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by

JENŐ SZÉP

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Jenő Szép

Department of Mathematics and Computer Science
University of Economics
Budapest, Hungary

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On the series

We are launching a series of mathematical books in conjunction with Akadémiai Kiadó — a publishing house with 170 years of expertise. With the present abundance of mathematical journals and publications of mathematical research centres, it is becoming increasingly difficult to keep up to date, even in special fields. Informed opinion has it that there should be *comprehensive surveys* available on many of the special topics that have not yet appeared in handbooks. Since researchers themselves are the best informed about the state of the art in their own field of interest, a number of outstanding researchers have already been approached as potential authors of the topics of this series.

The term "comprehensive" allows a certain amount of leeway in selection. One of the aims of the series is to help prevent unnecessary reproduction of results already achieved, it will help to settle questions of priority, and it will also be of great help in orientation.

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Introduction

The fundamental concepts are the *structure of parentheses*, the *setvectors* and the *ρ-multiplication* in the theory of *Coded structures* or *Vector-products*. For the structure of parentheses see Section 1.2. A setvector can be considered as a multiset-representation or a generalization of the concept of numbers, too (see Section 1.1) which can be used for the extension of rings (especially matrix rings) or other associative structures. Furthermore the introduced multiplication is able to get new semigroups starting with given semigroups.

The book consists of two parts: (I) the discrete case, (II) the continuous case.

In the discrete case, the components of a setvector are non negative integers. A setvector consists of two parts. The first part represents the non negative integers, the second part represents the non positive integers (or the anticomponents of the first part).

The addition of the setvectors is similar to the case of the "classical" vectors. The multiplication of the setvectors differs from the classical one in the result because the product of two setvectors is a setvector again. Therefore, the setvectors form an algebraic structure of two (or more) operations which is not associative (with respect to the several multiplications) in general (see Theorem 7 in Section 2).

Besides, for the multiplication, there are more possibilities: one operation, a set of operations, where the set of operations represents an operation in itself (see Sections 2.4. and 5.10). In Chapter 2, some simple properties of the multiplications with one operation will be shown.

The multiplicative structure is a semigroup (in the special case a group) for a given operation (see Sections 2.4 and 5.10) in the infinite case and in the finite case, too (see Section 6.12), but in the finite case some other conditions are necessary as well. In Chapter 6, necessary and sufficient conditions are given to produce finite semigroups in a simple way.

A special set of the setvectors (with addition and multiplication) is equivalent to the semiring of the non negative integers.

Using the multiplicative structure it is possible to introduce the concept of the *coded structure* in Chapter 7 and then to give a procedure to solve the decoding problem.

The generalisation of the concept of the setvector is introduced in Chapter 8, where the multiplicative structure of the vectors will be described if the components are elements of rings. Besides, we show an application for a finite algebra, too.

Finally, in part II (continuous structures, Chapter 10), we deal briefly with the vectors of real and complex components and will present some applications. In Section 10.1 a process is given to approximate an arbitrary distribution vector starting from the uniform distribution. Naturally, there can be many applications in several areas in mathematics (linear algebra, metric spaces, etc). In the frame of this book, we cannot touch upon these possibilities.

In the opinion of the author, the *coded structures* can be applied for the simulation of processes in the chemistry, nuclear physics and microbiology (for some simple examples, see in [3]) because it is possible to construct several formations using the structure $S(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$.

The theory of "coded structures" is related to *Lindenmayer-systems* and *Theory of automata* (see [1], [11], [12]). Besides, coded structures can be used to create data, text-protection procedures and digital signature of various complexity. Of these, for a few ones, softwares have already been developed.

The structure (described in the following chapters) has the advantage that it can easily be implemented on a computer.

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I
DISCRETE STRUCTURES

1. Basic ideas and properties

1.1. Setvector

Let

(1) $$H = \{h_1, \ldots, h_\mu; h_{-1}, \ldots, h_{-\mu}\}$$

be a finite set of symbols h_i, h_{-i} $(i = 1, \ldots, \mu)$.

Let $\mathcal{H}$ be the set of the finite multisets formed by the elements of H, that is, for any $s \in \mathcal{H}$, $s = \{h_{i_1}, \ldots, h_{i_r}\}$ ($i_j \in \{1, \ldots, \mu; -1, \ldots, -\mu\}$), where i_j and i_k are not necessarily different even if $j \neq k$. For example;

$$\{h_1, h_1, h_2, h_2, h_2, h_3, h_3\} \in \mathcal{H} \qquad (\mu \geq 3).$$

Let s be an arbitrary element of $\mathcal{H}$. The element s can be represented by a vector $[\alpha_1, \ldots, \alpha_\mu; \beta_1, \ldots, \beta_\mu]$ of 2μ components, where α_i and β_i $(i = 1, \ldots, \mu)$ are the number of occurrences of h_i respectively h_{-i} in s. If $\alpha_i = 0$ or $\beta_i = 0$, then it means that s does not contain the element h_i respectively h_{-i}. Thus the vector $[0, \ldots, 0; 0, \ldots, 0]$ represents the empty set of $\mathcal{H}$. Hence there exists a one-to-one correspondence between the elements of $\mathcal{H}$ and vectors of $\mathbf{N}_0^{2\mu}$, where $\mathbf{N}$ and $\mathbf{N}_0$ are the sets of positive, respectively nonnegative integers. This makes possible to identify the elements of $\mathcal{H}$ with the vectors representing them and simplify the description of operations defined bellow:

Addition:

Definition 1. If $s, s' \in \mathcal{H}$ and

$$s = [\alpha_1, \ldots, \alpha_\mu; \beta_1, \ldots, \beta_\mu], \qquad s' = [\alpha'_1, \ldots, \alpha'_\mu; \beta'_1, \ldots, \beta'_\mu]$$

then let

(2) $$s + s' = [\alpha_1 + \alpha'_1, \ldots, \alpha_\mu + \alpha'_\mu; \beta_1 + \beta'_1, \ldots, \beta_\mu + \beta'_\mu].$$

Obviously the defined addition is commutative and associative and the zerovector $0 = [0, \ldots, 0; 0, \ldots, 0]$ corresponding to the empty set is the zero element of $\mathcal{H}$ with respect to this addition, that is,

1) $s_1 + s_2 = s_2 + s_1$
2) $(s_1 + s_2) + s_3 = s_1 + (s_2 + s_3)$
3) $\forall s \in \mathcal{H} : s + 0 = 0 + s = s$.

Multiplication by scalars:

DEFINITION 2. If $\alpha \in \mathbf{N}_0$, and $s = [\alpha_1, \ldots, \alpha_\mu; \beta_1, \ldots, \beta_\mu] \in \mathcal{H}$, then

$$\alpha s = [\alpha\alpha_1, \ldots \alpha\alpha_\mu; \alpha\beta_1, \ldots, \alpha\beta_\mu].$$

The operation of multiplication by scalars has the following properties: if $\alpha, \beta \in \mathbf{N}_0$, and $s, s_1, s_2 \in \mathcal{H}$, then

a) $\alpha(s_1 + s_2) = \alpha s_1 + \alpha s_2 = (s_1 + s_2)\alpha = s_1\alpha + s_2\alpha$
b) $(\alpha + \beta)s = \alpha s + \beta s = s(\alpha + \beta) = s\alpha + s\beta$,
c) $(\alpha\beta)s = \alpha(\beta s) = s(\alpha\beta) = (s\alpha)\beta$,
d) $1s = s$.

The properties a) and c) remain valid even if the scalars are choosen from the set of integers **I**.

The representation $[\alpha_1, \ldots, \alpha_\mu; \beta_1, \ldots, \beta_\mu]$ of s is the *general form* of s in $\mathcal{H}$.

DEFINITION 3. The *normal form* of $s = [\alpha_1, \ldots, \alpha_\mu; \beta_1, \ldots, \beta_\mu] \in \mathcal{H}$ is $n(s) = [\bar{\alpha_1}, \ldots, \bar{\alpha_\mu}; \bar{\beta}_1, \ldots, \bar{\beta}_\mu] \in \mathcal{H}$ where

$$\bar{\alpha_i} = \begin{cases} \alpha_i - \beta_i & \text{if } \alpha_i > \beta_i \\ 0 & \text{if } \alpha_i \le \beta_i \end{cases} \qquad (i = 1, \ldots, \mu) \tag{3}$$
$$\bar{\beta}_i = \begin{cases} \beta_i - \alpha_i & \text{if } \beta_i > \alpha_i \\ 0 & \text{if } \beta_i \le \alpha_i . \end{cases}$$

It is clear that every element $s \in \mathcal{H}$ can be represented in normal form, and in $n(s)$ at least one of the components $\bar{\alpha_i}$ and $\bar{\beta}_i$ equals zero for all $i (= 1, \ldots \mu)$.

From the foregoing it is clear that in the case $\alpha_i = \beta_i$, $i = 1, \ldots, \mu$ the normal form of s is

$$s = [\alpha_1, \ldots, \alpha_\mu; \alpha_1, \ldots, \alpha_\mu] = [0, \ldots, 0; 0, \ldots, 0].$$

Let $\mathcal{N}(\mathcal{H})$ be the set of the elements $s \in \mathcal{H}$ written in normal form.

It is clear that

$$\mathcal{N}(\mathcal{H}) \subset \mathcal{H} \tag{4}$$

and there is a mapping

$$\varphi : \mathcal{H} \to \mathcal{N}(\mathcal{H})$$

$(s \to \varphi(s) = n(s))$ for which

(5) $$\operatorname{Ker}\varphi = \{s \in \mathcal{H} \mid s = [\alpha_1, \ldots, \alpha_\mu; \alpha_1, \ldots, \alpha_\mu]\},$$

that is, $\varphi(s) = 0\ \forall s \in \operatorname{Ker}\varphi$.

It can be seen that $s \in \mathcal{N}(\mathcal{H})$ if and only if $\alpha s \in \mathcal{N}(\mathcal{H})$ for all $\alpha \in I \setminus 0$ hold.

Let

(6) $$\begin{aligned} e_i &= [0, \ldots, 1, \ldots; 0, \ldots, 0], \quad \text{where } \alpha_i = 1, \\ -e_i &= [0, \ldots, 0; 0, \ldots, 1, \ldots, 0], \quad \text{where } \beta_i = 1. \end{aligned} \qquad (i = 1, \ldots, \mu)$$

Then $e_i, -e_i \in \mathcal{N}(\mathcal{H})$ and any element $s \in \mathcal{H}$ can be given as

(7) $$s = \sum_{i=1}^{\mu} \alpha_i e_i + \sum_{i=1}^{\mu} \beta_i(-e_i)$$

and

(8) $$n(s) = \sum_{i=1}^{\mu} (\alpha_i - \beta_i) e_i$$

is the normal form of s.

Let

(9) $$\mathcal{H}^+ = \Big\{s \in \mathcal{H} \mid s = \sum_{i=1}^{\mu} \alpha_i e_i,\ \alpha_i \in \mathbf{N}_0\Big\},$$

$$\mathcal{H}^- = \Big\{s \in \mathcal{H} \mid s = \sum_{i=1}^{\mu} \beta_i(-e_i),\ \beta_i \in \mathbf{N}_0\Big\}.$$

Using these notations we have, that

$$\mathcal{H}^+ \subset \mathcal{N}(\mathcal{H}), \qquad \mathcal{H}^- \subset \mathcal{N}(\mathcal{H}),$$

and it follows from equation (7) that

(10) $$\mathcal{H}^+ + \mathcal{H}^- = \mathcal{H}.$$

For the norm $\|s\|$ is a possible definition:

DEFINITION 4. Let $n(s) = [\alpha_1, ..., \alpha_\mu; \beta_1, ..., \beta_\mu] \in \mathcal{N}(\mathcal{H})$, then the *norm* of $n(s)$ is $\|n(s)\| = \sum_{i=1}^{\mu}(\alpha_i + \beta_i)$. We extend the domain of the norm function onto $\mathcal{H}$ as follows: if $s \in \mathcal{H}$, then let $\|s\| = \|n(s)\|$.

The $\|\cdot\|$ function defined above satisfies the following properties:
(i) $\|s\| \geq 0$,
(ii) $\|s\| = 0$ if and only if $n(s) = 0$,
(iii) $\|\alpha s\| = \|n(\alpha s)\| = \|\alpha n(s)\| = |\alpha|\|n(s)\| = |\alpha|\|s\| \ \forall \alpha \in \mathbf{I}$,
(iv) $\|s_1 + s_2\| \leq \|s_1\| + \|s_2\|$.

REMARK. To generalize the concept of the norm it can be useful to associate a not negative *weight* w_i to every position i of the vector s and $\|s\| = \sum_{i=1}^{\mu} w_i(\alpha_i + \beta_i)$. In this case the properties (i)–(iv) will not change.

1.2. Structures of parentheses

Let F be a groupoid with a multiplicative operation. A product $a_1 \cdots a_n$ of $n(>2)$ factors ($a_i \in F$; $i = 1, \ldots, n$) depends on the order of multiplications to be performed. Therefore we have to apply parentheses to make the product unambiguous.

DEFINITION 5. The *structure of parentheses* P_n $(n > 2)$ is a system of formal equations

$$\begin{aligned} P_n &= P_{n_1} P_{n_2} && (n_1 + n_2 = n) \\ P_{n_1} &= P_{n_{11}} P_{n_{12}} && (n_{11} + n_{12} = n_1) \\ P_{n_2} &= P_{n_{21}} P_{n_{22}} && (n_{21} + n_{22} = n_2) \\ &\ldots \\ P_3 &= P_1 P_2 \ \text{ or } \ P_2 P_1 \\ P_2 &= P_1 P_1, \end{aligned}$$

where P_k encloses k positions of elements.

EXAMPLE 1. The structure of parentheses applied to evaluate the product $((a_1 a_2) a_3)(a_4 a_5)$ is:

$$\begin{aligned} P_5 &= P_3 P_2, \\ P_3 &= P_2 P_1. \end{aligned}$$

DEFINITION 6. The *length* of a structure of parentheses P_n (means the number of positions enclosed by P_n) is n.

Let $\mathcal{P}_n$ be the set of all structures of parentheses with length n and let $|\mathcal{P}_n|$ be the number of structures of parentheses in $\mathcal{P}_n$.

THEOREM 1. *For any natural number n*

$$|\mathcal{P}_n| = \sum_{i=1}^{n-1} |\mathcal{P}_i| |\mathcal{P}_{n-i}| .$$

Proof. The statement of the theorem follows immediately from the fact, that for any $P_n \in \mathcal{P}_n$, $P_n = P_{n_1} P_{n_2}$ where $n_1 + n_2 = n$ holds. □

If $P_n = P_{n_1} P_{n_2}$ $(n_1 + n_2 = n)$, then P_n is said to be the product of P_{n_1} and P_{n_2}. If $P_n \in \mathcal{P}_n$, $P_m \in \mathcal{P}_m$, then $P_n P_m \in \mathcal{P}_{n+m}$. Thus we have

THEOREM 2. *For any natural number n*

$$\mathcal{P}_n = \bigcup_{i=1}^{n-1} \mathcal{P}_i \mathcal{P}_{n-i} .$$

Let $\mathcal{P}$ denote the set of all structures of parentheses of finite length. $\mathcal{P}$ becomes a groupoid with the previously defined multiplication $P_i P_j$ $(\in \mathcal{P}_{i+j})$ $(P_i \in \mathcal{P}_i,\ P_j \in \mathcal{P}_j)$.

$\mathcal{P}$ is called the *free structure of parentheses*. Obviously $\mathcal{P}$ is not commutative and not even associative. $\mathcal{P}$ is a free groupoid, generated by one element, namely by P_1.

Let F be a groupoid with a multiplicative operation, and $a_{i_1}, \ldots, a_{i_n} \in F$. The product of these elements according to the structure of parentheses P_n will be denoted by $P_n(a_{i_1} \ldots a_{i_n})$. F induces a relation Θ_F on $\mathcal{P}$ as follows:

$$P_n \ \Theta_F \ P'_n \quad \text{if and only if} \quad P_n(a_{i_1} \ldots a_{i_n}) = P'_n(a_{i_1} \ldots a_{i_n})$$
$$\forall a_{i_1}, \ldots, a_{i_n} \in F, \qquad P_n, P'_n \in \mathcal{P}_n .$$

It is easy to verify, that Θ_F is an equivalence relation on $\mathcal{P}$, furthermore if

$$P_n \ \Theta_F \ P'_n \qquad \text{and} \qquad P_m \ \Theta_F \ P'_m$$

then,

$$(P_n P_m) \ \Theta_F \ (P'_n P'_m)$$

holds, that is, Θ_F is a congruence relation.

The factorgroupoid $\mathcal{P}/\Theta_F$ will be denoted by $\mathcal{P}(F)$, characterizes — to a certain extent — the binary operation defined on F.

We summarize the previously observed results in the

THEOREM 3. *Every groupoid F induces a congruence relation Θ_F on the groupoid of all structures of parentheses $\mathcal{P}$, hence a factorgroupoid $\mathcal{P}/\Theta_F = \mathcal{P}(F)$ can be corresponded to F.*

THEOREM 4. *If F is a semigroup, then $\mathcal{P}(F)$ is an infinite cyclic semigroup.*

Proof. By definition $P_2 = P_1P_1 = P_1^2$, and $P_2P_1 \;\Theta_F\; P_1P_2$, because the operation defined on F is associative. Thus $(P_1P_1)P_1 \;\Theta_F\; P_1(P_1P_1)$, that is the only element of $\mathcal{P}_3(F)$ can be given in the form P_1^3. Let us assume that for any $i < n$ there is only one element of $\mathcal{P}_i(F)$, which can be given in the form P_1^i. Let P_n be an arbitrary element of $\mathcal{P}_n$. Then $P_n = P_iP_{n-i}$ for some i $(i < n)$, by the theorem 2. Then by the assumption

$$P_n \;\Theta_F\; (P_1^i)(P_1^{n-i}) \;\Theta_F\; P_1(P_1^{i-1})(P_1^{n-i}) \;\Theta_F\; P_1(P_1^{n-1})$$

and similarly

$$P_n \;\Theta_F\; (P_1^i)(P_1^{n-i}) \;\Theta_F\; (P_1^i)(P_1^{n-i-1})P_1 \;\Theta_F\; (P_1^{n-1})P_1 .$$

By the transitivity of Θ_F we obtain that

$$P_1P_{n-1} \;\Theta_F\; P_{n-1}P_1$$

and the only element of $\mathcal{P}_n$ can be written in the form P_1^n. By induction it follows that $\mathcal{P}(F)$ is cyclic and by the definition of P_n obviously structures of parentheses with different length are not Θ_F equivalent. □

The previous theorem shows that for a semigroup F, $|\mathcal{P}_n(F)| = 1$ holds for any positive integer n. The following one alleges that the condition $|\mathcal{P}_n(F)| = 1$ for some $n > 2$ is almost sufficient for F to be a semigroup.

THEOREM 5. *Let F be a groupoid and $F^2 = F$. If $|\mathcal{P}_n(F)| = 1$ for some $n \geq 3$, then F is a semigroup.*

Proof. (By induction). If the condition holds for $n = 3$, then the operation on F is associative, that is, F is a semigroup. Let us suppose that the statement of the theorem is true for $n-1$ $(n-1 \geq 3)$. Let $|\mathcal{P}_n(F)| = 1$. The elements of $\mathcal{P}_{n-1}$ can be obtained from those elements of $\mathcal{P}_n$ in which the last factor is P_2. To evaluate a product of n factors according to such a structure of parentheses the last two factors is multiplied in the first step. Let P_{n-1} and P'_{n-1} be two arbitrary elements of $\mathcal{P}_{n-1}$ and let P_n and P'_n be those elements of $\mathcal{P}_n$, from which P_{n-1} and P'_{n-1} can be obtained by replacing the last P_2 factor by P_1. Since any two elements in $\mathcal{P}_n$ are Θ_F equivalent, for arbitrary $a_1, \ldots, a_{n-1}, a_n \in F$

$$P_n(a_1, \ldots, a_{n-1}, a_n) = P'_n(a_1, \ldots, a_{n-1}, a_n) .$$

On the other hand, any elment $a'_{n-1} \in F$ is the product of suitably chosen elements $a_{n-1}, a_n \in F$, because $F^2 = F$ holds. Therefore for arbitrary $a_1, \ldots, a'_{n-1} \in F$

$$P_{n-1}(a_1, \ldots, a'_{n-1}) = P'_{n-1}(a_1, \ldots, a'_{n-1}) ,$$

verifying that any two elements of $\mathcal{P}_{n-1}$ are also Θ_F equivalent, that is, $|\mathcal{P}_{n-1}| = 1$ and by the hypothesis, F is a semigroup. □

COROLLARY 1. *If* $F^2 = F$ *and* $|\mathcal{P}_n(F)| = 1$ *for some* n (≥ 3) *then* $|\mathcal{P}_n(F)| = 1$ *for all positive integer* n.

COROLLARY 2. *If* $F^2 = F$ *and* F *is not associative, then* $|\mathcal{P}_n(F)| > 1$, *for all positive integer* n *greater than* 2.

COROLLARY 3. *The groupoid* F *is a semigroup if and only if* $\mathcal{P}(F)$ *is a semigroup.*

1.3. Finitely generated set products

Let X be a set of symbols $x_1, \ldots, x_n$. To each symbol x_i $(i = 1, \ldots, n)$ we correspond a vector s_i, representing a multiset in $\mathcal{H}$, such that $s_i \neq s_j$ if $i \neq j$, that is, the mapping $\phi : X \to \mathcal{H}$ is injective. We define an algebraic structure generated by X as follows:

Let $\underline{m}$ be a multiplication defined on $\mathcal{H}$, i.e., let $[\mathcal{H}; \underline{m}]$ be an algebraic structure. Let $x_i x_j = x_k$ if $s_i \ \underline{m}\ s_j = s_k$ and $s_k = \phi(x_k)$, or let $x_i x_j$ be a new element of the algebraic structure to be defined, if $s_k \notin \phi(X)$.

The obtained *multiplicative algebraic structure* is not associative and not commutative in general, therefore the evaluation of a product of n factors depends on the order of multiplications to be performed. If P_k is any structure of parentheses, then the product $P_k(x_{i_1}, \ldots, x_{i_k})$ $(i_j \leq n,$ $j = 1, \ldots, n)$ is evaluated according to the product $P_k(s_{i_1}, \ldots, s_{i_k})$.

The element s_i $(1 \leq i \leq n)$ is the *image* of x_i, $(1 \leq i \leq n)$, the element x_i is the *code* of s_i. The pair $[s_i; x_i]$ is the *coded image* of the element x_i. $P_k(x_{i_1} \ldots x_{i_k})$ is the *composed coded image* of the product $x_{i_1} \ldots x_{i_k}$ performed according to $[\mathcal{H}; \underline{m}]$ and P_k.

In the next sections we will see the central rule of the multiplication of the setvectors, applied for the finitely generated algebraic structures. In Chapter 7 the concept of the *coded structure* will be introduced.

2. Simple operation for setvectors

2.1. The ρ-product of the setvectors

If in the multiset s_i, each h_ρ, where ρ is fixed $(-\mu \le \rho \le \mu)$ element is replaced by the multiset s_j, then the obtained new multiset will be called the ρ-product of s_i and s_j.

Let s_i $(i = 1, \dots, n)$ be elements in $\mathcal{H}$.

Define the ρ-product (for $0 < \rho \le \mu$) as follows: for

$$s_i,\ s_j \quad \text{with} \quad (\alpha_\rho^{(i)} \neq 0)\,,$$

let

(1)
$$s_i\{\rho\}s_j =$$
$$= \left[\alpha_1^{(i)}, \dots, \alpha_\mu^{(i)}; \beta_1^{(i)}, \dots, \beta_\mu^{(i)}\right]\{\rho\}\left[\alpha_1^{(j)}, \dots, \alpha_\mu^{(j)}; \beta_1^{(j)}, \dots, \beta_\mu^{(j)}\right] =$$
$$= \Big[\alpha_1^{(i)} + \alpha_\rho^{(i)}\alpha_1^{(j)}, \dots, \alpha_\rho^{(i)}\alpha_\rho^{(j)}, \dots, \alpha_\mu^{(i)} + \alpha_\rho^{(i)}\alpha_\mu^{(j)};$$
$$\beta_1^{(i)} + \alpha_\rho^{(i)}\beta_1^{(j)}, \dots, \beta_\mu^{(i)} + \alpha_\rho^{(i)}\beta_\mu^{(j)}\Big].$$

The definition of the ρ-multiplication for $(-\mu \le \rho < 0)$ is analogous but we use the case $\rho > 0$ in general.

Remark 1. From the definition of the multiplication follows that $n(s_i\{\rho\}s_j) \neq n(n(s_i)\{\rho\}n(s_j))$ in general. There are cases when this equality holds (see Chapter 3, Theorems 6–8). The components α and β are not negative integers, therefore the products of these numbers are not negative integers and the product $\beta_i\beta_j$ is a β-component again.

If $\alpha_\rho^{(i)} = 0$ $(\rho > 0)$, respectively $\beta_\rho^{(i)} = 0$ $(\rho < 0)$, then we can say that the ρ-multiplication is *not realizable*. (At this point the definition differs from the usual algebraic concept. In this case the product $s_i\{\rho\}s_j$ remains in this form, or as a second possibility to write $s_i(\rho)s_j = s_i$ that is s_i is a left zero element in $\mathcal{H}$). The case "not realisable" can be necessity in some applications (see [1]). The elements s_i with $\alpha_\rho^{(i)} = 0$ ($\rho > 0$) form a left zero semigroup. The elements s_i with $\alpha_\rho^{(i)} \neq 0$ form a semigroup (Theorem 2) without left zero element.

We form products of multiple factors as well like

$$s_i\{\rho_1\}(s_j\{\rho_2\}s_k)$$

and this product will be said to be realizable, if both of the ρ_2-multiplication $s_j\{\rho_2\}s_k$ and the ρ_1-multiplication $s_i\{\rho_1\}(s_j\{\rho_2\}s_k)$ are realizable. In general, $P_k \in \mathcal{P}_k$ a structure of parentheses, and $\rho_1, \ldots, \rho_k$ are numbers satisfying $-\mu \leq \rho_i \leq \mu$ ($i = 1, \ldots, k$), then the product $P_k(s_{i_1}\{\rho_1\}\cdots\{\rho_k\}s_{i_k})$ is said to be realizable if the ρ_i-multiplications are realizable in the order determined by P_k.

REMARK 2. Obviously it may occur that for different structures of parentheses P_k and P'_k one of the products $P_k(s_{i_1}\{\rho_1\}\cdots\{\rho_k\}s_{i_k})$ and $P'_k(s_{i_1}\{\rho_1\}\cdots\{\rho_k\}s_{i_k})$ is realizable, while the other not.

DEFINITION 1. Because the direction of the substitution is from right to left (the right factor is "embedded" in the left factor) we say that the *supporting element* of the ρ-product $s_i\{\rho\}s_j$ is s_i. In general, the supporting element of the product $P_k(s_{i_1}\{\rho_1\}s_{i_2}\ldots\{\rho_k\}s_{i_k})$ is s_{i_1}, assuming that the multiplication is realizable.

The following theorem is about the evaluation process of a product of k factors.

THEOREM 1. *For every $P_k(s_{i_1}\{\rho_1\}s_{i_2}\{\rho_2\}\ldots\{\rho_k\}s_{i_k})$ the result of the multiplication is a unique product of some elements s_{i_t} $(t = 1, \ldots, k)$*

$$P_p(s_{i_1^{(1)}}s_{i_2^{(2)}}\ldots s_{i_p^{(p)}}) \qquad P_p \in \mathcal{P}_p \qquad (p \leq k)$$

with minimal p, where $s_{i_j^{(j)}}$ are supporting elements ($i_j^{(j)} \in \{i_1^{(1)}, \ldots, i_p^{(p)}\}$), and in $P_p(s_{i_1^{(1)}}s_{i_2^{(2)}}\cdots s_{i_p^{(p)}})$ further multiplications are not realizable.

In the following theorem we prove, that for any fixed ρ ($-\mu \leq \rho \leq \mu$) the ρ-multiplication is associative.

THEOREM 2. *Let s_i, s_j, s_l be elements of $\mathcal{H}$, with*

$$s_i = \left[\alpha_1^{(i)}, \ldots, \alpha_\mu^{(i)}; \beta_1^{(i)}, \ldots, \beta_\mu^{(i)}\right],$$

$$s_j = \left[\alpha_1^{(j)}, \ldots, \alpha_\mu^{(j)}; \beta_1^{(j)}, \ldots, \beta_\mu^{(j)}\right],$$

$$s_l = \left[\alpha_1^{(l)}, \dots, \alpha_\mu^{(l)}; \beta_1^{(l)}, \dots, \beta_\mu^{(l)}\right],$$

then

$$(s_i\{\rho\}s_j)\{\rho\}s_l = s_i\{\rho\}(s_j\{\rho\}s_l).$$

Proof. Consider the case $\rho > 0$. A simple calculation shows that

$$(2) \qquad (s_i\{\rho\}s_j)\{\rho\}s_l =$$

$$= \Big[\dots, (\alpha_r^{(i)} + \alpha_\rho^{(i)}\alpha_r^{(j)}) + \alpha_\rho^{(i)}\alpha_\rho^{(j)}\alpha_r^{(l)}, \dots, (\alpha_\rho^{(i)}\alpha_\rho^{(j)})\alpha_\rho^{(l)}, \dots; \\ \dots, (\beta_q^{(i)} + \alpha_\rho^{(i)}\beta_q^{(j)}) + \alpha_\rho^{(i)}\alpha_\rho^{(j)}\beta_q^{(l)}, \dots\Big] \quad r, q = 1, \dots, \mu, \ \ r \neq \rho$$

and

$$(3) \qquad s_i\{\rho\}(s_j\{\rho\}s_l) =$$

$$= \Big[\dots, \alpha_r^{(i)} + \alpha_\rho^{(i)}(\alpha_r^{(j)} + \alpha_\rho^{(j)}\alpha_r^{(l)}), \dots, \alpha_\rho^{(i)}(\alpha_\rho^{(j)}\alpha_\rho^{(l)}), \dots; \\ \dots, \beta_q^{(i)} + \alpha_\rho^{(i)}(\beta_q^{(j)} + \alpha_\rho^{(j)}\beta_q^{(l)}), \dots\Big] \quad r, q = 1, \dots, \mu, \ \ r \neq \rho\,.$$

Comparing (2) and (3) the statement of the theorem is obtained. The case $-\mu \leq \rho \leq -1$ is similar, therefore we omit the details. We say that the elements s_i are *ρ-associative* for the ρ-product. □

COROLLARY 1. *For any structures of parentheses* $P_k\,, P'_k \in \mathcal{P}_k$

$$P_k(s_{i_1}\{\rho\}s_{i_2}\{\rho\}\dots\{\rho\}s_{i_k}) = P'_k(s_{i_1}\{\rho\}s_{i_2}\{\rho\}\dots\{\rho\}s_{i_k})\,.$$

COROLLARY 2.

$$P_k(s_i\{\rho\}s_i\{\rho\}\dots\{\rho\}s_i) = s_i^{k(\rho)}, \quad \forall P_k \in \mathcal{P}_k.$$

COROLLARY 3. *If one of the components* $\alpha_\rho^{(i)}$, $\alpha_\rho^{(j)}$, $\alpha_\rho^{(l)}$ *is zero, then also the component* α_ρ *is zero in* $s = s_i\{\rho\}s_j\{\rho\}s_l$.

THEOREM 3. *For every* $s = [\alpha_1, \dots, \alpha_\mu; \beta_1 \dots, \beta_\mu]$, $(\alpha_\rho \neq 0)$ *there exists a unique minimal* s' *such that for* $s' = [\alpha'_1, \dots, \alpha'_\mu; \beta'_1 \dots, \beta'_\mu]$, $(\alpha'_\rho \neq 0)$

$$s'^{k(\rho)} = s\,.$$

(The minimality means that does not exist an s'' such that $s' = s''^{k'(\rho)}$ with $k'(\rho) > 1$.)

Proof. For s' and $(\rho > 0)$

$$s'^{k(\rho)} =$$

$$= \Big[\dots, \alpha'_r(1+\alpha'_\rho+\dots+\alpha'^{k-1}_\rho), \dots, \alpha'^k_\rho, \dots; \dots, \beta'_t(1+\alpha'_\rho+\dots+\alpha'^{k-1}_\rho), \dots\Big] =$$

$$= [\alpha_1, \dots, \alpha_\mu; \beta_1 \dots, \beta_\mu] = s \qquad (r \neq \rho,\ t = 1, \dots, \mu)\,.$$

This equation holds if and only if

$$\alpha'_r(1 + \alpha'_\rho + \dots + \alpha'^{k-1}_\rho) = \alpha_r\,, \qquad (r \neq \rho)$$

$$\alpha'^k_\rho = \alpha_\rho$$

$$\beta'_t(1 + \alpha'_\rho + \dots + \alpha'^{k-1}_\rho) = \beta_t\,, \qquad (t = 1, \dots, \mu)\,.$$

Case 1. If $\alpha_\rho = 1$ then $\alpha'_\rho = 1$ and $k\alpha'_r = \alpha_r$, $k\beta'_t = \beta_t$ ($r \neq \rho$, $t = 1, \dots, \mu$) must be satisfied. Therefore k is a common divisor of the integers $\alpha_1, \dots, \alpha_{\rho-1}, \alpha_{\rho+1}, \dots, \alpha_\mu, \beta_1, \dots, \beta_\mu$.

Case 2. If $\alpha_\rho > 1$, then $\alpha_{\rho'} > 1$ and with the notation $1 + \alpha'_\rho + \dots + \alpha'^{k-1}_\rho = \frac{\alpha'^k_\rho - 1}{\alpha'_\rho - 1} = q$, the equations can be written in the form

$$\alpha'_r q = \alpha_r \qquad (r \neq \rho)$$

$$\alpha'^k_\rho = \alpha_\rho$$

$$\beta'_t q = \beta_t \qquad (t = 1, \dots, \mu)\,.$$

Obviously for given α_t, β_t $(t = 1, \dots, \mu)$ these equations determine the integers α' and k (q) uniquely under the additional requirement that k and q must be maximal.

