline{W}}_N \underline{e}_N) = \underline{0} \qquad (A.5\text{-}10)$$

have to be fulfilled [57]. The subsequent train of thought agrees already with that applied in the equation (A.5-7). The properties of the GLS estimate agree with those of the LS estimate but the estimate with the minimum standard deviation requires the condition $\underline{\underline{W}} = \underline{\underline{Z}}^{-1}$.

On the basis of the conditions (A.5-7) and (A.5-10) we can attain the instrumental variables method. If, namely such instrumental variables can be constructed which participate in a $\underline{\underline{G}}_N$ instrumental matrix, where the conditions [57, 143]

$$\operatorname*{plim}_{N\to\infty} \frac{1}{N} (\underline{\underline{G}}_N^T \underline{\underline{F}}_N) = \underline{\underline{S}}_3 < \infty \quad \text{and} \quad \operatorname*{plim}_{N\to\infty} \frac{1}{N} (\underline{\underline{G}}_N^T \underline{e}_N) = \underline{0} \qquad (A.5\text{-}11)$$

are fulfilled, then also the estimate

$$\hat{\underline{p}} = (\underline{\underline{G}}_N^T \underline{\underline{F}}_N)^{-1} \underline{\underline{G}}_N^T \underline{y}_N \qquad (A.5\text{-}12)$$

will be asymptotically unbiased. In order to construct $\underline{\underline{G}}$ usually the noiseless output $v(t)$, resp. in the practice its estimate is employed, therefore the notation $\underline{\underline{G}}(u,v) \equiv \underline{\underline{G}}_{uv}$ is justified. Thus (A.5-12) will have the form

$$\hat{\underline{p}} = [\underline{\underline{G}}^T(\underline{u},\underline{v})\underline{\underline{F}}(\underline{u},\underline{y})]^{-1} \underline{\underline{G}}^T(\underline{u},\underline{v})\underline{y} \equiv$$

$$\equiv (\underline{\underline{G}}_{uv}^T \underline{\underline{F}}_{uy})^{-1} \underline{\underline{G}}_{uv}^T \underline{y} . \qquad (A.5\text{-}13)$$

By substituting $\underline{y}$ by (A.5-3) into (A.5-12) and considering (A.5-11), we obtain that

$$\operatorname*{plim}_{N\to\infty} \hat{\underline{p}}_N = \operatorname*{plim}_{N\to\infty} (\underline{\underline{G}}_N^T \underline{\underline{F}}_N)^{-1} \underline{\underline{G}}_N^T \underline{y}_N = \underline{p} +$$

$$+ \operatorname*{plim}_{N\to\infty} (\frac{1}{N} \underline{\underline{G}}_N^T \underline{\underline{F}}_N)^{-1} \operatorname*{plim}_{N\to\infty} (\frac{1}{N} \underline{\underline{G}}_N^T \underline{e}_N) = \underline{p} . \qquad (A.5\text{-}14)$$

The asymptotic properties of the IV method agree with those of the methods LS and GLS discussed above and it is easy to see [57, 143] that the optimal instrumental matrix $\underline{\underline{G}}$ is $\underline{\underline{G}} = \left[\underline{\underline{F}}^T \underline{\underline{Z}}^{-1}\right]^T$ that is it agrees with the ML estimate of general covariance matrix.

In what has gone before, the LS, GLS and IV methods were dealt with in detail in the case of a system model linear in parameters. The LS, GLS methods can, of course, be applied also in a case nonlinear in parameters, but the solution is much more complicated and usually obtained in a non-explicit form, but by some iterative minimization tehcnique. Actually, the same holds for the ML estimation nonlinear in parameters, too, since the relation to the methods LS, GLS have already been made clear. Let us now make some supplementary remarks on the maximum likelihood method.

In case of $\underline{\underline{Z}} = \lambda^2 \underline{\underline{I}}$, i.e. an uncorrelated error, the detailed form of the likelihood function (1.3-4) is [8]:

$$L(\underline{p},\lambda) = -\frac{N}{2} \ln 2\P - \frac{N}{2} \ln\lambda^2 - \frac{1}{2\lambda^2} \underline{e}^T \underline{e}\ , \qquad \text{(A.5-15)}$$

where we have indicated that L is the function of both $\underline{p}$ and λ. We obtain the loss function (1.3-7) simply by assuming λ known. It is shown now that (1.3-7) remains valid also in the case of an unknown λ and we are enable to estimate λ, too. Let us form the partial derivative of (A.5-15) by λ and make it equal to zero.

$$\frac{\partial L(\underline{p},\lambda)}{\partial \lambda} = -\frac{N}{\lambda} + \frac{\underline{e}^T \underline{e}}{\lambda^3} = 0, \qquad \text{(A.5-16)}$$

whence the estimate of λ^2

$$\hat{\lambda}^2 = \lambda^2(\hat{\underline{p}}) = \frac{1}{N} \underline{e}^T \underline{e}\Big|_{\hat{\underline{p}}} = \frac{2}{N} Q(\hat{\underline{p}}). \qquad \text{(A.5-17)}$$

Here it is denoted by $\hat{p}$ that the calculations have to be carried out with that p which ensures the maximum of L, i.e. with the estimated ML parameter vector. As now λ is a function of $\underline{p}$, let substitute it back into (A.5-15):

$$L(\underline{p},\lambda)\Big|_{\hat{\lambda}(\underline{p})} = L(\underline{p}) = -\frac{N}{2}\ln 2\P - \frac{N}{2}\ln\frac{\underline{e}^T\underline{e}}{N} - \frac{1}{2}\frac{N}{\underline{e}^T\underline{e}}\,\underline{e}^T\underline{e} =$$

$$= -\frac{N}{2}(\ln 2\P + 1) - \frac{N}{2}\ln\frac{\underline{e}^T\underline{e}}{N}. \qquad \text{(A.5-18)}$$

Hence it is already easy to see that the maximum of $L(\underline{p})$ can be achieved by minimizing the loss function

$$Q(\hat{\underline{p}}) = \min_{\underline{p}} Q(\underline{p}) = \min_{\underline{p}} \frac{1}{2}\,\underline{e}^T\underline{e} \qquad \text{(A.5-19)}$$

as we have stated it on the basis of simpler considerations.

In case of the likelihood function (1.3-15) set for the MIMO systems, the solution is no more so self- evident. By simply considering only that in case of a given $\underline{\underline{\Lambda}}$ the maximum of L is obtained by minimizing the expression $\sum_{t=1}^{N} \underline{e}^T(t)\,\underline{\underline{\Lambda}}^{-1}\,\underline{e}(t)$, then we would get quite another result, than by analyzing L in detail. Here the main problem is caused by $\underline{\underline{\Lambda}}$ being now a matrix. The likelihood function is now the function of $\underline{\underline{P}}$ and $\underline{\underline{\Lambda}}$ [108], i.e.

$$L(\underline{\underline{P}}, \underline{\underline{\Lambda}}) = -\frac{qN}{2}\ln 2\P - \frac{N}{2}\ln\left|\underline{\underline{\Lambda}}\right| - \frac{1}{2}\sum_{t=1}^{N}\underline{e}^T(t)\,\underline{\underline{\Lambda}}^{-1}\,\underline{e}(t). \qquad \text{(A.5-20)}$$

In order to determine the extremum value of L according to $\underline{\underline{\Lambda}}$, for the simplicity let us make the matrix derivated by $\underline{\underline{\Lambda}}^{-1}$ equal to zero matrix:

$$\frac{\partial L(\underline{\underline{P}},\underline{\underline{\Lambda}})}{\partial\, \underline{\underline{\Lambda}}^{-1}} = \frac{N}{2}\ \underline{\underline{\Lambda}} - \frac{1}{2}\ \sum_{t=1}^{N} \underline{e}(t)\underline{e}^{T}(t) = \underline{\underline{0}}, \qquad \text{(A.5-21)}$$

whence the estimate of $\underline{\underline{\Lambda}}$ is:

$$\hat{\underline{\underline{\Lambda}}} = \hat{\underline{\underline{\Lambda}}}(\hat{\underline{\underline{P}}}) = \frac{1}{N}\ \sum_{t=1}^{N} \underline{e}(t)\underline{e}^{T}(t)\Big|_{\hat{\underline{\underline{P}}}} . \qquad \text{(A.5-22)}$$

Here $\hat{\underline{\underline{P}}}$ denotes that the calculations have to be carried out with that $\underline{\underline{P}}$ which ensures the maximum of L, i.e. estimated ML parameter matrix. As now $\underline{\underline{\Lambda}}$ is the function of $\underline{\underline{P}}$, let it be substituted back into (A.5-20).

$$L(\underline{\underline{P}},\underline{\underline{\Lambda}})\Big|_{\hat{\underline{\underline{\Lambda}}}(\underline{\underline{P}})} = L(\underline{\underline{P}}) = -\frac{q.N}{2}\,(\ln 2\P + 1) - \frac{N}{2}\ln\left|\hat{\underline{\underline{\Lambda}}}(\underline{\underline{P}})\right|. \qquad \text{(A.5-23)}$$

