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ADDITION THEOREMS FOR THE SPHERICAL FUNCTIONS OF THE LORENTZ GROUP

M. HUSZÁR

Central Research Institute for Physics
H-1525 Budapest 114, P.O.B.49, Hungary

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ABSTRACT

Given a function $f(p_a p_b)$ of the scalar product of two timelike, light-like or spacelike four-vectors it can be expanded in terms of the products $Y^{\sigma}_{\vec{p}}(p_a)^* Y^{\sigma}_{\vec{p}}(p_b)$ of spherical functions on the $(p_a)^2$ and $(p_b)^2$ hyperboloids. Simple formulas for the evaluation of the expansion coefficients are derived. Six types of expansions exist according to whether p_a or p_b are timelike, lightlike or spacelike four-vectors.

АННОТАЦИЯ

Функция $f(p_a p_b)$ от скалярного произведения двух времени-, свето- или пространственноподобных 4-векторов разложима по произведениям $Y^{\sigma}_{\vec{p}}(p_a)^* Y^{\sigma}_{\vec{p}}(p_b)$ сферических функций на гиперболоидах $(p_a)^2$ и $(p_b)^2$. Выведены простые формулы для вычисления коэффициентов разложения. Имеется шесть типов разложений, соответствующих времени, свето- или простанственному характеру 4-векторов p_a и p_b.

KIVONAT

Két idõszerü, fényszerü vagy térszerü vektor skalárszorzatának $f(p_a p_b)$ függvénye kifejthetõ a $(p_a)^2$ ill. $(p_b)^2$ hiperboloidon értelmezett gömbfüggvények $Y^{\sigma}_{\vec{p}}(p_a)^* Y^{\sigma}_{\vec{p}}(p_b)$ alakú szorzata szerint. Egyszerü formulákat adunk a kifejezési együtthatók kiszámitására. Hatféle kifejtés létezik aszerint, hogy p_a és p_b idõszerü, fényszerü, vagy térszerü vektor.

1. INTRODUCTION

In a recent paper [1] a complete set of orthonormal functions on the timelike and spacelike hyperboloids as well as on the light cone have been derived. The homogeneous spaces considered are on the surfaces $u_\mu u^\mu = (u^0)^2-(u^1)^2-(u^2)^2-(u^3)^2 = \xi$ (with $\xi = 1,0,-1$) and can be specified as follows:

$H_+^\uparrow$: upper sheet of the double-sheeted hyperboloid,
$\xi = 1$, $u_\mu u^\mu = 1$, $u^0 \geq 1$,

$H_0^\uparrow$: forward light cone, $\xi = 0$, $u_\mu u^\mu = 0$, $u^0 > 0$,

H_- : single-sheeted hyperboloid, $\xi = -1$, $u_\mu u^\mu = -1$.

A parametrization introduced in [1] can be written in a unified form for all the three homogeneous spaces as

$$u^+ = \xi\ell + \frac{|\vec{z}|^2}{\ell}\,, \quad u^- = \frac{1}{\ell}\,, \quad \vec{u} = \frac{\vec{z}}{\ell} \qquad (\xi = 1,0,-1) \tag{1}$$

where $u^+ = u^0 + u^3$, $u^- = u^0-u^3$, $\vec{u} = (u^1, u^2)$, $\vec{z} = (x,y)$. The parameter ℓ ranges over $0<\ell<\infty$ for the hyperboloid $H_+^\uparrow$ and the cone $H_0^\uparrow$ and over $-\infty<\ell<\infty$ $(\ell\neq 0)$ for the hyperboloid H_-. The two-dimensional vector $\vec{z}$ covers the entire (x,y) plane in each case.

Denote the generators of spatial rotations and boosts by $\vec{M}$ and $\vec{N}$. The spherical functions on either of the above spaces satisfy then the eigenvalue equation of the Casimir operator,

$$(\vec{M}^2 - \vec{N}^2)Y = (j_0^2 - \sigma^2 - 1)Y \tag{2}$$

where $j_0 = 0$, σ is real continuous for the spherical functions on $H_+^\uparrow$, $H_0^\uparrow$ and on H_- for the continuous part of the spectrum. There is also a discrete spectrum on H_- for which $\sigma=0$, j_0=integer.

The basis is defined by the eigenvalue equation of the horospheric momenta

$$(N_1 + M_2)Y = P_1Y\,, \quad (N_2 - M_1)Y = P_2Y \tag{3}$$

which hold for all the three values of ξ.

The simultaneous eigenfunctions of Eqs. (2) and (3) derived in [1] are as follows:

1. Double-sheeted hyperboloid ($H_+^\uparrow$, $\xi=1$, $j_o=0$)

$$Y_{\vec{P}}^{\sigma}(\ell,\vec{z}) = \frac{2}{\pi}\sqrt{\frac{sh\pi\sigma}{\sigma}}\,\ell K_{i\sigma}(P\ell)\left(\frac{1}{2\pi}\,e^{-i\vec{P}\vec{z}}\right) \qquad (4)$$

$$(0<\sigma<\infty)$$

where $K_{i\sigma}$ is the third kind modified Bessel function [2], $\vec{P} = (P_1,P_2)$, $\vec{z} = (x,y)$, $\vec{P}\,\vec{z} = P_1x + P_2y$, $P = \sqrt{(P_1)^2 + (P_2)^2}$.