Similar statement can be proved for ρ-powers with $(\rho < 0)$. □

COROLLARY 4. *If $\alpha_\rho (> 1)$ is not a power of an integer, then s is minimal, that is $s \equiv s'$.*

THEOREM 4. *Let $s = [\alpha_1, \dots, \alpha_\mu; \beta_1, \dots, \beta_\mu]$, $s' = [\alpha'_1, \dots, \alpha'_\mu; \beta'_1, \dots, \beta'_\mu]$ be minimal elements with respect to the ρ-powering respectively ρ'-powering. If $s^{k(\rho)} = s'^{l(\rho')}$ $(\rho > 0,\ \rho' > 0,\ \rho \neq \rho')$, then*

1) $\alpha_\rho > 1$, $\alpha'_{\rho'} > 1$,

2) *from equation $\alpha^k_\rho = \alpha'^l_{\rho'}$ it follows that $\alpha_{\rho'}(\alpha'_{\rho'} - 1) = \alpha'_\rho(\alpha_\rho - 1)$,*

3) $\dfrac{\alpha_p}{\alpha'_p} = \dfrac{\alpha_q}{\alpha'_q}$, *where $p, q \neq \rho, \rho'$.*

Proof. If $\rho \neq \rho'$ then

$$s^{k(\rho)} = [\dots, \alpha_p(1 + \alpha_\rho + \dots + \alpha^{k-1}_\rho), \dots, \alpha^k_\rho, \dots;$$
$$\dots, \beta_q(1 + \alpha_\rho + \dots + \alpha^{k-1}_\rho), \dots] \qquad (p \neq \rho,\ q = 1, \dots, \mu)\,.$$

$$s^{\prime l(\rho')} = [\ldots, \alpha'_p(1+\alpha'_{\rho'}+\cdots+\alpha'^{l-1}_{\rho'}), \ldots, \alpha'^{l}_{\rho'}, \ldots; \\ \ldots, \beta'_q(1+\alpha'_{\rho'}+\cdots+\alpha'^{l-1}_{\rho'}), \ldots].$$

Therefore

$$(4) \qquad \alpha^k_\rho = \alpha'_\rho(1+\alpha'_{\rho'}+\cdots+\alpha'^{l-1}_{\rho'})$$

$$\alpha'^{l}_{\rho'} = \alpha_{\rho'}(1+\alpha_\rho+\cdots+\alpha^{k-1}_\rho)$$

$$\alpha_p(1+\alpha_\rho+\cdots+\alpha^{k-1}_\rho) = \alpha'_p(1+\alpha'_{\rho'}+\cdots+\alpha'^{l-1}_{\rho'}) \qquad (p \neq \rho, \rho').$$

It can be easily seen that $\alpha_\rho = 1$ implies that $\alpha'_\rho = 1$ and $\alpha'_{\rho'} = 0$ which is impossible. The case of $\alpha'_{\rho'} = 1$ is similar, which proves 1).

Now $\alpha_\rho > 1$ then $\alpha_{\rho'} > 1$ and for α_ρ and $\alpha'_{\rho'}$ we get the system of equations

$$(5) \qquad \alpha^k_\rho = \alpha'_\rho \frac{\alpha'^{l}_{\rho'}-1}{\alpha'_{\rho'}-1}, \qquad \alpha'^{l}_{\rho'} = \alpha_{\rho'}\frac{\alpha^k_\rho - 1}{\alpha_\rho - 1},$$

where all of α-s are positive integers. Written in other form

$$(6) \qquad \alpha'^{l}_{\rho'}[\alpha_{\rho'}\alpha'_\rho - (\alpha_{\rho'}-1)(\alpha_\rho-1)] = \alpha_{\rho'}\alpha'_\rho + \alpha_{\rho'}(\alpha'_{\rho'}-1)$$

$$\alpha^k_\rho[\alpha_{\rho'}\alpha'_\rho - (\alpha_{\rho'}-1)(\alpha_\rho-1)] = \alpha_{\rho'}\alpha'_\rho + \alpha'_\rho(\alpha_\rho-1).$$

From this it follows that $\alpha^k_\rho = \alpha'^{l}_\rho$ if and only if $\alpha_{\rho'}(\alpha'_{\rho'}-1) = \alpha'_\rho(\alpha_\rho-1)$ which proves 2). Furthermore from the symmetricity of ρ and ρ' $\alpha_{\rho'} = \alpha'_\rho \neq 1$, follows.

From (4) follows immediately 3). □

Example. If $\alpha'_{\rho'} = 2$, $\alpha_\rho = 3$, $\alpha'_\rho = 3$, $\alpha_{\rho'} = 1$, $l = 2$, $k = 2$, $4\alpha_p = 3\alpha'_p$ $(p \neq \rho, \rho')$, then $s^{2(\rho)} = s'^{2(\rho')}$.

If in the previous theorem $\rho > 0$ and $(\rho' < 0)$ is assumed, then $s^{k(\rho)} = s'^{l(\rho')}$ implies the following equations:

$$\alpha^k_\rho = \alpha_\rho(1+\beta'_{\rho'}+\cdots+\beta'^{l-1}_{\rho'})$$

$$\beta'^{l}_{\rho'} = \beta_{\rho'}(1+\alpha_\rho+\cdots+\alpha^{k-1}_\rho)$$

$$\alpha_p(1+\alpha_\rho+\cdots+\alpha^{k-1}_\rho) = \alpha'_p(1+\beta'_{\rho'}+\cdots+\beta'^{l-1}_{\rho'})$$

$$\beta_p(1+\alpha_\rho+\cdots+\alpha^{k-1}_\rho) = \beta'_p(1+\beta'_{\rho'}+\cdots+\beta'^{l-1}_{\rho'}) \qquad (p \neq \rho, \rho').$$

The analysis of this case is similar to the first one.

Remark 3. A complete analysis of this problem requires to solve the system of equations (5).

THEOREM 5. *Let*

$$s = [\alpha_1, \ldots, \alpha_\mu; \beta_1, \ldots, \beta_\mu], \qquad s' = [\alpha'_1, \ldots, \alpha'_\mu; \beta'_1, \ldots, \beta'_\mu].$$

The equation $s\{\rho'\}s' = s'\{\rho\}s$, $\rho \neq \rho'$ holds if and only if one of the following cases is satisfied:

1) *If $\rho > 0$ and $\rho' > 0$, $(\alpha_{\rho'} \neq 0, \alpha'_\rho \neq 0)$, then*
   a) $\alpha_{\rho'} = 4$, $\alpha'_\rho = 3$, $3\alpha'_r = 2\alpha_r$ $(r \neq \rho, \rho')$; $3\beta'_t = 2\beta_t$ $(t = 1, \ldots, \mu)$,
   b) $\alpha_{\rho'} = 3$, $\alpha'_\rho = 2$, $2\alpha'_r = \alpha_r$ $(r \neq \rho, \rho')$; $2\beta'_t = \beta_t$ $(t = 1, \ldots, \mu)$,
   c) $\alpha_{\rho'} = 2$, $\alpha'_\rho = 2$, $\alpha'_r = \alpha_r$ $(r \neq \rho, \rho')$; $\beta'_t = \beta_t$ $(t = 1, \ldots, \mu)$,

furthermore the cases that can be obtained from the above ones by interchanging the role of ρ and ρ'.

2) *The case of $\rho < 0$, $\rho' < 0$ is analogous to 1).*
3) *If $\rho > 0$ and $\rho' < 0$, $(\beta_{\rho'} \neq 0, \alpha'_\rho \neq 0)$ then*
   a) $\alpha_\rho = \alpha'_\rho = 4$, $\beta_{\rho'} = 3$, $\beta'_{\rho'} = 6$, $2\alpha'_r = 3\alpha_r$, $2\beta'_r = 3\beta_r$ $(r \neq \rho, \rho')$,
   b) $\alpha_\rho = \alpha'_\rho = 3$, $\beta_{\rho'} = 2$, $\beta'_{\rho'} = 6$, $\alpha_r = 2\alpha_r$, $\beta'_r = 2\beta_r$ $(r \neq \rho, \rho')$,
   c) $\alpha_\rho = 4$, $\alpha'_\rho = 2$, $\beta_{\rho'} = 2$, $\beta'_{\rho'} = 6$, $\alpha'_r = \alpha_r$, $\beta'_r = \beta_r$ $(r \neq \rho, \rho')$,

furthermore those cases that can be obtained from the above ones by interchanging the role of ρ and ρ'.

4) *The case of $\rho < 0$, $\rho' > 0$ is similar to 3).*

Proof.

1. Let $\rho > 0$, $\rho' > 0$, $\rho \neq \rho'$, and $s\{\rho'\}s' = s'\{\rho\}s$. Then

$$\alpha_{\rho'}\alpha'_{\rho'} = \alpha'_{\rho'} + \alpha'_\rho \alpha_{\rho'}, \tag{7}$$

$$\alpha'_\rho \alpha_\rho = \alpha_\rho + \alpha_{\rho'}\alpha'_\rho, \tag{8}$$

$$\alpha_r + \alpha_{\rho'}\alpha'_r = \alpha'_r + \alpha'_\rho \alpha_r \qquad (r \neq \rho, \rho'), \tag{9}$$

$$\beta_q + \alpha_{\rho'}\beta'_q = \beta'_q + \alpha'_\rho \beta_q \qquad (q = 1, \ldots, \mu).$$

It follows from (7) and (8) that $\alpha_{\rho'} \,|\, \alpha'_{\rho'}$, $\alpha'_\rho \,|\, \alpha_\rho$, thus $\alpha'_{\rho'} = k\alpha_{\rho'}$, $\alpha_\rho = k'\alpha'_\rho$. Therefore,we obtain from (7) that $k\alpha_{\rho'} = k + \alpha'_\rho$ and from (8) that $k'\alpha'_\rho = k' + \alpha_{\rho'}$ which shows that $k' \,|\, \alpha_{\rho'}$. Then we obtain the equation

$$k''(k'k - 1) = k + 1, \qquad (k, k', k'' \geq 1). \tag{10}$$

To satisfy the equation (10) the following cases are possible

a) $k = 1$, $k' = 2$ b) $k = 1$, $k' = 3$ c) $k = 2$, $k' = 2$

and the symmetrical ones. From these we get Case 1 of the theorem.

2. The result of the case $\rho < 0$, $\rho' < 0$ is similar to 1.

3. If $\rho > 0$, $\rho' < 0$, then $s\{\rho'\}s' = s'\{\rho\}s$ implies the following equations:

(11) $$\alpha_\rho + \beta_{\rho'}\alpha'_\rho = \alpha'_\rho\alpha_\rho\,,$$

(12) $$\beta'_{\rho'} + \alpha'_\rho\beta_{\rho'} = \beta_{\rho'}\beta'_{\rho'}\,,$$

(13) $$\alpha_r + \beta_{\rho'}\alpha'_r = \alpha'_r + \alpha_\rho\alpha_r\,,$$

(14) $$\beta_r + \beta_{\rho'}\beta'_r = \beta'_r + \alpha'_\rho\beta_r\,.$$

Similarly as in Case 1 we get $\alpha'_\rho \mid \alpha_\rho$ that is $\alpha_\rho = k\alpha'_\rho$, $\beta_{\rho'} \mid \beta'_{\rho'}$ that is $\beta'_{\rho'} = k'\beta_\rho$, $k' \mid \alpha'_\rho$ i.e., $\alpha'_\rho = k''k'$. Hence the equation $k''(kk'-1) = k+1$ holds, which is identical to (10). Therefore we have for k, k', k'' the subcases a), b), c), from which 3) follows.

4. The case $\rho < 0$, $\rho' > 0$ is analogous to 3) □

Similarly we get the following

THEOREM 6. *Let*

$$s = [\alpha_1, \ldots, \alpha_\mu; \beta_1, \ldots, \beta_\mu],$$

$$s' = [\alpha'_1, \ldots, \alpha'_\mu; \beta'_1, \ldots, \beta'_\mu]\,.$$

Then $s\{\rho\}s' = s'\{\rho\}s$ if and only if one of the following cases is satisfied:

I. *If $\rho > 0$,*

a) $\alpha_\rho = 1$, $\alpha'_\rho = 1$,

b) $\alpha_\rho = 1$, $\alpha'_\rho \neq 1$, $\alpha_r = 0$ $(r \neq \rho)$, α'_r *are arbitrary integers* $(r \neq \rho)$, $\beta_q = 0$ $(q = 1, \ldots, \mu)$, β'_q *are arbitrary integers. (And the case obtained by interchanging the role of α_ρ and α'_ρ).*

c) $\alpha_\rho \neq 1$, $\alpha'_\rho \neq 1$ *and*

$$\frac{\alpha'_r}{\alpha_r} = \frac{\beta'_q}{\beta_q} = \frac{\alpha'_\rho - 1}{\alpha_\rho - 1}\,.$$

II. *For $\rho < 0$ the conditions are analogous.*

Proof. If $s\{\rho\}s' = s'\{\rho\}s$ then

$$\alpha_r + \alpha_\rho\alpha'_r = \alpha'_r + \alpha'_\rho\alpha_r \qquad r \neq \rho$$

$$\beta_q + \alpha_\rho\beta'_q = p'_q + \alpha'_\rho p_q \qquad (q = 1, \ldots, \mu)$$

and from this equations the given conditions are easily obtained. □

THEOREM 7. *Let*

$$s_i = \left[\alpha_1^{(i)}, \ldots, \alpha_\mu^{(i)}; \beta_1^{(i)}, \ldots, \beta_\mu^{(i)}\right],$$

$$s_j = \left[\alpha_1^{(j)}, \dots, \alpha_\mu^{(j)}; \beta_1^{(j)}, \dots, \beta_\mu^{(j)}\right],$$
$$s_l = \left[\alpha_1^{(l)}, \dots, \alpha_\mu^{(l)}; \beta_1^{(l)}, \dots, \beta_\mu^{(l)}\right].$$

Assuming, that the indicated multiplications are realizable, the equation

(15) $$(s_i\{\rho'\}s_j)\{\rho''\}s_l = s_i\{\rho'\}(s_j\{\rho''\}s_l),$$

(ρ', ρ'')-associativity holds, if and only if one of the following conditions is satisfied:

Case 1) $\rho' = \rho''$.

Case 2) *If* $\rho' > 0$, $\rho'' > 0$, *and* $\rho' \neq \rho''$, *then*

a) $\alpha_{\rho''}^{(i)} = 0$,

b) $\alpha_{\rho''}^{(i)} \neq 0$ and $s_l \to [0, \dots, 1, \dots, 0; 0, \dots, 0]$ $(1 = \alpha_{\rho''}^{(l)})$.

Case 3) If $\rho' > 0$, $\rho'' < 0$,

a) $\beta_{\rho''}^{(i)} = 0$,

b) $\beta_{\rho''}^{(i)} \neq 0$ and $s_l \to [0, \dots, 0; 0, \dots, 1, \dots 0]$ $(1 = \beta_{\rho''}^{(l)})$,

and the symmetrical cases for both 2) *and* 3).

Proof. The condition of Case 1 is sufficient by Theorem 7.

Case 2: It is easy to see, that (15) implies

$$\alpha_r^{(i)} + \alpha_{\rho'}^{(i)}\alpha_r^{(j)} + \left(\alpha_{\rho''}^{(i)} + \alpha_{\rho'}^{(i)}\alpha_{\rho''}^{(j)}\right)\alpha_r^{(l)} = \alpha_r^{(i)} + \alpha_{\rho'}^{(i)}\left(\alpha_r^{(j)} + \alpha_{\rho''}^{(j)}\alpha_r^{(l)}\right) \quad (r \neq \rho', \rho''),$$

that is

(16) $$\alpha_{\rho''}^{(i)}\alpha_r^{(l)} = 0,$$

$$(\alpha_{\rho''}^{(i)} + \alpha_{\rho'}^{(i)}\alpha_{\rho''}^{(j)})\alpha_{\rho''}^{(l)} = \alpha_{\rho''}^{(i)} + \alpha_{\rho'}^{(i)}\alpha_{\rho''}^{(j)}\alpha_{\rho''}^{(l)}.$$

Then we have that

(17) $$\alpha_{\rho''}^{(i)}\alpha_{\rho''}^{(l)} = \alpha_{\rho''}^{(i)},$$

$$\alpha_{\rho'}^{(i)}\alpha_{\rho'}^{(j)} + (\alpha_{\rho''}^{(i)} + \alpha_{\rho'}^{(i)}\alpha_{\rho''}^{(j)})\alpha_{\rho'}^{(l)} = \alpha_{\rho'}^{(i)}(\alpha_{\rho'}^{(j)} + \alpha_{\rho''}^{(j)}\alpha_{\rho'}^{(l)}),$$

that is

(18) $$\alpha_{\rho''}^{(i)}\alpha_{\rho'}^{(l)} = 0,$$

$$\beta_q^{(i)} + \alpha_{\rho'}^{(i)}\beta_q^{(j)} + (\alpha_{\rho''}^{(i)} + \alpha_{\rho'}^{(i)}\alpha_{\rho''}^{(j)})\beta_q^{(l)} = \beta_q^{(i)} + \alpha_{\rho'}^{(i)}(\beta_q^{(j)} + \alpha_{\rho''}^{(j)}\beta_q^{(l)}),$$

i.e.,

(19) $$\alpha_{\rho''}^{(i)}\beta_q^{(l)} = 0 \qquad (q = 1, \dots, \mu).$$

From (16), (17), (18), (19) follows the second condition of the theorem.

In Case 3, if (15) holds, then

$$\alpha_r^{(i)}+\alpha_{\rho'}^{(i)}\alpha_r^{(j)}+(\beta_{\rho''}^{(i)}+\alpha_{\rho'}^{(i)}\beta_{\rho''}^{(j)})\alpha_r^{(l)}=\alpha_r^{(i)}+\alpha_{\rho'}^{(i)}(\alpha_r^{(j)}+\beta_{\rho''}^{(j)}\alpha_r^{(l)}) \quad (r\neq\rho',\rho''),$$

and

$$\beta_{\rho''}^{(i)}\alpha_r^{(l)}=0, \tag{20}$$

$$\alpha_{\rho'}^{(i)}\alpha_{\rho'}^{(j)}+(\beta_{\rho''}^{(i)}+\alpha_{\rho'}^{(i)}\beta_{\rho''}^{(j)})\alpha_{\rho'}^{(l)}=\alpha_{\rho'}^{(i)}(\alpha_{\rho'}^{(j)}+\beta_{\rho''}^{(j)}\alpha_{\rho'}^{(l)}),$$

and

$$\beta_{\rho''}^{(i)}\alpha_{\rho'}^{(l)}=0, \tag{21}$$

$$\beta_q^{(i)}+\alpha_{\rho'}^{(i)}\beta_q^{(j)}+(\beta_{\rho''}^{(i)}+\alpha_{\rho'}^{(i)}\beta_{\rho''}^{(j)})\beta_q^{(l)}=\beta_q^{(i)}+\alpha_{\rho'}^{(i)}(\beta_q^{(j)}+\beta_{\rho''}^{(j)}\beta_q^{(l)}) \quad (q\neq\rho',\rho''),$$

and

$$\beta_{\rho''}^{(i)}\beta_q^{(l)}=0, \tag{22}$$

$$(\beta_{\rho''}^{(i)}+\alpha_{\rho'}^{(i)}\beta_{\rho''}^{(j)})\beta_{\rho''}^{(l)}=\beta_{\rho''}^{(i)}+\alpha_{\rho'}^{(i)}\beta_{\rho''}^{(j)}\beta_{\rho''}^{(l)},$$

and last

$$\beta_{\rho''}^{(i)}\beta_{\rho''}^{(l)}=\beta_{\rho''}^{(i)}. \tag{23}$$

From (20), (21), (22), (23) follows the third condition of the theorem.

The cases ($\rho'<0$, $\rho''>0$) and ($\rho'<0$, $\rho''<0$) are analogous to 3) and 2). □

THEOREM 8. *Let*

$$s_i=\left[\alpha_1^{(i)},\dots,\alpha_\mu^{(i)};\beta_1^{(i)},\dots,\beta_\mu^{(i)}\right],$$

$$s_j=\left[\alpha_1^{(j)},\dots,\alpha_\mu^{(j)};\beta_1^{(j)},\dots,\beta_\mu^{(j)}\right],$$

$$s_l=\left[\alpha_1^{(l)},\dots,\alpha_\mu^{(l)};\beta_1^{(l)},\dots,\beta_\mu^{(l)}\right].$$

If $s_i\{\rho\}s_l=s_j\{\rho\}s_l$ *then* $s_i=s_j$.

Proof. To realize the multiplication $\alpha_l^{(i)}\neq 0$, $\alpha_l^{(j)}\neq 0$.

1. Let $\rho>0$, then

$$s_i\{\rho\}s_l=\left[\dots,\alpha_r^{(i)}+\alpha_\rho^{(i)}\alpha_1^{(l)},\dots,\alpha_\rho^{(i)}\alpha_\rho^{(l)},\dots;\dots,\beta_q^{(i)}+\alpha_\rho^{(i)}\beta_q^{(l)},\dots\right],$$

$$s_j(\rho)s_l=\left[\dots,\alpha_r^{(j)}+\alpha_\rho^{(j)}\alpha_r^{(l)},\dots,\alpha_\rho^{(j)}\alpha_\rho^{(l)},\dots;\dots,\beta_q^{(j)}+\alpha_\rho^{(j)}\beta_q^{(l)},\dots\right],$$

where $r\neq\rho$, $q=1,\dots,\mu$.

It is easily seen, that $s_i\{\rho\}s_l=s_j(\rho)s_l$ implies $\alpha_\rho^{(i)}=\alpha_\rho^{(j)}$, $\alpha_r^{(i)}=\alpha_r^{(j)}$, $\beta_q^{(i)}=\beta_q^{(j)}$.

2. In the case $\rho < 0$ the proof is similar. □

THEOREM 9. *Let*

$$s_i = \left[\alpha_1^{(i)}, \dots, \alpha_\mu^{(i)}; \beta_1^{(i)}, \dots, \beta_\mu^{(i)}\right],$$
$$s_l = \left[\alpha_1^{(l)}, \dots, \alpha_\mu^{(l)}; \beta_1^{(l)}, \dots, \beta_\mu^{(l)}\right].$$

The equation

$$s_i\{\rho\}s_l = s_i\{\rho'\}s_l \qquad (\rho \neq \rho')$$

holds if and only if one of the following conditions is satisfied:

a) $\alpha_r^{(l)} = \beta_q^{(l)} = 0$, $(r \neq \rho, \rho',\ q = 1, \dots, \mu)$, $\dfrac{\alpha_\rho^{(i)}}{\alpha_\rho^{(l)}} = \dfrac{\alpha_{\rho'}^{(i)}}{\alpha_{\rho'}^{(l)}}$,

b) $\alpha_r^{(l)} = \beta_q^{(l)} = 0$, $(r \neq \rho, \rho')$, $q \neq \rho, \rho'$, $\dfrac{\alpha_\rho^{(i)}}{\alpha_\rho^{(l)}} = \dfrac{\beta_{\rho'}^{(i)}}{\beta_{\rho'}^{(l)}}$.

Proof. Case 1. If $\rho > 0$ and $\rho' > 0$, then

$$s_i\{\rho\}s_l = \left[\dots, \alpha_r^{(i)} + \alpha_\rho^{(i)}\alpha_r^{(l)}, \dots, \alpha_\rho^{(i)}\alpha_\rho^{(l)}, \dots; \dots, \beta_q^{(i)} + \alpha_\rho^{(i)}\beta_q^{(l)}, \dots\right],$$
$$s_i\{\rho'\}s_l = \left[\dots, \alpha_r^{(i)} + \alpha_{\rho'}^{(i)}\alpha_r^{(l)}, \dots, \alpha_{\rho'}^{(i)}\alpha_{\rho'}^{(l)}, \dots; \dots, \beta_q^{(i)} + \alpha_{\rho'}^{(i)}\beta_q^{(l)}, \dots\right],$$

where $\alpha_\rho^{(i)} \neq 0$, $\alpha_{\rho'}^{(i)} \neq 0$, $r \neq \rho, \rho'$.

From the equation $s_i(\rho)s_l = s_i(\rho')s_l$ we can derive the subsequent equations:

$$\alpha_r^{(i)} + \alpha_\rho^{(i)}\alpha_r^{(l)} = \alpha_r^{(i)} + \alpha_{\rho'}^{(i)}\alpha_r^{(l)}, \tag{24}$$

$$\alpha_\rho^{(i)}\alpha_\rho^{(l)} = \alpha_\rho^{(i)} + \alpha_{\rho'}^{(i)}\alpha_\rho^{(l)}, \tag{25}$$

$$\alpha_{\rho'}^{(i)}\alpha_{\rho'}^{(l)} = \alpha_{\rho'}^{(i)} + \alpha_\rho^{(i)}\alpha_{\rho'}^{(l)}, \tag{26}$$

$$\beta_q^{(i)} + \alpha_\rho^{(i)}\beta_q^{(l)} = \beta_q^{(i)} + \alpha_{\rho'}^{(i)}\beta_q^{(l)}. \tag{27}$$

a) If $\alpha_r^{(l)} \neq 0$ or $\beta_q^{(l)} \neq 0$,
then (24) and (27) imply $\alpha_\rho^{(i)} = \alpha_{\rho'}^{(i)}$. Then $\alpha_\rho^{(i)}\alpha_\rho^{(l)} = \alpha_\rho^{(i)}\alpha_\rho^{(l)}$ follows from (25) and because $\alpha_\rho^{(i)} \neq 0$ we get that $\alpha_\rho^{(l)} = 1 + \alpha_\rho^{(l)}$ and using (26), $\alpha_{\rho'}^{(l)} = 1 + \alpha_{\rho'}^{(l)}$, which is a contradiction.

b) If $\alpha_r^{(l)} = 0$, $\beta_q^{(l)} = 0$ $(r \neq \rho, \rho')$,
then $\alpha_\rho^{(l)} \mid \alpha_\rho^{(i)}$, $\alpha_{\rho'}^{(l)} \mid \alpha_{\rho'}^{(i)}$ follows from (25) and (26), therefore again by (25) and (26) we get the equations $(\alpha_\rho^{(i)} = k\alpha_\rho^{(l)}, \alpha_{\rho'}^{(i)} = p\alpha_{\rho'}^{(l)})$

$$k\alpha_\rho^{(l)} = k + \alpha_{\rho'}^{(i)}, \qquad p\alpha_{\rho'}^{(l)} = p + \alpha_\rho^{(i)}. \tag{28}$$

Then (28) implies $k \mid \alpha_{\rho'}^{(i)}$, $p \mid \alpha_\rho^{(i)}$, that is $\alpha_{\rho'}^{(i)} = k'k$, $\alpha_\rho^{(i)} = p'p$. Therefore

(29) $$\alpha_\rho^{(l)} = 1 + k', \qquad \alpha_{\rho'}^{(l)} = 1 + p'.$$

Furthermore, $p'p = \alpha_\rho^{(i)} = k\alpha_\rho^{(l)} = k(1+k')$ and $kk' = \alpha_{\rho'}^{(i)} = p\alpha_{\rho'}^{(l)} = = p(1+p')$. From these equations we obtain that

$$k'k = p + k(1+k'),$$

$$p'p = k + p(1+p'),$$

that is, $k \mid p$ and $p \mid k$, therefore $p = k$ and the necessity of the conditions in the case $\rho > 0$, $\rho' > 0$ is proved.

Case 2. If $\rho > 0$ and $\rho' < 0$, then

$$s_i\{\rho\}s_l = \left[\ldots, \alpha_r^{(i)} + \alpha_\rho^{(i)}\alpha_r^{(l)}, \ldots, \alpha_\rho^{(i)}\alpha_\rho^{(l)}, \ldots; \ldots, \beta_q^{(i)} + \alpha_\rho^{(i)}\beta_q^{(l)}, \ldots\right],$$

$$s_i\{\rho'\}s_l = \left[\ldots, \alpha_r^{(i)} + \beta_{\rho'}^{(i)}\alpha_r^{(l)}, \ldots; \ldots, \beta_q^{(i)} + \beta_{\rho'}^{(i)}\beta_q^{(l)}, \ldots, \beta_{\rho'}^{(i)}\beta_{\rho'}^{(l)}, \ldots\right],$$

where $\alpha_\rho^{(i)} \neq 0$, $\beta_{\rho'}^{(i)} \neq 0$.

Thus, if $s_i(\rho)s_l = s_i(\rho')s_l$, then

(30) $$\alpha_r^{(i)} + \alpha_\rho^{(i)}\alpha_r^{(l)} = \alpha_r^{(i)} + \beta_{\rho'}^{(i)}\alpha_r^{(l)},$$

(31) $$\beta_q^{(i)} + \alpha_\rho^{(i)}\beta_q^{(l)} = \beta_q^{(i)} + \beta_{\rho'}^{(i)}\beta_q^{(l)},$$

(32) $$\alpha_\rho^{(i)}\alpha_\rho^{(l)} = \alpha_\rho^{(i)} + \beta_{\rho'}^{(i)}\alpha_\rho^{(l)},$$

(33) $$\beta_{\rho'}^{(i)}\beta_{\rho'}^{(l)} = \beta_{\rho'}^{(i)} + \alpha_\rho^{(i)}\beta_{\rho'}^{(l)}.$$

a) if $\beta_q^{(l)} \neq 0$ or $\alpha_r^{(l)} \neq 0$,
then $\alpha_\rho^{(i)} = \beta_{\rho'}^{(i)}$ and using (28) and (29) we obtain the contradictory equations $\alpha_\rho^{(l)} = 1 + \alpha_\rho^{(l)}$ and $\beta_{\rho'}^{(l)} = 1 + \beta_{\rho'}^{(l)}$.

b) if $\beta_q^{(l)} = \alpha_r^{(l)} = 0$,
then $\alpha_\rho^{(l)} \mid \alpha_\rho^{(i)}$, $\beta_{\rho'}^{(l)} \mid \beta_{\rho'}^{(i)}$ can be derived from (32) and (33). Therefore using (32) and (33) again we get $(\alpha_\rho^{(i)} = k\alpha_\rho^{(l)}, \beta_{\rho'}^{(i)} = p\beta_{\rho'}^{(l)})$

(34) $$k\alpha_\rho^{(l)} = k + \beta_{\rho'}^{(i)}, \qquad p\beta_{\rho'}^{(l)} = p + \alpha_\rho^{(i)}.$$

(34) implies that $k \mid \beta_\rho^{(i)}$ and $p \mid \alpha_\rho^{(i)}$. Let $\beta_{\rho'}^{(i)} = k'k$, $\alpha_\rho^{(i)} = p'p$, then

(35) $$\alpha_\rho^{(l)} = 1 + k' \qquad \beta_{\rho'}^{(l)} = 1 + p'.$$

As in Case 1) we obtain that $p = k$ an the necessity of the conditions in the case $\rho > 0$, $\rho' < 0$ is proved.

The cases $(\rho < 0,\ \rho' > 0)$ and $(\rho < 0,\ \rho' < 0)$ can be treated similarly.

The proof of the sufficiency of the conditions given in the theorem can be performed by straightforward calculation. □

3. Structures with simple operations

3.1. The structure $S(\rho_1, \ldots, \rho_\mu; -\rho_1, \ldots, -\rho_\mu)$

Let

$$s = [\alpha_1, \ldots, \alpha_\mu; \beta_1, \ldots, \beta_\mu]$$
$$s' = [\alpha'_1, \ldots, \alpha'_\mu; \beta'_1, \ldots, \beta'_\mu].$$

Then the ρ-product for $\rho > 0$ is (see Chapter 2.1.)

$$s\{\rho\}s' =$$
$$= [\alpha_1 + \alpha_\rho \alpha'_1, \ldots, \alpha_\rho \alpha'_\rho, \ldots, \alpha_\mu + \alpha_\rho \alpha'_\mu; \beta_1 + \alpha_\rho \beta'_1 \ldots, \beta_\mu + \alpha_\rho \beta'_\mu],$$

where α_ρ, $\alpha'_\rho \neq 0$.

The definition of the ρ-product for $\rho < 0$ is analogous.

Let us consider all the products

$$P_k(s_{i_1}\{\rho_{j_1}\}s_{i_2}\{\rho_{j_2}\} \cdots s_{i_{k-1}}\{\rho_{j_{k-1}}\}s_{i_k}$$

$k = 1, 2, \ldots$; $\rho_{i_k} \in \{\rho_1, \ldots, \rho_\mu; -\rho_1, \ldots, -\rho_\mu\}$, $P_k \in \mathcal{P}_k$.

The multiplicative structure getting by this way is said to be structure

$$S(\rho_1, \ldots, \rho_\mu; -\rho_1, \ldots, -\rho_\mu)$$

with operations $\rho_1, \ldots, \rho_\mu, -\rho_1, \ldots, -\rho_\mu$.

This structure in general is not an associative structure with respect to different operations. (Theorem 7, Chapter 2).

3.2. The structure $S(\rho)$

Let $\rho \in \{\rho_1, \ldots, \rho_\mu; -\rho_1, \ldots, -\rho_\mu\}$ be fixed and let us consider all the elements $s \in \mathcal{H}$ for which $\alpha_\rho \neq 0$ (if $\rho > 0$) or $\beta_\rho \neq 0$ (if $\rho < 0$).

The multiplicative structure $S\{\rho\}$ is the set of the products $s\rho s'$. It is clear that

$$S(\rho) \subset S\left(\rho_1, \ldots, \rho_\mu; -\rho_1, \ldots, -\rho_\mu\right).$$

Using Theorem 2 (Chapter 2) one gets

THEOREM 1. *The structure $S(\rho)$ is a semigroup.*

Let $\rho > 0$. We introduce the following special elements

(1) $g_{j0}(1) = [0,\dots,\overset{j}{\overbrace{1}},\dots,\overset{\rho}{\overbrace{1}},\dots,0;0,\dots,0]$ $\quad j \neq \rho,\ j = 1,\dots,\mu$

(2) $g_{0k}(1) = [0,\dots,\dots,\overset{\rho}{\overbrace{1}},\dots,0;0,\dots,\overset{k}{\overbrace{1}},\dots,0]$ $\quad k = 1,\dots,\mu$

(3) $g_{00}(p) = [0,\dots,\overset{\rho}{\overbrace{p}},\dots 0;0,\dots,0]$

where $p = 1$ or an arbitrary prime number.

THEOREM 2. *The elements $g_{j0}(1)$ $(j \neq \rho)$, $g_{0k}(1)$ $(k = 1,\dots,\mu)$, $g_{00}(p_l)$ $(p_0 = 1$, and p_l $(l > 0)$ runs over the set of the prime numbers) form an independent system of generators of $S(\rho)$.*

Proof. Considering the definition of the ρ-product, it is clear, that the system (1), (2), (3) is independent. It is necessary to prove that every element $[\alpha_1,\dots,\alpha_\mu;\beta_1,\dots,\beta_\mu]$ with $\alpha_\rho > 0$ $(\alpha_i, \beta_i \in \mathbf{N}^0)$ can be produced by the mentioned generators.