This relation is obtained by the following way. Let be

$$\underline{\underline{E}} = \left[\underline{e}(1),\ldots,\underline{e}(N)\right]. \qquad \text{(A.5-24)}$$

By this notation

$$\hat{\underline{\underline{\Lambda}}} = \frac{1}{N}\ \underline{\underline{E}}\ \underline{\underline{E}}^{T} \qquad \text{(A.5-25)}$$

as we can find out by a simple expansion. By taking advantage of the identities related to the trace of the matrix, the (A.5-20) can be written also in the following form by considering the notation (A.5-24):

$$L(\underline{\underline{P}},\underline{\underline{\Lambda}}) = -\frac{qN}{2}\ln 2\P - \frac{N}{2}\ln\left|\underline{\underline{\Lambda}}\right| - \frac{1}{2}\ \sum_{t=1}^{N} \mathrm{tr}\left[\underline{\underline{\Lambda}}^{-1}\ \underline{e}(t)\underline{e}^{T}(t)\right] =$$

$$= -\frac{qN}{2}\ln 2\P - \frac{N}{2}\ln\left|\underline{\underline{\Lambda}}\right| - \frac{1}{2}\ \mathrm{tr}(\underline{\underline{E}}^{T}\ \underline{\underline{\Lambda}}^{-1}\ \underline{\underline{E}}). \qquad \text{(A.5-26)}$$

Considering here $\hat{\underline{\underline{\Lambda}}}$ according to (A.5-25):

$$L(\underline{\underline{P}},\underline{\underline{\Lambda}})\Big|_{\hat{\underline{\underline{\Lambda}}}(\underline{\underline{P}})} = L(\underline{\underline{P}}) = -\frac{qN}{2}\ln 2\P - \frac{N}{2}\ln\left|\hat{\underline{\underline{\Lambda}}}(\underline{\underline{P}})\right| -$$

$$- \frac{N}{2}\,\mathrm{tr}\left[\underline{\underline{E}}^T(\underline{\underline{E}}^T\,\underline{\underline{E}})^{-1}\,\underline{\underline{E}}\right] =$$

$$= -\frac{qN}{2}\ln 2\P - \frac{N}{2}\ln\left|\hat{\underline{\underline{\Lambda}}}(\underline{\underline{P}})\right| - \frac{N}{2}\mathrm{tr}\left[(\underline{\underline{E}}^T\,\underline{\underline{E}})^{-1}\,\underline{\underline{E}}^T\,\underline{\underline{E}}\right] =$$

$$= -\frac{qN}{2}\ln 2\P - \frac{N}{2}\ln\left|\hat{\underline{\underline{\Lambda}}}(\underline{\underline{P}})\right| - \frac{N}{2}\,\mathrm{tr}(\underline{\underline{I}}_q) =$$

$$= -\frac{q\cdot N}{2}(\ln 2\P + 1) - \frac{N}{2}\ln\left|\hat{\underline{\underline{\Lambda}}}(\underline{\underline{P}})\right|, \qquad \text{(A.5-27)}$$

the unit matrix $\underline{\underline{I}}_q$ is now an (qxq) matrix.

With the MIMO systems therefore the loss function to be minimized ensuring the maximum of the likelihood function [108] is

$$Q(\hat{\underline{\underline{P}}}) = \min_{\underline{\underline{P}}}\left|\hat{\underline{\underline{\Lambda}}}(\underline{\underline{P}})\right| = \min_{\underline{\underline{P}}}\left|\frac{1}{N}\,\underline{\underline{E}}\,\underline{\underline{E}}^T\right| = \min_{\underline{\underline{P}}}\left|\frac{1}{N}\sum_{t=1}^{N}\underline{e}(t)\underline{e}^T(t)\right|, \qquad \text{(A.5-28)}$$

i.e. the determinant of a matrix (the covariance matrix of the source noise).

Appendix 6.

Definition:

The bounded $u(t)$ signal is called persistently exciting with order q (CESARO condition) if

$$\bar{u} = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t) \qquad (A.6\text{-}1)$$

and

$$r(k) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} [u(t)-\bar{u}][u(t+k)-\bar{u}] \qquad (A.6\text{-}2)$$

quantities exist and the (qxq) covariance matrix

$$\underset{\equiv q}{R} = \begin{bmatrix} r(0) & r(1) & \dots & r(q-1) \\ r(1) & r(0) & \dots & r(q-2) \\ \vdots & & & \\ r(q-1) & r(q-2) & \dots & r(0) \end{bmatrix} \qquad (A.6\text{-}3)$$

is posivite definite [12, 88].

Appendix 7.

According to Appendix 5, the LS estimation of the parameters is given by the solution

$$\hat{\underline{p}}_{ba} = (\underline{\underline{F}}_{uy}^T \underline{\underline{F}}_{uy})^{-1} \underline{\underline{F}}_{uy}^T \underline{y} = \left[\underline{\underline{F}}^T(\underline{u},\underline{y}) \underline{\underline{F}}(\underline{u},\underline{y}) \right]^{-1} \underline{\underline{F}}^T(\underline{u},\underline{y}) \underline{y} . \tag{A.7-1}$$

As the system equation has the form for N measurements

$$\underline{y} = \underline{\underline{F}}(\underline{u},\underline{y}) \underline{p}_{ba} + \underline{e} \equiv \underline{\underline{F}}_{uy} \underline{p}_{ba} + \underline{e} \ , \tag{A.7-2}$$

substituting it into (A.7-1) we get

$$\hat{\underline{p}}_{ba} = (\underline{\underline{F}}_{uy}^T \underline{\underline{F}}_{uy})^{-1} \underline{\underline{F}}_{uy}^T \underline{\underline{F}}_{uy} \underline{p}_{ba} + (\underline{\underline{F}}_{uy}^T \underline{\underline{F}}_{uy})^{-1} \underline{\underline{F}}_{uy}^T \underline{e} =$$

$$= \underline{p}_{ba} + (\underline{\underline{F}}_{uy}^T \underline{\underline{F}}_{uy})^{-1} \underline{\underline{F}}_{uy}^T \underline{e}. \tag{A.7-3}$$

Form now the probability boundary value of both sides

$$\operatorname*{plim}_{N\to\infty} \hat{\underline{p}}_{ba} = \underline{p}_{ba} + \operatorname*{plim}_{N\to\infty} \left(\frac{1}{N} \underline{\underline{F}}_{uy}^T \underline{\underline{F}}_{uy}\right)^{-1} \operatorname*{plim}_{N\to\infty} \left(\frac{1}{N} \underline{\underline{F}}_{uy}^T \underline{e}\right) . \tag{A.7-4}$$

In case of the fulfilment of the conditions (A.5-6), the second term of the right side would be a zero vector and thus the estimation asymptotically unbiased. The condition related to the matrix $\underline{\underline{S}}_1$ in (A.5-6) is fulfilled if the input signal is persistently exciting with order 2n [12, 88].

Therefore the question is whether the condition

$$\operatorname*{plim}_{N\to\infty} \left(\frac{1}{N} \underline{\underline{F}}_{uy}^T \underline{e}\right) = \underline{0} \tag{A.7-5}$$

is fulfilled? Let us use the definition of $\underline{\underline{F}}_{uy}$ according to (2.1-6). On this basis we can write that

$$\underline{\underline{F}}^T_{uy}\,\underline{e} \equiv \underline{\underline{F}}^T(\underline{u},\underline{y})\underline{e} = \underline{\underline{F}}^T(\underline{u},\underline{v})\underline{e} + \underline{\underline{F}}^T(\underline{0},\underline{\underline{A}}^{-1}\,\underline{e})\underline{e}\,. \qquad (A.7\text{-}6)$$

Here the equations (1.2-38), (2.1-5) and (2.1-6) have been considered and

$$\underline{v} = \underline{\underline{A}}^{-1}\,\underline{\underline{B}}\,\underline{u}\,, \qquad (A.7\text{-}7)$$

i.e. $\underline{v}$ is an N dimensional vector containing the values of the noiseless output of the process. Expand in detail the matrices $\underline{\underline{F}}^T(\underline{u},\underline{v})$ and $\underline{\underline{F}}^T(\underline{0},\underline{\underline{A}}^{-1}\,\underline{e})\underline{e}$

$$\underline{\underline{F}}^T(\underline{u},\underline{v}) = \begin{bmatrix} \underline{u}^T(\underline{\underline{S}}^T)^0 \\ \vdots \\ \underline{u}^T(\underline{\underline{S}}^T)^n \\ \hline -\underline{v}^T(\underline{\underline{S}}^T) \\ \vdots \\ -\underline{v}^T(\underline{\underline{S}}^T)^n \end{bmatrix} = \begin{bmatrix} \underline{u}^T(\underline{\underline{S}}^T)^0 \\ \vdots \\ \underline{u}^T(\underline{\underline{S}}^T)^n \\ \hline -\underline{u}^T\,\underline{\underline{B}}^T(\underline{\underline{A}}^{-1})^T(\underline{\underline{S}}^T) \\ \vdots \\ -\underline{u}^T\,\underline{\underline{B}}^T(\underline{\underline{A}}^{-1})^T(\underline{\underline{S}}^T)^n \end{bmatrix}, \qquad (A.7\text{-}8)$$

similarly

$$\underline{\underline{F}}^T(\underline{0},\underline{\underline{A}}^{-1}\underline{e}) = \begin{bmatrix} \underline{0}^T \\ \vdots \\ \underline{0}^T \\ \hline -\underline{e}^T(\underline{\underline{A}}^{-1})^T(\underline{\underline{S}}^T) \\ \vdots \\ -\underline{e}^T(\underline{\underline{A}}^{-1})^T(\underline{\underline{S}}^T)^n \end{bmatrix}. \qquad (A.7\text{-}9)$$