2. Light cone ($H_o^\uparrow$, $\xi=0$, $j_o=0$)

$$y_{\vec{P}}^{\sigma}(\ell,\vec{z}) = \frac{P^{-i\sigma}}{i\sigma\sqrt{\pi}}\,\ell^{1-i\sigma}\left(\frac{1}{2\pi}\,e^{-i\vec{P}\vec{z}}\right) \qquad (5)$$

$$(-\infty<\sigma<\infty).$$

3. Single-sheeted hyperboloid (H_-, $\xi = -1$)

a. Discrete spectrum ($\sigma = 0$, j_o = integer)

$$\mathbb{Y}_{\vec{P}}^{j_o}(\ell,\vec{z}) = i^n\sqrt{\frac{2}{|n|}}\,|\ell|J_n(P\ell)\left(\frac{1}{2\pi}\,e^{-i\vec{P}\vec{z}}\right) \qquad (6)$$

$$(n = j_o = \pm 1, \pm 2,\ldots)$$

where J_n is the Bessel function.

b. Continuous spectrum ($j_o=0$, σ continuous)

$$\mathbb{Y}_{\vec{P}}^{\sigma}(\ell,\vec{z}) = \frac{1}{\sqrt{\sigma sh\pi\sigma}}\,|\ell|J_{i\varepsilon\sigma}(P|\ell|)\left(\frac{1}{2\pi}\,e^{-i\vec{P}\vec{z}}\right) \qquad (7)$$

where $\varepsilon = sg(\ell) = \pm 1$ and $J_{i\varepsilon\sigma}$ is the Bessel function.

The functions given by Eqs. (4) and (5) form a complete set of functions on $H_+^\uparrow$ and on $H_o^\uparrow$. For a complete set on the single-sheeted hyperboloid H_- both the spherical functions of the discrete (6) and the continuous spectra (7) are needed.

The aim of the present paper is to give formulas for the expansion of a sufficiently well behaved function $f(u_au_b)$ of the scalar product of two four-vectors in terms of spherical functions of $Y_{\vec{P}}^{\sigma}(u_a)$ and $Y_{\vec{P}}^{\sigma}(u_b)$. Let e.g. $(u_a)^2=1$, $(u_b)^2=1$ be two timelike vectors. Then an expansion of this kind looks like

$$f(u_au_b) = \int_0^\infty \sigma^2 d\sigma \int_{-\infty}^{\infty} d^2\vec{P}\; a(\sigma)Y_{\vec{P}}^{\sigma}(u_a)^*Y_{\vec{P}}(u_b)\,, \qquad (8)$$

where the coefficient $a(\sigma)$ is determined by

$$\sigma a(\sigma) = \pi\int_{-\infty}^{\infty} d\omega \; sh\omega \sin(\sigma\omega) f(ch\omega) \qquad (9)$$

$$(ch\omega = (u_a u_b) \geq 1, \; -\infty < \omega < \infty).$$

The expansion (8) is conveniently presented in the form of the addition theorem

$$\int_{-\infty}^{\infty} d^2\vec{p} \; Y^{\sigma}_{\vec{p}}(u_a)^* \; Y^{\sigma}_{\vec{p}}(u_b) = \frac{1}{\pi^2 \sigma} \frac{\sin(\sigma\omega)}{sh\omega} \qquad (10)$$

and a subsequent expansion in terms of $\frac{1}{\pi^2 \sigma} \frac{\sin(\sigma\omega)}{sh\omega}$ as is given by

$$f(ch\omega) = \frac{1}{\pi^2} \int_{0}^{\infty} \sigma d\sigma \; a(\sigma) \frac{\sin(\sigma\omega)}{sh\omega} . \qquad (11)$$

This is a sine Fourier expansion whose inverse is given by Eq. (9).

A kind of addition theorem has been derived in [1]. By addition theorem in the present context the one given by Eq. (10) and by its counterparts for spacelike and lightlike vectors are meant. Analogous theorems in other, e.g. in angular momentum basis, can be found in literature, cf. [3], [4].

Due to the presence of the continuous spectrum a proper mathematical framework for treating the spherical functions is the theory of rigged Hilbert spaces. Its application to the present problem as well as the derivation and rigorous meaning of the addition theorems are outlined in the Appendix. It is recommended especially for understanding of the addition theorems where lightlike vectors are involved. In the next section addition theorems analogous to the one considered above are given for u_a, u_b timelike, lightlike and spacelike, which amounts to six cases.