The following properties are valid (for the ρ-product):

$$g_{j_1 0}(1) g_{j_2 0}(1) = g_{j_2 0}(1) g_{j_1 0}(1); \qquad g_{0k_1}(1) g_{0k_2}(1) = g_{0k_2}(1) g_{0k_1}(1)$$

$$g_{j0}(1) g_{0k}(1) = g_{0k}(1) g_{j0}(1),$$

$$g_{00}(p_{l_1}) g_{00}(p_{l_2}) = g_{00}(p_{l_2}) g_{00}(p_{l_1}) = [0,\dots,0;\overset{\rho}{\overbrace{p_{l_1} p_{l_2}}},0,\dots,0;0,\dots,0],$$

$$g_{j0}^{\alpha}(1) = [0,\dots,0,\overset{j}{\overbrace{\alpha}},0,\dots 0,\overset{\rho}{\overbrace{1}},0,\dots,0;0,\dots,0],$$

$$g_{0k}^{\beta}(1) = [0,\dots,0,\overset{\rho}{\overbrace{1}},0,\dots,0;0,\dots,0,\overset{k}{\overbrace{\beta}},0,\dots,0].$$

Therefore

(4) $$s = [\alpha_1,\dots,\alpha_\mu;\beta_1,\dots,\beta_\mu] = \prod_{j=1 j\neq\rho}^{\mu} g_{j0}^{\alpha_j}(1) \prod_{k=1}^{\mu} g_{0k}^{\beta_k}(1) \prod_{l=1}^{t} g_{00}(p_l)$$

where $\alpha_\rho = \prod_{l=1}^{t} p_l$. □

Let $e_\rho = g_{00}(1) = [0,\dots,\overset{\rho}{\overbrace{1}},\dots,0;0,\dots,0]$.

THEOREM 3. *$S(\rho) = AP$ $(A \cap P = e_\rho)$, where A and P are commutative semigroups. The semigroup A has 2μ generators, and semigroup P has infinite number of generators.*

Proof. The generators $g_{j0}(1)$, $g_{0k}(1)$ $(j \neq \rho;\ j,k = 1,\dots,\mu)$ with the element e_ρ form a commutative semigroup A and the generators $g_{00}(p_l)$ $(l = 0,1,2,\dots;\ g_{00}(1) = e_\rho)$ form a commutative semigroup which is isomorphic with the multiplicative semigroup of the positive integers generated by the prime numbers. The element e_ρ is the unit-element of $S(\rho)$. Using Theorem 2 one gets Theorem 3.

The case $\rho < 0$ is a similar one. □

COROLLARY. *Every element $s \in S(\rho)$ has a unique factorisation $s = s_1 s_2$ $(s_1 \in A,\ s_2 \in P)$ and s_1 has a unique factorisation with the generators $g_{j0}(1)$, $g_{0k}(1)$ (disregarding the order of the factors), and similarly s_2 has a unique factorisation with $g_{00}(p_l)$ because every natural number has a unique factorisation with the prime numbers.*

THEOREM 4. *If $\mu > 1$, then the product PA is a proper subset of AP and for $\rho > 0$, the following identities are valid*

$$(5)\qquad g_{00}(p_l)g_{j0}(1) = g_{j0}^{p_l}(1)g_{00}(p_l)$$

$$g_{00}(p_l)g_{0k}(1) = g_{0k}^{p_l}(1)g_{00}(p_l).$$

The case $\rho < 0$ is an analogue one.

Proof. Exercise. □

CONSEQUENCE. The following identities are valid:

$$(6)\qquad g_{00}^{\alpha}(p_l)g_{j0}^{\beta}(1) = g_{j0}^{\beta p_l^{\alpha}}(1)g_{00}^{\alpha}(p_l)$$

$$(7)\qquad g_{00}^{\alpha}(p_l)g_{0k}^{\beta}(1) = g_{0k}^{\beta p_l^{\alpha}}(1)g_{00}^{\alpha}(p_l),$$

(6) and (7) are a straightforward consequence of Theorem 3.

Consider the elements

$$(8)\qquad s_i = \left[\alpha_1^{(i)},\dots,\alpha_\rho^{(i)},\dots,\alpha_\mu^{(i)};\beta_1^{(i)},\dots,\beta_\mu^{(i)}\right] \qquad (i = 1,\dots,m)$$

where $\alpha_\rho^{(i)} \neq 0$. The multiplicative structure generated by s_i $(i = 1,\dots,m)$ is a semigroup:

$$(9)\qquad S(s_i(\rho)); \qquad (i = 1,\dots,m).$$

REMARK 1. If $(\alpha_\rho^{(i)},\alpha_\rho^{(j)}) = 1$ $(i \neq j;\ i,j = 1,\dots,m)$ then the elements $s_i(\rho)$ $(i = 1,\dots,m)$ are independent, that is a proper subset of s_i $(i = 1,\dots,m)$ does not generate $S(s_i(\rho))$; $i = 1,\dots,m)$.

It is clear that $S(s_i(\rho))$; $(i = 1, \dots, m)$ is a proper subsemigroup of $S(\rho)$ if the elements s_i $(i = 1, \dots, m)$ differ from the system of the generators (1), (2), (3).

Every element $s \in S\ (s_i(\rho);\ i = 1, \dots, m)$ can be represented as a product of the generators (7). Let

$$(10)\qquad s(\rho) = s_{i_1}\{\rho\}s_{i_2}\{\rho\}\cdots s_{i_v}\qquad i_1, \dots, i_v \in \{1, 2, \dots, m\}$$

then for $\rho > 0$

$$(11)\qquad s(\rho) =$$

$$\prod_{\substack{j=1\\ j\neq\rho}}^{\mu} g_{j0}^{\alpha_j^{(i_1)}}(1)\prod_{k=1}^{\mu} g_{0k}^{\beta_k^{(i_1)}}(1)g_{00}(\alpha_\rho^{(i_1)})\cdots\prod_{\substack{j=1\\ j\neq\rho}}^{\mu} g_{j0}^{\alpha_j^{(i_v)}}(1)\prod_{k=1}^{\mu} g_{0k}^{\beta_k^{(i_v)}}(1)g_{00}(\alpha_\rho^{(i_v)})$$

where $g_{00}(a) = g_{00}(p_1)\cdots g_{00}(p_r)$ if $a = p_1\cdots p_r$.

Using (5) and (6)

$$(12)\qquad s(\rho) = \prod_{j=1 j\neq\rho}^{\mu} g_{j0}^{\alpha_j^{(i_1)}+\alpha_j^{(i_1)}\alpha_\rho^{(i_1)}+\alpha_j^{(i_3)}\alpha_\rho^{(i_1)}\alpha_\rho^{(i_2)}+\cdots+\alpha_j^{(i_v)}\alpha_\rho^{(i_1)}\cdots\alpha_\rho^{(i_{v-1})}}\times$$

$$\times\prod_{k=1}^{\mu} g_{0k}^{\beta_k^{(i_1)}+\beta_k^{(i_2)}\alpha_\rho^{(i_1)}+\beta_k^{(i_3)}\alpha_\rho^{(i_1)}\alpha_\rho^{(i_2)}+\cdots+\alpha_k^{(i_v)}\alpha_\rho^{(i_1)}\quad\alpha_\rho^{(i_{v-1})}}\cdot g_{00}(\alpha_\rho^{(i_1)}\cdots\alpha_\rho^{(i_v)}).$$

Let $s'(\rho) \in S(s_i(\rho);\ i = 1, \dots, m)$ and

$$(13)\qquad s'(\rho) = s'_{j_1}\{\rho\}s'_{j_2}\{\rho\}\cdots s'_{j_z} =$$

$$= \prod_{\substack{j=1\\ j\neq\rho}}^{\mu} g_{j0}^{\alpha_j^{\prime(j_1)}}(1)\prod_{k=1}^{\mu} g_{0k}^{\beta_k^{\prime(j_1)}}g_{00}(\alpha_\rho^{\prime(j_1)})\cdots\prod_{\substack{j=1\\ j\neq\rho}}^{\mu} g_{j0}^{\alpha_j^{\prime(j_2)}}(1)\prod_{k=1}^{\mu} g_{0k}^{\beta_k^{\prime(j_2)}}(1)g_{00}(\alpha_\rho^{\prime(j_2)}) =$$

$$= \prod_{j=1 j\neq\rho}^{\mu} g_{j0}^{\alpha_j^{\prime(j_1)}+\alpha_j^{\prime(j_2)}\alpha_\rho^{\prime(j_1)}+\alpha_j^{\prime(j_3)}\alpha_\rho^{\prime(j_1)}\alpha_\rho^{\prime(j_2)}+\cdots+\alpha_j^{\prime(j_z)}\alpha_\rho^{\prime(j_1)}\cdots\alpha_\rho^{\prime(j_{z-1})}}\times$$

$$\times\prod_{k=1}^{\mu} g_{0k}^{\beta_k^{\prime(j_1)}+\beta_k^{\prime(j_2)}\alpha_\rho^{\prime(j_1)}+\beta_k^{\prime(j_3)}\alpha_\rho^{\prime(j_1)}\alpha_\rho^{\prime(j_2)}+\cdots+\beta_k^{\prime(j_z)}\alpha_\rho^{\prime(j_1)}\alpha_\rho^{\prime(j_{z-1})}}\cdot g_{00}(\alpha_\rho^{(j_1)}\cdots\alpha_\rho^{(j_z)}).$$

Then we obtain the following

THEOREM 5. *For $s(\rho), s'(\rho) \in S(s_i(\rho);\ i = 1, \dots, m)$ and for $\rho > 0$*

$$s(\rho) = s'(\rho)$$

if and only if

1) $\alpha_\rho^{(i_1)}\cdots\alpha_\rho^{(i_v)} = \alpha_\rho^{\prime(j_1)}\cdots\alpha_\rho^{\prime(j_z)}$.

2) $\alpha_j^{(i_1)} + \alpha_j^{(i_2)}\alpha_\rho^{(i_1)} + \cdots + \alpha_j^{(i_v)}\alpha_\rho^{(i_1)}\cdots\alpha_\rho^{(i_{v-1})} = \alpha_j'^{(j_1)} + \alpha_j'^{(j_2)}\alpha_\rho'^{(j_1)} + \cdots + \alpha_j'^{(j_2)}\alpha_\rho'^{(j_1)}\cdots\alpha_\rho'^{(j_{z-1})}$ $(j = 1,\ldots,\mu;\ j \neq \rho)$.

3) $\beta_k^{(i_1)} + \beta_k^{(i_2)}\alpha_\rho^{(i_1)} + \cdots + \beta_k^{(i_v)}\alpha_\rho^{(i_1)}\cdots\alpha_\rho^{(i_{z-1})} = \beta_k'^{(j_1)} + \beta_k'^{(j_2)}\alpha_\rho'^{(j_1)} + \cdots + \beta_k'^{(j_z)}\alpha_\rho'^{(j_1)}\cdots\alpha_\rho'^{(j_{z-1})}$ $(k = 1,\ldots,\mu)$.

Proof. Theorem 5 is a consequence of (10) and (11). □

THEOREM 6. *Let*

$$s = [\alpha_1,\ldots,\alpha_\mu;\ \beta_1,\ldots,\beta_\mu],$$

$$s' = [\alpha_1',\ldots,\alpha_\mu';\ \beta_1',\ldots,\beta_\mu'].$$

If $\alpha_\rho \neq 0$, $\alpha_\rho' \neq 0$ *and* $\beta_\rho = \beta_\rho' = 0$, *then*

$$n(s\{\rho\}s') = n(n(s)\{\rho\}n(s')). \tag{14}$$

Proof. By Definition 3 of Chapter 1, for the components of

$$n(s\{\rho\}s') = \left[\alpha_1'',\ldots,\alpha_\mu'';\beta_1'',\ldots,\beta_\mu''\right]$$

the followings are valid:

1) $\alpha_j + \alpha_\rho\alpha_j' - (\beta_j + \alpha_\rho\beta_j') = \alpha_j''$ if $\alpha_j + \alpha_\rho\alpha_j' > \beta_j + \alpha_\rho\beta_j'$, $\beta_j'' = 0$,

2) $\beta_j + \alpha_\rho\beta_j' - (\alpha_j + \alpha_\rho\alpha_j') = \beta_j''$ if $\alpha_j + \alpha_\rho\alpha_j' < \beta_j + \alpha_\rho\beta_j'$, $\alpha_j'' = 0$,

3) $\alpha_\rho'' = \alpha_\rho\alpha_\rho', \quad \beta_\rho'' = 0.$

In the product $n(s)\{\rho\}n(s') = [\overline{\alpha}_1,\ldots,\overline{\alpha}_\mu;\overline{\beta}_1,\ldots,\overline{\beta}_\mu]$ $\overline{\alpha}_\rho = \alpha_\rho\alpha_\rho'$ holds. We distinct four cases:

a) If $\alpha_j - \beta_j > 0$ $(j \neq \rho, 1 \leq j \leq \mu)$, $\alpha_j' - \beta_j' > 0$ $(j \neq \rho, 1 \leq j \leq \mu)$ then

$$\alpha_j - \beta_j + \alpha_\rho(\alpha_j' - \beta_j') = \alpha_j''.$$

b) If $\alpha_j - \beta_j > 0$ $(j \neq \rho,\ 1 \leq j \leq \mu)$, $\alpha_j' - \beta_j' < 0$ $(j \neq \rho,\ 1 \leq j \leq \mu)$, then

$$n(s) = [\ldots,\overset{j}{\overbrace{\alpha_j - \beta_j}},\ldots,\alpha_\rho,\ldots;\ldots,\overset{j}{\overbrace{0}},\ldots],$$

$$n(s') = [\ldots,\overset{j}{\overbrace{0}},\ldots,\alpha_\rho',\ldots;\ldots,\overset{j}{\overbrace{\alpha_j - \beta_j}},\ldots],$$

$$n(s)\{\rho\}n(s') = [\ldots,\overset{j}{\overbrace{\alpha_j - \beta_j}},\ldots,\alpha_\rho\alpha_\rho',\ldots,;\ldots,\overset{j}{\overbrace{\alpha_\rho(\beta_j' - \alpha_j')}},\ldots],$$

$$\alpha_j - \beta_j - \alpha_\rho(\beta_j' - \alpha_j') = \alpha_j + \alpha_\rho\alpha_j' - (\beta_j + \alpha_\rho\beta_j'),$$

which coincides with the cases 1) and 2) and results α_j'' or β_j''.

The cases c) $\alpha_i - \beta_j < 0$, $\alpha'_j - \beta'_j > 0$, and d) $\alpha_j - \beta_j < 0$, $\alpha'_j - \beta'_j < 0$ are analogous to b) and a). □

THEOREM 7. *Let S' be the set of the elements of S in which $\alpha_\rho \neq 0$, $\beta_\rho = 0$. Then $S'(\rho)$ is a semigroup. Let $n(S')$ be the set of the elements of S in normalform. Then*

$$n(S'(\rho)) = n(n(S')(\rho)). \tag{15}$$

The case $\alpha_\rho = 0$, $\beta_\rho \neq 0$ is an analogous one.

Proof. It is evident that $S'(\rho)$ is a semigroup. (15) is a consequence of Theorem 6 with

$$s \longrightarrow n(s), \qquad s' \longrightarrow n(s')$$

is

$$s\{\rho\}s' \longrightarrow n(s)\{\rho\}n(s') \longrightarrow n(n(s)\{\rho\}n(s')) = n(s\{\rho\}s') \quad (s, s' \in S'). \ \square$$

From Theorem 7 follows

THEOREM 8. *Let $n(s) \circ n(s') = n(n(s)\rho n(s'))$. Then the set $n(S'(\rho))$ is a semigroup with the (multiplicative) operation "$\circ$" . Besides with*

$$s \longrightarrow n(s), \qquad s' \longrightarrow n(s'), \qquad s\rho s' \longrightarrow n(s) \circ n(s') = n(s\rho s'),$$
$$S'(\rho) \sim n(S'(\rho))$$

follows.

Proof. It is easy to see that $n(S'(\rho))$ is a semigroup with the operation "$\circ$", because also $S'(\rho)$ is a semigroup. Then the Theorem 8 is a simple consequence of Theorem 7. □

THEOREM 9. *Let S'' be the set of the elements of S in which $\alpha_\rho = 1$, $\beta_\rho = 0$. Then $n(n(S'')(\rho))$ is a commutative group. The case $\alpha_\rho = 0$, $\beta_\rho = 1$ is an analogous one.*

Proof. Every element $n(s)$ $(s \in S'')$ has a unique inverse. With

$$n(s) = [\alpha_1, \ldots, \overset{\rho}{\overbrace{1}}, \ldots, \alpha_\mu; \beta_1, \ldots \overset{\rho}{\overbrace{0}}, \ldots, \beta_\mu],$$

$$n(s)^{-1} = [\beta_1, \ldots, \overset{\rho}{\overbrace{1}}, \ldots, \beta_\mu; \alpha_1, \ldots, \overset{\rho}{\overbrace{0}}, \ldots, \alpha_\mu],$$

and $n(n(s)\{\rho\}n^{-1}(s)) = n(n^{-1}(s)\{\rho\}n(s)) = e_\rho$. Besides S'' is an associative structure (semigroup) for the ρ-multiplication and e_ρ is the unit element of S''.

The case $\rho < 0$ is similar. □

Similarly as in the case of $S(\rho)$ we can introduce the semigroups $S(\rho_i)$ $(i = 1, \ldots, \mu)$ and $S(-\rho_i)$ $(i = 1, \ldots, \mu)$. $S(\rho_i)$ $(S(-\rho_i))$ are semigroups and

$$(16) \qquad S(\rho_i) = A_i P_i\,, \qquad S(-\rho_i) = A'_i P'_i$$

where $A_i \cap P_i = e_{\rho_i}$ is the unit element of $S(\rho_i)$ ($A'_i \cap P'_i = e'_{\rho_i}$ is the unit element of $S(-\rho_i)$.)

It is easy to see the

THEOREM 10. *For the semigroups $S(\rho_i)$ $(S(-\rho_i))$*

$$A_i \approx A_j \approx A'_j \qquad (i, j \in \{1, \ldots, \mu\})$$

$$P_i \approx P_j \approx P'_j \qquad (i, j \in \{1, \ldots, \mu\})$$

$$S(\rho_i) \approx S(\rho_j) \approx S(-\rho_j) \qquad (i, j \in \{1, \ldots, \mu\})$$

are valid.

Let $S^0(\rho)$ the set of those elements of S for which $\alpha_\rho = 0$ $\forall s \in S^0(\rho)$.

On the part of the applications we distinguish two cases:

1) $s\{\rho\}s'$ is not realizable if $s \in S^0(\rho)$ (see Section 2.1).

2) $s\{\rho\}s' = s$ if $s \in S^0(\rho)$, $s \in S$.

In the first case the pair $s\{\rho\}s'$ remains (the elements s, s' are "associated"), in the second case the elements s, s' are "united".

In the first case let us consider the free product $\overline{S}^0(\rho)$ of the elements of $S^0(\rho)$ for which we suppose the associativity. It is clear that $S(\rho)\{\rho\}S^0(\rho) \subseteq S^0(\rho)$ and $s\{\rho\}(s'\{\rho\}s'') = (s\{\rho\}s')\{\rho\}s''$ where $s \in S(\rho)$, $s', s'' \in S^0(\rho)$.

In the second case the set $S(\rho) \cup S^0(\rho) = \overline{S}^0(\rho)$ is a semigroup.

REMARK 2. The semigroups $S(\rho_i)$ $(S(-\rho_i))$ are suitable to use for certain coding and decoding problems.

3.2.1. A coding and decoding process

EXAMPLE. Using Theorem 5 the structure $S(\rho)$ is suitable for creating coding and decoding processes. In the following we describe a possible (very simple) process.

The coding process

Suppose that n stations send information to a center C. For this the stations can use the vectors $s_i(\rho) = [\alpha_1^{(i)}, \ldots, \alpha_\rho^{(i)}, \ldots, \alpha_\mu^{(i)}; \beta_1^{(i)}, \ldots, \beta_\mu^{(i)}]$, $i = 1, \ldots, n$ with 2μ positions (μ can be an arbitrary large number). The $\alpha_\rho^{(i)}$ is the code-number of the i-th station. The numbers α_j, β_j ($j \neq \rho$) contain other informations. The numbers $\alpha_\rho^{(i)}$ are different.

Suppose that the (elementary) information are formulated in binary form and $\alpha_j^{(i)}$ is the integer (in decimal system) of this information. So we have $2\mu - 1$ pieces of information. Let

$$\alpha_\rho^{(i)} > max(\alpha_j^{(i)}, \beta_j^{(i)}), \qquad (j \neq \rho).$$

n vectors (with 2μ components) arrive to the center C. The center C decides on the order of the multiplication of the n vectors (in $S(\rho)$). The result of this multiplication is $s(\rho)$, where

$$s(\rho) = s_{i_1}\{\rho\}s_{i_2}\{\rho\} \cdots \{\rho\}s_{i_n}.$$

The vector $s(\rho)$ will be stored in C.

The decoding process

The vector $s(\rho)$ can be decoded uniquely.

The code-numbers $\alpha_\rho^{(i)}$ are known in C. It means that α_ρ in $s(\rho)$ has a unique decomposition (as a product) ($= \alpha_\rho^{(i_1)} \cdots \alpha_\rho^{(i_n)}$).

The center C knows the ordering of the multiplication of the vectors, therefore by Theorem 5 (using property 2)) the sum (which is known in $s(\rho)$) is divisible by $\alpha_\rho^{(i_1)}$ except for $\alpha_j^{(i_1)}$. Thus the vector of the i_1-th station will be known uniquely because $\alpha_j^{(i_1)} < \alpha_\rho^{(i_1)}$. Repeating this process for the quotient

$$\frac{\alpha_j - \alpha_j^{(i_1)}}{\alpha_\rho^{(i_1)}}$$

(where α_j is the j-th component of $s(\rho)$) we get the components $\alpha_j^{(i_2)}$ etc. Components $\beta_j^{(i)}$ are similarly obtained.

In the decodings process it is necessary to know only the numbers $\alpha_\rho^{(i)}$, ($i = 1, \ldots, n$) and the ordering of the multiplication of the vectors. For this ordering stay $n!$ different possibilities are available.

3.2.2. The system of generators of $S(\rho_i)$

In Section 3.2 we have seen that the structures $S(\rho_i)$, $S(-\rho_i)$ are semigroups $(i = 1, \ldots, \mu)$, where the system of generators of $S(\rho)$ (ρ is one of ρ_i) is:

$$(17) \qquad g_{j0}(1) = [0, \ldots, \overset{j}{\breve{1}}, \ldots, \overset{\rho}{\breve{1}}, \ldots, 0; 0, \ldots, 0],$$

$$g_{0k}(1) = [0, \ldots, \overset{\rho}{\breve{1}}, \ldots, 0; 0, \ldots, \overset{k}{\breve{1}}, \ldots, 0],$$

$$g_{00}(p) = [0, \ldots, \overset{\rho}{\breve{p}}, \ldots, 0; 0, \ldots, 0].$$

To distinct the generators of $S(\rho_i)$, $S(-\rho_i)$, $(i = 1, \ldots, \mu)$ we write

$$(18) \qquad \begin{aligned} g_{j0}(1,\rho) &\equiv g_{j0}(1) \\ g_{0k}(1,\rho) &\equiv g_{0k}(1) \\ g_{00}(p,\rho) &\equiv g_{00}(p). \end{aligned}$$

instead of (17).

Therefore the generators of $S(\rho_i)$ are $g_{j0}(1, \rho_i)$, $g_{0k}(1, \rho_i)$, $g_{00}(p, \rho_i)$, and similarly for $S(-\rho_i)$.

REMARK 3. Every semigroup S has a disjoint left decomposition with components Λ_i, $i = 0, \ldots, 5$ and a dual right decomposition with components P_i, $i = 0, \ldots, 5$ (see [17]). In the case of $S(\rho_i)$ the following properties are true for the components:

$$\Lambda_0 = \emptyset, \quad \Lambda_1 = \emptyset, \quad \Lambda_2 = \emptyset, \quad \Lambda_3 = \emptyset, \quad \Lambda_4 \neq \emptyset, \quad \Lambda_5 = e_\rho,$$

$$P_0 = \emptyset, \quad P_1 = \emptyset, \quad P_2 \neq \emptyset, \quad P_3 = \emptyset, \quad P_4 \neq \emptyset, \quad P_5 = e_\rho.$$

4. Transformations and equations

4.1. Transformations in $S(\rho_1, \dots, \rho_\mu; -\rho_1, \dots, -\rho_\mu)$

Using the results of the previous chapter, it is easy to see the following properties (it is not a misunderstanding if we use ρ_i as operation belonging to the ith position and sometimes as ith position):

1) $$g_{\rho_i 0}(1, \rho_j) = g_{\rho_j 0}(1, \rho_i),$$

2) $$g_{j0}^{\alpha_j}(1, \rho_i)\{\rho_i\} g_{00}(\alpha_i, \rho_i) = g_{i0}^{\alpha_i}(1, \rho_j)\{\rho\}_j) g_{00}(\alpha_j, \rho_j),$$

3) $$g_{\rho_i 0}(1, -\rho_k) = g_{0 -\rho_k}(1, \rho_i),$$

4) $$g_{0k}^{\beta_k}(1, \rho_i)\{\rho_i\} g_{00}(\alpha_i, \rho_i) = g_{i0}^{\alpha_i}(1, -\rho_k)\{-\rho_k\} g_{00}(\beta_k, -\rho_k),$$

5) $$\prod_{\substack{j=1 \\ j \neq \rho_i, \rho_r}}^{\mu} g_{j0}^{\alpha_j}(1, \rho_i)\{\rho_i\} \prod_{k=1}^{\mu} g_{0k}^{\beta_k}(1, \rho_i)\{\rho_i\} g_{r0}^{\alpha_r}(1, \rho_i)\{\rho_i\} g_{00}(\alpha_i, \rho_i) =$$

$$= \prod_{\substack{j=1 \\ j \neq \rho_i, \rho_r}}^{\mu} g_{j0}^{\alpha_j}(1, \rho_r)\{\rho_r\} \prod_{k=1}^{\mu} g_{0k}^{\beta_k}(1, \rho_r)\{\rho_r\} g_{i0}^{\alpha_i}(1, \rho_r)\{\rho_r\} g_{00}(\alpha_r, \rho_r).$$

A similar equality for $\rho_i \longrightarrow -\rho_i$, or $\rho_r \longrightarrow -\rho_r$ is also true.

The relations 1–5) give a possibility for the transformation

$$s = [\alpha_1, \dots, \alpha_\mu; \beta_1, \dots, \beta_\mu] =$$

$$= \prod_{\substack{j=1 \\ j \neq \rho_i}}^{\mu} g_{j0}^{\alpha_j}(1, \rho_i)\{\rho_i\} \prod_{k=1}^{\mu} g_{0k}^{\beta_k}(1, \rho_i)\{\rho_i\} \prod_{\ell=1}^{t} g_{00}(p_\ell, \rho_i) =$$

$$= \prod_{\substack{j=1 \\ j \neq \rho_r}}^{\mu} g_{j0}^{\alpha_j}(1, \rho_r)\{\rho_r\} \prod_{k=1}^{\mu} g_{0k}^{\beta_k}(1, \rho_r)\{\rho_r\} \prod_{\ell=1}^{v} g_{00}(q_{\ell, \rho_r}) \qquad (\alpha_i \neq 0, \alpha_r \neq 0).$$

Let $s_i = [\alpha_1^{(i)}, \dots, \alpha_\mu^{(i)}; \beta_1^{(i)}, \dots, \beta_\mu^{(i)}]$ $i = 1, 2, \dots$.

THEOREM 1. *Consider the product*

$$P_k(s_1\{\rho_{j1}\}s_2\{\rho_{j2}\}\cdots s_{k-1}\{\rho_{j_{k-1}}\}s_k) \qquad k>1$$

where the product is realizable with $\mu > 1$ *and in every* s_i $(i=1,\ldots,k-1)$ $\alpha_{\rho_j} \neq 0$. *Suppose that none of* s_j $(j=1,\ldots,k)$ *is a unit element in* $S(\rho_j)$ $(S(-\rho_j))$. *Then for every* k $(P_k \in \mathcal{P}_k)$ *elements* $s_1^{(j_r)},\ldots,s_k^{(j_r)}$ *(different from unit elements) exist such that* $(1 \le r \le k-1)$

$$P_k\left(s_1\{\rho_{j1}\}\cdots s_{k-1}\left\{\rho_{j_{k-1}}\right\}s_k\right) = s_1^{(j_r)}\left\{\rho_{j_r}\right\}s_2^{(j_r)}\left\{\rho_{j_r}\right\}\cdots s_{k-1}^{(j_r)}\{\rho_{j_r}\}s_2^{(j_r)}.$$

Proof. By induction. Consider the product

$$(s_1\{\rho_{j_1}\}s_2(\rho_{j_2}))s_3 = P_2(s_1\{\rho_{j_k}\}s_2\{\rho_{j_2}\}s_3), \quad P_2 \in \mathcal{P}_2 \tag{1}$$

where s_1, s_2, s_3 are not unit elements. Using the properties 1)–5) there are elements s_1', s_2' (different from unit elements) such that $s_1\{\rho_{j_1}\}s_2 = s_1\{\rho_{j_2}\}s_2'$, therefore

$$(s_1\{\rho_{j_1}\}s_2\{\rho_{j_2}\})s_3 = (s_1'\{\rho_{j_2}\}s_2')\{\rho_{j_2}\}s_3 = s_1'\{\rho_{j_2}\}s_2'\{\rho_{j_2}\}s_3. \tag{2}$$

Similarly, in the case $s_1\{\rho_{j_1}\}(s_2\{\rho_{j_2}\}s_3) = P_2'(s_1\{\rho_{j_1}\}s_2\{\rho_{j_2}\}s_3)$ $(P_2' \in \mathcal{P}_2)$ instead of $s_2\{\rho_{j_2}\}s_3$ one can write $s_2''\{\rho_{j_1}\}s_3''$ and we obtain the result for $k=3$.

Let us suppose, that the theorem is true for k (>3) and consider the product

$$P_{k+1}(s_1\{\rho_{j_1}\}s_2\{\rho_{j_2}\}\cdots s_k\{\rho_{j_k}\}s_{k+1}), \qquad P_{k+1} \in \mathcal{P}_{k+1} \tag{3}$$

where $s_1,\ldots,s_{k+1}$ differ from unit elements.

It is known from Section 2.1 that

$$P_{k+1} = P_{n_1}P_{n_2} \qquad (n_1+n_2=k+1;\ n_1,n_2 \ge 1).$$

Suppose that for (3)

$$P_{k+1} = P_{n_1}\{\rho_j\}P_{n_2}$$

is valid. By induction $P_{n_1}(\cdots)$ and $P_{n_2}(\cdots)$ $(n_1,n_2 \le k)$ can be represented as product of n_1 (n_2 resp.) elements (different from unit elements) with operation ρ_j. It means that for $P_{k+1}(\cdots)$ the theorem is valid. □

It is easy to see the following

THEOREM 2. *If* $P_k, P_k' \in \mathcal{P}_k$ *then for arbitrary* $s_1,\ldots,s_k$ *(different from unit elements) there are* $s_1',\ldots,s_k'$ *(different from unit elements) such that*

$$P_k\left(s_1\left\{\rho_{j_1}\right\}s_2\left\{\rho_{j_2}\right\}\cdots s_{k-1}\left\{\rho_{j_{k-1}}\right\}s_k\right) =$$
$$= P_k'\left(s_1'\left\{\rho_{i_1}\right\}s_2'\left\{\rho_{i_2}\right\}\cdots s_{k-1}'\left\{\rho_{i_{k-1}}\right\}s_k'\right)$$

supposing that the products are realisable.

Proof. Using Theorem 1, the product $P_k(\cdots)$ can be transformed to a product with common operation and in the second step this product can be transformed in $P'_k(\cdots)$. □

4.2. Equations in $S(\rho_i)$

Let us consider the equations

$$\text{a)}\ \ s\{\rho_i\}x = s', \qquad \text{b)}\ \ x\{\rho_i\}s = s',$$

where

$$s = [\alpha_1, \ldots, \alpha_\mu;\ \beta_1, \ldots, \beta_\mu]$$
$$s' = [\alpha'_1, \ldots, \alpha'_\mu;\ \beta'_1, \ldots, \beta'_\mu]$$
$$x = [\chi_1, \ldots, \chi_\mu;\ \eta_1, \ldots, \eta_\mu].$$

and the goal is to solve the equations for x.

Definition. The element x is a solution of the equation $s\{\rho_i\}x = s'$ $(\alpha_i \neq 0)$ $(x\{\rho_i\}s = s'$ resp.) if an s'' exists such that $s\{\rho_i\}x = s''$ $(x\{\rho_i\}s = s''$ resp.) with $n(s'') = n(s')$.

Let

$$s'' = \left[\alpha'_1 + \tau'_1, \ldots, \alpha'_\mu + \tau'_\mu;\ \beta'_1 + \tau'_1, \ldots, \beta'_\mu + \tau'_\mu\right], \qquad \tau'_i \geq -\min(\alpha'_i, \beta'_i)$$

that is, $n(s') = n(s'')$.

Theorem 3. The equation

$$(4) \qquad s\{\rho_i\}x = s' \qquad (\alpha_i \neq 0)$$

has a solution if and only if

$$(5) \qquad \alpha_i \mid \alpha'_i - \beta'_i + \beta_j,$$

$$(6) \qquad \alpha_i \mid (\alpha'_j - \beta'_j) - (\alpha_j - \beta_j) \qquad j \neq i.$$

Proof. From $s\{\rho_i\}x = s''$ the equalities

$$(7) \qquad \begin{aligned} \alpha_i\chi_i &= \alpha'_i + \tau'_i \\ \beta_i + \alpha_i\eta_i &= \beta'_i + \tau'_i \end{aligned} \qquad \tau'_i \geq -\min(\alpha'_i, \beta'_i)$$

$$(8) \qquad \begin{aligned} \alpha_j + \alpha_i\chi_j &= \alpha'_j + \tau'_j \\ \beta_j + \alpha_i\eta_j &= \beta'_j + \tau'_j \end{aligned} \qquad j \neq i,\ \tau_j \geq -\min(\alpha'_j, \beta'_j)$$

follow.

From (5) one obtains $\alpha_i \,|\, \alpha'_i - \beta'_i + \beta_i$ therefore the condition (3) is necessary.

If $\alpha'_i - \beta'_j + \beta_i = 0$ then $\alpha_i \eta_i = \alpha'_i + \tau'_i$ hereby $\chi_i = \eta_i$ follows and $\alpha_i \eta_i = \tau'_i \geq 0$. It means that in this case (5) has a solution.

If $\alpha'_i - \beta'_i + \beta_i \neq 0$ then

$$\begin{aligned} \alpha_i \chi_i - \alpha'_i = \beta_i - \beta'_i + \alpha_i \eta_i = \tau'_i \\ \alpha_i(\chi_i - \eta_i) = \beta_i - \beta'_i + \alpha'_i \,. \end{aligned} \tag{9}$$

By condition (3) the equation (7) has a solution for $\chi_i - \eta_i$. For sufficiently large χ_i and η_i (5) has a solution with $\tau'_i \geq -\min(\alpha'_i, \beta'_i)$.

From (6) one obtains $\alpha_i \,|\, (\alpha'_j - \beta'_j) - (\alpha_j - \beta_j)$ therefore the condition (4) is necessary.

If $(\alpha'_j - \beta'_j) - (\alpha_j - \beta_j) = 0$ then $\alpha'_j - \alpha_j = \beta'_j - \beta_j$ and from (6) $x_j = \eta_j$ follows. Thus for sufficiently large x_j $(= \eta_j)$, (6) has a solution with $\tau'_j \geq -\min(\alpha'_j, \beta'_j)$.