On the basis of the above equations the probability limits of the quadratic forms

$$\underset{N\to\infty}{\text{plim}}\left[\frac{1}{N}\,\underline{u}^T\,(\underline{\underline{S}}^T)^j\,\underline{e}\right] \qquad \text{(A.7-10)}$$

and

$$\underset{N\to\infty}{\text{plim}}\left[\frac{1}{N}\,\underline{e}^T\,(\underline{\underline{S}}^T)^j\,\underline{e}\right] \qquad \text{(A.7-11)}$$

have to be formed in (A.7-5). The $\underline{\underline{B}}$, resp. $\underline{\underline{A}}^{-1}$ TOEPLITZ matrices in (A.7-8) and (A.7-9) can always be written as the sums (linear combinations) of the $\underline{\underline{S}}$ shift matrices, as it can be followed in App.2. In the formulas (A.7-10) and (A.7-11) the products of values of two signals shifted by j samples have to be added and averaged. This means therefore that

$$\underset{N\to\infty}{\text{plim}}\left[\frac{1}{N}\,\underline{u}^T(\underline{\underline{S}}^T)^j\,\underline{e}\right] = r_{ue}(j) \qquad \text{(A.7-12)}$$

and

$$\underset{N\to\infty}{\text{plim}}\left[\frac{1}{N}\,\underline{e}^T(\underline{\underline{S}}^T)^j\,\underline{e}\right] = r_{ee}(j) \qquad \text{(A.7-13)}$$

where $r_{ue}(j)$ is the value belonging to j shifts of the cross--correlation series of the signals u(t) and e(t), as well as $r_{ee}(j)$ is the value belonging to j shifts of the autocorrelation series of e(t). According to our assumption for the uncorrelatedness of the input signal and the source noise, $r_{ue}(j)$ is zero for all j-s. Likewise $r_{ee}(j)$ is also zero for the values $j \neq 0$ as e(t) is assumed to be a white noise. As j is never zero with the quadratic form (A.7-13), the condition (A.7-5) of the asymptotic unbiasedness is fulfilled.

It is easy to see that if the equation error contains, instead of $\underline{e}$, components of any other form (generally e.g. $\underline{\underline{C}}\underline{e}$), i.e. $\underline{\underline{S}}^i\underline{e}$ too, then because of the occurrence of the quadratic forms $\underline{e}^T(\underline{\underline{S}}^T)^j\,\underline{\underline{S}}^i\underline{e}$ the (A.7-5) is nc more fulfilled.

Appendix 8.

According to (2.6-10)

$$\hat{\underline{k}} = (\underline{\underline{M}}_u^T \underline{\underline{M}}_u)^{-1} \underline{\underline{M}}_u^T(\underline{y} - \underline{\underline{F}}_{uy} \underline{p}_{ba}). \qquad (A.8-1)$$

Substitute this into the term of the loss function

$$Q(\underline{p}_{ba}) = \frac{1}{2} \underline{k}^T \underline{k} \qquad (A.8-2)$$

then

$$Q(\underline{p}_{ba}) = \frac{1}{2} (\underline{y} - \underline{\underline{F}}_{uy} \underline{p}_{ba})^T \underline{\underline{M}}_u(\underline{\underline{M}}_u^T \underline{\underline{M}}_u)^{-1}(\underline{\underline{M}}_u^T \underline{\underline{M}}_u)^{-1} *$$

$$* \underline{\underline{M}}_u^T(\underline{y} - \underline{\underline{F}}_{uy} \underline{p}_{ba}), \qquad (A.8-3)$$

where it is used that $\underline{\underline{M}}_u^T \underline{\underline{M}}_u$ and thus its inverse, too, are symmetric matrices. The gradient of the loss function

$$\frac{dQ(\underline{p}_{ba})}{d\underline{p}_{ba}} = - \underline{\underline{F}}_{uy}^T \underline{\underline{M}}_u(\underline{\underline{M}}_u^T \underline{\underline{M}}_u)^{-1}(\underline{\underline{M}}_u^T \underline{\underline{M}}_u)^{-1} *$$

$$* \underline{\underline{M}}_u^T(\underline{y} - \underline{\underline{F}}_{uy} \underline{p}_{ba}), \qquad (A.8-4)$$

and by making it equal to zero vector we get

$$\hat{\underline{p}}_{ba} = [\underline{\underline{F}}_{uy}^T \underline{\underline{M}}_u(\underline{\underline{M}}_u^T \underline{\underline{M}}_u)^{-1}(\underline{\underline{M}}_u^T \underline{\underline{M}}_u)^{-1} \underline{\underline{M}}_u^T \underline{\underline{F}}_{uy}]^{-1} *$$

$$* \underline{\underline{F}}_{uy}^T \underline{\underline{M}}_u(\underline{\underline{M}}_u^T \underline{\underline{M}}_u)^{-1}(\underline{\underline{M}}_u^T \underline{\underline{M}}_u)^{-1} \underline{\underline{M}}_u^T \underline{y}. \qquad (A.8-5)$$

Appendix 9.

The maximization of the likelihood function according to (2.7-12) - as referred to above several times - is a task equivalent with the minimization of the loss function

$$Q(\underline{p}_{bla}) = \frac{1}{2} \sum_{t=1}^{N} \underline{w}^T(t) \underline{\underline{Z}}^{-1} \underline{w}(t). \qquad (A.9-1)$$

The minimization has to be performed in such a way, that

$$\underline{g}_o^T(u_o,y_o,t)\underline{p}_{bla} = \underline{g}_o^T(t)\underline{p}_{bla} = 0 \qquad (A.9-2)$$

should be fulfilled for t=1,2,...,N. The extremum value problem under this constraint can be solved in the simplest way by the introduction of LAGRANGE multiplicators [41, 141]. Let $2\Psi(t)$ be the multiplicator, then the original task can be reduced to the minimization of the modified loss function

$$Q'[\underline{g}_o(t),\Psi(t)] = \frac{1}{2} \sum_{t=1}^{N} [\underline{w}^T(t)\underline{\underline{Z}}^{-1} \underline{w}(t) + 2\Psi(t)\underline{g}_o^T(u_o,y_o,t)\underline{p}_{bla}] \qquad (A.9-3)$$

according to $\underline{g}_o(u_o,y_o,t)$ and $\Psi(t)$ [106]. Derive Q'

$$\frac{dQ'[\underline{g}_o(t),\Psi(t)]}{d\underline{g}_o(t)} = -\underline{\underline{Z}}^{-1} \underline{w}(t) + \Psi(t)\underline{p}_{bla}, \qquad (A.9-4)$$

where the relation (2.7-7) is also taken into consideration, i.e. the dependence of $\underline{w}(t)$ from $\underline{g}_o$. By making the gradient (A.9-4) equal to zero vector, we obtain that

$$\underline{w}(t) = \underline{g}(u,y,t) - \underline{g}_o(u_o,y_o,t) = \Psi(t)\underline{\underline{Z}}\,\underline{p}_{bla}. \qquad (A.9-5)$$

Hence multiplying by $\underline{p}_{bla}^T$

$$\underline{p}_{bla}^T \underline{g}(u,y,t) - \underline{p}_{bla}^T \underline{g}_o(u_o,y_o,t) = \psi(t)\underline{p}_{bla}^T \underline{\underline{Z}} \underline{p}_{bla} . \quad (A.9-6)$$

If we consider that

$$\underline{p}_{bla}^T \underline{g}_o(u_o,y_o,t) = \underline{g}_o^T(u_o,y_o,t)\underline{p}_{bla} = 0, \quad t=1,2,\ldots,N, \quad (A.9-7)$$

then

$$\underline{p}_{bla}^T \underline{g}(u,y,t) = \psi(t)\underline{p}_{bla}^T \underline{\underline{Z}} \underline{p}_{bla} \quad (A.9-8)$$

and finally $\psi(t)$ ensuring the minimum according to $\underline{g}_o(t)$:

$$\psi(t) = \frac{\underline{p}_{bla}^T \underline{g}(u,y,t)}{\underline{p}_{bla}^T \underline{\underline{Z}} \underline{p}_{bla}} ; \quad t=1,2,\ldots,N . \quad (A.9-9)$$