2. ADDITION THEOREMS

1. Timelike-timelike case

Let $(u_a)^2 = 1$, $(u_b)^2 = 1$, $(u_a, u_b \in H^{\dagger}_{+})$ be two timelike vectors. The spherical functions corresponding to them are given by Eq. (4). Consider now the integral

$$I^{\sigma} = \int d^2\vec{p} \; Y^{\sigma}_{\vec{p}}(u_a)^* \; Y^{\sigma}_{\vec{p}}(u_b) \qquad (12)$$

and denote the parameters of u_a and u_b by ℓ_a, $\vec{z}_a$ and ℓ_b, $\vec{z}_b$ as is given by Eq. (1) with $\xi=1$. The integration can be performed in polar coordinates either directly or by noticing that I^{σ} is a Lorentz invariant quantity thus depend-

ing on the scalar product $(u_a u_b)$ only in which case it can be evaluated in the frame $u_a = (1,0,0,0)$, $u_b = (u_b^0,0,0,u_b^3)$ which corresponds to the values of the coordinates, $\ell_a=1$, $\vec{z}_a=0$; $0<\ell_b<\infty$, $\vec{z}_b=0$.
The result is in any case

$$\int d^2\vec{p}\; Y^\sigma_{\vec{p}}(u_a)^*\; Y^\sigma_{\vec{p}}(u_b) = \frac{1}{\pi^2\sigma}\,\frac{\sin(\sigma\omega)}{\mathrm{sh}\omega} \tag{13}$$

with

$$\mathrm{ch}\omega = (u_a u_b) = \frac{1}{2\ell_a\ell_b}[\ell_a^2 + \ell_b^2 + (\vec{z}_a-\vec{z}_b)^2] \geq 1. \tag{14}$$

To obtain the expansion of a function $f(u_a u_b)$ in terms of spherical functions expand $\mathrm{sh}\omega f(\mathrm{ch}\omega)$ in Fourier integral,

$$\mathrm{sh}\omega f(\mathrm{ch}\omega) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\sigma d\sigma\; a(\sigma)\sin(\sigma\omega)\;. \tag{15}$$

As $\sigma\sin(\sigma\omega)$ is an even function of σ only the even part of $a(\sigma)$ contributes which means that $a(\sigma)$ can be supposed to be an even function, i.e.

$$\mathrm{sh}\omega f(\mathrm{ch}\omega) = \frac{1}{\pi^2}\int_0^{\infty}\sigma d\sigma\; a(\sigma)\sin(\sigma\omega)\;. \tag{16}$$

This can be easily reversed,

$$\sigma a(\sigma) = \pi\int_{-\infty}^{\infty}d\omega\; \mathrm{sh}\omega\sin(\sigma\omega)f(\mathrm{ch}\omega). \tag{17}$$

Substituting (13) into (16) one gets the expansion (8),

$$f(u_a u_b) = \int_0^{\infty}\sigma^2 d\sigma\int_{-\infty}^{\infty}d^2\vec{p}\; a(\sigma)\; Y^\sigma_{\vec{p}}(u_a)^*\; Y^\sigma_{\vec{p}}(u_b)\;. \tag{18}$$

2. <u>Timelike-lightlike case</u>

Let $u^2=1$, $k^2=0$ $(u\in H_+^\uparrow, k\in H_0^\uparrow)$ then the addition theorem is

$$\int d^2\vec{p}\; y^\sigma_{\vec{p}}(k)^* Y^\sigma_{\vec{p}}(u) = -\frac{1}{2\pi^2}\sqrt{\frac{\mathrm{sh}\pi\sigma}{\pi\sigma}}\;\Gamma(i\sigma)(ku)^{-1-i\sigma} \tag{19}$$

where the scalar product is expressed through the coordinates of $u=u(\ell,\vec{z})$, $k=k(\ell_0,\vec{z}_0)$ as

$$w = (ku) = \frac{1}{2\ell\ell_0}[\ell^2 + (\vec{z}-\vec{z}_0)^2] > 0.$$

The expansion of $f(ku)$ in terms of (19) is a Mellin transformation,

$$f(w) = -\frac{1}{2\pi^2}\int_{-\infty}^{\infty}\sigma^2\, d\sigma \sqrt{\frac{sh\pi\sigma}{\pi\sigma}}\Gamma(i\sigma)a(\sigma)w^{-1-i\sigma} \tag{20}$$

which has the inverse

$$\sigma^2 a(\sigma) = -\pi\sqrt{\frac{\pi\sigma}{sh\pi\sigma}}\frac{1}{\Gamma(i\sigma)}\int_0^{\infty}dw\, w^{i\sigma}f(w). \tag{21}$$

3. Timelike-spacelike case

Let now $u^2=1$, $q^2=-1$ ($u\in H_+^{\uparrow}$, $q\in H_-$) then the addition theorem reads

$$\int d^2\vec{P}\; Y_{\vec{P}}^{\sigma}(u)^* \mathbb{Y}_{\vec{P}}^{\sigma}(q) = \frac{1}{2\pi^2|\sigma|}\frac{e^{-i\sigma\omega}}{ch\omega} \tag{22}$$

with

$$(qu) = sh\omega = \frac{1}{2\ell_a\ell_b}[\ell_a^2-\ell_b^2+(\vec{z}_a-\vec{z}_b)^2] \qquad (-\infty<\omega<\infty)$$

where the coordinates of u and q are $u=u(\ell_a,\vec{z}_a)$, $q=q(\ell_b,\vec{z}_b)$. The expansion of $f(qu)$ in terms of (22) is again a Fourier expansion

$$f(sh\omega) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty}d\sigma\,\frac{a(\sigma)}{|\sigma|}\frac{e^{-i\sigma\omega}}{ch\omega} \tag{23}$$

with the inversion formula

$$a(\sigma) = \pi|\sigma|\int_{-\infty}^{\infty}d\omega\; ch\omega f(sh\omega)e^{i\sigma\omega}\;. \tag{24}$$

4. Lightlike-lightlike case

Let $(k_a)^2=0$, $(k_b)^2=0$ be two lightlike vectors (k_a, $k_b\in H_0^{\uparrow}$). The present and the next addition theorems differ from the previous ones in their form, as the label is extended to complex values. The Appendix provides some insight into this problem.