If $(\alpha'_i - \beta'_j) - (\alpha_i - \beta_j) \neq 0$, then from (8)

$$\alpha_1(\chi_j - \eta_j) = (\alpha'_j - \beta'_j) - (\alpha_j - \beta_j)$$

follows, which by condition (4) has a solution for $\chi_j - \eta_j$. For sufficiently large χ_j and η_j, (6) has a solution with $\tau'_j \geq -\min(\alpha'_j, \beta'_j)$.

Analogous result can be obtained for $\rho_i \longrightarrow -\rho_i$ also. □

THEOREM 4. *The equation*

$$x\{\rho_i\}s = s'(\chi_i \neq 0)$$

has a solution if and only if one of the following cases is valid:

1) $\alpha_i - \beta_i > 0$;
2) $\alpha_i - \beta_i = 0$ *and* $\alpha'_i - \beta'_i \leq 0$;
3) $\alpha_i - \beta_i < 0$ *and* $\alpha'_i - \beta'_i < \alpha_i - \beta_i$.

Proof. From $x\{\rho_i\}s = s'$ follows

$$\begin{aligned} \chi_i \alpha_i = \alpha'_i + \tau'_i \\ \eta_i + \chi_i \beta_i = \beta'_i + \tau'_i \end{aligned} \qquad \tau'_i \geq -\min(\alpha'_i, \beta'_i) \tag{10}$$

$$\begin{aligned} \chi_j + \chi_i \alpha_j = \alpha'_j + \tau'_j \\ \eta_j + \chi_i \beta_j = \beta'_j + \tau'_j . \end{aligned} \qquad j \neq i, \ \tau'_j \geq -\min(\alpha'_j, \beta_j) \tag{11}$$

From (8) one gets

$$\chi_i(\alpha_i - \beta_i) = \alpha'_i - \beta'_i + \eta_i \,. \tag{12}$$

Easily can be seen that for $\chi_i > 0$ only in the mentioned three cases exists a $\eta_i \geq 0$ for the validity of (10).

From (9)

$$\chi_j - \eta_j + \chi_i(\alpha_j - \beta_j) = \alpha'_j - \beta'_j \qquad (j \neq i) \tag{13}$$

follows. It is easy to see, that there exist always $\chi_j \geq 0$, $\eta_j \geq 0$ for which (11) is valid for arbitrary τ'_j.

Analogous result holds for $\rho_i \longrightarrow -\rho_i$ also. □

REMARK. From the proof of the Theorem 4 one sees, that if $x\{\rho_i\}s = s'$ has a solution for x, then also for $x\{\rho_i\}s = s''$ exists a solution where $n(s'') = n(s')$.

THEOREM 5. *If* $\alpha_i \neq 0$, $\chi_i \neq 0$ *and* x *is a solution of the equation* a) *or* b), *then also* $v\{\rho_i\}x$ *and* $x\{\rho_i\}v$ *are solutions with* $n(v) = e_{\rho_i}$.

Proof. Using Theorem 6 of Chapter 3 for $s\{\rho_i\}(v\{\rho_i\}x) = s'$

$$n(s\{\rho_i\}(v\{\rho_i\}x)) = n(n(s)\{\rho_i\}n(v\{\rho_i\}x\}) =$$

$$= n(n(s)\{\rho_i\}n(x)) = n(s\{\rho_i\}x) = n(s').$$

The proof can be done similarly also in the case $x\{\rho_i\}s = s'$ and for $x\{\rho_i\}v \longrightarrow v\{\rho_i\}x$. □

5. Composed operations for setvectors

5.1. The operation $\{\rho_{i_1}, \dots, \rho_{i_r}\}$

Let

$$s = \left[\alpha_1, \dots, \alpha_\mu;\ \beta_1, \dots, \beta_\mu\right]$$

$$s' = \left[\alpha'_1, \dots, \alpha'_\mu;\ \beta'_1, \dots, \beta'_\mu\right].$$

The operation $\{\rho_{i_1}, \dots, \rho_{i_r}\}$ with different ρ_{i_j} (> 0) $(j = 1, \dots, r)$ is defined in the following manner:

$$s\{\rho_{i_1}, \dots, \rho_{i_r}\}s' =$$

$$= \left[\alpha_1, \dots, \alpha_{i_1}s', \dots, \alpha_{i_r}s', \dots, \alpha_\mu; \beta_1, \dots, \beta_\mu\right] =$$

$$= \Big[\alpha_1 + (\alpha_{i_1} + \dots + \alpha_{i_r})\alpha'_1, \dots, (\alpha_{i_1} + \dots + \alpha_{i_r})\alpha'_{i_1}, \dots, (\alpha_{i_1} + \dots + \alpha_{i_r})\alpha'_{i_r},$$

$$\dots, \alpha_\mu + (\alpha_{i_1} + \dots + \alpha_{i_r})\alpha'_\mu \dots; \dots, \beta_k + (\alpha_{i_1} + \dots + \alpha_{i_r})\,\beta'_k, \dots\Big] =$$

$$= \Big[\alpha_1 + \Big(\sum_{j=1}^{r} \alpha_{i_j}\Big)\alpha'_1, \dots, \Big(\sum_{j=1}^{r} \alpha_{i_j}\Big)\alpha'_{i_1}, \dots, \Big(\sum_{j=1}^{r} \alpha_{i_j}\Big)\alpha'_{i_r}, \dots, \alpha_\mu +$$

$$+ \Big(\sum_{j=1}^{r} \alpha_{i_j}\Big)\alpha'_\mu; \dots, \beta_k + \Big(\sum_{j=1}^{j} \alpha_{i_j}\Big)\beta'_k, \dots\Big].$$

The definition is analogous in cases $\rho_{i_j} \longrightarrow -\rho_{i_j}$.

THEOREM 1. *For any fixed operation* $\{\rho_{i_1}, \dots, \rho_{i_r}\}$ *the structure*

$$S(\{\rho_{i_1}, \dots, \rho_{i_r}\})$$

is a semigroup.

Proof. Let

$$s = \left[\alpha_1, \ldots, \alpha_\mu;\ \beta_1, \ldots, \beta_\mu\right]$$

$$s' = \left[\alpha'_1, \ldots, \alpha'_\mu;\ \beta'_1, \ldots, \beta'_\mu\right]$$

$$s'' = \left[\alpha''_1, \ldots, \alpha''_\mu;\ \beta''_1, \ldots, \beta'_\mu\right].$$

Then

$$(s\{\rho_{i_1}, \ldots, \rho_{i_r}\}s')\{\rho_{i_1}, \ldots, \rho_{i_r}\}s'' =$$

$$=\Big[\alpha_1 + \Big(\sum_{j=1}^{r} \alpha_{i_j}\Big)\alpha'_1 + \Big(\sum_{j=1}^{r} \alpha_{i_j} \sum_{p=1}^{r} \alpha'_{i_p}\Big)\alpha''_1, \ldots \Big(\sum_{j=1}^{r} \alpha_{i_j} \sum_{p=1}^{r} \alpha'_{i_p}\Big)\alpha''_{i_1},$$

$$\ldots, \Big(\sum_{j=1}^{r} \alpha_{i_j} \sum_{p=1}^{r} \alpha'_{i_p}\Big)\alpha''_{i_r}, \ldots, \alpha_\mu + \Big(\sum_{j=1}^{r} \alpha_{i_j}\Big)\alpha'_\mu + \Big(\sum_{j=1}^{r} \alpha_{i_j} \sum_{p=1}^{r} \alpha'_{i_p}\Big)\alpha''_\mu;$$

$$\ldots, \beta_k + \Big(\sum_{j=1}^{r} \alpha_{i_j}\Big)\beta'_j + \Big(\sum_{j=1}^{r} \alpha_{i_j} \sum_{p=1}^{r} \alpha'_{i_p}\Big)\beta''_j, \ldots,\Big]$$

and

$$s\{\rho_{i_1}, \ldots, \rho_{i_r}\}(s'\{\rho_{i_1}, \ldots, \rho_{i_r}\}s'') =$$

$$=\Big[\alpha_1 + \Big(\sum_{j=1}^{r} \alpha_{i_j}\Big)\Big(\alpha'_1 + (\sum_{p=1}^{r} \alpha'_{i_p})\alpha''_1\Big), \ldots, \Big(\sum_{j=1}^{r} \alpha_{i_p}\Big)\Big(\sum_{p=1}^{r} \alpha'_{i_p}\Big)\alpha''_{i_1},$$

$$\ldots, \Big(\sum_{j=1}^{r} \alpha_{i_j}\Big)\Big(\sum_{p=1}^{r} \alpha'_{i_p}\Big)\alpha''_{i_r}, \ldots, \alpha_\mu + \Big(\sum_{j=1}^{r} \alpha_{i_j}\Big)\Big(\alpha'_\mu + \Big(\sum_{p=1}^{r} \alpha'_{i_p}\Big)\alpha''_\mu\Big);$$

$$\ldots, \beta_k + \Big(\sum_{j=1}^{r} \alpha_{i_j}\Big)\Big(\beta'_k + \Big(\sum_{p=1}^{k} \alpha'_{i_p}\Big)\beta''_k\Big), \ldots\Big]$$

from which the theorem follows. □

5.2. The semigroup $S(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$

Let us consider the elements $s = [\alpha_1, \ldots, \alpha_\mu;\ \beta_1, \ldots, \beta_\mu]$, where at least one of the elements $\alpha_{i_1}, \ldots, \alpha_{i_r}$ differs from 0.

Consider the set H of the elements s, where $\alpha_{j_1} = \cdots = \alpha_{j_h} = 0$ and $(j_1, \ldots, j_h) \subset (i_1, \ldots, i_r)$. It is clear that for $s, s' \in H$, $\alpha''_{j_1} = \cdots = \alpha''_{j_h} = 0$ also holds in the product

$$s\{\rho_{i_1}, \ldots, \rho_{i_r}\}s' = s'' = [\alpha''_1, \ldots, \alpha''_\mu;\ \beta''_1, \ldots, \beta''_\mu].$$

Therefore we get the following:

THEOREM 2. *The set H forms a semigroup for the operation*

$$\{\rho_{i_1},\ldots,\rho_{i_r}\},$$

where

$$H(\{\rho_{i_1},\ldots,\rho_{i_r}\})\approx S(\{\rho_{k_1},\ldots,\rho_{k_{r-h}}\})$$

with

$$(k_1,\ldots,k_{r-h})=(i_1,\ldots,i_r)\setminus(j_1,\ldots,j_h).$$

Proof. Theorem 2 is a consequence of Theorem 1. □

Simple calculation shows

THEOREM 3.

$$H(\{\rho_{i_1},\ldots,\rho_{i_r}\})\{\rho_{i_1},\ldots,\rho_{i_r}\}S(\{\rho_{i_1},\ldots,\rho_{i_r}\})\not\subset H(\{\rho_{i_1},\ldots,\rho_{i_r}\}),$$

$$S(\{\rho_{i_1},\ldots,\rho_{i_r}\})\{\rho_{i_1},\ldots,\rho_{i_r}\}H(\{\rho_{i_1},\ldots,\rho_{i_r}\})\subset H(\{\rho_{i_1},\ldots,\rho_{i_r}\}),$$

that is $H(\{\rho_{i_1},\ldots,\rho_{i_r}\})$ is a left ideal in the semigroup $S(\{\rho_{i_1},\ldots,\rho_{i_r}\})$.

Let H_1 be the set of the elements s, in which

$$\alpha_{j_1}=\cdots=\alpha_{j_h}=0,\qquad (j_1,\ldots,j_h)\subset(i_1,\ldots,i_r)$$

and for $(i_1,\ldots,i_r)\setminus(j_1,\ldots,j_h)=K_1$, $\alpha_i\neq 0$, $i\in K_1$.
Similarly let H_2 be the set of the elements, in which

$$\alpha_{\ell_1}=\cdots=\alpha_{\ell_p}=0,\quad (\ell_1,\ldots,\ell_p)\subset(i_1,\ldots,i_r),$$

$$(\ell_1,\ldots,\ell_p)\neq(j_1,\ldots,j_h),$$

and for $(i_1,\ldots,i_r)\setminus(\ell_1\ldots,\ell_p)=K_2$, $\alpha_j\neq 0$, $j\in K_2$. Then it is easy to see the following

THEOREM 4.

$$H_1(\{\rho_{i_1},\ldots,\rho_{i_r}\})\cap H_2(\{\rho_{i_1},\ldots,\rho_{i_r}\})=\emptyset.$$

Let

$$g_0\Big(1;\Big\{\rho_{i_1},\ldots,\rho_{i_r}\Big\}\Big)=\Big[0,\ldots,\overset{i_1}{\overbrace{1}},\ldots,\overset{i_r}{\overbrace{1}},\ldots,0;0,\ldots,0\Big],$$

$$g_{j0}\Big(1;\Big\{\rho_{i_1},\ldots,\rho_{i_r}\Big\}\Big)=\Big[0,\ldots,\overset{i_1}{\overbrace{1}},\ldots,\overset{i_r}{\overbrace{1}},\ldots,\overset{j}{\overbrace{1}},\ldots;0,\ldots,0\Big],$$

$$g_{0k}\Big(1;\Big\{\rho_{i_\ell},\ldots,\rho_{i_r}\Big\}\Big)=\Big[0,\ldots,\overset{i_1}{\overbrace{1}},\ldots,\overset{i_r}{\overbrace{1}},\ldots,0;0,\ldots\overset{k}{\overbrace{1}},\ldots,0\Big].$$

Let $Z_0(1;\{\rho_{i_1},\ldots,\rho_{i_r}\})$ be the cyclic semigroup generated by the element $g_0(1;\{\rho_{i_1},\ldots,\rho_{i_r}\})$. Then

$$g_0^n(1;\{\rho_{i_1},\ldots,\rho_{i_r}\}) = \Big[\ldots,\underbrace{r^{n-1}}_{i_1},\ldots,\underbrace{r^{n-1}}_{i_r},\ldots,0;0,\ldots,0\Big].$$

Let $Z_{j0}(1;\{\rho_{i_1},\ldots,\rho_{i_r}\})$ be the cyclic semigroup generated by the element $g_{j0}(1;\{\rho_{i_1},\ldots,\rho_{i_r}\})$. Then

$$g_{j0}^m(1;\{\rho_{i_1},\ldots,\rho_{i_r}\}) = \Big[0,\ldots,\underbrace{r^{m-1}}_{i_1},\ldots,\underbrace{r^{m-1}}_{i_r},\ldots,\frac{r^m-1}{r-1},\ldots,0;0,\ldots,0\Big].$$

Let $Z_{0k}(1;\{\rho_{i_1},\ldots,\rho_{i_r}\})$ be the cyclic semigroup generated by the element $g_{0k}(1;\{\rho_{i_1},\ldots,\rho_{i_r}\})$. Then

$$g_{0k}^p(1;\{\rho_{i_1},\ldots,\rho_{i_r}\}) = \Big[0,\ldots,\underbrace{r^{n-1}}_{i_1},\ldots,\underbrace{r^{p-1}}_{i_r},\ldots,0;0,\ldots,\underbrace{\frac{r^m-1}{r-1}}_{k},\ldots,0\Big].$$

It is easy to see that

(1) $$g_0^n(1;\{\rho_{i_1},\ldots,\rho_{i_r}\})\{\rho_{i_1},\ldots,\rho_{i_r}\}g_{j0}^m(1;\{\rho_{i_1},\ldots,\rho_{i_r}\}) =$$

$$= \Big[0,\ldots,\underbrace{r^{n+m-1}}_{i_1},\ldots,\underbrace{r^{n+m-1}}_{i_r},\ldots,\underbrace{r^n\frac{r^n-1}{r-1}}_{j},\ldots,0;0,\ldots,0\Big],$$

(2) $$g_{j0}^m(1;\{\rho_{i_1},\ldots,\rho_{i_r}\})\{\rho_{i_1},\ldots,\rho_{i_r}\}g_0^n(1;\{\rho_{i_1},\ldots,\rho_{i_r}\}) =$$

$$= \Big[0,\ldots,\underbrace{r^{n+m-1}}_{i_1},\ldots,\underbrace{r^{n+m-1}}_{i_r},\ldots,\underbrace{\frac{r^m-1}{r-1}}_{j},\ldots,0;0,\ldots,0\Big],$$

(3) $$g_{j0}^m(1;\{\rho_{i_1},\ldots,\rho_{i_r}\})\{\rho_{i_1},\ldots,\rho_{i_r}\}g_{\ell0}^n(1;\{\rho_{i_1},\ldots,\rho_{i_r}\}) =$$

$$= \Big[\ldots,\underbrace{r^{n+m-1}}_{i_1},\ldots,\underbrace{r^{n+m-1}}_{i_r},\ldots,\underbrace{\frac{r^m-1}{r-1}}_{j},\ldots,\underbrace{r^m\frac{r^n-1}{r-1}}_{\ell},\ldots,0;0,\ldots,0\Big].$$

One also gets similar results for the elements $g_{0k}(1;\{\rho_{i_1},\ldots,\rho_{i_r}\})$ and for $\rho_{i_\ell}\longrightarrow-\rho_{i_\ell}$.

THEOREM 5. For different $j_1,\ldots,j_k$ with $(i_1,\ldots,i_r)\cap(j_1,\ldots,j_k)=\emptyset$

$$g_{j_10}^{n_1}(1;\{\rho_{i_1},\ldots,\rho_{i_r}\})\ldots g_{j_k}^{n_k}(1;\{\rho_{i_1},\ldots,\rho_{i_r}\}) =$$

$$=\Big[\ldots,\underbrace{r^{\sum_{i=1}^{k} n_i-1}}_{i_1},\ldots,\underbrace{r^{\sum_{i=1}^{k} n_i-1}}_{i_r},\ldots,\underbrace{\frac{r^{n_1}-1}{r-1}}_{j_1},\ldots,$$

$$\ldots,\underbrace{r^{n_1}\frac{r^{n_2}-1}{r-1}}_{j_2},\ldots,\underbrace{r^{\sum_{i=1}^{k-1} n_i}\frac{r^{n_k}-1}{r-1}}_{j_k},\ldots,0;0,\ldots,0\Big].$$

Proof. By induction using (3). □

Using Theorem 3 the following theorems are obtained.

THEOREM 6. For different $j_1,\ldots,j_k$ with $(i_1,\ldots i_r)\cap(j_1,\ldots,j_k)=\emptyset$ all the elements are different in the product

$$Z_{j_1 0}(1;\{\rho_{i_1}\ldots\rho_{i_r}\})\cdots Z_{j_k 0}(1;\{\rho_{i_1},\ldots,\rho_{i_r}\}).$$

THEOREM 7. *For different $j_1,\ldots,j_k$ with $(i_1,\ldots,i_r)\cap(j_1,\ldots,j_k)=\emptyset$ all the elements are different in the union*

$$\bigcup_{t_1\ldots t_k\in\Pi} Z_{t_1 0}(1;\{\rho_{i_1},\ldots,\rho_{i_r}\})\{\rho_{i_1},\ldots,\rho_{i_r}\}\cdots Z_{t_k 0}(1;\{\rho_{i_1},\ldots,\rho_{i_k}\}), \tag{4}$$

where $(t_1\ldots t_k)$ runs through all the permutations of $(j_1\ldots j_k)$.

REMARK 1. Theorems 4 and 5 give an opportunity to produce a process for the problem of inversion (decoding problem), that is to produce uniquely the original components (generators) and the ordering of the components from the result of a product (4).

REMARK 2. It is easy to see that several analogous results (to Theorem 5) can be derived by changing the generators g_0, g_{j0}, g_{0k} and μ.

There are easy to see the following theorems:

THEOREM 8. *Let us consider the elements s with $\alpha_{i_j}\geq 0$ for a j and $\alpha_{i_k}=0$ for $k\neq j$. The set $S_{i_j}(\{\rho_{i_1},\ldots,\rho_{i_r}\})$ of these elements is a semigroup which is a right-ideal in the semigroup $S(\{\rho_{i_1},\ldots,\rho_{i_r}\})$.*

THEOREM 9. *For the semigroup $S(\{\rho_{i_1},\ldots,\rho_{i_r}\})$ the decomposition*

$$S(\{\rho_{i_1},\ldots,\rho_{i_r}\})=\bigcup_{j=1}^{r} S_{i_j}(\{\rho_{i_1},\ldots,\rho_{i_r}\}),$$

$$S_{i_k}(\{\rho_{i_1},\ldots,\rho_{i_r}\})\cap S_{i_l}(\{\rho_{i_1},\ldots,\rho_{i_r}\})=[0,\ldots;0,\ldots],\qquad k\neq l$$

is valid.

5.3. Semigroups with modified vectors

Let us consider the set of the vectors

(5) $$s = [\alpha_1, \dots, \alpha_n]$$

where the components α_i, $(i = 1, \dots, n)$ can be negative integers, too.

The introduction of the operation $\{\rho_{i_1}, \dots, \rho_{i_r}\}$ in this set can be done similarly as in Chapter 5.1 (resulting the associativity), because in Theorem 1 the nonnegativity of the components was not used. Therefore the elements (5) form a semigroup $S'(\{\rho_{i_1}, \dots, \rho_{i_r})$.

Let $r > 1$. In this case $\sum_{j=1}^{r}$ can be 0 or 1 without being the components α_{i_j}, $(j = 1, \dots, r)$ 0 or 1.

It is easy to see that the semigroup $S'(\{\rho_{i_1}, \dots, \rho_{i_r}\}$ has not (two sided) zero element and unit element. The elements s with $\sum_{j=1}^{r} \alpha_{\rho_j} = 1$, $\alpha_i = 0$ $(i \neq \rho_j,\ j = 1, \dots, r)$ are left unit elements. The elements s with $\sum_{j=1}^{r} \alpha_{\rho_j} = 0$, $\alpha_i = 0$, $(i \neq \rho_j,\ j = 1, \dots, r)$ are left zero elements.

Remark 3. Every semigroup has disjoint decompositions Λ_i and P_i $i = 0, \dots, 5$ (see [17]).

It is not difficult to see that in $S'(\{\rho_{i_1}, \dots, \rho_{i_r}\})$

$$\Lambda_0 = \emptyset, \quad \Lambda_1 = \emptyset, \quad \Lambda_2 = \emptyset, \quad \Lambda_3 = \emptyset, \quad \Lambda_4 \neq \emptyset, \quad \Lambda_5 \neq \emptyset$$

$$P_0 = \emptyset, \quad P_1 = \emptyset, \quad P_2 \neq \emptyset, \quad P_3 \neq \emptyset, \quad P_4 \neq \emptyset, \quad P_5 = \emptyset.$$

are valid.

Remark 4. Every semigroup S has a disjoint decomposition with components Λ_i, $i = 1, \dots, 5$ and a dual right decomposition with components P_i, $i = 1, \dots, 5$ ([17]). In the case of $S(\{\rho_{i_1}, \dots, \rho_{i_r}\})$, $r > 1$ for the components the following properties are true:

$$\Lambda_0 = \emptyset, \quad \Lambda_1 = \emptyset, \quad \Lambda_2 = \emptyset, \quad \Lambda_3 = \emptyset, \quad \Lambda_4 \neq \emptyset, \quad \Lambda_5 \neq \emptyset$$

$$P_0 = \emptyset, \quad P_1 = \emptyset, \quad P_2 \neq \emptyset, \quad P_3 \neq \emptyset, \quad P_4 = \emptyset, \quad P_5 = \emptyset.$$

6. Finite structures

6.1. Finite semigroups

Let us consider the semigroup $S(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$. This semigroup, introduced in Chapter 5, is an infinite semigroup in which every element has infinite order (except for the identity if it exists).

We show how we can produce finite semigroups from a semigroup $S(\{\rho_{i_1}, \ldots \ldots, \rho_{i_r}\})$ derived.

Consider the elements

$$s = [\alpha_1, \ldots, \alpha_\mu; \beta_1, \ldots, \beta_\mu]$$

of the semigroups $S(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$. To every position i $(i = 1, \ldots, \mu)$ we associate an (positive) integer $n_i > 1$ for α and $m_i > 0$ for β (to position i_j for ρ_{i_j} belongs n_{i_j}) and consider the components α_i, β_i $\mathrm{mod}(n_i), \mathrm{mod}(m_i)$ resp. that is, instead α_i, β_i (and instead classes) we write $\alpha_i + k_i n_i$, $\beta_i + l_i m_i$ with $0 \leq \alpha_i < n_i$, $0 \leq \beta_i < m_i$.

$$s = [\alpha_1 + k_1 n_1, \ldots, \alpha_\mu + k_\mu n_\mu; \beta_1 + l_1 m_1, \ldots, \beta_\mu + l_\mu m_\mu].$$

DEFINITION 1. Two elements s, s' are considered to be equivalent if and only if

$$\alpha_i = \alpha_i', \quad \beta_i = \beta_i'. \quad (i = 1, \ldots, \mu)$$

This relation is an *equivalence* $E(n_1, \ldots, n_\mu; m_1, \ldots, m_\mu) = E(\mathbf{n}; \mathbf{m})$.

For the sake of simplicity we assume, that ρ_j, $(j = 1, \ldots, r)$ are positive (that is they belong to α).

THEOREM 1. *The equivalence relation* $E(\mathbf{n}; \mathbf{m})$ *is a congruence relation* $C(\mathbf{n}; \mathbf{m})$ *if and only if*

1) $n_i, m_q \,|\, n_{i_j}$ $(j = 1, \ldots, r;\ i \notin \{i_1, \ldots, i_r\};\ q = 1, \ldots, \mu)$
2) $n_{i_j} = n_{i_k} = n$ $(j, k = 1, \ldots, r)$.

Proof. Let $sE(\mathbf{n},\mathbf{m})s'$ and $s'' \in S(\rho_{i_1},\ldots,\rho_{i_r})$ be arbitrary elements

$$s=[\ldots,\alpha_i+k_in_i,\ldots,\alpha_{i_j}+k_{i_j}n_{i_j},\ldots;\ldots,\beta_q+l_qm_q,\ldots],$$
$$s'=[\ldots,\alpha_i+k'_in_i,\ldots,\alpha_{i_j}+k'_{i_j}n_{i_j},\ldots;\ldots,\beta_q+l'_qm_q,\ldots],$$
$$s''=[\ldots,\alpha''_i+k''_in_i,\ldots,\alpha''_{i_j}+k''_{i_j}n_{i_j},\ldots;\ldots,\beta''_q+l''_qm_q,\ldots].$$

1. (*Necessity*).

$$s\{\rho_{i_1},\ldots,\rho_{i_r}\}s''=$$
$$=\Big[\ldots,\alpha_i+k_in_i+(\alpha''_i+k''_in_i)\sum_{j=1}^{r}(\alpha_{i_j}+k_{i_j}n_{i_j}),\ldots$$
$$\ldots,(\alpha''_{i_k}+k''_{i_k}n_{i_k})\sum_{j=1}^{r}(\alpha_{i_j}+k_{i_j}n_{i_j}),\ldots;$$
$$\ldots,\beta_q+l_qm_q+(\beta''_q+l''_qm_q)\sum_{j=1}^{r}(\alpha_{i_j}+k_{i_j}n_{i_j}),\ldots\Big],$$

similarly

$$s'\{\rho_{i_1},\ldots,\rho_{i_r}\}s''=$$
$$=\Big[\ldots,\alpha_i+k'_in_i+(\alpha''_i+k''_in_i)\sum_{j=1}^{r}(\alpha_{i_j}+k'_{i_j}n_{i_j}),\ldots$$
$$\ldots,(\alpha''_{i_k}+k''_{i_k}n_{i_k})\sum_{j=1}^{r}(\alpha_{i_j}+k'_{i_j}n_{i_j}),\ldots;$$
$$\ldots,\beta_q+l'_qm_q+(\beta''_q+l''_qm_q)\sum_{j=1}^{r}(\alpha_{i_j}+k'_{i_j}n_{i_j}\Big].$$

One sees that (right congruence property)

$$s\{\rho_{i_1},\ldots,\rho_{i_r}\}s''E(\mathbf{n};\mathbf{m})s'\{\rho_{i_1},\ldots,\rho_{i_r}\}s''$$

if and only if

$$\alpha''_i\sum_{j=1}^{r}n_{i_j}(k'_{i_j}-k_{i_j})\equiv 0 \qquad \operatorname{mod}(n_i)$$
$$\beta''_q\sum_{j=1}^{r}n_{i_j}(k'_{i_j}-k_{i_j})\equiv 0 \qquad \operatorname{mod}(m_q)$$

for every $0\le k_{i_j},k'_{i_j},l_q,l'_q$; $0\le\alpha''_i<n_i$, $0\le\beta_q,\beta''_q<m_q$, that is, $n_i,m_q\,|\,n_{i_j}$, $(j=1,\ldots,r)$.

Besides

$$(\alpha''_{i_k}+k''_{i_k}n_{i_k})\sum_{j=1}^{r}(\alpha_{i_j}+k_{i_j}n_{i_j})\equiv(\alpha''_{i_k}+k''_{i_k}n_{i_k})\sum_{j=1}^{r}(\alpha_{i_j}+k'_{i_j}n_{i_j}) \quad \operatorname{mod}(n_{i_k})$$

from where

$$\alpha''_{i_k} \sum_{j=1}^{r} (k_{i_j} - k'_{i_j}) n_{i_j} \equiv 0; \qquad \mathrm{mod}(n_{i_k})$$

for $0 \le k_{i_j}, k'_{i_j}$; $0 \le \alpha''_{i_k} < n_{i_k}$ and $n_{i_k} \,|\, n_{i_j}$ $(j, k = 1, \ldots, r)$ follows, that is $n_{i_k} = n_{i_j}$.

After what has been said, one gets easily that the (left) congruence property

$$s''\{\rho_{i_1}, \ldots, \rho_{i_r}\} s E(\mathbf{n}; \mathbf{m}) s''\{\rho_{i_1}, \ldots, \rho_{i_r}\} s'$$

is true.

2. (*Sufficiency*). It follows immediately from 1). □

THEOREM 2. *If*
1) $n_{i_j} = n_{i_k} = n$ $(j, k = 1, \ldots, r)$ *and*
2) $n_i, m_q \,|\, n$ $(i, q = 1, \ldots, \mu)$, *then*

$$S(\{\rho_{i_1}, \ldots, \rho_{i_r}\})/C(\mathbf{n}; \mathbf{m}) = S(\{\rho_{i_1}, \ldots, \rho_{i_r}\})(\mathbf{n}; \mathbf{m})$$

is a finite semigroup.

Proof. It is a simple consequence of Theorem 1. □

For shortness let

$$S(\{\rho_{i_1}, \ldots, \rho_{i_r}\})(n_1, \ldots, n_\mu; m_1, \ldots, m_\mu) = (\{\rho_{i_1}, \ldots, \rho_{i_r}\})(\mathbf{n}; \mathbf{m})$$

be the above introduced finite maximal semigroup with given $\mathbf{n}$ and $\mathbf{m}$.

It is clear that the order $|S|$ of the semigroup $S(\{\rho_{i_1}, \ldots, \rho_{i_r}\})(\mathbf{n}; \mathbf{m})$ is

$$|S| = n^r \prod_{\substack{i=1 \\ i \ne i_j}}^{\mu} n_i \prod_{k=1}^{\mu} m_k$$

Let $P(\mathbf{n}, \mathbf{m})$ be a P permutation of the components of $\mathbf{n}$ and $\mathbf{m}$, resp., and $P(i)$ the component of $\mathbf{n}$, $\mathbf{m}$, resp., in which the component i is transformed by $P(\mathbf{n})$, $P(\mathbf{m})$, resp. Then it is easy to see the following theorem of isomorphism:

THEOREM 3.

$$S(\{\rho_{P(i_1)}, \ldots, \rho_{P(i_r)}\})(P(\mathbf{n}); P(\mathbf{m})) \approx S(\{\rho_{i_1}, \ldots, \rho_{i_r}\})(\mathbf{n}; \mathbf{m}).$$

Proof. The mapping $s \to s'$ with $(\mathbf{n}; \mathbf{m}) \to P((\mathbf{n}; \mathbf{m}))$ is an isomorphic mapping. □

6.2. The idempotent and identity elements

THEOREM 4. *The element*

$$s = [\alpha_1 + k_1 n_1, \dots, \alpha_\mu + k_\mu n_\mu; \dots, \beta_q + l_q m_q, \dots] \in S(\{\rho_{i_1}, \dots, \rho_{i_r}\})(\mathbf{n}; \mathbf{m})$$

is idempotent if and only if

1) $$\alpha_i \sum_{j=1}^{r} \alpha_{i_j} \equiv 0 \quad \mod(n_i), \quad i \neq i_j \ (j = 1, \dots, r)$$

2) $$\beta_q \sum_{j=1}^{r} \alpha_{i_j} \equiv 0 \quad \mod(m_q), \quad (q = 1, \dots, \mu)$$

3) $$\alpha_{i_k} \Big(\sum_{j=1}^{r} \alpha_{i_j} - 1 \Big) \equiv 0. \quad \mod(n). \quad (k = 1, \dots, r)$$

Proof. It is easily seen (for $E(\mathbf{n}; \mathbf{m})$)

$$\begin{aligned} s^2 = \Big[&\dots, \alpha_i + k_i n_i + (\alpha_i + k_i n_i) \sum_{j=1}^{r} (\alpha_{i_j} + k_{i_j} n), \dots, \\ &\dots, (\alpha_{i_k} + k_{i_k} n) \sum_{j=1}^{r} (\alpha_{i_j} + k_{i_j} n), \dots; \\ &\dots, \beta_q + l_q m_q + (\beta_q + l_q m_q) \sum_{j=1}^{r} (\alpha_{i_j} + k_{i_j} n), \dots \Big] = \end{aligned}$$

$$= s = [\dots, \alpha_i + k_i n_i, \dots, \alpha_{i_k} + k_{i_k} n, \dots; \dots, \beta_q + l_q m_q, \dots]$$

if and only if 1), 2), and 3) hold. □

THEOREM 5. *Let*

$$s \in S(\{\rho_{i_1}, \dots, \rho_{i_r}\})(\mathbf{n}; \mathbf{m}).$$

a) *The element s is a left identity in the semigroup* $S(\{\rho_{i_1}, \dots, \rho_{i_r})(\mathbf{n}; \mathbf{m})$ *if and only if*
   1) $\alpha_i \equiv 0 \mod(n_i)$, $(i \neq i_j; j = 1, \dots, r)$,
   2) $\beta_q \equiv 0 \mod(m_q)$, $(q = 1, \dots, \mu)$,
   3) $\sum_{j=1}^{r} \alpha_{i_j} \equiv 1 \mod(n)$.

b) *The element s is a right identity in the semigroup* $S(\{\rho_{i_1}, \dots, \rho_{i_r}\})(\mathbf{n}; \mathbf{m})$ *if and only if*
   1) $\alpha_i = 0$, $(i \neq i_j; j = 1, \dots, r)$,
   2) $\beta_q = 0$, $(q = 1, \dots, \mu)$,

3) $\alpha_{i_k} = 1$, $(k = 1, \ldots, r)$,

4) $r = 1$.

c) *The element s is the identity in the semigroup* $S(\{\rho_{i_1}, \ldots, \rho_{i_r})(\mathbf{n}; \mathbf{m})$ *if and only if s is a right identity.*

Proof. Let

$$s = [\ldots, \alpha_i + k_i n_i, \ldots, \alpha_{i_k} + k_{i_k} n, \ldots; \ldots, \beta_q + l_q m_q, \ldots]$$

and

$$s' = [\ldots, \alpha'_i + k'_i n_i, \ldots, \alpha'_{i_k} + k'_{i_k} n, \ldots; \ldots, \beta'_q + l'_q m_q, \ldots]$$

an arbitrary element.