On the other hand by expressing $\underline{g}_o(u_o,y_o,t)$ from (A.9-5)

$$\underline{g}_o(u_o,y_o,t) = \underline{g}(u,y,t) - \psi(t)\underline{\underline{Z}} \underline{p}_{bla} \quad (A.9-10)$$

and substituting it into (A.9-3), after the arrangement we obtain the expression

$$Q' = \frac{1}{2} \sum_{t=1}^{N} \psi(t)\underline{g}^T(u,y,t)\underline{p}_{bla} , \quad (A.9-11)$$

where the relation (A.9-8) is also taken account. As neither (A.9-9) nor (A.9-11) depend upon $\underline{g}_o(t)$, thus by substituting the $\psi(t)$ value belonging to the minimum according to (A.9-9) into (A.9-11), we obtain the minimum value Q^* according to $\underline{g}_o(t)$ of Q' (and thus of Q, too, for we have considered the constraint). Therefore

$$Q^{*} = \min_{g_o(t)} Q' = \frac{1}{2} \sum_{t=1}^{N} \frac{\underline{p}_{bla}^{T}\, \underline{g}(u,y,t)\underline{g}^{T}(u,y,t)\, \underline{p}_{bla}}{\underline{p}_{bla}^{T}\, \underline{\underline{Z}}\, \underline{p}_{bla}} =$$

$$= \frac{1}{2} \frac{\underline{p}_{bla}^{T} \left[\sum_{t=1}^{N} \underline{g}(u,y,t)\underline{g}^{T}(u,y,t) \right] \underline{p}_{bla}}{\underline{p}_{bla}^{T}\, \underline{\underline{Z}}\, \underline{p}_{bla}}$$

$$= \frac{1}{2} \frac{\underline{p}_{bla}^{T}\, \underline{\underline{G}}\, \underline{p}_{bla}}{\underline{p}_{bla}^{T}\, \underline{\underline{Z}}\, \underline{p}_{bla}} = Q^{*}(\underline{p}_{bla}), \qquad \text{(A.9-12)}$$

where

$$\underline{\underline{G}} = \sum_{t=1}^{N} \underline{g}(u,y,t)\underline{g}^{T}(u,y,t). \qquad \text{(A.9-13)}$$

Appendix 10.

The KOOPMANS-LEVIN method leads to the minimization according to $\underline{p}_{bla}$ of the form

$$Q^{*}(\underline{p}_{bla}) = \frac{1}{2} \frac{\underline{p}_{bla}^{T} \underline{\underline{G}} \underline{p}_{bla}}{\underline{p}_{bla}^{T} \underline{\underline{Z}} \underline{p}_{bla}} \qquad (A.10\text{-}1)$$

as written in (2.7-15).

It is well-known [105] that the solution of the extremum value problem

$$\frac{\underline{p}^{T} \underline{\underline{G}} \underline{p}}{\underline{p}^{T} \underline{\underline{Z}} \underline{p}} = \min_{\underline{p}} \qquad (A.10\text{-}2)$$

is given by the eigenvector belonging to the least eigenvalue of the generalized eigenvalue problem

$$(\underline{\underline{G}} \underline{p} - \mu \underline{\underline{Z}} \underline{p}) = \underline{0} . \qquad (A.10\text{-}3)$$

$\underline{\underline{G}}$ resolution:

Let the diagonal matrix formed from the eigenvalues of the symmetric matrix $\underline{\underline{G}}$ be $\underline{\underline{L}}$, the matrix of its orthonormed eigenvectors be $\underline{\underline{A}}$. On the basis of the well-known theorems of the main diagonal transformation of the quadratic forms [105] it can be written that

$$\underline{\underline{A}}^{T} \underline{\underline{G}} \underline{\underline{A}} = \underline{\underline{L}} , \qquad (A.10\text{-}4)$$

where because of orthonormality

$$\underline{\underline{A}}^T \underline{\underline{A}} = \underline{\underline{I}} \qquad \text{(A.10-5)}$$

i.e.

$$\underline{\underline{A}}^{-1} = \underline{\underline{A}}^T . \qquad \text{(A.10-6)}$$

By expressing $\underline{\underline{G}}$ from (A.10-4)

$$\underline{\underline{G}} = (\underline{\underline{A}}^T)^{-1} \underline{\underline{L}} \underline{\underline{A}}^{-1} = (\underline{\underline{A}}^{-1})^{-1} \underline{\underline{L}} \underline{\underline{A}}^{-1} = \underline{\underline{A}} \underline{\underline{L}} \underline{\underline{A}}^T =$$

$$= \underline{\underline{A}} \underline{\underline{L}}^{1/2} \underline{\underline{L}}^{1/2} \underline{\underline{A}}^T = \underline{\underline{D}} \underline{\underline{D}}^T , \qquad \text{(A.10-7)}$$

where

$$\underline{\underline{D}} = \underline{\underline{A}} \underline{\underline{L}}^{1/2} . \qquad \text{(A.10-8)}$$

The calculation of the square root matrix $\underline{\underline{L}}^{1/2}$ is now very simple because of the diagonality. By taking the resolution (A.10-7) in the original equation (A.10-3) into account, we obtain

$$(\underline{\underline{D}} \underline{\underline{D}}^T - \mu \underline{\underline{Z}}) \underline{p} = \underline{0} . \qquad \text{(A.10-9)}$$

With a rearrangement:

$$\underline{\underline{D}} [\underline{\underline{I}} - \mu \underline{\underline{D}}^{-1} \underline{\underline{Z}} (\underline{\underline{D}}^{-1})^T] \underline{\underline{D}}^T \underline{p} = \underline{0} , \qquad \text{(A.10-10)}$$

whence with the substitution

$$\underline{q} = -\underline{\underline{D}}^T \underline{p} \qquad \text{(A.10-11)}$$

we obtain that

$$[\underline{\underline{D}}^{-1} \underline{\underline{Z}} (\underline{\underline{D}}^{-1})^T - \frac{1}{\mu} \underline{\underline{I}}] \underline{q} = (\underline{\underline{P}} - \nu \underline{\underline{I}}) \underline{q} = \underline{0} . \qquad \text{(A.10-12)}$$

Here already $\underline{\underline{P}} = \underline{\underline{P}}^T$ so that the original general eigenvalue problem was reduced to the seeking of the eigenvalue of a symmetric matrix. The course of the solution is therefore the following:

1. Determine the eigenvalues and eigenvectors of the symmetric $\underline{\underline{G}}$ matrix, thereupon form the $\underline{\underline{L}}$ and $\underline{\underline{A}}$ matrices,
2. Generate matrix $\underline{\underline{D}}$ resulting from the resolution of $\underline{\underline{G}}$ and determine the symmetric matrix $\underline{\underline{P}} = \underline{\underline{D}}^{-1}\ \underline{\underline{Z}}(\underline{\underline{D}}^{-1})^T$.
3. As there is a reciprocal relation between μ and ν, we determine the eigenvector $\underline{q}^*$ belonging to the greatest eigenvalue ν.
4. On the basis of (A.10-11) we retransform $\underline{q}^*$

$$\underline{p}^* = -(\underline{\underline{D}}^{-1})^T\ \underline{q}^* . \qquad \text{(A.10-13)}$$

5. As $\underline{p}^*$ yields only an optimal direction in the parameter space, we have to ensure also that the (n+2)-nd component of $\underline{p}_{bla}$ should be made equal to 1 according to the definition (2.7-3) to obtain the appropriate normalization of the vector. Therefore the final parameter estimation will have the form

$$\hat{\underline{p}}_{bla} = \frac{1}{p^*_{n+2}}\ \underline{p}^* , \qquad \text{(A.10-14)}$$

where p^*_{n+2} denotes the (n+2)nd component of $\underline{p}^*$. The above described scheme therefore requires the determination of the eigenvalues, eigenvectors of only a <u>symmetric</u> matrix.

<u>$\underline{\underline{Z}}$ resolution:</u>

Similarly to the $\underline{\underline{G}}$ resolution according to (A.10-7), the $\underline{\underline{Z}}$ resolution can also be carried out. Also in this case we would get to the preceding five points, resp. their application according to sense. Now, of course, the eigenvalues

and eigenvectors of $\underline{\underline{Z}}$ have to be determined in the first step. The problem of the $\underline{\underline{Z}}$ resolution arises just with the above initial step. In the practice it is a natural requirement to allow for certain "degeneration" of $\underline{\underline{Z}}$ when in (2.7-9) $\underline{\underline{Z}}_{uu}$ or $\underline{\underline{Z}}_{yy}$ is a zero matrix, or at least partially is. If the input signal is measured without error, the $\underline{\underline{Z}}_{uu} = \underline{\underline{0}}$. The application of the $\underline{\underline{Z}}$ resolution would cause a great problem for the performance of the main diagonal transformation. The choice of the $\underline{\underline{G}}$ resolution seems to be more reasonable because of such and similar problems.

Note that Eq. (A.10-3) multiplied by the inverse of the invertible matrix can be reduced to a simple eigenvalue problem but, of course, the eigenvalue of a non-symmetric matrix has to be found.

Appendix 11.