It has been shown in [1] that for real values of σ $y_{\vec{P}}^{\sigma}(k)$ satisfies orthogonality and completeness relations. These can be generalized as follows

$$\int d^2z\frac{d\ell}{2\ell^3}\, y_{-\vec{P}'}^{-\sigma'}(\ell,\vec{z})\; y_{\vec{P}}^{\sigma}(\ell,\vec{z}) = \frac{1}{\sigma^2}\delta(\sigma_1'-\sigma_1)\delta^2(\vec{P}'-\vec{P}) \tag{25}$$

$$\int_{-\infty+i\sigma_2}^{\infty+i\sigma_2}\sigma^2 d\sigma d^2\vec{P}\; y_{-\vec{P}}^{-\sigma}(\ell',\vec{z}')y_{\vec{P}}^{\sigma}(\ell,z) = 2\ell^3\delta(\ell'-\ell)\delta^2(\vec{z}'-\vec{z}). \tag{26}$$

$$(\sigma = \sigma_1+i\sigma_2,\; \sigma' = \sigma_1'+i\sigma_2)$$

For real values of σ $y^{\sigma}_{\vec{p}}(k)^* = y^{-\sigma}_{-\vec{p}}(k)$, i.e. these equations reproduce the usual orthogonality and completeness relations.

The addition theorem is given in the form

$$\int d^2\vec{p}\; y^{\sigma}_{\vec{p}}(k_a) y^{\sigma}_{-\vec{p}}(k_b) = M_\sigma (k_a k_b)^{-1+i\sigma} \tag{27}$$

$$(-1<\mathrm{Im}\sigma<-\tfrac{1}{4})$$

with

$$M_\sigma = -\frac{2^{-1-i\sigma}}{\pi^2\sigma^2}\frac{\Gamma(1-i\sigma)}{\Gamma(i\sigma)} .$$

The scalar product is expressed through the coordinates simply as

$$(k_a k_b) = \frac{(\vec{z}_a-\vec{z}_b)^2}{2\ell_a\ell_b} .$$

Expand $f(k_a k_b)$ in terms of (27),

$$f(k_a k_b) = \int_{-\infty+i\sigma_2}^{\infty+i\sigma_2} d\sigma\; M_\sigma a(\sigma)(k_a k_b)^{-1+i\sigma} . \tag{28}$$

This can be reversed as

$$M_\sigma a(\sigma) = \frac{1}{2\pi}\int_0^\infty dw\; f(w) w^{-i\sigma} \qquad (\sigma=\sigma_1+i\sigma_2,\quad -1<\sigma_2<-\tfrac{1}{4}) .$$

5. Lightlike-spacelike case

Let $k^2=0$ be a lightlike and $q^2=-1$ a spacelike vector ($k\in H_0^{\uparrow}$, $q\in H_-$). Add a positive imaginary part to σ, i.e. $\sigma=\sigma_1+i\sigma_2$ then the following addition theorems hold in the strip $0<\sigma_2<1$

$$\int d^2\vec{p}\; y^{-\sigma}_{-\vec{p}}(k) Y^{\sigma}_{\vec{p}}(q) = N_\sigma (kq)_+^{-1-i\sigma} \tag{29}$$

$$\int d^2\vec{p}\; y^{-\sigma}_{-\vec{p}}(k) Y^{-\sigma}_{\vec{p}}(q) = N_\sigma (kq)_-^{-1-i\sigma} \tag{30}$$

$$(0<\mathrm{Im}\sigma<1)$$

where

$$N_\sigma^{-1} = 2\pi\sqrt{\pi\sigma\,\mathrm{sh}\,\pi\sigma}\;\Gamma(1-i\sigma)$$

and

$$(kq)_+^{\mu} = \begin{cases} (kq)^{\mu} & \text{if } (kq) > 0 \\ 0 & \text{if } (kq) < 0 \end{cases}$$

$$(kq)_-^{\mu} = \begin{cases} 0 & \text{if } (kq) > 0 \\ |kq|^{\mu} & \text{if } (kq) < 0 \end{cases} .$$