Then

$$s\{\rho_{i_1}, \ldots, \rho_{i_r}\}s' =$$

$$= \Big[\ldots, \alpha_i + k_i n_i + (\alpha'_i + k'_i n_i) \sum_{j=1}^{r} (\alpha_{i_j} + k_{i_j} n), \ldots$$

$$\ldots, (\alpha'_{i_k} + k'_{i_k} n) \sum_{j=1}^{r} (\alpha_{i_j} + k_{i_j} n), \ldots;$$

$$\ldots, (\beta'_q + l'_q m_q) \sum_{j=1}^{r} (\alpha_{i_j} + k_{i_j} n), \ldots \Big],$$

$$s'\{\rho_{i_1}, \ldots, \rho_{i_r}\}s =$$

$$= \Big[\ldots, \alpha'_i + k'_i n_i + (\alpha_i + k_i n_i) \sum_{j=1}^{r} (\alpha'_{i_j} + k'_{i_j} n), \ldots$$

$$\ldots, (\alpha_{i_k} + k_{i_k} n) \sum_{j=1}^{r} (\alpha'_{i_j} + k'_{i_j} n), \ldots; \ldots \Big].$$

For a)

$$\alpha_i + k_i n_i + (\alpha'_i + k'_i n_i) \sum_{j=1}^{r} (\alpha_{i_j} + k_{i_j} n) \equiv$$

$$\equiv \alpha'_i + k'_i n_i \qquad \mathrm{mod}(n_i), \qquad (i \neq i_j),$$

$$\beta_q + l_q m_q + (\beta'_q + l'_q m_q) \sum_{j=1}^{r} (\alpha_{i_j} + k_{i_j} n) \equiv$$

$$\equiv \beta'_q + l'_q m_q \qquad \mathrm{mod}(m_q), \qquad (q = 1, \ldots, \mu)$$

$$(\alpha'_{i_k} + k'_{i_k} n) \sum_{j=1}^{r} (\alpha_{i_j} + k_{i_j} n) \equiv \alpha'_{i_k} + k'_{i_k} n, \qquad \mathrm{mod}(n)$$

from where

$$\alpha_i+\alpha_i'(\sum_{j=1}^{r}\alpha_{i_j}-1)\equiv 0 \mod(n_i),\qquad \alpha_{i_k}'(\sum_{j=1}^{r}\alpha_{i_j}-1)\equiv 0 \mod(n)$$

$$\beta_q+\beta_q'(\sum_{j=1}^{r}\alpha_{i_j}-1)\equiv 0 \mod(m_q),\qquad (0\le\beta_q,\beta_q'<m_q;\ q=1,\dots,\mu)$$

$$(0\le\alpha_i'<n_i,\qquad i\ne i_j;\qquad 0\le\alpha_{i_k}<n)$$

and — as consequence — the statement a) follows.

For b)

$$\alpha_i'+k_i'n_i+(\alpha_i+k_in_i)\sum_{j=1}^{r}(\alpha_{i_j}'+k_{i_j}'n)\equiv$$

$$\equiv\alpha_i'+k_i'n_i \mod(n_i),\qquad (i\ne i_j),$$

$$\beta_q'+l_qm_q+(\beta_q+l_qm_q)\sum_{j=1}^{r}(\alpha_{i_j}'+k_{i_j}'n)\equiv$$

$$\equiv\beta_q'+l_q'm_q \mod(m_q),\qquad (q=1,\dots,\mu),$$

$$(\alpha_{i_k}+k_{i_k}n)\sum_{j=1}^{r}(\alpha_{i_j}'+k_{i_j}'n)\equiv\alpha_{i_k}'+k_{i_k}'n \mod(n)$$

from where

$$\alpha_i\sum_{j=1}^{r}\alpha_{i_j}'\equiv 0 \mod(n_i),\qquad \alpha_{i_k}\sum_{j=1}^{r}\alpha_{i_j}'\equiv\alpha_{i_k}' \mod(n),$$

$$(0\le\alpha_i'<n_i,\qquad 0\le\alpha_{i_j}'<n),$$

$$\beta_q\sum_{j=1}^{r}\alpha_{i_j}'\equiv 0 \mod(m_q),\qquad (q=1,\dots,\mu).$$

Let $\alpha_{i_k}'=1$, $\alpha_{i_j}'=0$ $(j\ne k)$, then $\alpha_{i_k}=1$ $(k=1,\dots,r)$ follows. If $\alpha_{i_k}=1$ $(k=1,\dots,r)$, then $\alpha_{i_k}'=\sum_{j=1}^{r}\alpha_{i_j}'$ $(k=1,\dots,r)$ and

$$(r-1)\sum_{j=1}^{r}\alpha_{i_j}'\equiv 0 \mod(n)$$

follows, that is $n\,|\,r-1$. Besides, for $r\ne 1$, $\alpha_{i_k}=0$ $(k=1,\dots,r)$. So the statement b) holds.

Besides it follows that, if s is a right identity, then it is the identity element, too.

The sufficiency of a), b), and c) is clear. □

Consequence 1. If $n>1$ then the semigroup $S(\{\rho_{i_1},\dots,\rho_{i_r}\})(\mathbf{n};\mathbf{m})$ has left identity.

6.3. The trace of the element s

DEFINITION 2. Let $s = [\alpha_1, \ldots, \alpha_\mu; \beta_1, \ldots, \beta_\mu]$. For the operation $\{\rho_{i_1}, \ldots, \rho_{i_r}\}$ (with positive ρ) the *trace* $\operatorname{tr}(s)$ of the element s is $\sum_{j=1}^{r} \alpha_{i_j}$, $\operatorname{mod}(n)$.

The following theorem is straightforward.

THEOREM 6. *For two elements $s, s' \in S(\{\rho_{i_1}, \ldots, \rho_{i_r}\})(\mathbf{n}; \mathbf{m})$*

$$\operatorname{tr}(s\{\rho_{i_1}, \ldots, \rho_{i_r}\}s') = \operatorname{tr}(s)\operatorname{tr}(s').$$

COROLLARY. *If* $(\operatorname{tr}(s), n) = (\operatorname{tr}(s'), n) = 1$, *then* $(\operatorname{tr}(ss'), n) = 1$.

From this corollary follows that the elements s with $(\operatorname{tr}(s), n) = 1$ form a semigroup in $S(\{\rho_{i_1}, \ldots, \rho_{i_r}\})(\mathbf{n}, \mathbf{m})$. Let $S_P(\{\rho_{i_1}, \ldots, \rho_{i_r}\})(\mathbf{n}; \mathbf{m})$ be this semigroup. Let $S_{P_0}(\{\rho_{i_1}, \ldots, \rho_{i_r}\})(\mathbf{n}, \mathbf{m})$ be the semigroup of the left identities ($\operatorname{tr}(s) \equiv 1 \operatorname{mod}(n)$). This semigroup ($S_{P_0}$) is a right zero semigroup.

THEOREM 7. *In the semigroup $S_P(\{\rho_{i_1}, \ldots, \rho_{i_r}\})(\mathbf{n}, \mathbf{m})$ every element s has a unique right inverse s' for a given left identity e_l.*

Proof. Let $e_l = [0, \ldots, \alpha^0_{i_1}, \ldots, \alpha^0_{i_r}, 0, \ldots; 0, \ldots, 0]$ ($\sum_{j=1}^{r} \alpha^0_{i_j} \equiv 1 \operatorname{mod}(n)$) be the given left identity. Then $ss' = e_l$ if and only if

$$\alpha_i + \alpha'_i \sum_{j=1}^{r} \alpha_{i_j} \equiv 0 \qquad \operatorname{mod}(n_i), \qquad (i \neq i_j),$$

$$\beta_q + \beta'_q \sum_{j=1}^{r} \alpha_{i_j} \equiv 0 \qquad \operatorname{mod}(m_q), \qquad (q = 1, \ldots, \mu),$$

$$\alpha'_{i_k} \sum_{j=1}^{r} \alpha_{i_j} \equiv \alpha^0_{i_k} \qquad \operatorname{mod}(n).$$

From this follows that for given s and e_l the element s' is determined uniquely. □

CONSEQUENCE 2. The semigroup $S_P(\{\rho_{i_1}, \ldots, \rho_{i_r}\})(\mathbf{n}; \mathbf{m})$ is a regular semigroup (moreover a right group), because from $ss' = e_l$ the equality $ss's = s$ follows $\forall s, e_l \in S_P(\{\rho_{i_1}, \ldots, \rho_{i_r}\})(\mathbf{n}; \mathbf{m})$ and s' is determined uniquely.

THEOREM 8. *Let $r \equiv 1 \operatorname{mod}(n)$. Let $S_{P_1}(\{\rho_{i_1}, \ldots, \rho_{i_r}\})(\mathbf{n}; \mathbf{m})$ be the semigroup of the elements $s \in S_P(\{\rho_{i_1}, \ldots, \rho_{i_r}\})(\mathbf{n}; \mathbf{m})$, where $\alpha_{i_1} = \cdots = \alpha_{i_r} = 1$. Let $S_{P_{01}}$ the semigroup of the elements $s \in S_P(\{\rho_{i_1}, \ldots, \rho_{i_r}\})(\mathbf{n}; \mathbf{m})$, where $\alpha_i = 0$, $(i \neq i_j;\ j = 1, \ldots, r)$; $\beta_q = 0$, $(q = 1, \ldots, \mu)$.*

Then

$$S_P = S_{P_1} S_{P_{01}}, \qquad S_{P_1} \cap S_{P_{01}} = e_{01}$$

where in e_{01} $\alpha_i = 0$, $i \neq i_j$; $\alpha_{i_j} = 1$ $(j = 1, \dots, r)$; $\beta_q = 0$ $(q = 1, \dots, \mu)$. S_{P_1} *is a commutative group,* $S_{P_{01}}$ *is a semigroup in which the elements* s *for which* $\mathrm{tr}(s) \equiv 1 \bmod(n)$ *are left identities.*

Proof. The statements of the theorem are simple consequences of Theorems 4, 5 and 6. □

THEOREM 9. *If* $r = 1$, *then the semigroup* $S_P(\rho_{i_1}, \dots, \rho_{i_r})(\mathbf{n}, \mathbf{m})$ *is a group* $G = AP$, $A \cap P = e$ *with* $A \lhd G$, *where* A *is an abelian group and* P *is a cyclic group.*

Proof . In the case $r = 1$ the semigroup S_P has only one left identity e which is the identity element in S_P. Moreover using Theorem 4, in S_P every element has a unique right inverse. It is easy to see that there is only one right inverse, which coincides with the left inverse. To see that $G = AP$ see Theorems 3 and 4 in Chapter 3. □

Let $S_1(\{\rho_{i_1}, \dots, \rho_{i_r}\})(\mathbf{n}; \mathbf{m})$ (briefly $S_1(\mathbf{n}; \mathbf{m})$) be the set of the elements s (with positive ρ), in which $\mathrm{tr}(s) \equiv 1 \bmod(n)$. It is clear that this set is a semigroup.

Let $S_T(\{\rho_{i_1}, \dots, \rho_{i_r}\})(\mathbf{n}; \mathbf{m})$ (briefly $S_T(\mathbf{n}; \mathbf{m})$) be the set of the elements s in which $\alpha_k = 0$, $k \neq i_j$, $j = 1, \dots, r$; $\beta_l = 0$, $l = 1, \dots, \mu$. It is clear that S_T is a semigroup.

Similarly one gets easily the following

THEOREM 10. *The semigroup* $S(\{\rho_{i_1}, \dots, \rho_{i_r}\})(\mathbf{n}; \mathbf{m})$ *has the factorisation*

$$S(\{\rho_{i_1}, \dots, \rho_{i_r}\})(\mathbf{n}; \mathbf{m}) = S_1(\mathbf{n}; \mathbf{m})\{\rho_{i_1}, \dots, \rho_{i_r}\} S_T(\mathbf{n}; \mathbf{m}),$$

where

$$S_1(\mathbf{n}; \mathbf{m}) \cap S_T(\mathbf{n}; \mathbf{m}) = S_{P_0}(\mathbf{n}; \mathbf{m}).$$

It is not difficult to see the following theorems, too:

THEOREM 11. *Let* S_0 *be the set of the elements* s *in the semigroup* $S = S(\{\rho_{i_1}, \dots, \rho_{i_r}\}(\mathbf{n}; \mathbf{m})$ *with* $n \mid \mathrm{tr}(s)$, *that is,* $\mathrm{tr}(s) \equiv 0 \bmod(n)$. *Then* S_0 *is a right ideal in* S.

THEOREM 12. *Let* S_2 *be the set of the elements* s *in the semigroup* $S = S(\{\rho_{i_1}, \dots, \rho_{i_r}\})(\mathbf{n}; \mathbf{m})$ *with* $(\mathrm{tr}(s), n) > 1$. *Then* S_2 *is a right ideal in* S *and contains the subsemigroup* S_0.

THEOREM 13. *For every semigroup $S = S(\{\rho_{i_1}, \dots, \rho_{i_r}\})$ the decomposition*

$$S = S_P \cup S_2, \qquad S_P \cap S_2 = \emptyset$$

is true.

REMARK. The semigroup S_1 is not regular in general (see example 4).

Examples

1. Consider the semigroup $S(\{\rho_{i_1}, \dots, \rho_{i_r}\})(\mathbf{n}; \mathbf{m})$ with the elements $s = [\alpha_1, \alpha_2, \alpha_3; 0, 0, 0]$ where $n_1 = 2$, $n_2 = n_3 = n = 6$, $m_1 = m_2 = m_3 = 1$ and $r = 2$, $\rho_1 = 2$, $\rho_2 = 3$.

For the element α_1 there are 2 possibilities (0,1), for the pair (α_2, α_3) there are 36 possibilities. Therefore $|S| = 72$.

This semigroup is too large to give the table of Cayley.

2. Consider the subsemigroup of this semigroup with $(\text{tr}(s), 6) = 1$. In this case for α_2, α_3 the possibilities are

α_2	α_3
0	1
1	0
0	5
5	0
1	4
4	1
2	3
3	2
3	4
4	3
2	5
5	2

The elements s are the following

$s_1 = [0, 0, 1; 0, 0, 0]$
$s_2 = [0, 1, 0; 0, 0, 0]$
$s_3 = [0, 0, 5; 0, 0, 0]$
$s_4 = [0, 5, 0; 0, 0, 0]$
$s_5 = [0, 1, 4; 0, 0, 0]$
$s_6 = [0, 4, 1; 0, 0, 0]$
$s_7 = [0, 2, 3; 0, 0, 0]$
$s_8 = [0, 3, 2; 0, 0, 0]$
$s_9 = [0, 3, 4; 0, 0, 0]$
$s_{10} = [0, 4, 3; 0, 0, 0]$
$s_{11} = [0, 2, 5; 0, 0, 0]$
$s_{12} = [0, 5, 2; 0, 0, 0]$
$s_{13} = [1, 0, 1; 0, 0, 0]$
$s_{14} = [1, 1, 0; 0, 0, 0]$
$s_{15} = [1, 0, 5; 0, 0, 0]$
$s_{16} = [1, 5, 0; 0, 0, 0]$
$s_{17} = [1, 1, 4; 0, 0, 0]$
$s_{18} = [1, 4, 1; 0, 0, 0]$

$$s_{19} = [1,2,3;0,0,0] \qquad s_{22} = [1,4,3;0,0,0]$$
$$s_{20} = [1,3,2;0,0,0] \qquad s_{23} = [1,2,5;0,0,0]$$
$$s_{21} = [1,3,4;0,0,0] \qquad s_{24} = [1,5,2;0,0,0]$$

The semigroup described above is a right group with order 24. The elements s_1, s_2, s_9, s_{10}, s_{11}, s_{12} are left identities.

3. Let us consider now the following elements of the semigroup introduced in 1.

$$s_0 = [0,0,0;0,0,0] \qquad s_5 = [0,4,0;0,0,0]$$
$$s_1 = [0,0,2;0,0,0] \qquad s_6 = [0,2,4;0,0,0]$$
$$s_2 = [0,2,0;0,0,0] \qquad s_7 = [0,4,2;0,0,0]$$
$$s_3 = [0,2,2;0,0,0] \qquad s_8 = [0,4,4;0,0,0]$$
$$s_4 = [0,0,4;0,0,0]$$

These elements form a semigroup (of order 9) with the following product table with $s_i \longrightarrow i$

	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	4	5	8	1	2	7	6	3
2	0	4	5	8	1	2	7	6	3
3	0	1	2	3	4	5	6	7	8
4	0	1	2	3	4	5	6	7	8
5	0	1	2	3	4	5	6	7	8
6	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0
8	0	4	5	8	1	2	7	6	3

4. Consider now the elements $s = [\alpha_1, \alpha_2, \alpha_3; 0,0,0]$ of the semigroup S where $n_1 = 3$, $n_2 = n_3 = 6$, $m_1 = m_2 = m_3 = 1$ and $r = 2$, $\rho_1 = 2$, $\rho_2 = 3$ with $\mathrm{tr}(s) = 3$.

In this case the elements of the semigroup are $s_1 = [0,0,3;0,0,0]$, $s_2 = [0,1,2;0,0,0]$, $s_3 = [0,2,1;0,0,0]$, $s_4 = [0,3,0;0,0,0]$, $s_5 = [3,0,3;0,0,0]$, $s_6 = [3,1,2;0,0,0]$, $s_7 = [3,2,1;0,0,0]$, $s_8 = [3,3,0;0,0,0]$.

In this semigroup

$$s_is_1 = s_1, \quad s_is_2 = s_4, \quad s_is_3 = s_1, \quad s_is_4 = s_4, \quad s_is_5 = s_5,$$

$$s_is_6 = s_8, \quad s_is_7 = s_5, \quad s_is_8 = s_8, \qquad (i = 1, \ldots, 8).$$

This semigroup is not regular, because $s_2s_is_2 \neq s_2$, $(i = 1, \ldots, 8)$.

7. Coded structures

7.1. The code and the kernel of the semigroup $S(\{\rho_{i_1},\dots,\rho_{i_r}\})$

For the sake of simplicity we suppose that the operations ρ_i are positive. Let $H(\{\rho_{i_1},\dots,\rho_{i_r}\}) \subseteq S(\{\rho_{i_1},\dots,\rho_{i_r}\})$ be an arbitrary subsemigroup.

Let $s = [\alpha_1,\dots,\alpha_{i_1},\dots,\alpha_{i_r},\dots,\alpha_\mu;\beta_1,\dots,\beta_\mu] \in H(\{\rho_{i_1},\dots,\rho_{i_r}\})$.

DEFINITION 1. For the operation $\{\rho_{i_1},\dots,\rho_{i_r}\}$ the *code* of the element s is the vector

$$c(s) = [0,\dots,\alpha_{i_1},\dots,\alpha_{i_r},\dots,0;0,\dots,0] \tag{1}$$

that is $\alpha_i = 0$ for $i \neq i_j$ $(j = 1,\dots,r)$, $\beta_k = 0$ $(k = 1,\dots,\mu)$.

The vectors $c(s)$ are elements of $S(\{\rho_{i_1},\dots,\rho_{i_r}\})$.

For $c(s)$ and $c(s')$

$$\begin{aligned} c(s)\{\rho_{i_1},\dots,\rho_{i_r}\}c(s') &= \\ &= [0,\dots,\alpha'_{i_1}\|c(s)\|,\dots,\alpha'_{i_r}\|c(s)\|,\dots,0;0,\dots,0] = \\ &= \|c(s)\|[0,\dots,\alpha'_{i_1},\dots,\alpha'_{i_r},\dots,0;0,\dots,0]. \end{aligned} \tag{2}$$

For $\|\cdot\|$ see Section 1.1.

REMARK 1. It is evident that $\|c(s)\| = \mathrm{tr}(s)$ (see Section 6.3).

This means that the codes of the elements s form a semigroup for the operation $\{\rho_{i_1},\dots,\rho_{i_r}\}$ and

$$c(s)\{\rho_{i_1},\dots,\rho_{i_r}\}c(s') = \|c(s)\|c(s') = c(s\{\rho_{i_1},\dots,\rho_{i_r}\}s'). \tag{3}$$

DEFINITION 2. The subsemigroup $K(S)$ of the codes $c(s)$ in $S(\{\rho_{i_1},\dots$ $\dots,\rho_{i_r}\})$ is the *kernel* of $S(\{\rho_{i_1},\dots,\rho_{i_r}\})$.

To every subsemigroup H of $S(\{\rho_{i_1}, \dots, \rho_{i_r}\})$ belongs a $K(H)$ but H does not necessarily contain the semigroup $K(H)$.

DEFINITION 3. Two elements $c(s)$, $c(s') \in K(S)$ are *similar* (in sign: $c(s) \sim c(s')$) if there are two integers ($\in \mathbf{N}$) p, p' such that $pc(s) = p'c(s')$.

THEOREM 1. *For similar elements the following properties hold:*
1) $c(s) \sim c(s)$ *(reflexivity).*
2) *If* $c(s) \sim c(s')$ *then* $c(s') \sim c(s)$ *(symmetricity).*
3) *If* $c(s) \sim c(s')$ *and* $c(s') \sim c(s'')$ *then* $c(s) \sim c(s'')$ *(transitivity).*
4) *If* $c(s) \not\sim c(s')$, $c(s) \not\sim c(s'')$ *then* $c(s) \not\sim c(s')\{\rho_{i_1}, \dots, \rho_{i_r}\}c(s'')$.
5) *If* $c(s) \not\sim c(s')$, $c(s) \sim c(s'')$ *then* $c(s) \sim c(s')\{\rho_{i_1}, \dots, \rho_{i_r}\}c(s'')$, $c(s) \not\sim c(s'')\{\rho_{i_1}, \dots, \rho_{i_r}\}c(s')$.

Proof. The properties 1), 2), and 3) are evident. The properties 4) and 5) follow from (2). □

COROLLARY. *A consequence of property* 5) *is*

$$c(s) \sim c(s')\{\rho_{i_1}, \dots, \rho_{i_r}\}c(s) \qquad \forall c(s') \in K(S). \tag{4}$$

THEOREM 2. *The elements* $s \in S(\{\rho_{i_1}, \dots, \rho_{i_r}\})$ *with similar codes form a subsemigroup of* $S(\{\rho_{i_1}, \dots, \rho_{i_r}\})$, *which is a left ideal in* $S(\{\rho_{i_1}, \dots, \rho_{i_r}\})$.

Proof. The theorem is a simple consequence of (3) and (4). □

THEOREM 3. *If* $s_1, s_2, s_3 \in S(\{\rho_{i_1}, \dots, \rho_{i_r}\})$ *and*
1) $c(s_1) = c(s_2)$
2) $s_1\{\rho_{i_1}, \dots, \rho_{i_r}\}s_3 = s_2\{\rho_{i_1}, \dots, \rho_{i_r}\}s_3$

then $s_1 = s_2$.

Proof. Let

$$s_i = [\alpha_1^{(i)}, \dots, \alpha_{i_1}^{(i)}, \dots, \alpha_{i_r}^{(i)}, \dots, \alpha_\mu^{(i)}; \beta_1^{(i)}, \dots, \beta_\mu^{(i)}], \qquad i = 1, 2, 3$$

then

$$\begin{aligned} s_1\{\rho_{i_1}, \dots, \rho_{i_r}\}s_3 = \Big[\alpha_1^{(1)} + \alpha_1^{(3)}\|c(s_1)\|, \dots, \alpha_{i_1}^{(3)}\|c(s_1)\|, \dots, \alpha_{i_r}^{(3)}\|c(s_1)\|, \\ \dots; \beta_1^{(1)} + \beta_1^{(3)}\|c(s_1)\|, \dots\Big], \end{aligned}$$

$$\begin{aligned} s_2\{\rho_{i_1}, \dots, \rho_{i_r}\}s_3 = \Big[\alpha_1^{(2)} + \alpha_1^{(3)}\|c(s_2)\|, \dots, \alpha_{i_1}^{(3)}\|c(s_2)\|, \dots, \alpha_{i_r}^{(3)}\|c(s_2)\|, \\ \dots; \beta_1^{(2)} + \beta_1^{(3)}\|c(s_2)\|, \dots\Big]. \end{aligned}$$

If $s_1\{\rho_{i_1}, \dots, \rho_{i_r}\}s_3 = s_2\{\rho_{i_1}, \dots, \rho_{i_r}\}s_3$, then the theorem easily follows. □

THEOREM 4. *Suppose that*

$$s_1\{\rho_{i_1},\ldots,\rho_{i_r}\}s_2\cdots\{\rho_{i_1},\ldots,\rho_{i_r}\}s_n = s'_1\{\rho_{i_1},\ldots,\rho_{i_r}\}s'_2\cdots\{\rho_{i_1},\ldots,\rho_{i_r}\}s'_n$$

and

1) $\|c(s_i)\| = \|c(s'_i)\|$ $i = 1,\ldots n$
2) $\|c(s_i)\| > \alpha_j^{(i)}, \beta_k^{(i)}, \alpha_l'^{(i)}, \beta_k'^{(i)}$ $i = 1,\ldots,n,\ j \neq i_j,\ k = 1,\ldots,\mu$

then $s_i = s'_i$ $i = 1,\ldots,n$.

Proof. Let

$$s_i = \left[\alpha_1^{(i)},\ldots,\alpha_{i_1}^{(i)},\ldots\alpha_{i_r}^{(i)},\ldots,\alpha_\mu^{(i)};\beta_1^{(i)},\ldots,\beta_\mu^{(i)}\right],$$

$$s'_i = \left[\alpha_1'^{(i)},\ldots,\alpha_{i_1}'^{(i)},\ldots\alpha_{i_r}'^{(i)},\ldots;\beta_1'^{(i)},\ldots,\beta_\mu'^{(i)}\right],$$

$i = 1,\ldots,n$. Then

$$s_1\{\rho_{i_1},\ldots,\rho_{i_r}\}\cdots\{\rho_{i_1},\ldots,\rho_{i_r}\}s_n =$$

$$=\Big[\alpha_1^{(1)} + \alpha_1^{(2)}\|c(s_1)\| + \alpha_1^{(3)}\|c(s_1)\|\|c(s_2)\| + \cdots + \alpha_1^{(n)}\|c(s_1)\|\cdots\|c(s_{n-1})\|,$$

$$\ldots,\alpha_{i_1}^{(n)}\|c(s_1)\|\cdots\|c(s_{n-1})\|,\ldots,\alpha_{i_r}^{(n)}\|c(s_1)\|\cdots\|c(s_{n-1})\|,\ldots;$$

$$\beta_1^{(1)} + \beta_1^{(2)}\|c(s_1)\| + \cdots + \beta_1^{(n)}\|c(s_1)\|\cdots\|c(s_{n-1})\|,\ldots\Big],$$

$$s'_1\{\rho_{i_1},\ldots,\rho_{i_r}\}\cdots\{\rho_{i_1},\ldots,\rho_{i_r}\}s'_n =$$

$$=\Big[\alpha_1'^{(1)} + \alpha_1'^{(2)}\|c(s'_1)\|\|c(s'_2)\| + \cdots + \alpha'^{(n)}\|c(s'_1)\|\cdots\|c(s'_{n-1})\|,$$

$$\ldots,\alpha_{i_1}'^{(n)}\|c(s'_1)\|\cdots\|c(s'_{n-1})\|,\ldots,\alpha_{i_r}'^{(n)}\|c(s'_1)\|\cdots\|c(s'_{n-1})\|,\ldots;$$

$$\beta_1'^{(1)} + \beta_1'^{(2)}\|c(s'_1)\| + \cdots + \beta_1'^{(n)}\|c(s'_1)\|\cdots\|c(s'_{n-1})\|,\ldots\Big].$$

If $s_1\{\rho_{i_1},\ldots,\rho_{i_r}\}\cdots s_n = s'_1\{\rho_{i_1},\ldots,\rho_{i_r}\}\cdots s'_n$ then

$$\|c(s_1)\| \mid (\alpha_1^{(1)} - \alpha_1'^{(1)}), \qquad \|c(s_2)\| \mid (\alpha_1^{(2)} - \alpha_1'^{(2)}),$$

and so forth.

Using condition 2) one gets the theorem. □

THEOREM 5. *Consider the product*

$$s_1\{\rho_{i_1},\ldots,\rho_{i_r}\}\cdots\{\rho_{i_1},\ldots,\rho_{i_r}\}s_n$$

where

1) $\|c(s_i)\| > \alpha_k^{(i)}, \beta_k^{(i)}$ $(i = 1,\ldots,n)$
2) $c(s_i) \not\sim c(s_j)$ $(i \neq j)$.

Then all the products

$$s_{j_1}\{\rho_{i_1},\ldots,\rho_{i_r}\}\cdots\{\rho_{i_1},\ldots,\rho_{i_r}\}s_{j_n}$$

are different, where $(j_1,\ldots,j_n)$ *runs through the permutations of* $(1,\ldots,n)$.

Proof. Using Theorem 4 the theorem follows from condition 2). □

Suppose that

$$s_1\{\rho_{i_1},\dots,\rho_{i_r}\}\cdots\{\rho_{i_1},\dots,\rho_{i_r}\}s_n = s$$

If we know

1) the element s,
2) the number n,
3) the norms $\|c(s_i)\|$ $(i = 1,\dots,n)$,

then it is possible to determine uniquely the components $\alpha_j^{(i)}$, $j \neq i_j$, $\beta_k^{(i)}$, $(k = 1,\dots,\mu)$ for which

$$\|c(s_i)\| > \alpha_j^{(i)}, \beta_k^{(i)} \qquad (i = 1,\dots,n)\ j \neq i_j,\ k = 1,\dots,\mu.$$

One gets the decoding process easily from Theorem 4, besides, any components $c(s_i)$ satisfying condition 2), can be used.

7.2. Coded structures in $S(\{\rho_{i_1},\dots,\rho_{i_r}\})$

Consider the mapping

$$c(s_i) \longrightarrow s_i \qquad s_i \in S(\{\rho_{i_1},\dots,\rho_{i_r}\}).$$

DEFINITION 4. The product $s_1\{\rho_{i_1},\dots,\rho_{i_r}\}\cdots\{\rho_{i_1},\dots,\rho_{i_r}\}s_n$ with the mapping

$$c(s_1)\cdots c(s_n) \longrightarrow s_1\{\rho_{i_1},\dots,\rho_{i_r}\}\cdots\{\rho_{i_1}\cdots\rho_{i_r}\}s_n$$

is a *coded structure* with the *chain of code* $c(s_1)\cdots c(s_n)$.

REMARK 2. From the theorems proved before followes that from the code $c(s_1)\cdots c(s_n)$ and from the result s one can decode the factors s_1, s_2, …, s_n using some simple conditions.

7.3. Coded structures in $S(O_1,\dots,O_m)$

Consider several $\{\rho_{i_{r_1}},\dots,\rho_{i_{r_j}}\}$ operations $(i_{r_h} \neq i_{r_l}\ r_h \neq r_l)$. For a given operation the structure S is a semigroup. Let O_k be such an operation, and O_1, $O_2,\dots$ are different operations.

To get the product $s_1\{O_1\}s_2\{O_2\}\cdots\{O_m\}s_m$ it is also necessary to have also the structure in the parenthesis, because (see Sections 1.2, 2.4) for several operations the multiplication is not associative in general. Therefore $P_m(s_1\{O_1\}\cdots\{O_m\}s_m)$ is an uniquely determined product.

Definition 5. Let $S(O_{i_1}, \ldots, O_{i_m})$ be the (not associative in general) structure in which one uses only the given $O_{i_1}, \ldots, O_{i_m}$ operations.

In this case $c(s, O_i)$ is the code of the element s for the operation O_i.

Definition 6. For given elements $s_1, s_2, \ldots, s_n$ the product

$$P_n(s_1\{O_{i_1}\}s_2\{O_{i_2}\}\cdots\{O_{i_{n-1}}\}s_n)$$

is a coded structure with the chain of code $c(s_1, O_{i_1})c(s_2, O_{i_2})\cdots c(s_n, O_{i_n})$

If we know the product

$$P_n(s_1\{O_{i_1}\}s_2\{O_{i_2}\}\cdots\{Oi_{n-1}\}s_n) = s$$

and the chain of codes

$$P_n(c(s_1\{O_{i_1}\})c(s_2\{O_{i_2}\})\cdots\{c(s_n\{O_{i_n}\}))$$

then using the decomposition (see Section 1.2)

$$P_n = P_{n_1}P_{n_2}, \qquad (n_1 + n_2 = n)$$

one can derive the decoding process for s in a similar way as it was done in the previous section.

8. The structure $S^{[n]}(\{\rho_{i_1}, \dots, \rho_{i_r}\})$

8.1. The distributivity in $S(\{\rho_{i_1}, \dots, \rho_{i_r}\})$

For the sake of simplicity suppose that the operations ρ_{i_j} are positive.

Let us consider the elements

$$s = [\alpha_1, \dots, \alpha_\mu; \beta_1, \dots, \beta_\mu],$$
$$s' = [\alpha'_1, \dots, \alpha'_\mu; \beta'_1, \dots, \beta'_\mu],$$
$$s'' = [\alpha''_1, \dots, \alpha''_\mu; \beta''_1, \dots, \beta''_\mu],$$
$$\vdots$$
$$s^{(k)} = [\alpha_1^{(k)}, \dots, \alpha_\mu^{(k)}; \beta_1^{(k)}, \dots, \beta_\mu^{(k)}]$$

with $s, s', s'', \dots, s^{(k)} \in S(\{\rho_{i_1}, \dots, \rho_{i_r}\})$. Let $s^0 \in S(\{\rho_{i_1}, \dots, \rho_{i_r}\})$ be the element obtained from s with $\alpha_{i_j} = 0$ $(j = 1, \dots, r)$ and $c(s) \in S(\{\rho_{i_1}, \dots, \rho_{i_r}\})$ the element (obtained from s) such that $\alpha_i = 0$, $\beta_k = 0$ with $i \neq i_j$ $(j = 1, \dots, r;\ k = 1, \dots, \mu)$, (see Chapter 7).

For simplicity, let O_r be the operation $\{\rho_{i_1}, \dots, \rho_{i_r}\}$.