Examine now the estimation of the parameter matrix $\underline{\underline{P}}_{BA}$ of the system equation (3.1-8) according to the least squares method. The direct generalization of the LS estimation discussed with the SISO systems would be obtained by the minimization of the loss function

$$\underline{\underline{W}} = \sum_{t=1}^{N} \underline{e}(t)\underline{e}^T(t) = \underline{\underline{E}}\ \underline{\underline{E}}^T . \qquad (A.11-1)$$

where

$$\underline{\underline{E}} = \underline{\underline{Y}} - \underline{\underline{P}}_{BA}\ \underline{\underline{F}} . \qquad (A.11-2)$$

Thus $\underline{\underline{W}}$ is really proportional to the matrix of the moments of the calculated residuals $\underline{e}(t)$. As $\underline{\underline{W}}$ is a matrix, several kinds of scalar measures can really be used for minimization and this set of problems makes the essential difference compared to the SISO systems. After the choice of the appropriate scalar measure, it is usually intended with the MIMO systems that only the derivation of the scalar according to a vector should have to be used for minimization and the application of the derivative according to the matrix possibly avoided. Now the explicit solution of the LS parametric identification of the MIMO systems is given also via various approaches.

Choose as a scalar measure (loss function) the logarithm of the determinant of $\underline{\underline{W}}$ [57]. Then the condition

$$\frac{d \ln|\underline{\underline{W}}|}{d\ \underline{\underline{P}}_{BA}} = 2(\underline{\underline{Y}}\ \underline{\underline{F}}^T - \underline{\underline{P}}_{BA}\ \underline{\underline{F}}\ \underline{\underline{F}}^T)\ \underline{\underline{W}}^{-1} = \underline{\underline{0}} \qquad (A.11-3)$$

has to be ensured for minimization. By expressing $\underline{\underline{P}}_{BA}$ from the matrix equation, we obtain the LS estimate of the parameters of the MIMO systems

$$\hat{\underline{\underline{P}}}_{BA} = \underline{\underline{Y}}\ \underline{\underline{F}}^T\ (\underline{\underline{F}}\ \underline{\underline{F}}^T)^{-1}. \qquad (A.11\text{-}4)$$

(In the relation (A.11-3) the rules of the derivation of the determinant according to the matrix have been used.)

We have shown in Appendix 5 that $|\hat{\underline{\underline{\Lambda}}}(\underline{\underline{P}})|$ determinant is to be minimized with the ML parameter estimation of the MIMO systems. It is easy to see from the comparison of (A.5-28) and (A.11-1) that with the condition (A.11-3) we essentially solve the minimization of (A.5-28), too. This means that - alike the SISO systems - in case of $\underline{e}(t)$ source noise of normal distribution the LS and ML estimations coincide also with the MIMO systems and lead to the explicit solution according to (A.11-4).

Examine also another possible mode of approach. According to Eq. (3.1-12), the joint (matrix) system equation (3.1-18) can be arranged also in a vectorial form:

$$\underline{Y}_M = \underline{\underline{G}}\ \underline{p}_{BA} + \underline{e}_M . \qquad (A.11\text{-}5)$$

As the measuring error appears now in the form of the vector $\underline{e}_M$, the ML estimation technique discussed with the SISO systems can be applied also now. The loss function formed in analogous way with the expression (1.3-5):

$$Q(\underline{p}_{BA}) = \frac{1}{2}\ \underline{e}_M^T\ (\underline{\underline{I}}_N \otimes \underline{\underline{\Lambda}}^{-1})\underline{e}_M . \qquad (A.11\text{-}6)$$

Here we have considered that the covariance matrix of an

e(t) is $\underline{\underline{\Lambda}}$, thus for the all set is $\underline{\underline{I}}_N \otimes \underline{\underline{\Lambda}}$. The N subscript indicates the size of the unit matrix. By further rewriting of (A.11-6), we obtain that

$$Q(\underline{p}_{BA}) = \frac{1}{2}\left[\underline{y}_M^T \left(\underline{\underline{I}}_N \otimes \underline{\underline{\Lambda}}^{-1}\right) \underline{y}_M - 2\underline{y}_M^T \left(\underline{\underline{I}}_N \otimes \underline{\underline{\Lambda}}^{-1}\right)\underline{\underline{G}}\, \underline{p}_{BA} + \right.$$

$$\left. + \underline{p}_{BA}^T\, \underline{\underline{G}}^T \left(\underline{\underline{I}}_N \otimes \underline{\underline{\Lambda}}^{-1}\right)\underline{\underline{G}}\, \underline{p}_{BA}\right]. \qquad \text{(A.11-7)}$$

Let us form the gradient of Q according to $\underline{p}_{BA}$:

$$\frac{\partial Q(\underline{p}_{BA})}{\partial \underline{p}_{BA}} = -\underline{\underline{G}}^T\left(\underline{\underline{I}}_N \otimes \underline{\underline{\Lambda}}^{-1}\right)\underline{y}_M + \underline{\underline{G}}^T \left(\underline{\underline{I}}_N \otimes \underline{\underline{\Lambda}}^{-1}\right)\underline{\underline{G}}\, \underline{p}_{BA}\ , \qquad \text{(A.11-8)}$$

and making it equal to zero, we obtain:

$$\hat{\underline{p}}_{BA} = \left[\underline{\underline{G}}^T \left(\underline{\underline{I}}_N \otimes \underline{\underline{\Lambda}}^{-1}\right)\underline{\underline{G}}\right]^{-1} \underline{\underline{G}}^T \left(\underline{\underline{I}}_N \otimes \underline{\underline{\Lambda}}^{-1}\right) \underline{y}_M\ . \qquad \text{(A.11-9)}$$

By taking the definition of $\underline{\underline{G}}$ in (3.1-13) into account:

$$\hat{\underline{p}}_{BA} = \left[\left(\underline{\underline{F}} \otimes \underline{\underline{I}}_q\right) \left(\underline{\underline{I}}_N \otimes \underline{\underline{\Lambda}}^{-1}\right) \left(\underline{\underline{F}}^T \otimes \underline{\underline{I}}_q\right)\right]^{-1} \left(\underline{\underline{F}} \otimes \underline{\underline{I}}_q\right)\left(\underline{\underline{I}}_N \otimes \underline{\underline{\Lambda}}^{-1}\right) \underline{y}_M =$$

$$= \left(\underline{\underline{F}}\, \underline{\underline{F}}^T \otimes \underline{\underline{\Lambda}}^{-1}\right)^{-1} \left(\underline{\underline{F}} \otimes \underline{\underline{\Lambda}}^{-1}\right)\underline{y}_M = \left[\left(\underline{\underline{F}}\, \underline{\underline{F}}^T\right)^{-1} \underline{\underline{F}} \otimes \underline{\underline{I}}_q\right] \underline{y}_M\ . \qquad \text{(A.11-10)}$$

Appendix 12.

Consider the loss function

$$\tilde{Q}(\underline{\underline{P}}_{BAC}) = \frac{N}{2} \ln \left| \hat{\underline{\underline{\Lambda}}} \, \underline{\underline{P}}_{BAC} \right| \qquad (A.12\text{-}1)$$

to be minimized. Form the scalar derivative according to the p_k-th component:

$$\frac{\partial \tilde{Q}(\underline{\underline{P}}_{BAC})}{\partial p_k} = \frac{\partial}{\partial p_k} \frac{N}{2} \ln \left| \hat{\underline{\underline{\Lambda}}}(\underline{\underline{P}}_{BAC}) \right| =$$

$$= \frac{N}{2} \sum_{i=1}^{q} \sum_{j=1}^{q} \frac{\partial \ln \left| \hat{\underline{\underline{\Lambda}}} \right|}{\partial \hat{\lambda}_{ij}} \frac{\partial \hat{\lambda}_{ij}}{\partial p_k} , \qquad (A.12\text{-}2)$$

where the rule of differentiation of the indirect function is applied, namely first the derivatives according to the elements $\hat{\lambda}_{ij}$ of the matrix $\hat{\underline{\underline{\Lambda}}}$ have been formed. By utilizing that

$$\frac{\partial \ln \underline{\underline{X}}}{\partial \underline{\underline{X}}} = \underline{\underline{X}}^{-1} \qquad (A.12\text{-}3)$$

and denoting by $\hat{\lambda}^{ij}$ the i,j-th element of matrix $\hat{\underline{\underline{\Lambda}}}^{-1}$, we obtain the expression

$$\frac{\partial \tilde{Q}(\underline{\underline{P}}_{BAC})}{\partial p_k} = \sum_{i=1}^{q} \sum_{j=1}^{q} \hat{\lambda}^{ij} \frac{\partial}{\partial p_k} \left[\frac{1}{2} \sum_{t=1}^{N} e_i(t) e_j(t) \right] =$$

$$= \sum_{t=1}^{N} \underline{e}^T(t) \hat{\underline{\underline{\Lambda}}}^{-1} \frac{\partial \underline{e}(t)}{\partial p_k} \qquad (A.12\text{-}4)$$

which is only one element of the gradient.

On this basis the complete gradient vector [108] is:

$$\frac{\partial \tilde{Q}(\underline{\underline{P}}_{BAC})}{\partial \underline{p}_{BAC}} = \sum_{t=1}^{N} \frac{d\underline{e}^T(t)}{d\underline{p}_{BAC}} \hat{\underline{\underline{\Lambda}}}^{-1} \underline{e}(t) . \qquad (A.12\text{-}5)$$

Examine now the various expanded forms of the filter equation (3.2-22) in order to construct a recursive formula for their easy computation.