The scalar product is expressed in terms of $k = k(\ell_a, \vec{z}_a)$, $q = q(\ell_b, \vec{z}_b)$ as

$$(kq) = \frac{1}{2\ell_a\ell_b}\,[(\vec{z}_a-\vec{z}_b)^2-\ell_b^2].$$

The expansion in terms of $(kq)_{\pm}^{-1-i\sigma}$ is again a Mellin transformation

$$f(kq) = \int_{-\infty+i\sigma_2}^{\infty+i\sigma_2} d\sigma\; N_\sigma[a_+(\sigma)(kq)_+^{-1-i\sigma} + a_-(\sigma)(kq)_-^{-1-i\sigma}] \tag{31}$$

which has the inverse

$$N_\sigma a_{\pm}(\sigma) = \frac{1}{2\pi}\int_0^\infty dw\; f(\pm w)w^{i\sigma} \qquad (0<\mathrm{Im}\sigma<1). \tag{32}$$

6. Spacelike-spacelike case

As a consequence of the contribution from the supplementary series this case is somewhat more complicated than the previous ones. Let $(q_a)^2=-1, (q_b)^2=-1$ be two spacelike vectors with the corresponding spherical functions $\Upsilon_{\vec{p}}^{\sigma}(q_a)$, $\Upsilon_{\vec{p}}^{\sigma}(q_b)$ of the continuous spectrum. For these two kinds of addition theorems are needed one of which is conveniently stated as

$$\int d^2\vec{p}\,[\Upsilon_{\vec{p}}^{\sigma}(q_b)^*\,\Upsilon_{\vec{p}}^{\sigma}(q_a)+\Upsilon_{\vec{p}}^{-\sigma}(q_b)^*\,\Upsilon_{\vec{p}}^{-\sigma}(q_a)] = \begin{cases} -\dfrac{1}{\pi^2\sigma}\dfrac{\sin(\sigma\Theta)}{\mathrm{sh}\Theta} & \text{if } (q_aq_b) = -\mathrm{ch}\Theta<-1 \\[2ex] \dfrac{1}{\pi^2\sigma}\dfrac{\mathrm{ch}(\sigma\varphi)}{\mathrm{sh}\pi\sigma\sin\varphi} & \text{if } -1<(q_aq_b)=\cos\varphi<1 \\[2ex] 0 & \text{if } (q_aq_b) = \mathrm{ch}\Theta>1 \end{cases} \tag{33}$$

and the other one as

$$\int d^2\vec{p}\,[\,\mathbb{Y}^{\sigma}_{\vec{p}}(q_b)^*\,\mathbb{Y}^{-\sigma}_{\vec{p}}(q_a)+\mathbb{Y}^{-\sigma}_{\vec{p}}(q_b)^*\,\mathbb{Y}^{\sigma}_{\vec{p}}(q_a)\,] = \begin{cases} 0 \text{ if } (q_a q_b) = -\mathrm{ch}\theta<-1 \\[2ex] \dfrac{1}{\pi^2\sigma}\,\dfrac{\mathrm{ch}\sigma(\pi-\varphi)}{\mathrm{sh}\pi\sigma\sin\varphi} \text{ if } -1<(q_a q_b)=\cos\varphi<1 \\[2ex] -\dfrac{1}{\pi^2\sigma}\,\dfrac{\sin(\sigma\theta)}{\mathrm{sh}\theta} \text{ if } (q_a q_b) = \mathrm{ch}\theta>1 \end{cases}$$

$$(\theta>0,\ 0<\varphi<\pi). \tag{34}$$

A similar relation holds for the spherical functions of the discrete spectrum,

$$\int d^2\vec{p}\,\mathbb{Y}^{n}_{\vec{p}}(q_b)^*\,\mathbb{Y}^{n}_{\vec{p}}(q_a) = \begin{cases} 0 & \text{if } (q_a q_b)<-1 \\[2ex] \dfrac{(-)^n}{\pi^2|n|}\,\dfrac{\cos n\varphi}{\sin\varphi} & \text{if } -1<(q_a q_b) = \cos\varphi<1 \\[2ex] 0 & \text{if } (q_a q_b)>1 \end{cases} \tag{35}$$

$$(0<\varphi<\pi).$$

It is seen that when expanding a function $f(q_a q_b)$ in terms of the above formulas only Eq. (33) (Eq. (34)) contributes in the range $(q_a q_b)<-1$ ($(q_a q_b)>1$) while all the three types of functions given by Eqs. (33),(34),(35) show up in the range $-1<(q_a q_b)<1$.

The expansion in the first range is

$$f(q_a q_b) = f(-\mathrm{ch}\theta) = -\frac{1}{2\pi^2}\int_{-\infty}^{\infty} d\sigma\; c_-(\sigma)\,\frac{\sin(\sigma\theta)}{\mathrm{sh}\theta}$$

$$\text{if } (q_a q_b) = -\mathrm{ch}\theta<-1 \quad (\theta>0).$$

Since only the odd part of $c_-(\sigma)$ contributes it can be supposed to be an odd function of σ that implies the expansion of the form

$$f(q_a q_b) = f(-\mathrm{ch}\theta) = -\frac{1}{\pi^2}\int_{0}^{\infty} d\sigma\; c_-(\sigma)\,\frac{\sin(\sigma\theta)}{\mathrm{sh}\theta} \quad ((q_a q_b)<-1) \tag{36}$$

which is a sine Fourier expansion of the odd function $\mathrm{sh}\theta f(-\mathrm{ch}\theta)$.