Then

1.

$$sO_r s'' =$$

$$= \Big[\dots, \alpha_i + \Big(\sum_{j=1}^{r} \alpha_{i_j}\Big)\alpha''_i, \dots, \Big(\sum_{j=1}^{r} \alpha_{i_j}\Big)\alpha''_{i_k}, \dots; \dots, \beta_l + \Big(\sum_{j=1}^{r} \alpha_{i_j}\Big)\beta''_l, \dots\Big]$$

$$s'O_r s'' =$$

$$= \Big[\dots, \alpha'_i + \Big(\sum_{j=1}^{r} \alpha'_{i_j}\Big)\alpha''_i, \dots, \Big(\sum_{j=1}^{r} \alpha'_{i_j}\Big)\alpha''_{i_k}, \dots; \dots, \beta'_l + \Big(\sum_{j=1}^{r} \alpha'_{i_j}\Big)\beta''_l, \dots\Big]$$

$$(s+s')O_r s'' =$$

$$=\Big[\dots,\alpha_i+\alpha'_i+\Big(\sum_{j=1}^{r}(\alpha_{i_j}+\alpha'_{i_j})\Big)\alpha''_i,\dots,\Big(\sum_{j=1}^{r}(\alpha_{i_j}+\alpha'_{i_j})\Big)\alpha''_{i_k},\dots;$$

$$\dots,\beta_l+\beta'_l+\Big(\sum_{j=1}^{r}(\alpha_{i_j}+\alpha'_{i_j})\Big)\beta''_l,\dots,\Big]$$

from which

$$(s+s')O_r s'' = sO_r s'' + s'O_r s''$$

follows and by induction one gets

$$\Big(\sum_{k=1}^{q} s^{(k)}\Big)O_r\hat{s} = \sum_{k=1}^{q}(s^{(k)}O_r\hat{s})$$

for $\forall\hat{s}\in S(\{\rho_{i_1},\dots,\rho_{i_r}\})$. This means that right distributivity holds.

2.

$$s''O_r s =$$

$$=\Big[\dots,\alpha''_i+\Big(\sum_{j=1}^{r}\alpha''_{i_j}\Big)\alpha_i,\dots,\Big(\sum_{j=1}^{r}\alpha''_{i_j}\Big)\alpha_{i_k},\dots;\dots,\beta''_l+\Big(\sum_{j=1}^{r}\alpha''_{i_j}\Big)\beta_l,\dots\Big]$$

$$s''O_r s' =$$

$$=\Big[\dots,\alpha''_i+\Big(\sum_{j=1}^{r}\alpha''_{i_j}\Big)\alpha'_i,\dots,\Big(\sum_{j=1}^{r}\alpha''_{i_j}\Big)\alpha'_{i_k},\dots;\dots,\beta''_l+\Big(\sum_{j=1}^{r}\alpha''_{i_j}\Big)\beta'_l,\dots\Big]$$

$$s''O_r(s+s') =$$

$$=\Big[\dots,\alpha''_i+\Big(\sum_{j=1}^{r}\alpha''_{i_j}\Big)(\alpha_i+\alpha'_i),\dots,\Big(\sum_{j=1}^{r}\alpha''_{i_j}\Big)(\alpha_{i_k}+\alpha'_{i_k}),\dots;$$

$$\dots,\beta''_l+\Big(\sum_{j=1}^{r}\alpha''_{i_j}\Big)(\beta_l+\beta'_l),\dots\Big]$$

implying

$$s''O_r(s+s') + s''^0 = s''O_r s + s''O_r s'$$

or

$$s''O_r(s+s') = c(s'')O_r s + s''O_r s' = s''O_r s + c(s'')O_r s'$$

and by induction

$$\hat{s}O_r\sum_{k=1}^{q}s^{(k)} + (q-1)\hat{s}^0 = \sum_{k=1}^{q}(\hat{s}O_r s^{(k)})$$

is obtained and can be written as

$$\hat{s}O_r \sum_{k=1}^{q} s^{(k)} = \sum_{k=1}^{q-1}(c(\hat{s})O_r s^{(k)}) + \hat{s}O_r s^{(k)} = \cdots$$

for $\forall \hat{s} \in S(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$.

This means that the left distributivity does not hold. Cases 1 and 2 show that left and right distributivities differ in the structure $S(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$ in general.

8.2. The structure $S^{[n]}(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$

Let $S^{[1]}(\{\rho_{i_1}, \ldots, \rho_{i_r}\}) = S(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$ be the structure with the elements (vectors) $s^{[1]} = [\alpha_1, \ldots, \alpha_\mu; \beta_1, \ldots, \beta_\mu]$. The components α_i, β_i are elements (integers) of type $s^{[0]}$.

Let us consider the vectors

$$s^{[n]} = \left[s_1^{[n-1]}, \ldots, s_\mu^{[n-1]}; t_1^{[n-1]}, \ldots, t_\mu^{[n-1]}\right], \qquad (n = 1, 2, \ldots)$$

where $s_i^{[n-1]}, t_i^{[n-1]} \in S^{[n-1]}(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$, $(i = 1, \ldots, \mu)$, that is the components of the vectors $s^{[n]}$ are elements of type $s^{[n-1]}$. Let

$$s^{[n]} = \left[s_1^{[n-1]}, \ldots, s_\mu^{[n-1]}; t_1^{[n-1]}, \ldots, t_\mu^{[n-1]}\right],$$

$$s'^{[n]} = \left[s_1'^{[n-1]}, \ldots, s_\mu'^{[n-1]}; t_1'^{[n-1]}, \ldots, t_\mu'^{[n-1]}\right].$$

The addition for the vectors $s \in S$ and the multiplication of these vectors with non negative integers have been introduced in Chapter 1. Starting from this case the sum of $s^{[n]}$ and $s^{[n]'}$ is defined as

$$s^{[n]} + s'^{[n]} = \left[s_1^{[n-1]} + s_1'^{[n-1]}, \ldots, s_\mu^{[n-1]} + s_\mu'^{[n-1]};\right.$$

$$\left. t_1^{[n-1]} + t_1'^{[n-1]}, \ldots, t_\mu^{[n-1]} + t_\mu'^{[n-1]}\right] = s'^{[n]} + s^{[n]}$$

and for the product with an integer α

$$\alpha s^{[n]} = \left[\alpha s_1^{[n-1]}, \ldots, \alpha s_\mu^{[n-1]}; \alpha t_1^{[n-1]}, \ldots, \alpha t_\mu^{[n-1]}\right] =$$

$$= \left[s_1^{[n-1]}\alpha, \ldots, s_\mu^{[n-1]}\alpha; t_1^{[n-1]}\alpha, \ldots, t_\mu^{[n-1]}\alpha\right] = s^{[n]}\alpha.$$

The $\{\rho_{i_1}, \ldots, \rho_{i_r}\}$ $(= O_r)$ -product of two vectors $s^{[n]}$ and $s'^{[n]}$ is defined in the following way:

$$s^{[n]}O_r s'^{[n]} = \Big[s_1^{[n-1]} + \sum_{j=1}^{r} s_{i_j}^{[n-1]} O_r s_1'^{[n-1]}, \ldots, \sum_{j=1}^{r} s_{i_j}^{[n-1]} O_r s_{i_1}'^{[n-1]}, \ldots,$$

$$\sum_{j=1}^{r} s_{i_j}^{[n-1]} O_r s_{i_r}'^{[n-1]}, \ldots, s_\mu^{[n-1]} + \sum_{j=1}^{r} s_{i_j}^{[n-1]} O_r s_\mu'^{[n-1]};$$

$$\ldots, t_j^{[n-1]} + \sum_{j=1}^{r} s_{i_j}^{[n-1]} O_r t_j'^{[n-1]}, \ldots\Big].$$

The structure $S^{[n]}(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$ is the set of the vectors $s^{[n]}$ with the above defined operations of addition and multiplication.

THEOREM 1. *In the structure $S^{[n]}(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$ there exist a 0-element and left identities. The set of the left identities form a right 0-semigroup, that is, $e_i^{[n]} O_r e_j^{[n]} = e_j^{[n]}$.*

Proof. The statements are clear in the case $n = 1$ and it is easily to see that the theorem holds by induction using the rules introducted above for addition and multiplication. □

In the case of $s^{[1]}$-vectors the components are non negative integers and for these numbers left and right distributivity (i.e. the product) coincide. By the previous section, for the components of the vectors $s^{[2]}$ this is not true. A consequence of this fact is: the multiplicative structure in $S^{[n]}\{\rho_{i_1}, \ldots, \rho_{i_r}\}$ is not associative in general.

THEOREM 2. *In the structure $S^{[n]}(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$ the right distributivity holds.*

Proof. Using the results of Section 8.1, it can be easily seen that right distributivity is true in the structure $S^{[2]}(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$ and by induction, this also holds in the structure $S^{[n]}(\rho_{i_1}, \ldots, \rho_{i_r})$. □

REMARK. For the left distributivity the properties are similar as in the case of $n = 2$ (mutatis mutandis).

8.3. The algorithm of Euclid in $S^{[n]}(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$

Let

$$s = [\alpha_1, \ldots, \alpha_\mu; \beta_1, \ldots, \beta_\mu], \qquad s' = [\alpha_1', \ldots, \alpha_\mu'; \beta_1', \ldots, \beta_\mu']$$

DEFINITION 1. $s \geq s'$ if and only if $\alpha_i \geq \alpha_i'$, $\beta_i \geq \beta_i'$, $(i = 1, \ldots, \mu)$. $s > s'$ if and only if $s \geq s'$ and $\exists i$ with $\alpha_i > \alpha_i'$ or $\beta_i > \beta_i'$.

Definition 2. $s^{[n]} \geq s'^{[n]}$ if and only if $s_i^{[n-1]} \geq s_i'^{[n-1]}$, $t_i^{[n-1]} \geq t_i'^{[n-1]}$, $(i = 1, \ldots, \mu)$. $s^{[n]} > s'^{[n]}$ if and only if $s^{[n]} \geq s'^{[n]}$ and $\exists i$ with $s_i^{[n-1]} > $ $> s_i'^{[n-1]}$ or $t_i^{[n-1]} > t_i'^{[n-1]}$.

Theorem 3. *If $s^{[n]} \geq s'^{[n]}$, then for O_r there exists exactly one $s''^{[n]}$ and $q^{[n]}$ such that*

$$s^{[n]} = s'^{[n]} O_r s''^{[n]} + q^{[n]}$$

with

$$q^{[n]} < \Big[\sum_{j=1}^{r} s_{i_j}'^{[n-1]}, \ldots, \sum_{j=1}^{r} s_{i_j}'^{[n-1]}; \sum_{j=1}^{r} s_{i_j}'^{[n-1]}, \ldots, \sum_{j=1}^{r} s_{i_j}'^{[n-1]} \Big],$$

where $\sum_{j=1}^{r} s_{i_j}'^{[n-1]} \neq \sum_{j=1}^{r} 0^{[n-1]}$.

Proof. The theorem clearly holds for $n = 1$. Starting from the case $n = 1$ and using Theorem 1, the assertion of the theorem is easily seen to hold by induction. □

Consider the product

$$s^{[n](1)} O_r s^{[n](2)} = s^{[n]}.$$

Similarly to the case of the structure $S(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$, and using the previous theorem the following is derived.

Theorem 4. *Given*

1) *element $s^{[n]}$,*
2) *sums $\sum_{j=1}^{r} s_{i_j}^{[n-1](i)}$, $(i = 1, 2)$,*

then components $s_j^{[n-1](i)}$, $j \neq i_j$, $t_l^{[n-1](i)}$, $i = 1, 2$; $j = 1, \ldots, \mu$ can be uniquely determined so that

$$\sum_{j=1}^{r} s_{i_j}^{[n-1](i)} > s_j^{[n-1](i)}, t_l^{[n-1](i)} \qquad (i = 1, 2)\ j \neq i_j,\ l = 1, \ldots, \mu.$$

Proof. A simple consequence of Theorem 3. □

8.4. Coded structures in $S^{[n]}(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$

Consider the mapping

$$c(s^{[n](i)}) \longrightarrow s^{[n](i)}, \qquad s^{[n](i)} \in S^{[n]}(\{\rho_{i_1}, \ldots, \rho_{i_r}\}).$$

Let P_m be a structure of parentheses (of m factors). Then the product $P_m(s^{[n](1)}O_r s^{[n](2)}O_r \cdots O_r s^{[n](m)}) = s^{[n]}$ is determined uniquely. This product is a *coded structure* with the chain of code

$$P_m(c(s^{[n](1)}), O_r)c(s^{[n](2)}), O_r) \cdots c(s^{[n](m)}, O_r))$$

and

$$P_m(c(s^{[n](1)})O_r \cdots O_r c(s^{[n](m)}) \longrightarrow P_m(s^{[n](1)}O_r \cdots O_r s^{[n](m)}) = s^{[n]}.$$

If we know the product $s^{[m]}$, then using the decomposition successively

$$P_m = P_{m_1}P_{m_2}, \qquad (m_1 + m_2 = m)$$

the decoding process for $s^{[n]}$ can be derived determining uniquely the m factors which meet the conditions of Theorem 3.

8.5. The structure $S_+^{[n]}(\{\rho_{i_1}, \dots, \rho_{i_\mu}\})$

Let us consider the vectors $s = [\alpha_1, \dots, \alpha_\mu; 0, \dots, 0]$, that is, $\beta_i = 0$ for $i = 1, \dots, \mu$. For the simplicity we write $s = [\alpha_1, \dots, \alpha_\mu]$. For the operation O_r (introduced in this chapter) we suppose that $r = \mu$. This structure is $S_+(\{\rho_1, \dots, \rho_r\})$. In this case $c(s) = s$. Using the results of the Section 8.1 in the structure $S_+(\{\rho_1, \dots, \rho_r\}$ the left distributivity is also true. It means that $S_+(\{\rho_1, \dots, \rho_r\})$ is a distributive structure, where the additive structure and the multiplicative structure is a semigroup.

Similarly as in Section 8.2 one can introduce the structure $S_+^{[n]}(\{\rho_1, \dots, \rho_r\})$ which for $\mu = r$ is a distributive structure. In this structure

$$s^{[n]}O_r s'^{[n]} = \Big[\sum_{j=1}^{r} s_j^{[n-1]}O_r s_1'^{[n-1]}, \dots, \sum_{j=1}^{r} s_j^{[n-1]}O_r s_r'^{[n-1]}\Big].$$

Because $S_+(\{\rho_1, \dots, \rho_r\})$ is a distributive structure it follows

THEOREM 5. *The structure $S_+^{[n]}(\{\rho_1, \dots, \rho_r\})$ is a distributive structure.*

Proof. By induction for n. □

THEOREM 6. *The multiplicative structure of $S_+^{[n]}(\{\rho_1, \dots, \rho_r\})$ is a semigroup.*

Proof. Using Theorem 1, one sees easily that the O_r-product is associative. □

9. The abstract $S_n(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$-structure

9.1. The product $\{\rho_{i_1}, \ldots, \rho_{i_r}\}$ over distributive structures

Let us consider the vectors $s = [\alpha_1, \ldots, \alpha_n]$ which do not contain "anti"-components, that is do not contain "β" components.

For these vectors we define the $\{\rho_{i_1}, \ldots, \rho_{i_r}\}$-product in a similar way as before, that is

$$s\{\rho_{i_1}, \ldots, \rho_{i_r}\}s' = [\alpha_1, \ldots, \alpha_n]\{\rho_{i_1}, \ldots, \rho_{i_r}\}[\alpha'_1, \ldots, \alpha'_n] =$$

$$= \Big[\alpha_1 + \Big(\sum_{j=1}^{r} \alpha_{i_j}\Big)\alpha'_1, \ldots, \Big(\sum_{j=1}^{r} \alpha_{i_j}\Big)\alpha'_{i_1}, \ldots,$$

$$\ldots, \Big(\sum_{j=1}^{r} \alpha_{i_j}\Big)\alpha'_{i_r}, \ldots, \alpha_n + \Big(\sum_{j=1}^{r} \alpha_{i_j}\Big)\alpha'_n\Big].$$

Let R be an algebraic structure with two operations ("+", "·"), where these operations in R are (in itself) associative, besides in R the distributive laws are valid. It means that the "+"-structure and the "·"-structure are semigroups. For the operation "+" the commutativity is not prescribed.

One sees by a simple calculation (similarly as in Section 5.1) that the multiplicative structure $S_n(R, \{\rho_{i_1}, \ldots, \rho_{i_r}\})$ — generated by the elements s — is associative, using the introduced $\{\rho_{i_1}, \ldots, \rho_{i_r}\}$-product and the assumptions for R, where the components α_i, α'_i, $i = 1, \ldots, n$ are elements of R. Therefore we get the

THEOREM 1. *The multiplicative structure $S_n(R, \{\rho_{i_1}, \ldots, \rho_{i_r}\})$ is a semigroup determined uniquely by R.*

THEOREM 2. *The structure $S_n(R, \{\rho_{i_1}, \ldots, \rho_{i_r}\})$ is a ring (not necessarily with commutative addition) if and only if the additive structure R^+ is an idempotent semigroup.*

Proof. Because the multiplicative structure and the additive structure are semigroups it is necessary to prove that the distributive laws are true.

Analogously as in Section 8.1, one sees easily that

$$(s+s')\{\rho_{i_1},\ldots,\rho_{i_r}\}s'' = s\{\rho_{i_1},\ldots,\rho_{i_r}\}s'' + s'\{\rho_{i_1},\ldots,\rho_{i_r}\}s'',$$

but the "left distributivity" is valid if and only if for the components of the elements s the equality $2\alpha_j = \alpha_j$, $j = 1,\ldots,n$ is valid. □

9.2. The structure of the semigroup $S_n(R,\rho_i)$ over a ring R

Let R be a ring and consider the structure with the only operation ρ_i (instead $\{\rho_i\}$ we write only ρ_i). Let $R^{(+)}$ be the additive structure of R and $R^{(\cdot)}$ the multiplicative structure of R.

In this case the product of two elements s, s' is

$$s\rho_i s' = [\alpha_1,\ldots,\alpha_i,\ldots.\alpha_n]\rho_i[\alpha'_1,\ldots,\alpha'_i,\ldots,\alpha'_n] =$$

$$= [\alpha_1+\alpha_i\alpha'_1,\ldots,\alpha_i\alpha'_i,\ldots,\alpha_n+\alpha_i\alpha'_n].$$

Because R is a ring, the semigroup R^+ has a 0-element, which is the 0-element in R. From this follows that

$$s_0 = [0,\ldots,0]$$

is a left zero element of the $S_n(R,\rho_i)$.

If R has a right unit element e_r, then the vector

$$s_{e_r} = [0,\ldots,e_r,\ldots,0], \qquad e_r = \alpha_i$$

is a right unit element in $S_n(R,\rho_i)$. Similarly, if e_l is a left unit element in R, then

$$s_{e_l} = [0,\ldots,e_l,\ldots,0], \qquad e_l = \alpha_i$$

is a left unit element in $S_n(R,\rho_i)$. Therefore it follows that if e is the unit element of R, then

$$s_e = [0,\ldots,e,\ldots,0], \qquad 1 = \alpha_i$$

is the unit element in $S_n(R,\rho_i)$.

The semigroup $S_n(R,\rho_i)$ is not commutative in general, because in $s\rho_i s'$ the component $\alpha_k + \alpha_i\alpha'_k$ differs from the component $\alpha'_k + \alpha'_i\alpha_k$ of the product $s'\rho_i s$ in general. This is also true if R is a commutative ring. This means that by a ring it is possible to produce not commutative semigroups.

Let e_l be a left unit element of the ring R. Consider the vectors

$$s = [\alpha_1, \ldots, e_l, \ldots, \alpha_n],$$

where $e_l = \alpha_i$ and $\alpha_j \in R; j \neq i$ are arbitrary elements of R. These vectors form a commutative semigroup $S_n^{(+)}(e_l^{(i)})$in $S_n(R, \rho_i)$. Besides it is easy to see that the vectors

$$s = [0, \ldots, \alpha_j, \ldots, e_l, \ldots, 0], \qquad e_l = \alpha_i;\ j \neq i$$

with fixed j form a semigroup $\overline{R}_j^{(+)}$, for which

$$\overline{R}_j^{(+)} \approx R^{(+)}, \qquad j \neq i.$$

THEOREM 3. *$S_n^{(+)}(e_l, i)$ is the direct product of $n-1$ commutative semigroups:*

$$S_n^{(+)}(e_l, i) \approx \overline{R}_1^{(+)} \rho_i \cdots \rho_i \overline{R}_{i-1}^{(+)} \rho_i \overline{R}_{i+1}^{(+)} \rho_i \cdots \rho_i \overline{R}_n^{(+)}$$

where $\overline{R}_j^{(+)} \approx R^{(+)}$, $j \neq i$.

Proof. Using the facts written above and considering the products

$$[0, \ldots, \alpha_j, 0, \ldots, \alpha_k, 0, \ldots, e_l, 0, \ldots, 0]\rho_i[0, \ldots, \alpha_j', 0, \ldots, \alpha_k', 0, \ldots,$$

$$\ldots, e_l, 0, \ldots, 0] = [0, \ldots, \alpha_j + \alpha_j', 0, \ldots, \alpha_k + \alpha_k', \ldots, e_l, 0, \ldots, 0]$$

one gets the theorem. □

Consider the vectors

$$s = [0, \ldots, \alpha_i, \ldots, 0], \qquad \alpha_i \in R$$

one sees easily that for the ρ_i operation these vectors form a semigroup $\overline{R}_n^{(\cdot)}(i)$ for which $\overline{R}_n^{(\cdot)}(i) \approx R^{(\cdot)}$

THEOREM 4. *If R is a ring, then $S_n(R, \rho_i)$ has the following factorisation*

$$S_n(R, \rho_i) = S_n^{(+)}(e_l, i)\rho_i \overline{R}_n^{(\cdot)}(i), \qquad S_n^{(+)}(e_l, i) \cap \overline{R}^{(\cdot)}(i) = s_0, \qquad l \neq i$$

where e_l is a left unit element of R.

REMARK. $S_n^{(+)}(e_l, i)\rho_i \overline{R}_n^{(\,)}(i) \neq \overline{R}_n^{(\cdot)} \rho_i S_n^{(+)}(e_l, i)$ in general.

If $e_r \in R$ is a right unit element, then the elements $s = [\alpha_1, \ldots, e_r, \ldots, \alpha_n]$, $\alpha_i = e_r$ form a semigroup $S_n(e_r, i)$ and in this case the following theorem is true:

THEOREM 5. *If e_r is a right unit element of R, then $S_n(R,\rho_i)$ has the following factorisation:*

$$S_n(R,\rho_i) = \overline{R}_n^{(\,)}(i)\rho_i S_n(e_r,i), \qquad \overline{R}_n^{(\cdot)}(i) \cap S_n(e_r,i) = s_0.$$

A consequence of these theorems

THEOREM 6. *If R is a ring and e is the unit element of R, then $S_n(R,\rho_i)$ has the following factorisation:*

$$S_n(R,\rho_i) = S_n^{(+)}(e,i)\rho_i \overline{R}_n^{(\,)}(i) = \overline{R}_n^{(\,)}(i)\rho S_n^{(+)}(e,i).$$

If R is a field, then $R^{(\cdot)}$ (without 0-element) is a multiplicative group with unit element e. In this case $[0,\dots,e,\dots,0]$ is the unit element of $S_n(R,\rho_i)$.

Consider the elements $s=[0,\dots,\alpha_i,\dots,0]$, $\alpha_i \neq 0$ in the field R. The set of these elements for the ρ_i-product forms a group $\overline{S}_n^{(\cdot)}(\rho_i,\alpha_j=0,j\neq i)$, which is isomorphic to $R^{(\cdot)}$.

Consider the elements $s=[\alpha_1,\dots,\alpha_{i-1},e,\alpha_{i+1},\dots,\alpha_n]$ with the α-components of the field R. The set of these elements form a semigroup, which is a commutative group. Let $\overline{S}_n^{(+)}(R,e,\rho_i)$ be this group.

THEOREM 7. *If R is a field with unit element $e=1$, then the semigroup of the elements $s=[\alpha_1,\dots,\alpha_i,\dots,\alpha_n]$, $\alpha_i\neq 0$ is a group $S_n(R,\rho_i,\alpha_i\neq 0)$ and has the factorisation*

$$S_n(R,\rho_i,\alpha_i\neq 0) = \overline{S}_n^{(+)}(R,e,\rho_i)\rho_i\overline{S}_n^{(\cdot)}(\rho_i,\alpha_j=0,j\neq 0)$$

where

a) $\overline{S}_n^{(+)}(R,e,\rho_i) = \mathop{\times}\limits_{k=1}^{n-1} \overline{R}_k^{(\,)}$ *with* $\overline{R}_k^{(\cdot)} \approx R^{(\,)}$, $k=1,\dots,n-1$.

b) $\overline{S}_n^{(+)}(R,e,\rho_i)$ *is a normal subgroup in the group* $S_n(R,\rho_i,\alpha_i\neq 0)$.

Proof. The assertions for the factorisation and for b) are simple calculation. The assertion a) is a simple consequence of the fact that the set of the elements s forms a semigroup in which $\alpha_i=e$, α_k with fixed k run through the elements of R and $\alpha_j=0$, $j\neq i,k$. This semigroup is isomorphic to $R^{(\cdot)}$. For $k=1,\dots,i-1,i+1,\dots,n$ one gets the direct product mentioned in a).

$S_n(R,\rho_i,\alpha_i\neq 0)$ is a group, because it is a semigroup with unit element, and the inverse of the element s is

$$s^{-1} = [-\alpha_i^{-1}\alpha_1,\dots,\alpha_i^{-1},\dots,-\alpha_i^{-1}\alpha_n]$$

which is determined uniquely. □

9.3. The structure of the semigroup $S_n(R, \{\rho_{i_1}, \dots, \rho_{i_r}\})$ over a ring R

Let $e_l^{(1)}, \dots, e_l^{(r)}$ be left unit elements of R. Consider the vectors

$$s = [\alpha_1, \dots, q_1 e_l^{(1)}, \dots, q_r e_l^{(r)}, \dots, \alpha_n],$$

$$s' = [\alpha'_1, \dots, q'_1 e_l^{(1)}, \dots, q'_r e_l^{(r)}, \dots, \alpha'_n],$$

where $q_j e_l^{(j)} = \alpha_{i_j}$, $j = 1, \dots, r$ are integers and the product

$$s\{\rho_{i_1}, \dots, \rho_{i_r}\}s' =$$

$$= \Big[\alpha_1 + \Big(\sum_{j=1}^{r} q_j e_l^{(j)}\Big)\alpha'_1, \dots, \Big(\sum_{j=1}^{r} q_j e_l^{(j)}\Big) q'_1 e_l^{(1)}, \dots,$$

$$\dots, \Big(\sum_{j=1}^{r} q_r e_l^{(j)}\Big) q'_r e_l^{(r)}, \dots, \alpha_n + \Big(\sum_{j=1}^{r} q_j e_l^{(j)}\Big)\alpha'_n\Big] =$$

$$= \Big[\alpha_1 + \Big(\sum_{j=1}^{r} q_j\Big)\alpha'_1, \dots, \Big(\sum_{j=1}^{r} q_j\Big) q'_1 e_l^{(1)}, \dots,$$

$$\dots, \Big(\sum_{j=1}^{r} q_j\Big) q'_r e_l^{(r)}, \dots, \alpha_n + \Big(\sum_{j=1}^{r} q_j\Big)\alpha'_n\Big].$$

One sees that these elements form a semigroup $\overline{S}_n^{(+)}(e_l^{(1)}, \dots, e_l^{(r)})$, which is not commutative in general.

Consider the elements

$$s = [0, \dots, 0, \alpha_{i_1}, 0, \dots, 0, \alpha_{i_r}, 0, \dots, 0],$$

$$s' = [0, \dots, 0, \alpha'_{i_1}, 0, \dots, 0, \alpha'_{i_r}, 0, \dots, 0],$$

where $\alpha_k, \alpha'_k = 0$, $k \neq i_j$, $(j = 1, \dots, r)$ then

$$s\{\rho_1, \dots, \rho_r\}s' = \Big[0, \dots, 0, \Big(\sum_{j=1}^{r} \alpha_{i_j}\Big)\alpha'_{i_1}, 0, \dots, \Big(\sum_{j=1}^{r} \alpha_{i_j}\Big)\alpha'_{i_r}, 0, \dots, 0\Big]$$

therefore these elements form a semigroup.

The product $(\sum_{j=1}^{r} \alpha_{i_j})\alpha'_{i_k}$, $k = 1, \dots, r$ for fixed k run through all the elements of R if the components α_{i_j}, α'_{i_k} run through the elements of R, (because $\sum_{j=1}^{r} \alpha_{i_j}$ contains the case $\alpha_{i_p} = e$, $\alpha_{i_j} = 0, (j \neq p)$). Let $\overline{S}_n^{(\cdot)}(i_1, \dots, i_r)$ be this semigroup.

Let $s \in \overline{S}_n^{(+)}(e_l^{(1)}, \dots, e_l^{(r)})$ and $s' \in S_n^{(\cdot)}(i_1, \dots, i_r)$, then

$$s\{\rho_1,\dots,\rho_r\}s' =$$
$$= [\alpha_1,\dots,q_1 e_l^{(i_1)},\dots,q_r e_l^{(i_r)},\dots,\alpha_n]\{\rho_1,\dots,\rho_r\}[0,\dots,\alpha'_{i_1},\dots,\alpha'_{i_r},\dots,0] =$$
$$= \Big[\alpha_1,\dots,\Big(\sum_{j=1}^{r} q_j\Big)\alpha'_{i_1},\dots,\Big(\sum_{j=1}^{r} q_j\Big)\alpha'_{i_r},\dots,\alpha_n\Big].$$

From these facts follows the following

THEOREM 8. *If R is a ring with left unit elements $e_l^{(t)}$, $t = 1,\dots$, then the semigroup $S_n(R,\{\rho_{i_1},\dots,\rho_{i_r}\})$ has the factorisation*

$$S_n(R,\{\rho_{i_1},\dots,\rho_{i_r}\}) = \overline{S}_n^{(+)}(e_l^{(i_1)},\dots,e_l^{(i_r)})\overline{S}_n^{(\,)}(i_1,\dots,i_r),$$
$$\overline{S}_n^{(+)}(e_l^{(i_1)},\dots,e_l^{(i_r)}) \cap \overline{S}_n^{(\,)}(i_1,\dots,i_r) = s_0.$$

Let $e_r^{(i_j)}$, $j = 1\dots,r$ be right unit elements of the ring R.

Consider the vectors

$$s = \Big[\alpha_1,\dots,\sum_{j=1}^{r} q_j e_r^{(i_j)},\dots,\sum_{j=1}^{r} q_j e_r^{(i_j)},\dots,\alpha_n\Big]$$

where q_j, $j = 1,\dots,r$ are integers and $\alpha_{i_j} = \sum_{j=1}^{r} q_j e_l^{(i_j)}$, $j = 1,\dots,r$.

One sees by a simple calculation that these elements form a semigroup for the $\{\rho_1,\dots,\rho_r\}$-product. Let $\overline{R}^{(+)}(e_r^{(i_1)},\dots,e_r^{(i_r)})$ be this semigroup. Besides consider the semigroup $S_n^{(\cdot)}(i_1,\dots,i_r)$ introduced before. Then it is true the following

THEOREM 9. *If R is a ring with right unit elements $e_r^{(t)}$, $t = 1,\dots$, then the semigroup $S_n(R,\{\rho_{i_1},\dots,\rho_{i_r}\})$ has the factorisation*

$$S_n(R,\{\rho_{i_1},\dots,\rho_{i_r}\}) = \overline{S}_n^{(\,)}(i_1,\dots,i_r)\{\rho_{i_1},\dots,\rho_{i_r}\}\overline{S}_n^{(+)}(e_r^{(i_1)},\dots,e_r^{(i_r)}),$$
$$\overline{S}_n^{(\,)}(i_1,\dots,i_r) \cap \overline{S}_n^{(+)}(e_r^{(i_1)},\dots,e_r^{(i_r)}) = s_0\,.$$

Proof. Executing the multiplications

$$[0,\dots,\alpha_{i_1},\dots,\alpha_{i_r},\dots,0]\{\rho_{i_1},\dots,\rho_{i_r}\} \times$$
$$\times \Big[\alpha'_1,\dots,\sum_{j=1}^{r} q_{1j} e_r^{(j)},\dots,\sum_{j=1}^{r} q_{rj} e_r^{(j)},\dots,\alpha_n\Big]$$

one gets all the elements of the semigroup $S_n(R,\{\rho_{i_1},\dots,\rho_{i_r}\})$. It is clear that the common element of the factors is s_0. □

If $R^{(\,)}$ (the multiplicative semigroup of R) has unit element e, then the following theorem is true:

THEOREM 10. *If $R^{(\cdot)}$ has the unit element e, then the semigroup $S_n(R, \{\rho_{i_1}, \dots, \rho_{i_r}\})$ has the following factorisation:*

$$S_n(R, \{\rho_{i_1}, \dots, \rho_{i_r}\}) = \overline{S}_n^{(\cdot)}(i_1, \dots, i_r)\{\rho_{i_1}, \dots, \rho_{i_r}\}\overline{S}_n^{(+)}(e^{(i_1)}, \dots, e^{(i_r)}) =$$

$$= \overline{S}_n^{(+)}(e^{(1)}, \dots, e^{(r)})\{\rho_{i_1}, \dots, \rho_{i_r}\}\overline{S}_n^{(\cdot)}(i_1, \dots, i_r)$$

where $e^{(j)} = \alpha_{i_j}$, $j = 1, \dots, r$.

Proof. A simple consequence of the Theorem 5. □

Suppose that R has the unit element e. Consider the elements $s = [\alpha_1, \dots, \alpha_{i_1}, \dots, \alpha_{i_r}, \dots, \alpha_n] \in R$ for which $\sum_{j=1}^r \alpha_{i_j} = e$. These elements form a semigroup $S_n(\sum_{j=1}^r \alpha_{i_j} = e, \{\rho_{i_1}, \dots, \rho_{i_r}\})$ which can be proved by a simple calculation.

THEOREM 11. *If R^+ does not contain an element with order 2 (excepted 0), then the semigroup $S_n(R, \sum_{j=1}^r \alpha_{i_j} = e, \{\rho_{i_1}, \dots, \rho_{i_r}\})$ is a right group.*

Proof.

1) The idempotent elements e_k, $k = 1, \dots$ are the elements $s = [0, \dots, \alpha_{i_1}, \dots, \alpha_{i_r}, \dots, 0]$, because from

$$s\{\rho_{i_1}, \dots, \rho_{i_r}\}s = [\alpha_1 + \alpha_1, \dots, \alpha_{i_1}, \dots, \alpha_{i_r}, \dots, \alpha_n + \alpha_n] = s$$

follows that $\alpha_i = 0$, $i \neq i_j$, $(j = 1, \dots, r)$.

2) The set of the elements e_k is a right zero semigroup, that is, $e_k\{\rho_{i_1}, \dots, \rho_{i_r}\}e_l = e_l$, $\forall e_k$ (exercise).