Arrange the partial derivatives the following way into a vector (i,j=1,...,q)

$$\underline{g}^B_{ij}(t) = \left[\frac{\partial \underline{e}^T(t)}{\partial B_{oij}} , \ldots , \frac{\partial \underline{e}^T(t)}{\partial B_{nij}} \right]^T , \qquad (A.12\text{-}6)$$

$$\underline{g}^A_{ij}(t) = \left[\frac{\partial \underline{e}^T(t)}{\partial A_{lij}} , \ldots , \frac{\partial \underline{e}^T(t)}{\partial A_{nij}} \right]^T , \qquad (A.12\text{-}7)$$

$$\underline{g}^C_{ij}(t) = \left[\frac{\partial \underline{e}^T(t)}{\partial C_{lij}} , \ldots , \frac{\partial \underline{e}^T(t)}{\partial C_{nij}} \right]^T . \qquad (A.12\text{-}8)$$

By considering the relations (3.2-23) - (3.2-26), the following recursive equations can be written:

$$\underline{g}^B_{ij}(t) = \underline{\underline{S}} \, \underline{g}^B_{ij}(t-1) + \underline{h}^B_{ij}(t) , \qquad (A.12\text{-}9)$$

$$\underline{g}^A_{ij}(t) = \underline{\underline{S}} \, \underline{g}^A_{ij}(t-1) + \underline{h}^A_{ij}(t) , \qquad (A.12\text{-}10)$$

$$\underline{g}^C_{ij}(t) = \underline{\underline{S}} \, \underline{g}^C_{ij}(t-1) + \underline{h}^C_{ij}(t) , \qquad (A.12\text{-}11)$$

where

$$\underline{\underline{S}} = \begin{bmatrix} -\underline{\underline{C}}_1 & -\underline{\underline{C}}_2 & \cdots & -\underline{\underline{C}}_{n-1} & -\underline{\underline{C}}_n \\ \underline{\underline{I}} & \underline{\underline{0}} & & \underline{\underline{0}} & \underline{\underline{0}} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \underline{\underline{0}} & \underline{\underline{0}} & & \underline{\underline{I}} & \underline{\underline{0}} \end{bmatrix} \qquad \text{(A.12-12)}$$

furthermore

$$\underline{h}^{B}_{ij}(t) = [\,0 \ \ldots -u_j(t) \ldots 0,\ \underline{0}^T, \ldots, 0^T]^T\,, \qquad \text{(A.12-13)}$$

($-u_j(t)$: i-th)

$$\underline{h}^{A}_{ij}(t) = [\,0 \ \ldots y_j(t) \ldots\ 0, \underline{0}^T, \ \ldots\ , \underline{0}^T]^T, \qquad \text{(A.12-14)}$$

($y_j(t)$: i-th)

$$\underline{h}^{C}_{ij}(t) = [\,0 \ \ldots -e_j(t) \ldots\ 0, \underline{0}^T, \ \ldots\ , \underline{0}^T]^T. \qquad \text{(A.12-15)}$$

($-e_j(t)$: i-th)

Here it has to be noted that $\underline{\underline{S}}$ in the equation (A.12-9) contains really by one more hypermatrices consisting of zeros under the fictive coefficient matrix $\underline{\underline{C}}_{n+1} \equiv \underline{\underline{0}}$. This has to be taken into account because of the coeffient $\underline{\underline{B}}_0$. This slight inaccuracy in the notation does not cause any particular problem because we never perform in reality the multiplication by $\underline{\underline{S}}$ in its totality (let us think of the many zero elements) but we multiply only according to the first hyper row and set the other elements by shift. The relations presented now correspond to the generalization of the equation (2.2-26) for MIMO systems.

In order to facilite computer storage, for the storage of the elements of $g^{B}_{ij}(t)$ we suggest to use the matrix

$$\underline{\underline{G}}^{B}_{ij}(t)=\left[\frac{\partial \underline{e}(t)}{\partial B_{oij}},\ldots,\frac{\partial \underline{e}(t)}{\partial B_{nij}}\right] \qquad \text{(A.12-16)}$$

resp. the matrix constructed from these:

$$\underline{\underline{G}}^{B}(t)=\begin{bmatrix} \underline{\underline{G}}^{B}_{11}(t) \\ \vdots \\ \underline{\underline{G}}^{B}_{1q}(t) \\ \underline{\underline{G}}^{B}_{21}(t) \\ \vdots \\ \underline{\underline{G}}^{B}_{q1}(t) \\ \vdots \\ \underline{\underline{G}}^{B}_{qq}(t) \end{bmatrix} \qquad \text{(A.12-17)}$$

The same can be carried out for the matrices $\underline{\underline{G}}^{A}(t)$ and $\underline{\underline{G}}^{C}(t)$ and all partial derivatives are contained in the matrix

$$\underline{\underline{G}}(t)=\begin{bmatrix} \underline{\underline{G}}^{B}(t) \\ \underline{\underline{G}}^{A}(t) \\ \underline{\underline{G}}^{C}(t) \end{bmatrix} . \qquad \text{(A.12-18)}$$

By following the equations (A.12-16) - (A.12-18) , it can be found that by the recursive formation of the elements it is also very easy to calculate the gradient vector and the

$\underline{\underline{T}}_1$ since in the summation according to t the partial derivations are available in an easily accessible form, only the programming solution of the arrangement into vector and arrangement back has to be skilfully achieved.
The calculation of the second order partial derivatives $\partial^2\underline{e}(t)/\partial p_i \partial p_j$ is fairly complicated so their computer programming can seldom occur. In spite of this, we give the relations. It is easy to see from the relations (3.2-23) - (3.2-25) that the second derivatives according to the elements of the coefficient matrices $\underline{\underline{B}}$ and $\underline{\underline{A}}$ have zero values, in consequence there is no interaction, while the second order derivatives formed by the elements of $\underline{\underline{C}}$ are not zeroes. Thus, the construction of $\underline{\underline{T}}_2$ similar to what was used with the SISO systems. The formation of the second order derivatives will be shown for the elements C_{kij} and C_{fgh}. Derive first both sides of the system equation (3.2-18) according to C_{kij}

$$\frac{\partial \underline{\underline{C}}(z^{-1})}{\partial C_{kij}} \underline{e}(t) + \underline{\underline{C}}(z^{-1}) \frac{\partial \underline{e}(t)}{\partial C_{kij}} = \underline{0} \qquad \text{(A.12-19)}$$

i.e.

$$z^{-k} \underline{\underline{\delta}}_{ij} \underline{e}(t) + \underline{\underline{C}}(z^{-1}) \frac{\partial \underline{e}(t)}{\partial C_{kij}} = \underline{0}, \qquad \text{(A.12-20)}$$

whence

$$\underline{\underline{C}}(z^{-1}) \frac{\partial \underline{e}(t)}{\partial C_{kij}} = -z^{-k} \underline{\underline{\delta}}_{ij} \underline{e}(t) = \begin{bmatrix} 0 \\ \vdots \\ -z^{-k} e_j(t) \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{i-th} . \qquad \text{(A.12-21)}$$

The Eqs. (3.2-22) - (3.2-25) can be directly obtained by a deduction of such type. Here $\underline{\underline{\delta}}_{ij}$ is the so-called KRONECKER matrix whose ij-th element has the value 1, all others have the value of zero. Derive now (A.12-21) according to C_{fgh}:

$$\frac{\partial \underline{\underline{C}}(z^{-1})}{\partial C_{fgh}} \frac{\partial \underline{e}(t)}{\partial C_{kij}} + \underline{\underline{C}}(z^{-1}) \frac{\partial^2 \underline{e}(t)}{\partial C_{fgh} \partial C_{kij}} = -z^{-k} \underline{\underline{\delta}}_{ij} \frac{\partial \underline{e}(t)}{\partial C_{fgh}} , \tag{A.12-22}$$

resp.

$$z^{-f} \underline{\underline{\delta}}_{gh} \frac{\partial \underline{e}(t)}{\partial C_{kij}} + \underline{\underline{C}}(z^{-1}) \frac{\partial^2 \underline{e}(t)}{\partial C_{fgh} C_{kij}} = -z^{-k} \underline{\underline{\delta}}_{ij} \frac{\partial \underline{e}(t)}{\partial C_{fgh}} . \tag{A.12-23}$$

Hence the moving average filter equation serving for the generation of the second order derivative can be simply obtained:

$$\underline{\underline{C}}(z^{-1}) \frac{\partial^2 \underline{e}(t)}{\partial C_{fgh} \partial C_{kij}} = -z^{-k} \underline{\underline{\delta}}_{ij} \frac{\partial \underline{e}(t)}{\partial C_{fgh}} - z^{-f} \underline{\underline{\delta}}_{gh} \frac{\partial \underline{e}(t)}{\partial C_{kij}} . \tag{A.12-24}$$

Here it is very difficult to find a solution as simple as for the SISO systems where the formation of the second order derivative coincided - regarding the technique - with that of the first order ones. Otherwise (A.12-24) can be written also in the form

$$\underline{C}(z^{-1})\frac{\partial^2 \underline{e}(t)}{\partial C_{fgh}\partial C_{kij}} = \begin{bmatrix} 0 \\ \vdots \\ -\dfrac{\partial e_j(t-k-f+1)}{\partial C_{lgh}} \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ -\dfrac{\partial e_h(t-k-f+1)}{\partial C_{lij}} \\ \vdots \\ 0 \end{bmatrix} \qquad \text{(A.12-25)}$$

i-th row g-th row

by taking into account the identities (3.2-26). On the basis of this latter equation, the rules of recursive formation similar to the equations (A.12-9) - (A.12-11) can also be given for the generation of second order derivatives. The results can be extended, of course, to the formation of the other non-zero second order partial derivatives.