Similarly, in the range $(q_a q_b)>1$ one gets

$$f(q_a q_b) = f(\mathrm{ch}\theta) = -\frac{1}{\pi^2}\int_{0}^{\infty} d\sigma\; c_+(\sigma)\,\frac{\sin(\sigma\theta)}{\mathrm{sh}\theta} \quad ((q_a q_b)>1) \tag{37}$$

and in the intermediate range $-1<(q_a q_b)<1$

$$f(q_a q_b) = f(\cos\varphi) = \frac{1}{\pi^2}\int_0^\infty d\sigma \frac{1}{\mathrm{sh}\pi\sigma\sin\varphi} [c_-(\sigma)\mathrm{ch}(\sigma\varphi)+c_+(\sigma)\mathrm{ch}\sigma(\pi-\varphi)] +$$

$$+\frac{1}{2\pi^2} d_o \frac{1}{\sin\varphi} + \frac{1}{\pi^2}\sum_{n=1}^\infty d_n(-)^n \frac{\cos n\varphi}{\sin\varphi} \tag{38}$$

$$(-1<(q_a q_b)<1).$$

The expansions (36) and (37) can easily be inverted,

$$-c_-(\sigma) = 2\pi\int_0^\infty d\Theta\ \mathrm{sh}\Theta\sin(\sigma\Theta)f(-\mathrm{ch}\Theta) \tag{39}$$

$$-c_+(\sigma) = 2\pi\int_0^\infty d\Theta\ \mathrm{sh}\Theta\sin(\sigma\Theta)f(\mathrm{ch}\Theta)\ . \tag{40}$$

Thus the integral term along with $f(\cos\varphi)$ becomes a known function whose Chebishev expansion is Eq. (38). Denoting $\frac{1}{2\pi}(-)^n d_n = f_n+g_n$ $(n=0,1,2,\ldots)$ one gets the final form of the expansion in the intermediate range

$$\sin\varphi f(\cos\varphi) = -\frac{1}{\pi}\int_0^\infty d\Theta\mathrm{sh}\Theta \left[\frac{\mathrm{sh}\Theta}{\mathrm{ch}\Theta-\cos\varphi} f(\mathrm{ch}\Theta) + \frac{\mathrm{sh}\Theta}{\mathrm{ch}\Theta+\cos\varphi} f(-\mathrm{ch}\Theta)\right] +$$

$$+\frac{2}{\pi}\sum_{n=1}^\infty (f_n+g_n)\cos n\varphi + \frac{1}{\pi}(f_o+g_o) \tag{41}$$

$$(\Theta>0,\ 0<\varphi<\pi)$$

where

$$f_n = \int_0^\infty d\Theta\ \mathrm{sh}\Theta e^{-n\Theta}\ [f(\mathrm{ch}\Theta) + (-)^n f(-\mathrm{ch}\Theta)] \tag{42}$$

$$g_n = \int_0^\pi d\varphi\ \sin\varphi\cos n\varphi f(\cos\varphi) \tag{43}$$

$$(n = 0,1,2,\ldots)\ .$$

APPENDIX

Mathematical aspects

The Hilbert space we start with is the space of square integrable functions on the hyperboloid or cone, $\mathcal{H} = L^2(H,dp)$, where H stands for one of the spaces $H_+^\uparrow$, $H_o^\uparrow$, H_- and dp is the invariant measure on H,

$$dp = \frac{d\ell}{2\ell^3}\, d^2\vec{z} \ .$$

The scalar product is defined by

$$(f,g) = \int dp\ f^*(p)g(p) \qquad (f,g\in\mathcal{H}).$$

The operators $\vec{M}^2 - \vec{N}^2$, $N_1 + M_2$, $N_2 - M_1$ whose simultaneous eigenvalue equations have been solved are formally Hermitean provided suitable boundary conditions are imposed on the functions in their domain. Nevertheless, apart from the discrete spectrum, they do not have any nonzero common eigenfunctions in $\mathcal{H}$, as the spherical functions belonging to the continuous spectrum (4),(5),(7) are not normalizable. A proper mathematical framework for treating such eigenfunctions is the theory of rigged Hilbert spaces [5]. Consider a continuous linear operator $K: \mathcal{H} \to \mathcal{H}$ with a dense range and for which $\mathrm{Ker}K=0$. Denote by $\mathcal{H}_+$ the space $\mathcal{H}_+ = K\mathcal{H} \subset \mathcal{H}$ which becomes a separable Hilbert space itself if a scalar product is defined in it by