3) The set of the elements $s = [\alpha_1, \dots, \alpha_{i_1}, \dots, \alpha_{i_r}, \dots, \alpha_n]$ with fixed $\{\alpha_{i_1}, \dots, \alpha_{i_r}\}$ is a commutative group $G_n(\alpha_{i_1}, \dots, \alpha_{i_r})$ with the unit element $e_k = [0, \dots, \alpha_{i_1}, \dots, \alpha_{i_r}, \dots, 0]$. It is easy to see that the set of the elements s with fixed e_k forms a commutative semigroup with the unit element e_k and in this semigroup an other idempotent does not exist. Besides every element

$$s = [\alpha_1, \dots, \alpha_{i_1}, \dots, \alpha_{i_r}, \dots, \alpha_n]$$

has only one inverse

$$s^{-1} = [-\alpha_1, \dots, \alpha_{i_1}, \dots, \alpha_{i_r}, \dots, -\alpha_n].$$

It means that the set of the elements s with fixed $\{\alpha_{i_1}, \dots, \alpha_{i_r}\}$ is a commutative group. From this follows that

$$S_n\Big(\sum_{j=1}^{r} \alpha_{i_j} = e, \{\rho_{i_1}, \dots, \rho_{i_r}\}\Big) = \bigcup_{\sum \alpha_{i_j}} G_n(\alpha_{i_1}, \dots, \alpha_{i_r}),$$

$$G_n(\alpha_{i_1}, \ldots, \alpha_{i_r}) \cap G_n(\alpha'_{i_1}, \ldots, \alpha'_{i_r}) = \emptyset,$$
$\{\alpha_{i_1}, \ldots, \alpha_{i_r}\} \neq \{\alpha'_{i_1}, \ldots, \alpha'_{i_r}\}$; with $\sum \alpha_{i_j} = e$.

4) The group $G_n(\alpha_{i_1}, \ldots, \alpha_{i_r})$ is a direct product of $n - r$ commutative groups, which are isomorphic to $R^{(+)}$. This property follows from the fact that the elements in which in one position, e.g., in the k-th $(k \neq i_j, j = 1, \ldots, r)$ the components run through the elements of the R, and the others $\alpha_i = 0$, $i \neq i_j, k$ form a commutative group which is isomorphic to $S^{(+)}$. The direct product of these groups produces the group $G_n(\alpha_{i_1}, \ldots, \alpha_{i_r})$.
5) The set of the elements $s = [0, \ldots, \alpha_{i_1}, \ldots, \alpha_{i_r}, \ldots, 0]$ with $\sum_{j=1}^{r} \alpha_{i_j} = e$ is a right zero semigroup.

The points 1)–5) proves that $S_n(\sum_{j=1}^{r} \alpha_{i_j} = e, \{\rho_{i_1}, \ldots, \rho_{i_r}\})$ is a right group. □

9.4. The semigroup $S_n^{[2]}(R, \rho_i)$

Consider the semigroup $S_n(R, \rho_i)$ and the vectors
$$s^{[2]} = [s_1, \ldots, s_i, \ldots, s_n], \quad s_k \in S_n(R, \rho_i).$$
The set of these vectors is $S_n^{[2]}(R, \rho_i)$.

We remark that $s^{[1]} = s \in S_n(R, \rho_i)$.

THEOREM 12. *In the semigroup $S_n(R, \rho_i)$ the right distributivity is valid:*
$$(s + s')\rho_i s'' = s\rho_i s'' + s'\rho_i s''.$$
The left distributivity
$$s''\rho_i(s + s') = s''\rho_i s + s''\rho_i s'$$
is valid if and only if in s'' the components $\alpha_k = 0$, $k \neq i$.

Proof. The assertion for the right distributivity is a simple calculation. For the left distributivity
$$s''\rho_i(s + s') = [\alpha''_1 + \alpha''_i(\alpha_1 + \alpha'_1), \ldots, \alpha''_i(\alpha_i + \alpha'_i), \ldots, \alpha''_n + \alpha''_i(\alpha_n + \alpha'_n)],$$
$$s''\rho_i s = [\alpha''_1 + \alpha''_i\alpha_1, \ldots, \alpha''_i\alpha_i, \ldots, \alpha''_n + \alpha''_i\alpha_n],$$
$$s''\rho_i s' = [\alpha''_1 + \alpha''_i\alpha'_1, \ldots, \alpha''_i\alpha'_i, \ldots, \alpha''_n + \alpha''_i\alpha'_n].$$
From these products one sees that
$$s''\rho_i(s + s') = s''\rho_i s + s''\rho_i s'$$
if and only if $2\alpha''_k = \alpha''_k$, $k \neq i$, that is $\alpha_k = 0$, $k \neq i$. □

CONSEQUENCE 1. The set $S_n^{[2]}(R, \rho_i)$ is a semigroup if and only if in $s^{[2]} = [s_1, \ldots, s_i, \ldots, s_n]$ the components s_k have the form $[0, \ldots, \alpha_i^{(k)}, \ldots, 0]$, $k \neq i$.

9.5. The structure of the semigroup $S_n(A, \rho_i)$ over a finite algebra A

Consider the structures A with two operations "+" and "·", where $A^{(+)}$ and $A^{(\cdot)}$ are semigroups and the distributive laws are valid. Using such structures and the $\{\rho_{i_1}, \ldots, \rho_{i_r}\}$-products it is possible to produce several types of semigroups which differ from the semigroups $A^{(+)}$ and $A^{(\cdot)}$. In some special cases these structures are described perfectly. If $A^{(+)}$ is a group (which is not necessarily a commutative group), see [15] and [16]. If $A^{(+)}$ is a semigroup and $A^{(\cdot)}$ is a group, see [2], [13], [14].

In the next parts 9.5.1.–9.5.5. we show some examples for the last mentioned case.

9.5.1. A finite algebra with two operations

It will be introduced a special algebra with two operations the structure of which can be described perfectly.

Let $a, b, c, \ldots$ be a finite set A of symbols. On this set we define two operations "+" and "·". The operation "+" defines on A a semigroup A_2 (which is not a group) and the operation "·" defines a group A_1 (in which instead $a \cdot b$ we will write ab).

The given operations are connected by the following distributive law:

$$(1) \qquad (a+b)c = (ac) + (bc); \qquad c(a+b) = (ca) + (cb).$$

The analysis of this structure is given in eleven steps:

1) Let 1 be the unit element of A_1. The semigroup A_2 is finite, therefore it has an idempotent element: $a + a = a$. Let a^{-1} be the inverse of a in A_1. Using (1) we get $1 + 1 = (a+a)a^{-1} = 1$ and it follows $b + b = b$ for every $b \in A_2$, that is, A_2 is an idempotent semigroup.
2) Consider the element $a + 1 = b$. It follows $b + 1 = a + 1 + 1 = a + 1 = b$, that is, $b + 1 = b$. If $c + 1 = c$ and $c' + 1 = c'$, then $cc' + 1 = cc'$. Namely from $c + 1 = c$ it follows $cc' + c' = cc'$, whence $cc' + 1 = [(c+1)c'] + 1 = cc' + c' + 1 = cc' + c' = cc'$. Because A_1 is finite from $c + 1 = c$ it follows $c^{-1} + 1 = c^{-1}$.

3) From $c+1=c$ we get $a^{-1}ca+1=a^{-1}ca$ for every $a \in A_1$. $\{c^{A_1}\}$ (the subgroup generated by the conjugates of c in A_1) is a normal subgroup of A_1. By 2) we get $a+1=a$ for every $a \in \{c^{A_1}\}$.
4) If $c+1=c$ holds, then $\{c^{A_1}\}$ is a left zerosemigroup in A_2. For, if $a, b \in \{c^{A_1}\}$, then $a+b=(ab^{-1}+1)b=ab^{-1}b=a$.
5) Similarly, if $1+b=b$, then properties analogous to 2), 3), 4) hold and we get that $\{b^{A_1}\}$ is a right zerosemigroup in A_2.
6) If $x+1=x$, then $1+x=1$. In fact, $x=x+1=(1+x^{-1})x=x$ that is, $1+x^{-1}=1$ for every $x \in \{a^{A_1}\}$ and because $\{a^{A_1}\}$ is a group $1+x=1$ follows. Similarly if $1+y=y$, then $y+1=1$ for every $y \in \{b^{A_1}\}$.
7) Let us consider all the elements $a_1, a_2, \ldots$ for which $a_i+1=a_i$ $(i=1,2,\ldots)$. Then
$$\{\{a_1^{A_1}\},\{a_2^{A_1}\},\ldots\}=G_1$$
is a normal subgroup of A_1 (using 2) and 3)). Similarly, consider the elements $b_1, b_2, \ldots$ for which $1+b_i=b_i$ $(i=1,2,\ldots)$, then
$$\{\{b_1^{A_1}\},\{b_2^{A_1}\},\ldots\}=G_2$$
is a normal subgroup of A_1.
8) $G_1 \cap G_2=1$. In order to prove this assertion, suppose $a \in G_1 \cap G_2$. Then $a+1=a$ and $1+a=a$. But from the first equality we get $1+a=1$ by 6). Hence $a=1$.
9) $A_1=G_1 \times G_2$ (where $\times$ means the direct product of G_1 and G_2). Consider an arbitrary element $a \in A_1$. Because of $(1+a^{-1})a=a+1=c \in G_1$ we have $ca^{-1} \in G_2$, that is, $a \in G_2c \subseteq G_2G_1=G_2 \times G_1$.
10) $1+g_1g_2=g_2$, $g_1g_2+1=g_1$, $g_1 \in G_1$, $g_2 \in G_2$. In fact $1+g_1g_2 \in G_2$ (using 2)), therefore from $g_1g_2(g_1^{-1}g_2^{-1}+1) \in G_2$ it follows $g_1^{-1}g_2^{-1}+1 \in g_1^{-1}G_2$. On the other hand $g_1^{-1}g_2^{-1}+1 \in G_1$ from which we get $g_1^{-1}g_2^{-1}+1=g_1^{-1}$ for any $g_1 \in G_1$, $g_2 \in G_2$. This means that $g_1g_2+1=g_1$ for every $g_1 \in G_1$, $g_2 \in G_2$. Similarly we get the equality $1+g_1g_2=g_2$ $(g_1 \in G_1,\ g_2 \in G_2)$.
11) $g_1g_2+g_1'g_2'=g_1g_2'$ $(g_1, g_1' \in G_1,\ g_2, g_2' \in G_2$. In fact $g_1g_2+g_1'g_2'= =(1+g_1'g_1^{-1}g_2'g_2^{-1})g_1g_2=g_2'g_2^{-1}g_1g_2=g_2'g_1=g_1g_2'$.

We have the following

THEOREM 13. *The finite algebra A has the following structure:*

$$A_1=G_1 \times G_2 \tag{2}$$

and for A_2

$$g_1g_2+g_1'g_2'=g_1g_2' \qquad (g_1, g_1' \in G_1,\ g_2, g_2' \in G_2).$$

9.5.2. The structure of the semigroup $S_n(A, \rho_i)$

Consider the structure with the only operation ρ_i. Let A be the finite algebra introduced in Section 9.5.1. The vectors of $S_n(A, \rho_i)$ we write in the form

$$s = \left[g_1^{(1)} g_2^{(1)}, \ldots, g_1^{(i)} g_2^{(i)}, \ldots, g_1^{(n)} g_2^{(n)}\right],$$
$$g_1^{(j)} \in G_1, \qquad g_2^{(j)} \in G_2, \qquad j = 1, \ldots, n.$$

THEOREM 14. *The elements s with* $g_1^{(i)} g_2^{(i)} = 1$ *for the* ρ_i*-product form an idempotent semigroup with the operation "+" for the components:*

$$g_1^{(j)} g_2^{(j)} + g_1'^{(j)} g_2'^{(j)} = g_1^{(j)} g_2'^{(j)}, \qquad j \neq i.$$

Proof. A simple consequence of the fact that A_2 is an idempotent semigroup (see (2), (3) in Section 9.5.1). □

CONSEQUENCE 2. The vectors $s = [g_1^{(1)}, \ldots, g_1^{(i)}, \ldots, g_1^{(n)}]$ with $g_1^{(i)} = 1$ form a left zero semigroup.

CONSEQUENCE 3. The vectors $s = [g_2^{(1)}, \ldots, g_2^{(i)}, \ldots, g_2^{(n)}]$ with $g_2^{(i)} = 1$ form a right zero semigroup.

THEOREM 15. *With fixed components* $g_1^{(j)}$, $j \neq i$ *the set of the elements* $s(g_1^{(j)}, j \neq i) = [g_1^{(1)}, \ldots, g_1^{(i)}, \ldots, g_1^{(n)}]$ *for the* ρ_i*-product forms a group* $\overline{G}_1(g_1^{(j)}, j \neq i)$ *for which*

$$\overline{G}_1(g_1^{(j)}, j \neq i) \approx G_1.$$

Proof. Executing the ρ_i-multiplication for two elements s, s' and using the properties (2), (3) in Section 9.5.1, from the bijective mapping

$$s = [g_1^{(1)}, \ldots, g_1^{(i)}, \ldots, g_1^{(n)}] \leftrightarrow g_1^{(i)}, \qquad s' = [g_1'^{(1)}, \ldots, g_1'^{(i)}, \ldots, g_1'^{(n)}] \leftrightarrow g_1'^{(i)}$$

the following product and mapping

$$s \rho_i s' = [g_1^{(1)}, \ldots, g_1^{(i)} g_1'^{(i)}, \ldots, g_1^{(n)}] \longleftrightarrow g_1^{(i)} g_1'^{(i)}$$

follows from which the assertion of the theorem follows. □

Let $H_1(i)$ be the set of all the vectors $(g_1^{(1)}, \ldots, g_1^{(i-1)}, g_1^{(i+1)}, \ldots, g_1^{(n)})$, where the elements $g_1^{(j)}$ $(j \neq i)$ run through all the elements of G_1.

THEOREM 16. *Let $\overline{G}_1(i)$ be the semigroup of the elements*

$$s = [g_1^{(1)}, \dots, g_1^{(i)}, \dots, g_1^{(n)}],$$

then

$$\overline{G}_1(i) = \bigcup_{H_1(i)} \overline{G}_1(g_1^{(j)}, j \neq i)$$

and $\overline{G}_1(i)$ is a left group because the different components $\overline{G}_1(g_1^{(j)}, j \neq i)$ are disjunctive.

Proof. A simple consequence of the Theorem 14 considering the equality $g_1^{(j)} + g_1'^{(j)} = g_1^{(j)}$. □

THEOREM 17. *For fixed components $g_2^{(j)}$, $j \neq i$ the set of the elements $s = [g_2^{(i)} g_2^{(1)}, \dots, g_2^{(i)}, \dots, g_2^{(i)} g_2^{(n)}]$ for the ρ_i-product forms a group $\overline{G}_2(g_2^{(j)}, j \neq i)$ for which*

$$\overline{G}_2(g_2^{(j)}, j \neq i) \approx G_2.$$

Proof. Executing the ρ_i-multiplication for two elements s, s' and using the properties (2), (3) in Section 9.5.1, from the bijective mapping

$$s = [g_2^{(i)} g_2^{(1)}, \dots, g_2^{(i)}, \dots, g_2^{(i)} g_2^{(n)}] \longleftrightarrow g_2^{(i)}$$

$$s' = [g_2'^{(i)} g_2^{(1)}, \dots, g_2'^{(i)}, \dots, g_2'^{(i)}, \dots, g_2'^{(i)} g_2^{(n)}] \longleftrightarrow g_2'^{(i)}$$

the following product and mapping

$$s\rho_i s' = [g_2^{(i)} g_2'^{(i)} g_2^{(1)}, \dots, g_2^{(i)} g_2'^{(i)}, \dots, g_2^{(i)} g_2'^{(i)} g_2^{(n)}] \longleftrightarrow g_2^{(i)} g_2'^{(i)}$$

follows, which gives the assertion of the theorem. □

Let $H_2(i)$ the set of all the vectors $(g_2^{(1)}, \dots, g_2^{(i-1)}, g_2^{(i+1)}, \dots, g_2^{(n)})$, where the elements $g_2^{(j)}$ $(j \neq i)$ run through all the elements of G_2.

THEOREM 18. *Let $\overline{G}_2(i)$ be the semigroup of the elements*

$$s = [g_2^{(i)} g_2^{(1)}, \dots, g_2^{(i)}, \dots, g_2^{(i)} g_2^{(n)}]$$

then

$$\overline{G}_2(i) = \bigcup_{H_2(i)} \overline{G_2}(g_2^{(j)}, j \neq i)$$

and $\overline{G}_2(i)$ is a right zero semigroup with $g_2^{(i)} g_2^{(j)} + g_2^{(i)} g_2'^{(i)} g_2^{(j)} = g_2^{(i)} g_2'^{(i)} g_2^{(j)}$ and the different components in the union are disjunctive.

Proof. A simple consequence of the Theorem 13 and of the properties (2), (3) of Section 9.5.1. □

THEOREM 19. *With fixed elements* $g_1^{(j)}$, $g_2^{(j)}$, $j \neq i$ *for the set of the products*

$$s_1\rho_i s_2 = \left[g_1^{(1)}, \ldots, g_1^{(i)}, \ldots, g_1^{(n)}\right]\rho_i\left[g_2^{(i)} g_2^{(1)}, \ldots, g_2^{(i)}, \ldots, g_2^{(i)} g_2^{(n)}\right] =$$

$$= \left[g_1^{(1)} g_2^{(i)} g_2^{(1)}, \ldots, g_1^{(i)} g_2^{(i)}, \ldots, g_1^{(n)} g_2^{(i)} g_2^{(n)}\right]$$

forms a semigroup $\overline{G}(g_1^{(j)}, g_2^{(j)}, j \neq i)$ *for which*

$$\overline{G}(g_1^{(j)}, g_2^{(j)}, j \neq i) \approx G_1 \times G_2$$

is valid.

Proof. Consider the bijective mapping

$$s = \left[g_1^{(1)} g_2^{(i)} g_2^{(1)}, \ldots, g_1^{(i)} g_2^{(i)}, \ldots, g_1^{(n)} g_2^{(i)} g_2^{(n)}\right] \longleftrightarrow g_1^{(i)} g_2^{(i)}$$

$$s' = \left[g_1^{(1)} g_2'^{(i)} g_2^{(1)}, \ldots, g_1'^{(i)} g_2'^{(i)}, \ldots, g_1^{(n)} g_2'^{(i)} g_2^{(n)}\right] \longleftrightarrow g_1'^{(i)} g_2'^{(i)}$$

then

$$s\rho_i s' = \left[g_1^{(1)} g_2^{(i)} g_2'^{(i)} g_2^{(1)}, \ldots, g_1^{(i)} g_1'^{(i)} g_2^{(i)} g_2'^{(i)}, \ldots, g_1^{(n)} g_2^{(i)} g_2'^{(i)} g_2^{(n)}\right]$$

$$\longleftrightarrow g_1^{(i)} g_1'^{(i)} g_2^{(i)} g_2'^{(i)}$$

from which the theorem follows. □

Let $\overline{G}^{(l)}$ be one of the semigroups $\overline{G}(g_1^{(j)}, g_2^{(j)}, j \neq i)$.

THEOREM 20.

$$S_n(A, \rho_i) = \cup_l^{|G_1 \times G_2|^{n-1}} \overline{G}^{(l)}, \qquad \overline{G}^{(l)} \cap \overline{G}^{(k)} = \emptyset \qquad l \neq k,$$

$$\overline{G}_l \approx G_1 \times G_2, \qquad l = 1, \ldots, |G_1 \times G_2|^{n-1},$$

that is, the semigroup $S_n(A, \rho_i)$ *is a disjunctive union of groups, where for the multiplication in* $S_n(A, \rho_i)$ *the properties* (2), (3) *of Section* 9.5.1 *are valid. It means that* $S_n(A, \rho_i)$ *is a completely regular semigroup.*

Proof. A simple consequence of Theorem 18. □

THEOREM 21. *The semigroup* $S_n(A, \rho_i)$ *has the following factorisation:*

$$S_n(A, \rho_i) = \overline{G}_1(i)\overline{G}_2(i), \qquad \overline{G}_1(i) \cap \overline{G}_2(i) = [1, \ldots, 1, \ldots, 1].$$

Proof. A simple calculation proves the theorem. □

9.5.3. The structure of the semigroup $S_n(A, \{\rho_{i_1}, \dots, \rho_{i_r}\})$

If $s \in S_n(A, \{\rho_{i_1}, \dots, \rho_{i_r}\})$, then s has the following form

$$s = \left[g_1^{(1)} g_2^{(1)}, \dots, g_1^{(i_1)} g_2^{(i_1)}, \dots, g_1^{(i_r)} g_2^{(i_r)}, \dots, g_1^{(n)} g_2^{(n)}\right].$$

THEOREM 22. *The set of the elements*

$$s = \left[g_1^{(1)}, \dots, g_1^{(i_1)}, \dots, g_1^{(i_r)}, \dots, g_1^{(n)}\right]$$

with $g_1^{(i_j)} = 1$, $j = 1, \dots, r$ *for the* $\{\rho_{i_1}, \dots, \rho_{i_r}\}$*-product forms a left zero semigroup.*

Proof. Using the properties (2), (3) of Section 9.5.1 one gets the theorem. □

THEOREM 23. *For fixed* $g_1^{(k)}$, $k \neq i_j$, $j = 1, \dots, r$ *the set of the elements*

$$s = \left[g_1^{(1)}, \dots, g_1^{(i_1)}, \dots, g_1^{(i_r)}, \dots, g_1^{(n)}\right]$$

for the $\{\rho_{i_1}, \dots, \rho_{i_r}\}$*-product forms a group* $\overline{G}_1$ *for which*

$$\overline{G}_1 = G_1^{(i_1)} \times \dots \times G_1^{(i_r)},$$

where $G_1^{(i_j)} \approx G_1$, $j = 1, \dots, r$ *and* $G_1^{(i_j)}$ *is the group of the elements* s *in which* $g_1^{(i_l)} = 1$, $i_l \neq i_j$.

Proof. One sees easily with simple calculation in an analogous way as in the case of the Theorem 14. □

THEOREM 24. *For fixed* $g_2^{(k)}$, $k \neq i_j$, $j = 1, \dots, r$ *the set of the elements*

$$s = \left[g_2^{(i_r)} g_2^{(1)}, \dots, g_2^{(i_1)}, \dots, g_2^{(i_r)}, \dots, g_2^{(i_r)} g_2^{(n)}\right]$$

for the $\{\rho_{i_1}, \dots, \rho_{i_r}\}$*-product forms a group* $\overline{G_2}$ *for which*

$$\overline{G_2} = G_2^{(i_1)} \times \dots \times G_2^{(i_r)}$$

where $G_2^{(i_j)} \approx G_2$, $j = 1, \dots, r$ *and* $G_2^{(i_j)}$ *is the group of the elements* s *in which* $g_2^{(i_l)} = 1$, $i_l \neq i_j$.

Proof. One sees easily with simple calculation in an analogous way as the case of Theorem 15. □

There are true analogous theorems in Sections 9.5.2 and 9.5.3.

Let $H_1(i_1, \dots, i_r)$ be the set of all the vectors

$$(g_1^{(1)}, \dots, g_1^{(i_1-1)}, g_1^{(i_1+1)}, \dots, g_1^{(i_r-1)}, g_1^{(i_r+1)}, \dots, g_1^{(n)}).$$

THEOREM 25. *Let* $\overline{G}_1(i_1,\ldots,i_r)$ *be the semigroup of all the elements*

$$s = [g_1^{(1)},\ldots,g_1^{(i_1)},\ldots,g_1^{(i_r)},\ldots,g_1^{(n)}],$$

then

$$\overline{G}_1(i_1,\ldots,i_r) = \bigcup_{H_1(i_1,\ldots,i_r)} \overline{G}_1(g_1^{(k)}, k \neq i_j; j = 1,\ldots,r)$$

and $\overline{G}_1(i_1,\ldots,i_r)$ *is a left group in which the group-components are isomorphic to* $G_1^{(i_1)} \times \cdots \times G_1^{(i_r)}$.

Let $H_2(i_1,\ldots,i_r)$ be the set of all the vectors

$$(g_2^{(1)},\ldots,g_2^{(i_1-1)}, g_2^{(i_1+1)},\ldots,g_2^{(i_r-1)}, g_2^{(i_r+1)},\ldots,g_2^{(n)}).$$

THEOREM 26. *Let* $\overline{G}_2(i_1,\ldots,i_r)$ *be the semigroup of all the elements*

$$s = [g_2^{(i_r)}g_2^{(1)},\ldots,g_2^{(i_1)},\ldots,g_2^{(i_r)},\ldots,g_2^{(i_r)}g_2^{(n)}],$$

then

$$\overline{G}_2(i_1,\ldots,i_r) = \bigcup_{H_2(i_1,\ldots,i_r)} \overline{G}_2(g_2^{(k)}, k \neq i_j; j = 1,\ldots,r)$$

and $\overline{G}_2(i_1,\ldots,i_r)$ *is a right group in which the group-components are isomorphic to* $G_2^{(i_1)} \times \cdots \times G_2^{(i_r)}$.

THEOREM 27. *For fixed elements* $g_1^{(k)}$, $g_2^{(k)}$, $k \neq i_j$, $j = 1,\ldots,r$ *the set of the products*

$$s_1\{\rho_1,\ldots,\rho_r\}s_2 = [g_1^{(1)},\ldots,g_1^{(i_1)},\ldots,g_1^{(i_r)},\ldots,g_1^{(n)}]\{\rho_1,\ldots,\rho_r\},$$

$$\left[g_2^{(i_r)}g_2^{(1)},\ldots,g_2^{(i_1)},\ldots,g_2^{(i_r)},\ldots,g_2^{(i_r)}g_2^{(n)}\right] =$$

$$= \left[g_1^{(1)}g_2^{(r)}g_2^{(1)},\ldots,g_1^{(i_1)}g_2^{(i_1)},\ldots,g_1^{(i_r)}g_2^{(i_r)},\ldots,g_1^{(n)}g_2^{(r)}g_2^{(n)}\right]$$

forms a group $\overline{G}(g_1^{(k)}, g_2^{(k)}, k \neq i_j)$ *for which*

$$\overline{G}(g_1^{(k)}, g_2^{(k)}, k \neq i_j) \approx G^{(1)} \times \cdots \times G^{(r)},$$

where $G^{(j)} \approx G_1 \times G_2$.

Let $H(i_1,\ldots,i_r)$ be the set of all the vectors

$$\Big(g_1^{(1)}g_2^{(1)},\ldots,g_1^{(i_1-1)}g_2^{(i_1-1)}, g_1^{(i_1+1)}g_2^{(i_1+1)},$$
$$\ldots,g_1^{(i_r-1)}g_2^{(i_r-1)}, g_1^{(i_r+1)}g_2^{(i_r+1)},\ldots,g_1^{(n)}g_2^{(n)}\Big).$$

THEOREM 28. *The set of the vectors*

$$[g_1^{(1)}g_2^{(r)}g_2^{(1)},\dots,g_1^{(i_1)}g_2^{(i_1)},\dots,g_1^{(i_r)}g_2^{(i_r)},\dots,g_1^{(n)}g_2^{(r)}g_2^{(n)}]$$

forms a completely regular semigroup $\overline{G}(i_1,\dots,i_r)$ *for which*

$$\overline{G}(i_1,\dots,i_r)=\bigcup_{H(i_1,\ \ ,i_r)}\overline{G}_H^{(j)}$$

with the properties (2), (3) *of section* 9.5.1. $\overline{G}_H^{(j)}\approx\overline{G}(g_1^{(k)},g_2^{(k)},k\neq i_j)$.

9.5.4. The semigroup $S_n^{[2]}(A,\rho_i)$

Consider the semigroup $S_n(A,\rho_i)$ and the vectors

$$s^{[2]}=[s_1,\dots,s_i,\dots,s_n],\qquad s_k\in S_n(A,\rho_i)$$

The set of these vectors is $S_n^{[2]}(A,\rho_i)$.
We remark that $s^{[1]}=s\in S_n(A,\rho_i)$.

THEOREM 29. *The set* $S_n^{[2]}(A,\rho_i)$ *is a semigroup for the operation* ρ_i.

Proof. Consider the elements of $S_n(A,\rho_i)$

$$s=\left[g_1^{(1)}g_2^{(1)},\dots,g_1^{(i)}g_2^{(i)},\dots,g_1^{(n)}g_2^{(n)}\right],$$

$$s'=\left[g_1'^{(1)}g_2'^{(1)},\dots,g_1'^{(i)}g_2'^{(i)},\dots,g_1'^{(n)}g_2'^{(n)}\right],$$

$$s''=\left[g_1''^{(1)}g_2''^{(1)},\dots,g_1''^{(i)}g_2''^{(i)},\dots,g_1''^{(n)}g_2''^{(n)}\right].$$

With an easy calculation (using Theorem 11) one sees the following properties

$$(s+s')\rho_i s''=s\rho_i s''+s'\rho_i s'';\qquad s''\rho_i(s+s')=s''\rho_i s+s''\rho_i s'.$$

By Theorem 13 follows that the vectors $s^{[2]}=[s_1,\dots,s_i,\dots,s_n]$ $s_k\in S_n(A,\rho_i)$, $k=1,\dots,n$ form a semigroup. □

THEOREM 30. *The set* $S_n^{[2]}(A,\rho_i)$ *is an algebra for the operations "+" and "*ρ_i*" with the following properties:*

a) $S_n^{[2](+)}(A,\rho_i)$ *is an idempotent semigroup for the operation "+"*

b) $S_n^{[2]}(A,\rho_i)$ *is a semigroup for the operation "*ρ_i*" and has the factorisation*

$$S_n^{[2]}(A,\rho_i)=S_n^{[2]}(G_1,A,\rho_i)\rho_i S_n^{[2]}(G_2,A,\rho_i).$$

It is easy to see in an analogue way as before the

THEOREM 31. *The set* $S_n^{[m]}(A, \rho_i)$, $m > 2$ *is an algebra for the operation "+" and "*ρ_i*" with the following properties:*

a) $S_n^{[m]}(A, \rho_i)$ *is an idempotent semigroup for the operation "+",*

b) $S_n^{[m]}(A, \rho_i)$ *is a semigroup for the operation "*ρ_i*" and has the factorisation*

$$S_n^{[m]}(A, \rho_i) = S_n^{[m]}(G_1, A, \rho_i)\rho_i S_n^{[m]}(G_2, A, \rho_i).$$

Proof. With induction for m. □

9.5.5. The semigroup $S_n^{[2]}(A, \{\rho_{i_1}, \ldots, \rho_{i_r}\})$

Consider the semigroup $S_n(A, \{\rho_{i_1}, \ldots, \rho_{i_r}\})$ and the vectors

$$s^{[2]} = [s_1, \ldots, s_{i_1}, \ldots, s_{i_r}, \ldots, s_n], \quad s_k \in S_n(A, \{\rho_{i_1}, \ldots, \rho_{i_r}\}).$$

The set of these vectors is $S_n^{[2]}(A, \{\rho_{i_1}, \ldots, \rho_{i_r}\})$.

We remark that $s^{[1]} = s \in S_n(A, \{\rho_{i_1}, \ldots, \rho_{i_r}\})$.

THEOREM 32. a) *The set* $S_n^{[2]}(A, \{\rho_{i_1}, \ldots, \rho_{i_r}\})$ *is a semigroup for the operation* $\{\rho_{i_1}, \ldots, \rho_{i_r}\}$.

b) *The semigroup* $S_n^{[2]}(A, \{\rho_{i_1}, \ldots, \rho_{i_r}\})$ *has the following factorisation:*

$$S_n^{[2]}(A, \{\rho_{i_1}, \ldots, \rho_{i_r}\}) =$$

$$= S_n^{[2]}(G_1, A, \{\rho_{i_1}, \ldots, \rho_{i_r}\})\{\rho_{i_1}, \ldots, \rho_{i_r}\}S_n^{[2]}(G_2, A, \{\rho_{i_1}, \ldots, \rho_{i_r}\}).$$

Proof. For a) consider the elements of the semigroup $S_n(A, \{\rho_{i_1}, \ldots, \rho_{i_r}\})$

$$s = [g_1^{(1)} g_2^{(1)}, \ldots, g_1^{(n)} g_2^{(n)}],$$

$$s' = [g_1'^{(1)} g_2'^{(1)}, \ldots, g_1'^{(n)} g_2'^{(n)}],$$

$$s'' = [g_1''^{(1)} g_2''^{(1)}, \ldots, g_1''^{(n)} g_2''^{(n)}].$$

Using Theorem 14,

$$s + s' = [g_1^{(1)} g_2'^{(1)}, \ldots, g_1^{(n)} g_2'^{(n)}]$$

and

$$\sum_{j=1}^{r} g_1^{(i_j)} g_2'^{(i_j)} = g_1^{(i_1)} g_2'^{(i_r)},$$

$$\sum_{j=1}^{r} g_1''^{(i_j)} g_2''^{(i_j)} = g_1''^{(i_1)} g_2''^{(i_r)},$$

one sees easily that

$$(s+s')\{\rho_{i_1},\dots,\rho_{i_r}\}s'' = s\{\rho_{i_1},\dots,\rho_{i_r}\}s'' + s'\{\rho_{i_1},\dots,\rho_{i_r}\}s'',$$

$$s''\{\rho_{i_1},\dots,\rho_{i_r}\}(s+s') = s''\{\rho_{i_1},\dots,\rho_{i_r}\}s + s''\{\rho_{i_1},\dots,\rho_{i_r}\}s',$$

that is the distributive laws are valid.

For b) one gets the assertion by a simple calculation using Theorem 13 and part a).

From these properties the theorem follows. □

In an analogous way as before one sees by induction for m the analogous theorems in Section 9.5.4 for the set $S_n^{[m]}(A,\{\rho_{i_1},\dots,\rho_{i_r}\})$ too.

9.5.6. A noncommutative semigroup with unit roots

An example

Consider the roots of the equation $x^n = 1$. Let $\varepsilon_1 = 1, \varepsilon_2, \dots, \varepsilon_n$ be the roots of this equation and the vectors $s = [\alpha_1, \dots, \alpha_i, \dots, \alpha_n]$ with the following components

$$s = \Big[\sum_{j_1} \varepsilon_{j_1}, \dots, \varepsilon_k, \dots, \sum_{j_n} \varepsilon_{j_n}\Big],$$

where $\alpha_i = \varepsilon_k$.

Let S_1 the set of the mentioned vectors with $\alpha_i = 1$ and Z the set of the mentioned vectors with $\alpha_j = 0$, $j \neq i$ and $\alpha_i = \varepsilon_k$, $k = 1, \dots, n$.

The element e with $\alpha_i = 1$ and $\alpha_k = 0$, $k \neq i$ is the unit element in S_1 and in Z.