Appendix 13.

The loss function to be minimized is

$$Q(\underline{\underline{P}}_{BA}) = \frac{1}{2}\left[\mathrm{vec}(\underline{\underline{P}}_K)\right]^T\left[\mathrm{vec}(\underline{\underline{P}}_K)\right] = \frac{1}{2}\,\mathrm{tr}(\underline{\underline{P}}_K^T\,\underline{\underline{P}}_K)\,, \qquad \text{(A.13-1)}$$

where $\underline{\underline{P}}_k$ is considered which ensures the LS fit and whose formula is

$$\hat{\underline{\underline{P}}}_k = (\underline{\underline{Y}} - \underline{\underline{P}}_{BA}\,\underline{\underline{F}})\underline{\underline{G}}^T(\underline{\underline{G}}\,\underline{\underline{G}}^T)^{-1}\,. \qquad \text{(A.13-2)}$$

Form the arrangement of both sides of (A.13-2) into vector

$$\mathrm{vec}(\hat{\underline{\underline{P}}}_k) = \left[(\underline{\underline{G}}\,\underline{\underline{G}}^T)^{-1}\,\underline{\underline{G}}\otimes\underline{\underline{I}}\right]\mathrm{vec}(\underline{\underline{Y}} - \underline{\underline{P}}_{BA}\underline{\underline{F}})\,, \qquad \text{(A.13-3)}$$

consequently

$$Q(\underline{\underline{P}}_{BA}) = \frac{1}{2}\left\{\left[(\underline{\underline{G}}\,\underline{\underline{G}}^T)^{-1}\,\underline{\underline{G}}\otimes\underline{\underline{I}}\right]\mathrm{vec}(\underline{\underline{Y}} - \underline{\underline{P}}_{BA}\,F)\right\}^T *$$

$$* \left\{\left[(\underline{\underline{G}}\,\underline{\underline{G}}^T)^{-1}\,\underline{\underline{G}}\otimes\underline{\underline{I}}\right]\mathrm{vec}(\underline{\underline{Y}} - \underline{\underline{P}}_{BA}\,\underline{\underline{F}})\right\}. \qquad \text{(A.13-4)}$$

By performing th transposition and multiplication, we obtain

$$Q(\underline{\underline{P}}_{BA}) = \{\mathrm{vec}(\underline{\underline{P}}_{BA})^T(\underline{\underline{F}}\otimes\underline{\underline{I}})\left[\underline{\underline{G}}^T(\underline{\underline{G}}\,\underline{\underline{G}}^T)^{-1}(\underline{\underline{G}}\,\underline{\underline{G}}^T)^{-1}\,\underline{\underline{G}}\otimes\underline{\underline{I}}\right](\underline{\underline{F}}^T\otimes\underline{\underline{I}})\mathrm{vec}(\underline{\underline{P}}_{BA}) -$$

$$- \mathrm{vec}(\underline{\underline{Y}})^T\left[\underline{\underline{G}}^T(\underline{\underline{G}}\,\underline{\underline{G}}^T)^{-1}(\underline{\underline{G}}\,\underline{\underline{G}}^T)^{-1}\,\underline{\underline{G}}\otimes\underline{\underline{I}}\right](\underline{\underline{F}}^T\otimes\underline{\underline{I}})\mathrm{vec}(\underline{\underline{P}}_{BA}) -$$

$$- \mathrm{vec}(\underline{\underline{P}}_{BA})^T(\underline{\underline{F}}\otimes\underline{\underline{I}})\left[\underline{\underline{G}}^T(\underline{\underline{G}}\,\underline{\underline{G}}^T)^{-1}(\underline{\underline{G}}\,\underline{\underline{G}}^T)^{-1}\,\underline{\underline{G}}\otimes\underline{\underline{I}}\right]\mathrm{vec}(\underline{\underline{Y}}) +$$

$$+ \mathrm{vec}(\underline{\underline{Y}})^T\left[\underline{\underline{G}}^T(\underline{\underline{G}}\,\underline{\underline{G}}^T)^{-1}(\underline{\underline{G}}\,\underline{\underline{G}}^T)^{-1}\,\underline{\underline{G}}\otimes\underline{\underline{I}}\right]\mathrm{vec}(\underline{\underline{Y}})\}\,\frac{1}{2}\,, \qquad \text{(A.13-5)}$$

where the identity $\mathrm{vec}(\underline{\underline{P}}\ \underline{\underline{F}}) \equiv (\underline{\underline{F}}^T \otimes \underline{\underline{I}})\mathrm{vec}(\underline{\underline{P}})$ is used. Derive the expression (A.13-5) according to vec $(\underline{\underline{P}}_{BA})$

$$\frac{dQ(\underline{\underline{P}}_{BA})}{d\underline{\underline{P}}_{BA}} = \{ 2(\underline{\underline{F}}\otimes\underline{\underline{I}})[\underline{\underline{G}}^T(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}\ \underline{\underline{G}}\otimes\underline{\underline{I}}](\underline{\underline{F}}^T\otimes\underline{\underline{I}})\mathrm{vec}(\underline{\underline{P}}_{BA}) -$$

$$- (\underline{\underline{F}}\otimes\underline{\underline{I}})[\underline{\underline{G}}^T(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}\ \underline{\underline{G}}\otimes\underline{\underline{I}}](\underline{\underline{F}}^T\otimes\underline{\underline{I}})\mathrm{vec}(\underline{\underline{Y}}) -$$

$$- (\underline{\underline{F}}\otimes\underline{\underline{I}})[\underline{\underline{G}}^T(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}\ \underline{\underline{G}}\otimes\underline{\underline{I}}](\underline{\underline{F}}^T\otimes\underline{\underline{I}})\mathrm{vec}(\underline{\underline{Y}})\} \frac{1}{2} . \qquad (A.13\text{-}6)$$

By making the gradient equal to zero vector, we obtain

$$\mathrm{vec}(\hat{\underline{\underline{P}}}_{BA}) = \{(\underline{\underline{F}}\otimes\underline{\underline{I}})[\underline{\underline{G}}^T(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}\ \underline{\underline{G}}\otimes\underline{\underline{I}}](\underline{\underline{F}}^T\otimes\underline{\underline{I}})\}^{-1} *$$

$$* (\underline{\underline{F}}\otimes\underline{\underline{I}})[\underline{\underline{G}}^T(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}\ \underline{\underline{G}}\otimes\underline{\underline{I}}]\mathrm{vec}(\underline{\underline{Y}}). \qquad (A.13\text{-}7)$$

By the simplification of this latter equation:

$$\mathrm{vec}(\hat{\underline{\underline{P}}}_{BA}) = [\underline{\underline{F}}\ \underline{\underline{G}}^T(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}\ \underline{\underline{G}}\ \underline{\underline{F}}^T\otimes\underline{\underline{I}}]^{-1} *$$

$$* [\underline{\underline{F}}\ \underline{\underline{G}}^T(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}\ \underline{\underline{G}}]\mathrm{vec}(\underline{\underline{Y}}) =$$

$$= \{\underline{\underline{G}}^T(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}\ \underline{\underline{G}}\ \underline{\underline{F}}^T[\underline{\underline{F}}\ \underline{\underline{G}}^T(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}(\underline{\underline{G}}\ \underline{\underline{G}}^T)^{-1}$$

$$\underline{\underline{G}}\ \underline{\underline{F}}^T]^{-1}\otimes\underline{\underline{I}}\} * \mathrm{vec}(\underline{\underline{Y}}) . \qquad (A.13\text{-}8)$$

On the basis of (A.4-4) this expression can already be written back into the form of a matrix relation

$$\hat{\underline{\underline{P}}}_{BA} = \underline{\underline{Y}}\,\underline{\underline{G}}^T(\underline{\underline{G}}\,\underline{\underline{G}}^T)^{-1}(\underline{\underline{G}}\,\underline{\underline{G}}^T)^{-1}\,\underline{\underline{G}}\,\underline{\underline{F}}^T\big[\underline{\underline{F}}\,\underline{\underline{G}}^T(\underline{\underline{G}}\,\underline{\underline{G}}^T)^{-1}(\underline{\underline{G}}\,\underline{\underline{G}}^T)^{-1}\,\underline{\underline{G}}\,\underline{\underline{F}}^T\big]^{-1}. \qquad \text{(A.13-9)}$$

Appendix 14.