$$(f_+,g_+)_+ = (K^{-1}f_+,\ K^{-1}g_+) \qquad (f_+,g_+\subset\mathcal{H}_+).$$

Consider the space of continuous antilinear functionals on $\mathcal{H}_+$: $\ell_{g_-}(f_+) = (g_-,f_+) = (f_+,g_-)^*$; $f_+\in\mathcal{H}_+$. The linear space of g_- vectors becomes a ilbert space, too, if a scalar product is defined in it by $(g_-,f_-)_- = (K^+g_-,K^+f_-)$, $(g_-,f_-\in\mathcal{H}_-)$ where K^+ is the adjoint operator. Thus the embedding $\mathcal{H}_+\subset\mathcal{H}\subset\mathcal{H}_-$ is obtained where $\mathcal{H}_+$ is *dense* in $\mathcal{H}$ (in the topology of $\mathcal{H}$) and $\mathcal{H}$ is *dense* in $\mathcal{H}_-$ (in the topology of $\mathcal{H}_-$). The operators K and K^+ map onto each other these spaces according to $\mathcal{H}_+ \xleftarrow{K} \mathcal{H} \xleftarrow{K^+} \mathcal{H}_-$. The scalar products in $\mathcal{H}_+$ and $\mathcal{H}_-$ are defined by pulling back the elements into $\mathcal{H}$ by means of K^{-1} and K^+. The point is that when $\mathcal{H}$ is restricted to $\mathcal{H}_+$ then the space of antilinear functionals on $\mathcal{H}_+$, (g_-,f_+), becomes larger than the original $\mathcal{H}$ and it is large enough to contain the spherical functions of the continuous spectrum. The spherical functions are thus viewed as functionals $(Y^\sigma_{\vec{p}},\ f_+)$ on $\mathcal{H}_+$. The subspace $\mathcal{H}_+$ can be conceived as a space of test functions for the elements of $\mathcal{H}_-$.

The eigenvalue equation $A\psi_\lambda = \lambda\psi_\lambda$ of a selfadjoint operator A in $\mathcal{H}$, $\psi_\lambda\in D_A\subset\mathcal{H}$ holds whenever $(A\varphi,\psi_\lambda) = \lambda(\varphi,\psi_\lambda)$ holds for any $\varphi\in D_A$. This equation can be extended to the vectors $\psi_\lambda\in\mathcal{H}_-$ for $\varphi\in D_A\cap\mathcal{H}_+\neq 0$, $A\varphi\in\mathcal{H}_+$. This provides a generalization to those eigenfunctions of the continuous spectrum which are elements of the Hilbert space $\mathcal{H}_-$.

Thus the rigorous meaning of the eigenvalue equations $(\vec{M}^2-\vec{N}^2)Y^\sigma_{\vec{P}} = -(\sigma^2+1)Y^\sigma_{\vec{P}}$, $(N_1+M_2)Y^\sigma_{\vec{P}} = P_1Y^\sigma_{\vec{P}}$, $(N_2-M_1)Y^\sigma_{\vec{P}} = P_2Y^\sigma_{\vec{P}}$ is that $(A_k\varphi, Y^\sigma_{\vec{P}}) = \lambda_k(\varphi,Y^\sigma_{\vec{P}})$ holds for $\varphi\in D$, $D\equiv D_{A_o}\cap D_{A_1}\cap D_{A_2}\cap \mathcal{H}_+$, $k=0,1,2$, where $A_o = (\vec{M}^2-\vec{N}^2)$, $A_1 = N_1+M_2$, $A_2 = N_2-M_1$ and λ_o, λ_1, λ_2 are the corresponding eigenvalues.

Define the Fourier components of $f\in\mathcal{H}_+$ by

$$c^\sigma(\vec{P}) = (Y^\sigma_{\vec{P}},f) \ . \tag{A1}$$

Due to the Plancherel formula this equation can be extended from $f\in\mathcal{H}_+$ to $f\in\mathcal{H}$. The expansion of $f(p)$ which is the inverse to (A1) has been written in [1] in the form

$$f(p) = \int\sigma^2 d\sigma d^2\vec{P}\ c^\sigma(\vec{P})\ Y^\sigma_{\vec{P}}(p)^*$$

which should be understood as

$$(f,\varphi) = \int\sigma^2 d\sigma d^2\vec{P}\ c^\sigma(\vec{P})(Y^\sigma_{\vec{P}},\varphi) \ .$$

With these preliminary notions at hand addition theorems can be obtained in the following manner.

Consider a bounded linear operator $F: \mathcal{H}_a \rightarrow \mathcal{H}_b$ with a dense domain D_F. Here $\mathcal{H}_a(\mathcal{H}_b)$ is the Hilbert space of the square integrable functions on one of the hyperboloids $H = (H_+^\uparrow, H_o^\uparrow, H_-)$ and F maps $\varphi(p_a)\rightarrow\varphi(p_b)$, $p_a^2 = \xi_a$, $p_b^2 = \xi_b$ with ξ_a and $\xi_b = -1,0,1$, independently.

Suppose that F commutes with the unitary operator of the left displacement,

$$[T_g, F] = 0$$

where T_g is defined as $(T_g\varphi)(p) = \varphi(g^{-1}p)$. Let, furthermore, F has an integral representation of the form

$$(F\varphi)(p_a) = \int dp_b F(p_a,p_b)\varphi(p_b), \quad (p_a,p_b\in H,\varphi(p_b)\in D_F).$$

Then

$$(T_gF\varphi)(p_a) = \int dp_b F(g^{-1}p_a,p_b)\varphi(p_b),$$

$$(FT_g\varphi)(p_a) = \int dp_b F(p_a,p_b)\varphi(g^{-1}p_b) = \int dp_b F(p_a,gp_b)\varphi(p_b).$$

The domain of F beeing dense, the commutativity $[T_g,F] = 0$ implies $F(p_a,gp_b) = F(g^{-1}p_a,p_b)$ or $F(gp_a,gp_b) = F(p_a,p_b)$ which means that $F(p_a,p_b)$ is a function of the scalar product only,

$$F(p_a,p_b) = f(p_ap_b).$$

The condition of the boundedness of F is satisfied for $p_a^2 = p_b^2 = 1$ provided

$$\int_0^\infty d\omega |f(ch\omega)|^2 sh^2\omega < \infty .$$

Similar conditions can be given for the remaining cases.