It is easy to see that S_1 is an infinite commutative semigroup, and Z is a finite cyclic group with order n_k, where n_k is a divisor of n. Then the product

$$S = S_1 \rho_i Z, \qquad S_1 \cap Z = e$$

is a noncommutative semigroup, where the multiplicative order of the elements s with $\alpha_i \neq 1$ is a divisor of n.

9.6. Semigroups over a semiring of finite semigroup

Let F a finite semigroup with elements f_q, $(q = 1, \dots, m)$. Consider the set H of the elements $\sum_{q=1}^{m} j_q f_q$, where j_q are nonnegative numbers. It is clear that H forms a semiring (the coefficients cannot be negative numbers in R^+)!) for the operations "+" and "·" in the usual way. The left (right) distributivity are valid in H.

Consider the vectors

$$s = \Big[\sum_{q=1}^{m} j_q^{(1)} f_q, \ldots, \sum_{q=1}^{m} j_q^{(n)} f_q\Big]$$

and the $\{\rho_{i_1}, \ldots, \rho_{i_r}\}$-products of the vectors s.

The set of the vectors s forms a semigroup $S(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$ for the operation $\{\rho_{i_1}, \ldots, \rho_{i_r}\}$.

There the following properties of the semigroup $S(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$ are true:

1) the set S_0 of the elements with $\sum_{q=1}^{m} j_q^{(i_j)} f_q = 0$, $(j = 1, \ldots, r)$ is a semigroup and it is an ideal in $S\{\rho_{i_1}, \ldots, \rho_{i_r}\}$,
2) S_0 is a left zero semigroup,
3) the set S_u of the elements with $\sum_{q=1}^{m} j_q^{(i_j)} f_q = \sum_{q=1}^{m} j_q^{(i_k)} f_q$, $(j, k = = 1, \ldots, r)$ forms a semigroup, which is a left ideal in $S(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$,
4) let S_1 be the set of the vectors $s = [\alpha_1, \ldots, \alpha_n] \in S(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$ for which $\alpha_i = 0$, $(i \neq i_j)$. Then

$$S(\{\rho_{i_1}, \ldots, \rho_{i_r}\}) = S_u\{\rho_{i_1}, \ldots, \rho_{i_r}\} S_1.$$

9.6.1. Finite semigroups

To every position i, $(i = 1, \ldots, n)$ we associate an (positive) integer $n_i > 1$, to position i_j for ρ_{i_j} belongs n_{i_j}. Consider the i-th component $\sum_{q=1}^{m} j_q^{(i)} f_q$ with coefficients $\operatorname{mod}(n_i)$.

In a similar way as in Section 6.1 one gets

THEOREM 33. *If*
1) $n_{i_j} = n_{i_k} = N$, $(j, k = 1, \ldots, r)$ *and*
2) $n_i \mid N$, $(i = 1, \ldots, n)$, *then*

$$S(\{\rho_{i_1}, \ldots, \rho_{i_r}\})$$

is a finite semigroup for which the properties 1)-4) *of Section* 9.6 *are valid.*

II
CONTINUOUS STRUCTURES

10. The multiplicative structure of real or complex vectors

Let us consider the vectors

$$s = [\alpha_1, \dots, \alpha_n]$$

with real or complex components. Let S_n be the set of these vectors.

In S_n we define two operations, the addition and the multiplication.

1) The addition: the sum $s + s'$ of the vectors s and s' is defined as in the classical case.

2) The multiplication of a vector s with a scalar c as in the classical case.

3) The ρ_i-product of two vectors s and s'

$$s\rho_i s' = [\alpha_1, \dots, \alpha_i, \dots, \alpha_n]\rho_i[\alpha_1', \dots, \alpha_i', \dots, \alpha_n'] =$$
$$= [\alpha_1 + \alpha_i\alpha_1', \dots, \alpha_i\alpha_i', \dots, \alpha_n + \alpha_i\alpha_n'].$$

4) The $\{\rho_{i_1}, \dots, \rho_{i_r}\}$-product of two vectors $(1 \le i_j \le n,\ i_j \ne i_k,\ j \ne k)$ s and s'

$$s\{\rho_{i_1}, \dots, \rho_{i_r}\}s' =$$
$$= \Big[\alpha_1 + \Big(\sum_{j=1}^{r}\alpha_{i_j}\Big)\alpha_1', \dots, \Big(\sum_{j=1}^{r}\alpha_{i_j}\Big)\alpha_{i_1}', \dots, \Big(\sum_{j=1}^{r}\alpha_{i_j}\Big)\alpha_{i_r}',$$
$$\dots, \alpha_n + \Big(\sum_{j=1}^{r}\alpha_{i_j}\Big)\alpha_n'\Big].$$

One sees that the case 4) is generalization of the case 3. Further the vector s is a generalization of the number concept, because in the case of the ρ_i-operation if $\alpha_j = 0$, $j \ne i$, then one gets the ring which is isomorphic to the ring of the real (or complex) numbers. In spite of this one sees that the scalar α_i and the vector $[0, \dots .\alpha_i, \dots, 0] = \bar{s}$ are not equivalent, because $\bar{s}\rho_i s \ne s\rho_i\bar{s}$.

It is easy to see the following

THEOREM 1. *The multiplicative structure of the vectors $s \in S_n$ with given operation $\{\rho_{i_1}, \dots, \rho_{i_r}\}$ is an associative structure, therefore it is a semigroup.*

Proof. By simple realization of the products $s(s's'')$ and $(ss')s''$. □

Besides it is easy to see the following properties of the $\{\rho_1, \ldots, \rho_r\}$-products:

1) For every $\{i_1, \ldots, i_r\}$, $i_j \neq i_k$ $(j \neq k)$, the $\{\rho_{i_1}, \ldots, \rho_{i_r}\}$-product exists.
2) The $\{\rho_{i_1}, \ldots, \rho_{i_r}\}$-product is not commutative in general, that is, $s\{\rho_{i_1}, \ldots, \rho_{i_r}\}s' \neq s'\{\rho_{i_1}, \ldots, \rho_{i_r}\}s$ in general.
3) The product $s_1 O_1 s_2 O_2 \ldots O_{m-1} s_m$ exists for every structure of parentheses applied for the product of m elements and for operations $O_j = \{\rho_{j_1}, \ldots, \rho_{j_r}\}$ $1 \leq r \leq m$.
4) In the case $r = 1$ the vector $e = [0, \ldots, 0, 1, 0, \ldots, 0]$ is a multiplicative unit element.

Let $S_n(+, \{\rho_{i_1}, \ldots, \rho_{i_r}\})$ be the structure introduced before.

Let $S_n(\{\rho_1, \ldots, \rho_r\})$ the multiplicative structure of the vectors of the set S_n with the multiplicative operations $\{\rho_1, \ldots, \rho_r\}$.

Remark. Every semigroup S has a disjoint left decomposition with components Λ_i, $i = 1, \ldots, 5$ and a dual right decomposition with components P_i, $i = 1, \ldots, 5$ (see [17]). In the case of $S_n(\{\rho_1, \ldots, \rho_r\})$, $r > 1$ for the components the following properties are true:

$$\Lambda_0 = \emptyset, \quad \Lambda_1 = \emptyset, \quad \Lambda_2 = \emptyset, \quad \Lambda_3 = \emptyset, \quad \Lambda_4 \neq \emptyset, \quad \Lambda_5 \neq \emptyset,$$
$$P_0 = \emptyset, \quad P_1 = \emptyset, \quad P_2 \neq \emptyset, \quad P_3 \neq \emptyset, \quad P_4 = \emptyset, \quad P_5 = \emptyset.$$

Theorem 2. *The multiplicative structure $S_n(\rho_i)$ of the vectors with $\alpha_i \neq 0$ is a group with fixed ρ_i. The element $e_i = [0, \ldots, \overset{i}{\overbrace{1}}, \ldots, 0]$ is the unit element of $S(\rho_i)$.*

Proof. Let $s = [\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n]$ be an arbitrary element of $S_n(\rho_i)$, $\alpha_i \neq 0$. One sees that the inverse element of s is the vector

$$s^{-1} = \Big[-\frac{\alpha_1}{\alpha_i}, \ldots, \frac{1}{\alpha_i}, \ldots, -\frac{\alpha_n}{\alpha_i}\Big]$$

and e_i is the unit element of $S_n(\rho_i)$. □

Let $G(\rho_i)$ be the group introduced in Theorem 2.

Theorem 3.

$$G(\rho_i) = A\rho_i Z$$

where A is the commutative group of the elements s with $\alpha_i = 1$ and Z is the commutative group of the elements s' with $\alpha'_j = 0$, $(j \neq i)$, $\alpha'_i \neq 0$. Furthermore the subgroup A is normal subgroup of $G(\rho_i)$.

Proof. One sees the theorem by a simple calculation. □

DEFINITION. A vector $s = [\alpha_1, \dots, \alpha_n]$ with real components is an *1-vector*, if $\sum_{j=1}^{n} \alpha_j = 1$.

THEOREM 4. *The ρ_i-product of two 1-vectors is a 1-vector again.*

Proof. Simple calculation. □

Let $S_n(\rho_i, 1)$ the set of the 1-vectors with $\alpha_i \neq 0$. Then similarly as before one gets the following

THEOREM 5. *$S_n(\rho_i, 1)$ is a group.*

Proof. The inverse of an 1-vector an 1-vector again and using Theorem 2 one gets Theorem 5. □

The distributivity in $S_n(+, \{\rho_{i_1}, \dots, \rho_{i_r}\})$ differs from the classical case. Similarly as in Chapter 8 one sees that

$$(s + s')\{\rho_{i_1}, \dots, \rho_{i_r}\}s'' = s\{\rho_{i_1}, \dots, \rho_{i_r}\}s'' + s'\{\rho_{i_1}, \dots, \rho_{i_r}\}s''$$

which is the classical right distributivity and is also true for more elements.

For the "left distributivity"

$$s''\{\rho_{i_1}, \dots, \rho_{i_r}\}(s + s') = s''\{\rho_{i_1}, \dots, \rho_{i_r}\}s + c(s'')\{\rho_{i_1}, \dots, \rho_{i_r}\}s',$$

where in the case of any vector $s = [\alpha_1, \dots, \alpha_n]$ the vector $c(s)$ is defined

$$c(s) = [0, \dots, \alpha_{i_1}, \dots, \alpha_{i_r}, \dots, 0] \qquad (\alpha_i = 0,\ i \neq i_j,\ j = 1, \dots, r).$$

10.1. The ρ_i-product of distribution vectors (an application)

A distribution vector (or distribution) is a vector

$$s = [\alpha_1, \dots, \alpha_n],$$

where the components α_i $(i = 1, \dots, n)$ are nonnegative real numbers with $\sum_{i=1}^{n} \alpha_i = 1$.

Let

$$s = [\alpha_1, \dots, \alpha_n], \qquad s' = [\alpha'_1, \dots, \alpha'_n]$$

be distribution vectors.

Using the properties mentioned before it is easy to see the following properties of the ρ_i-products of distribution vectors:

1) For every i $(1 \leq i \leq n)$, the ρ_i-product exists.

2) The ρ_i-product is not commutative in general, that is, $s\rho_i s' \neq s'\rho_i s$ in general.

3) $s\rho_i s' \neq s\rho_j s'$ for $i \neq j$ in general.

4) The product $s_1\rho_{i_1}s_2\rho_{i_2}\cdots\rho_{i_{n-1}}s_n$ exists for every structure of parentheses applied for the product of n elements $(1 \leq i_j \leq n, j = 1, \ldots, n)$.

5) If s and s' are distribution vectors, then $s\rho_i s'$ is also an distribution vector.

6) The ρ_i-product of three vectors s, s' and s'' has the associative property, that is, $(s\rho_i s')\rho_i s'' = s\rho_i(s'\rho_i s'')$ for a fixed i.

7) $(s\rho_i s')\rho_j s'' \neq s\rho_i(s'\rho_j s'')$ $(i \neq j)$ in general.

From the properties listed above it follows:

THEOREM 6. *If $D(\rho_i)$ is the set of the distribution vectors, then $D(\rho_i)$ is a semigroup with a unit element for the operation ρ_i (ρ_i is fixed).*

Proof. A simple consequence of property 5). □

THEOREM 7. *Let $s^{m(\rho_i)} = s\rho_i s\rho_i \cdots \rho_i s$ with m factors. If $s = [\alpha_1, \ldots, \alpha_n]$ is a distribution vector with $\alpha_i \neq 0$ and $s \neq e_i$, then*

$$s' = \lim_{m\to\infty} s^{m(\rho_i)} = [\alpha'_1, \ldots, \underbrace{0}^{i}, \ldots, \alpha'_n]$$

and s' is a distribution vector.

Proof. A simple consequence of the ρ_i-product and property 5). □

THEOREM 8. *If $s = [\alpha_1, \ldots, \alpha_n]$ is a distribution vector with $\alpha_i \neq 0$, then*

$$\lim_{m_n\to\infty}\Big(\cdots\Big(\lim_{m_{i+1}\to\infty}\Big(\lim_{m_{i-1}\to\infty}\Big(\cdots\Big(\lim_{m_1\to\infty}\Big(s^{m_1(\rho_1)}\Big)\Big)\cdots$$
$$\cdots\Big)^{m_{i-1}(\rho_{i-1})}\Big)^{m_{i+1}(\rho_{i+1})}\Big)\cdots\Big)^{m_n(\rho_n)} = [0, \ldots, \underbrace{1}^{i}, \ldots, 0].$$

Proof. If $\alpha_j = \alpha'_j = 0$ is in the vectors $s = [\alpha_1, \ldots, \alpha_n]$, $s' = [\alpha'_1, \ldots, \alpha'_n]$, then in the product $s\rho_i s'$ the j-th component is also 0. Therefore, it is possible to get 0 successively for every position excepted for the i-th one, which is 1, always using Theorem 7 and property 5). □

From Theorem 8 we get

THEOREM 9. *There exists a structure of parentheses P_q in $S(\rho_1, \ldots, \rho_n)$ (with length q) such that for the distribution vector s with $\alpha_i \neq 0$*

$$P_q(s\rho_{i_1}\cdots\rho_{i_{q-1}}s) = [\varepsilon_1, \ldots, \hat{\alpha}_i, \ldots, \varepsilon_n]$$

and for a given $\varepsilon > 0$ the inequality $1 - \hat{\alpha}_i < \varepsilon$ holds.

Proof. Using Theorem 8 one gets Theorem 9 for sufficiently great powers (without lines). □

Consider the distribution vector $s^{(n)} = [\frac{1}{n}, \dots, \frac{1}{n}]$, which represents a uniform distribution.

THEOREM 10. *Let $s = [\alpha_1, \dots, \alpha_n]$ be an arbitrary distribution vector. Then for a given $\varepsilon > 0$, there exists a structure of parentheses P_p in $S(\rho_1, \dots, \rho_n)$ such that for the distribution*

$$P_p(s^{(n)}\rho_{i_1} \dots \rho_{i_{p-1}} s^{(n)}) = [\alpha'_1, \dots, \alpha'_n]$$

the inequality $|\alpha_j - \alpha'_j| < \varepsilon$ $(j = 1, \dots, n)$ holds.

Proof. Consider the element $s = [\alpha_1, \dots, \alpha_n]$. In the first step for convenience we suppose that $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$ and the steps of the process follow this order.

For the sake of simplicity we will use the unit vectors (instead of $[\varepsilon_1, \dots, \hat{\alpha}_i, \dots, \varepsilon_n]$) $e_i = [0, \dots, \overset{i}{1}, \dots, 0]$, $(i = 1, \dots, n)$, because it is possible to approximate it with arbitrary precision by Theorem 8.

Step 1. One sees easily that

$$\overline{s}_1 = (\cdots((s^{(n)}\rho_n \overbrace{e_1)\rho_{n-1})\cdots)\rho_{n-k_1+1}e_1}^{k_1-1} = \Big[\frac{k_1}{n}, \overbrace{\frac{1}{n}, \dots, \frac{1}{n}}^{n-k_1}, 0, \dots, 0\Big],$$

where $\frac{k_1-1}{n} < \alpha_1 \leq \frac{k_1}{n}$. If $\alpha_2 < \frac{1}{n}$ then $|\alpha_i - \overline{\alpha}_i| < \frac{1}{n}$, $i > 2$. If $\frac{k_2-1}{n} < \alpha_2 \leq \frac{k_2}{n}$ $(k_2 > 1)$, then (similarly as before) the product

$$\overline{s}_2 = (\cdots((\overline{s}_1\rho_{n-k_1}e_2)\rho_{n-k_1-1}e_2)\cdots\rho_{n-k_1-k_2-1})e_2 = \Big[\frac{k_1}{n}, \frac{k_2}{n}, \dots\Big],$$

where$(k_1 + k_2 \leq n)$. If $k_1 + k_2 = n$, then the third component 0 and $\overline{s}_2 = s$. If $k_1 + k_2 < n$ then the next $n - (k_1 + k_2)$ components are $\frac{1}{n}$ in $\overline{s}_2$ and we continue the process. Suppose that

$$\overline{s}_{r-1} = \Big[\frac{k_1}{n}, \frac{k_2}{n}, \dots, \frac{k_{r-1}}{n}, \dots\Big],$$

is the (distribution) vector for which $\alpha_r < \frac{1}{n}$. If $\sum_{j=1}^{r-1} k_j = n$ then $\overline{s}_{r-1} = s$. If $\sum_{j=1}^{r-1} k_j < n$, then $k_r = \dots = k_t = 1$ such that $\sum_{j=1}^{t} k_j = n$.

For the differences the inequality $|\alpha_i - \frac{k_i}{n}| < \frac{1}{n}$ $(i = 1, \dots, n)$ holds and $\alpha_i \geq \frac{k_i}{n}$ $(i = 1, \dots, r-1)$.

If the unit vectors are given, then for the Step 1, $p \leq \sum_i k_i + 1$ is the length of P_p.

Suppose that $\overline{s}_{r-1} \neq s$.

If we want an approximation of precision $< \frac{1}{n^2}$, then we continue the process with

Step 2. Suppose that

$$\frac{k'_i+1}{n^2} > \alpha_i - \frac{k_i}{n} \geq \frac{k'_i}{n^2}, \qquad i \neq r$$

where some k_j, k'_l can be 0, besides

$$\frac{1}{n} - \sum_{\substack{i=1 \\ i\neq r}}^{n} \frac{k'_i}{n^2} = \frac{k'_r}{n^2}.$$

Similarly as in Step 1, one will get the distribution

$$s'_2 = \Big[\frac{k'_1}{n}, \ldots, \frac{k'_r}{n}, \ldots, \frac{k'_n}{n}\Big].$$

The vector (distribution) s'_2 is determined uniquely for the approximation vector $s' = [\beta_1, \ldots, \beta_n]$ of the original s, where

$$\beta_i = \alpha_i - (\frac{k_i}{n} + \frac{k'_i}{n^2}), \qquad i \neq r$$

and $|\alpha_r - (\frac{1}{n} + \frac{k'_r}{n^2})| \leq \frac{1}{n^2}$.

In this case

$$\overline{s}_{r-1}\rho_r s'_2 = \Big[\frac{k_1}{n} + \frac{k'_1}{n^2}, \frac{k_2}{n} + \frac{k'_2}{n^2}, \ldots, \frac{k'_r}{n^2}, \frac{k'_{r+1}}{n^2}, \ldots, \frac{k'_n}{n^2}\Big],$$

which coincides with s or it is not equal with s and in this case at least one component (e.g. uth) is $\frac{1}{n^2}$ greater then α_u.

If the unit vectors are given, then considering Steps 1 and 2, $p \leq \sum_i (k_i + k'_i) + 1$ is the length of P_p.

Then, it is possible to continue the process similarly as in the Step 2 for the precision $\frac{1}{n^3}$ etc.

By Theorem 8, Theorem 10 has been proved. □

10.2. The structure of the group $S_n(\rho_i, 1)$

Similarly as in Theorem 3 one sees easily

THEOREM 11. *The elements $s \in S_n(\rho_i, 1)$ with $\alpha_i = 1$ form a commutative normal subgroup of $S_n(\rho_i, 1)$.*

Let $NC(1)$ be this group.

Let $s' = [\alpha'_1, \ldots, \alpha'_n] \in S_n(\rho_i, 1)$ be an fixed element and $s = [\alpha_1, \ldots, \alpha_n] \in S_n(\rho_i, 1)$ an arbitrary element.

Because $S_n(\rho_i, 1)$ is a group, the conjugates of the element s' in this group are

$$ss's^{-1} = [\alpha_1 + \alpha_i\alpha'_1 - \alpha'_i\alpha_1, \ldots, \alpha'_i, \ldots, \alpha_n + \alpha_i\alpha'_n - \alpha'_i\alpha_n] =$$
$$= [\alpha_1(1 - \alpha'_i) + \alpha_i\alpha'_1, \ldots, \alpha'_i, \ldots, \alpha_n(1 - \alpha'_i) + \alpha_i\alpha'_n].$$

From this, it follows that for a given s', the group $G(s')$ generated by the elements

$$ss's^{-1} \qquad (s' \in S_n(\rho_i, 1))$$

has elements of the form $[\ldots, \alpha_i'^k, \ldots]$, $k = 0, \pm 1, \ldots$. From this property, it follows

THEOREM 12. *$G(s')$ is a normal subgroup of $S_n(\rho_i, 1)$.*

THEOREM 13. *The normal subgroup $NC(1)$ of $S_n(\rho_i, 1)$ contains the commutator subgroup of $S_n(\rho_i, 1)$.*

Proof. One sees easily that for every element $s' \in S_n(\rho_i, 1)$ in the product $\overline{s} = ss's^{-1}s'^{-1}$, $s \in S_n(\rho_i, 1)$ the component $\overline{\alpha}_i$ is 1, therefore $\overline{\alpha} \in NC(1)$. □

10.3. The structure of the semigroup $S_n^+(\rho_i)$

The semigroup $S_n^+(\rho_i)$ is the set of the elements $s \in S_n(\rho_i)$ for which $\alpha_j \geq 0$, $j = 1, \ldots, n$ and $\alpha_i \neq 0$.

Let $\|s\| = \sum_{j=1}^n \alpha_j$.

By simple calculation, one sees the following

THEOREM 14. *If $s \in S_n^+(\rho_i)$ and $s' \in S_n(\rho_i, 1)$, then*

$$\|s\| = \|s\rho_i s'\|.$$

COROLLARY 1. *The semigroup $S_n^+(\rho_i)$ has the disjunctive decomposition*

$$S_n^+(\rho_i) = \bigcup_{\|s\|} sS_n^+(\rho_i, 1)$$

where $S_n^+(\rho_i, 1) = D(\rho_i)$ (see section 10.1).

REMARK. A similar decomposition is valid also for the group $S_n(\rho_i)$.

10.4. The space S_n^m

Similarly as in the case of the real space R^m, it is possible to introduce the concept of the S_n^m space.

The vector

$$\mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{bmatrix} \qquad s_j \in S_n, \; j = 1, \ldots, m$$

is a "point" of S_n^m. The vectors s_j are real vectors with n components. The addition of such vectors goes in the classical way. The right ρ_i-multiplication with a $\lambda \in S_n(\rho_i)$ is

$$\mathbf{s}\rho_i\lambda = \begin{bmatrix} s_1\rho_i\lambda \\ s_2\rho_i\lambda \\ \vdots \\ s_m\rho_i\lambda \end{bmatrix}.$$

The ρ_i-right linear combination of the vectors $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k (\in S_n)$ is

$$\mathbf{s}_1\rho_i\lambda_1 + \cdots + \mathbf{s}_k\rho_i\lambda_k,$$

where $\lambda_j \in S_n$, $j = 1, \ldots, k$.

A $\lambda \in S_n$ is nonnegative ($\lambda \geq 0$) if every component of λ is nonnegative.

The right linear combination is *convex* if $\lambda_j \geq 0$ and $\sum_{j=1}^{k} \lambda_j = [1, \ldots, 1]$.

10.5. Functions in S_n^m space

A function is a mapping

$$f : S_n^{(1)} \longrightarrow S_n^{(2)},$$

where $S_n^{(1)}, S_n^{(2)}$ are subsets of S_n^m. The continuity of a function f is defined in the classical way.

1. Fixed point theorem

Let

$$\mathbf{s}_j = \begin{bmatrix} s_1^{(j)} \\ s_2^{(j)} \\ \vdots \\ s_m^{(j)} \end{bmatrix}, \qquad j = 1, \ldots, m$$

be given vectors of S_n^m.

The convex ρ_i right linear combination of the vectors $\mathbf{s}_j$ forms a *"polytope"* $\mathcal{S}(\rho_i)$ in the space S_n^m.

We will use the following special vectors:

If $s = [\alpha_1, \ldots, \alpha_n]$, $s \in S_n$, then $s(\alpha_k)$ is the vector, where $\alpha_l = 0$, $l \neq k$. Then it is clear that $\sum_{k=1}^n s(\alpha_k) = s$.

The polytop $\mathcal{S}(\rho_i)$ is bounded if every (real) component in $\mathbf{s}_j$, $(j = 1, \ldots, m)$ is bounded.

THEOREM 15. *Let f be a continuous mapping of the bounded $\mathcal{S}(\rho_i)$ into itself. Then f has a fixed point, that is, there exists an $\mathbf{s} \in \mathcal{S}(\rho_i)$ such that $f(\mathbf{s}) = \mathbf{s}$.*

Proof. Consider the convex ρ_i right linear combinations of the vectors $\mathbf{s}_j$

$$\sum_{j=1}^m \mathbf{s}_j \rho_i \lambda_j.$$

This sum produces the points of the polytope $\mathcal{S}(\rho_i)$. Here

$$\mathbf{s_j}\rho_i\lambda_j = \begin{bmatrix} s_1^{(j)} \\ s_2^{(j)} \\ \vdots \\ s_m^{(j)} \end{bmatrix} \rho_i\lambda_j = \begin{bmatrix} \alpha_{11}^{(j)} + \alpha_{1i}^{(j)}\lambda_{j1}, \ldots \alpha_{1i}^{(j)}\lambda_{ji}, \ldots, \alpha_{1n}^{(j)} + \alpha_{1i}^{(j)}\lambda_{jn} \\ \alpha_{21}^{(j)} + \alpha_{2i}^{(j)}\lambda_{j1}, \ldots, \alpha_{2i}^{(j)}\lambda_{ji}, \ldots, \alpha_{2n}^{(j)} + \alpha_{2i}^{(j)}\lambda_{jn} \\ \vdots \\ \alpha_{m1}^{(j)} + \alpha_{mi}^{(j)}\lambda_{j1}, \ldots, \alpha_{mi}^{(j)}\lambda_{ji}, \ldots, \alpha_{mn}^{(j)} + \alpha_{mi}^{(j)}\lambda_{jn} \end{bmatrix} =$$

$$= \begin{bmatrix} \sum_{k=1,k\neq i}^n s_1^{(j)}(\alpha_{1k}) \\ \vdots \\ \sum_{k=1,k\neq i}^n s_m^{(j)}(\alpha_{mk}) \end{bmatrix} + \begin{bmatrix} s_1^{(j)}(\alpha_{1i}^{(j)}) \\ \vdots \\ s_m^{(j)}(\alpha_{mi}^{(j)}) \end{bmatrix} \lambda_{j1} + \cdots + \begin{bmatrix} s_1^{(j)}(\alpha_{1i}^{(j)}) \\ \vdots \\ s_m^{(j)}(\alpha_{mi}^{(j)}) \end{bmatrix} \lambda_{jn},$$

$$\sum_{j=1}^m \mathbf{s}_j\rho_i\lambda_j =$$

$$= \sum_{j=1}^m \begin{bmatrix} \sum_{k=1,k\neq i}^n s_1^{(j)}(\alpha_{1k}^{(j)}) \\ \vdots \\ \sum_{k=1,k\neq i}^n s_m^{(j)}(\alpha_{mk}^{(j)}) \end{bmatrix} + \sum_{j=1}^m \begin{bmatrix} s_1^{(j)}(\alpha_{1i}^{(j)}) \\ \vdots \\ s_m^{(j)}(\alpha_{1i}^{(j)}) \end{bmatrix} \lambda_{j1} + \sum_{j=1}^m \begin{bmatrix} s_1^{(j)}(\alpha_{1i}^{(j)}) \\ \vdots \\ s_m^{(j)}(\alpha_{mi}^{(j)}) \end{bmatrix} \lambda_{jn}.$$

A vector

$$\begin{bmatrix} s_1^{(j)}(\alpha_{1i}^{(j)}) \\ \vdots \\ s_m^{(j)}(\alpha_{mi}^{(j)}) \end{bmatrix} \in S_n^m(\rho_i)$$

is equivalent to the (real) vector

$$\begin{bmatrix} \alpha_{1i}^{(j)} \\ \vdots \\ \alpha_{mi}^{(j)} \end{bmatrix}$$

because the other components in the vectors $s_k^{(j)}(\alpha_{ki}^{(j)})$ are 0.

Because $\sum_{j=1}^n \lambda_{jl} = 1$, $l = 1, \ldots, n$, then the members in the above-written sum are convex linear combinations of these (real) vectors, which form real polytope. These vectors are given by the vectors $\mathbf{s}_j$ and contain the i-th components of the vectors $\mathbf{s}_j$. Therefore the function f contains m independent mappings of the polytope determined by the real vectors

$$\text{(1)} \qquad \mathbf{v}_{ji} = \begin{bmatrix} s_1^{(j)}(\alpha_{1i}^{(j)}) \\ \vdots \\ s_m^{(j)}(\alpha_{mi}^{(j)}) \end{bmatrix}, \qquad j = 1, \ldots, m.$$

Because f is a $f(\lambda_1, \ldots, \lambda_n)$ continuous mapping of the original polytope, therefore it is a continuous mapping of the above real polytopes (into themselves). Using the classical fixed point theorem of Brouwer for these real polytopes, one gets the fixpoint theorem for the $S_n^m(\rho_i)$ polytope. It means that there exist parameters λ_j^* for which

$$f\Big(\sum_{j=1}^n \mathbf{s}_j \rho_i \lambda_j^*\Big) = \sum_{j=1}^n \mathbf{s}_j \rho_i \lambda_j^*. \qquad \square$$

COROLLARY 2. *Because every bounded, closed and convex set in R^n can be approximated by polytopes, therefore Theorem* 15 *is valid in the case when $\mathcal{S}$ is an arbitrary bounded, closed and convex set in $S^m(\rho_i)$ and f is a continuous mapping of this set into itself.*

2. Fixed point theorem

If we define the right $(\rho_{i_1}, \ldots, \rho_{i_m})$ linear combination with the sum

$$\sum_{j=1}^m \mathbf{s}_j \rho_{i_j} \lambda_j$$

where ρ_{i_k} are not different necessarily, then we get for the polytope $\mathcal{S}(\rho_{i_1}, \ldots, \rho_{i_m})$ an analogous fixed point theorem as before. In this case, one gets the linear combination of not only the vectors (1) but the linear combinations of all the vectors

$$\mathbf{v}_{jj} = \begin{bmatrix} \alpha_{1i_j}^{(j)} \\ \vdots \\ \alpha_{mi_j}^{(j)} \end{bmatrix}, \qquad j = 1, \ldots, m.$$

Because the operations ρ_{i_j}, $j = 1, \ldots, m$ are not necessarily different, therefore one gets at least m different polytopes for which the function f gives a continuous mapping and the conditions ensure the fixed points for every polytope.

THEOREM 16. *Let f be a continuous mapping of the bounded and closed $\mathcal{S}(\rho_{i_1}, \ldots, \rho_{i_m})$ into itself. Then f has a fixed point, that is, there exists an $\mathbf{s} \in \mathcal{S}(\rho_{i_1}, \ldots, \rho_{i_m})$ such that $f(\mathbf{s}) = \mathbf{s}$.*

An application

In the theory of games, the Nikaido–Isoda theorem plays an important role. It is possible to give a generalization of this theorem. The proof can be done analogously to the original proof of the Nikaido–Isoda theorem. To this it is necessary to use section 10.4. and the Theorem 16.

Let $K(f) \longrightarrow \mathcal{R}$ a mapping to the set of the real numbers introduced for f. It means that to every $\mathbf{s} \in S_n^m$ belongs a real number $K(f(\mathbf{s}))$.

THEOREM 17. *Consider the $\Gamma = \{\Sigma_1, \ldots, \Sigma_t; f_1, \ldots, f_t\}$ t-person game with the following conditions:*

1) *Σ_i are $S_i(\rho_{i_1}, \ldots, \rho_{i_{m_i}})$ bounded, convex, closed polytopes of $S_{n_i}^{m_i}$.*
2) *The functions $f_i(\mathbf{s_1}, \ldots, \mathbf{s_t})$, $(i = 1, \ldots, t)$ are concave in the variable $\mathbf{s}_i$ (with respect to $K(f)$ and Section 10.4).*
3) *The functions f_i are continuous in every variable $\mathbf{s}_j$, $(j = 1, \ldots, t)$.*

Then the game Γ has equilibrium point.

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Manuscripts and editorial correspondence should be addressed to
Dr. József Merza
Studia Scientiarum Mathematicarum Hungarica
P.O.Box 127
H-1364 Budapest
Hungary
Phone: (36 1) 318 2875
Fax: (36 1) 317 7166
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GYÖRGY MICHALETZKY,
JÓZSEF BOKOR and PÉTER VÁRLAKI

Representability in Stochastic Systems

The purpose of this book is to investigate modeling and representation approaches of stationary stochastic phenomena. These approaches have their origins in the theory of multivariate stochastic processes, time series analysis and in the algebraic geometric theory of stochastic systems. The stochastic representations most frequently used are the auto-regressive-moving-average (ARMA), matrix-fraction-descriptions (MFD) and the state-space representations. It is shown how to derive these — forward and backward — representations and their dual forms from the analytic and co-analytic spectral factors. These representations are parametrized by system invariants reflecting the four basic Kalmanian principles of controllability, observability, reachability and reconstructibility.
Detailed structure of the state-space realizations is provided using geometric (Hilbert-space) principles including the analysis of the zero structure and balanced realizations.
The structure of generalized Wiener-Hopf factorization is studied in details, first using geometric consideration, then computing the system matrices.
The perspective provided by this can be interesting for those who are doing research in signal processing, stochastic modeling and system identification, or in control system design. Parts of the text can also be useful in courses on stochastic systems, filtering, prediction or on realization theory.

AKADÉMIAI KIADÓ, BUDAPEST

The fundamental concepts dealt with here are the *structure of parentheses* and the *ρ-multiplications* of vectors, in the latter case the product of two vectors is again a vector and the stucture (for a given ρ) is a semigroup.

By means of a multiplicative structure it is possible to introduce the concept of the *coded structure* and a procedure for solving coding and decoding problems can then be given.

In addition, applications for rings and finite algebras are illustrated. To demonstrate other applications, a process is given for approximating an arbitrary distribution vector starting from uniform distribution, a novel theorem is given for the theory of games.

An extremely useful feature of the *structure of vectorproducts* is that it can easily be implemented on a computer.

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JENŐ SZÉP *Vectorproducts and Applications*