The second order TAYLOR series of the loss function (4-5) at $\underline{p}(t)$ for the reference point $\underline{p}(t-1)$ is [17]:

$$J[\underline{p}(t)] \cong J[\underline{p}(t-1)] + \Delta\underline{p}^T(t-1) \frac{dJ[\underline{p}(t-1)]}{d\underline{p}(t-1)} +$$

$$+ \frac{1}{2} \Delta\underline{p}^T(t-1) \underline{H}\{J[\underline{p}(t-1)]\} \Delta\underline{p}(t-1), \qquad (A.14-1)$$

where

$$\Delta\underline{p}(t-1) = \underline{p}(t) - \underline{p}(t-1) . \qquad (A.14-2)$$

During the seeking optimal in the second order sense we are looking for the minimum of (A.14-1) in every step, that is such a $\Delta\underline{p}(t-1)$ step for which the necessary condition of the minimum

$$\frac{dJ[\underline{p}(t)]}{d\underline{p}(t)} = \frac{dJ[\underline{p}(t-1)]}{d\underline{p}(t-1)} + H\{J[\underline{p}(t-1)]\} \Delta\underline{p}(t-1) = \underline{0} \qquad (A.14-3)$$

will be fulfilled.

Determine the terms in (A.14-3). As the distribution or time function of Q is not known, (4-5) will be approximated in the following form:

$$J[\underline{p}(t)] = \frac{1}{t} \sum_{j=1}^{t} Q[\underline{x}(j), \underline{p}(t)] . \qquad (A.14-4)$$

Consider that although in (4-6) the argument is $\underline{p}(t-1)$ but $\underline{x}(t)$ has already arrived and $\underline{x}(j)=\underline{x}(t)$ is in Q. On the basis of these and taking (A.14-4) into account:

$$\underline{\underline{H}}\{J[\underline{p}(t-1)]\} = \frac{1}{t} \sum_{j=1}^{t} \underline{\underline{H}}\{Q[\underline{x}(j), \underline{p}(t-1)]\} . \qquad (A.14-5)$$

It is easy to see that we still do not know the new $\underline{p}$ (on the other hand we already know $\underline{x}(t)$), therefore exists $\underline{p}(t-1)$.

Likewise we obtain that

$$\frac{dJ[\underline{p}(t-1)]}{d\underline{p}(t-1)} = \frac{1}{t} \sum_{j=1}^{t} \frac{dQ[\underline{x}(j),\underline{p}(t-1)]}{d\underline{p}(t-1)} . \qquad (A.14-6)$$

If we have performed the steps until now in the on-line STA algorithm optimally, then the condition

$$\frac{1}{t} \sum_{j=1}^{t-1} \frac{dQ[\underline{x}(j),\underline{p}(t-1)]}{d\underline{p}(t-1)} = \underline{0} \qquad (A.14-7)$$

must hold. Therefore

$$\frac{dJ[\underline{p}(t-1)]}{d\underline{p}(t-1)} = \frac{1}{t} \frac{dQ[\underline{x}(t),\underline{p}(t-1)]}{d\underline{p}(t-1)} . \qquad (A.14-8)$$

In connection with the summations in the relations (A.14-5) and (A.14-6), not the following: Here the running index j in the argument of $\underline{x}$ and $\underline{p}$ is always the last known value. This circumstance makes the calculations very difficult, since the summation had to be repeated after the determination of every $\underline{p}$ value from the beginning of the data collection. In connection with (A.14-6) this problem will be solved due to the assumption (A.14-7), but we can obtain the better computational form (A.14-5) if $\underline{\underline{H}}$ is only the function of the series $\underline{x}(j)$ and does not depend upon $\underline{p}$. (This is only a nec-

essary condition!)

According to the above, Eq. (A.14-3) becomes

$$\frac{1}{t}\ \frac{dQ[\underline{x}(t),\underline{p}(t-1)]}{d\underline{p}(t-1)} + \left[\frac{1}{t}\ \sum_{j=1}^{t} \underline{\underline{H}}\{Q[\underline{x}(j),\underline{p}(t-1)]\}\right] \Delta\underline{p}(t-1) = \underline{0}\ . \tag{A.14-9}$$

Hence

$$\Delta\underline{p}(t-1) = \left[\sum_{j=1}^{t} \underline{\underline{H}}\{Q[\underline{x}(j),\underline{p}(t-1)]\}\right]^{-1}\ \frac{dQ[\underline{x}(t),\underline{p}(t-1)]}{d\underline{p}(t-1)}\ . \tag{A.14-10}$$

As the on-line algorithm has the form

$$\Delta\underline{p}(t-1) = -\ \underline{\underline{R}}(t)\ \frac{dQ[\underline{x}(t),\underline{p}(t-1)]}{d\underline{p}(t-1)}\ , \tag{A.14-11}$$

thus the optimal value of $\underline{\underline{R}}(t)$ from the comparison of the two equations is

$$\underline{\underline{R}}(t) = \left[\sum_{j=1}^{t} \underline{\underline{H}}\{Q[\underline{x}(j),\underline{p}(t-1)]\}\right]^{-1}\ . \tag{A.14-12}$$

Appendix 15.

On the basis of (4.1-6) we can write that

$$\underline{\underline{R}}(t)=\left[\underline{\underline{R}}^{-1}(t-1)+\underline{f}(t)\underline{f}^{T}(t)\right]^{-1}. \qquad \text{(A.15-1)}$$

Apply now the general identity relating to the partitioned matrices (proved among others also in [80]):

$$(\underline{\underline{A}}+\underline{\underline{B}}\,\underline{\underline{D}}\,\underline{\underline{B}}^{T})^{-1}=\underline{\underline{A}}^{-1}-\underline{\underline{A}}^{-1}\,\underline{\underline{B}}(\underline{\underline{D}}^{-1}+\underline{\underline{B}}^{T}\,\underline{\underline{A}}^{-1}\,\underline{\underline{B}})^{-1}\,\underline{\underline{B}}^{T}\,\underline{\underline{A}}^{-1}. \qquad \text{(A.15-2)}$$

Let be

$$\underline{\underline{A}} \triangleq \underline{\underline{R}}^{-1}(t-1)\ , \qquad \text{(A.15-3)}$$

$$\underline{\underline{B}} \triangleq \underline{f}(t) \qquad \text{(A.15-4)}$$

and

$$\underline{\underline{D}} \triangleq 1\ . \qquad \text{(A.15-5)}$$

Thus

$$\underline{\underline{R}}(t)=\underline{\underline{R}}(t-1)-\underline{\underline{R}}(t-1)\underline{f}(t)*$$

$$*\left[1+\underline{f}^{T}(t)\underline{\underline{R}}(t-1)\underline{f}(t)\right]^{-1}\ \underline{f}^{T}(t)\underline{\underline{R}}(t-1)=$$

$$=\underline{\underline{R}}(t-1)\ -\ \frac{\underline{\underline{R}}(t-1)\underline{f}(t)\underline{f}^{T}(t)\underline{\underline{R}}(t-1)}{1+\underline{f}^{T}(t)\underline{\underline{R}}(t-1)\underline{f}(t)}\ . \qquad \text{(A.15-6)}$$

Appendix 16.

Substitute the relation (4.1-11) into the formula (4.1-10) of the parameter estimation:

$$\underline{p}_{ba}(N+1) = \underline{\underline{R}}(N)\underline{\underline{F}}^{T}_{uy}(N)\underline{y}_N - \frac{\underline{\underline{R}}(N)\underline{f}(N+1)\,\underline{f}^{T}(N+1)\,\underline{\underline{R}}(N)}{1+\underline{f}^{T}(N+1)\,\underline{\underline{R}}(N)\,\underline{f}\,(N+1)}\,\underline{\underline{F}}^{T}_{uy}(N)\underline{y}_N +$$

$$+\ \underline{\underline{R}}(N)\underline{f}(N+1)\,y(N+1) - \frac{\underline{\underline{R}}(N)\underline{f}(N+1)\,\underline{f}^{T}(N+1)\,\underline{\underline{R}}(N)}{1+\underline{f}^{T}(N+1)\,\underline{\underline{R}}(N)\,\underline{f}(N+1)}\,\underline{f}(N+1)\,y(N+1)\,, \tag{A.16-1}$$

where the notation (4.1-12) is taken into account. By using the relations (4.1-9) and (4.1-12)

$$\underline{p}_{ba}(N+1) = \underline{p}_{ba}(N) + \frac{\underline{\underline{R}}(N)\underline{f}(N+1)}{1+\underline{f}^{T}(N+1)\,\underline{\underline{R}}(N)\,\underline{f}(N+1)}\left[y(N+1) - \underline{f}^{T}(N+1)\,\underline{p}_{ba}(N)\right]. \tag{A.16-2}$$

By multiplying both sides of the relation (4.1-7) by $\underline{f}(t)$ it is easy to see that in the latter equation

$$\frac{\underline{\underline{R}}(N)\underline{f}(N+1)}{1+\underline{f}^{T}(N+1)\,\underline{\underline{R}}(N)\,\underline{f}(N+1)} = \underline{\underline{R}}(N+1)\,\underline{f}(N+1)\,, \tag{A.16-3}$$

therefore

$$\underline{p}_{ba}(N+1) = \underline{p}_{ba}(N) + \underline{\underline{R}}(N+1)\,\underline{f}(N+1)\left[y(N+1) - \underline{f}^{T}(N+1)\,\underline{p}_{ba}(N)\right]. \tag{A.16-4}$$

R E F E R E N C E S

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