It is a consequence of $[T_g, F] = 0$ that the operators A_k $(k=0,1,2)$ defining the spherical functions commute with $F, [A_k, F] = 0$ in the sense that for each $\varphi \in \mathcal{L}_+$

$$(\varphi, [A_k, F] Y^\sigma_{\vec{p}}) = (\varphi, A_k(FY^\sigma_{\vec{p}}) - \lambda_k(FY^\sigma_{\vec{p}})) = 0 .$$

This implies that $FY^\sigma_{\vec{p}}$ is also a solution of the eigenvalue equations $A_k Y^\sigma_{\vec{p}} = \lambda_k Y^\sigma_{\vec{p}}$ in the sense of Eq. (A1). It follows from this that

$$(FY^\sigma_{\vec{p}})(p_a) = \int dp_b f(p_a p_b) Y^\sigma_{\vec{p}}(p_b) = c_+(\sigma) Y^\sigma_{\vec{p}}(p_a) + c_-(\sigma) Y^{-\sigma}_{\vec{p}}(p_a)$$

must hold in a weak sense, again. Then as a consequence of the expansion formulas for any $\varphi \in D_F$

$$(F\varphi)(p_a) = \int \sigma^2 d\sigma d^2\vec{p} (c_+(\sigma) Y^\sigma_{\vec{p}}(p_a) + c_-(\sigma) Y^\sigma_{\vec{p}}(p_a))(Y^\sigma_{\vec{p}}, \varphi), \qquad (p_a \in H_+^\uparrow) .$$

Actually, $Y^\sigma_{\vec{p}}$ denotes the spherical functions on the $H_+^\uparrow$ hyperboloid. These are symmetric under the change of the sign of σ, therefore, $c_+(\sigma) + c_-(\sigma) = c(\sigma)$ can be put, i.e.

$$(F\varphi)(p_a) = \int \sigma^2 d\sigma d^2\vec{p}\; c(\sigma) Y^\sigma_{\vec{p}}(p_a)(Y^\sigma_{\vec{p}}, \varphi) \qquad (\varphi \in D_F,\; p_a \in H_+^\uparrow). \tag{A2}$$

An analogous formula holds for the spherical functions on the cone. It turns out that there $c_+(\sigma) = 0$ and the counterpart to (A2) takes the form

$$(F\varphi)(p_a) = \int \sigma^2 d\sigma d^2\vec{p}\; c_-(\sigma) y^{-\sigma}_{\vec{p}}(p_a)(y^\sigma_{\vec{p}}, \varphi) \qquad (\varphi \in D_F,\; p_a \in H_0^\uparrow) . \tag{A3}$$

Rewrite Eq. (A2) conditionally by omitting $\varphi(p_b)$ as

$$f(p_a p_b) = \int \sigma^2 d\sigma d^2\vec{p}\; c(\sigma) Y^\sigma_{\vec{p}}(p_a) Y^\sigma_{\vec{p}}(p_b)^* .$$

Whether or not the integration over $\vec{p}$ can or cannot be performed in advance and the integration in $(Y^\sigma_{\vec{p}}, \varphi)$ over p_b can be left to the subsequent step depends on which spherical function is concerned. Thus e.g.

$$\int d^2\vec{p}\; Y^\sigma_{\vec{p}}(p_a) Y^\sigma_{\vec{p}}(p_b)^* = \frac{1}{\pi^2 \sigma} \frac{\sin(\sigma\omega)}{sh\omega}$$

holds as is given by Eq. (10) and the integrations with respect to σ and p_b can be performed subsequently. On the other hand, the integral

$$\int d^2\vec{p}\, y_{\vec{p}}^{-\sigma}(p_a)\, y_{\vec{p}}^{\sigma}(p_b)^* \tag{A4}$$

does not exist at all. Anyway, the rigorous form of the addition theorems as given by Eqs. (A2),(A3) is always valid. Analogous formulas involving the single-sheeted hyperboloid can be derived in a similar manner.

There exists, however, a quite different viewpoint for treating the integral (A4), namely, by considering $y_{\vec{p}}^{\sigma}(p)$ as an ordinary function no matter whether or not it is an element of any Hilbert space. It turns out that if extending σ to the complex domain the integral (A4) aquires a meaning and the change of the order of integrations is legitimate. In Sect. II. the addition theorem is given in its complex form (cf. in particular, Sects. II.4,5) as for practical purposes it seems more useful.

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63. 774

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Nyelvi lektor: Sebestyén Ákos
Gépelte: Simándi Józsefné
Példányszám: 375 Törzsszám: 85-450
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