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TWO DIMENSIONAL N-COMPONENT
CUBIC MODEL. II.

THE CUBIC TRANSITION

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# HAMILTONIAN STUDIES OF THE TWO DIMENSIONAL N-COMPONENT CUBIC MODEL, II.

## THE CUBIC TRANSITION

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# ABSTRACT

The phase transition behaviour of the Hamiltonian version of the two dimensional n-component cubic model is studied along the cubic transition line. 1/n expansion and anisotropy expansion around the 2n-state Potts point are used to determine the phase transition line and the latent heat. The latent heat depends on the value of the coupling whereas the crossover value of n, where the transition changes from second to first order, does not (it is $n_c = 2$). The latent heat has essential singularity at n=2 along the cubic transition line.

# АННОТАЦИЯ

Исследованы вдоль линии кубического перехода свойства фазового перехода гамильтоновского варианта двумерной n-компонентной кубической модели. Для определения линии перехода и скрытой теплоты применены методы 1/n разложения и разложения по степеням анизотропии в окрестности точки Поттса 2n-ого состояния. Скрытая теплота зависит от константы связи, а критическое значение n, при котором фазовый переход второго рода преобразуется в переход первого рода, не зависит от константы связи ($n_c = 2$). Скрытая теплота имеет существенную особенность при n=2 вдоль кубической линии перехода.

# KIVONAT

A két dimenziós n-komponensü köbös modell Hamilton változatának fázisátalakulási tulajdonságait vizsgáljuk a köbös átalakulási vonal mentén. 1/n sorfejtést és a 2n-állapotu Potts pont körül anizotrópia sorfejtést használunk a fázisátalakulási vonal és a látens hő meghatározására. A látens hő függ a csatolástól, mig n kritikus értéke, ahol az átalakulás másod-rendüből első-rendübe változik, nem függ a csatolástól (ez $n_c = 2$). A látens hő n=2-nél lényeges szingularitást mutat a köbös átalakulási vonalon.

# 1. Introduction

In the first part of this work (Iglói 1984, hereafter referred to as Paper I) the (1+1) dimensional n-component cubic model was investigated by mean-field calculation and by several renormalization-group (RG) transformations. The phase diagram and the critical properties of the (1+1) dimensional model were found to be identical with those of the two dimensional (2D) model.

In this part of the work the properties of the cubic transition are investigated, when this transition is of first order. We use two methods of expansion both of which supply complementary information. The $1/n$ expansion for large values of n is a good approximation, while the coefficients of the anisotropy expansion around the 2n-state Potts point have small values when n is close to 2. These expansion methods, are unfortunately not suitable for investigating the neighbourhood of the multicritical point because at this point all terms of the perturbational expressions become equally important, signalling the role of competing effects.

The $1/n$ series is determined up to second order for the phase transition line, for the latent heat and for the crossover value of $n_c$, where the order of the phase transition changes from first to second order. While in the second part of the paper the 2n-state Potts model was taken as a reference system, and the expansion was made with respect to the anisotropy. This series was determined in first order only

because calculation of the higher order terms would need a
knowledge of the properties of the 2n-state Potts model
beyond the critical point, and these are not known.

The paper is arranged as follows: §.2. contains the
formalism and the duality properties of the model; §§.3. and
4, respectively give the results of the 1/n expansion and
the anisotropy expansion; §.5. contains a summary. The
details of the calculations are presented in the Appendix.

## 2. Duality properties

The time-continuum limit of the 2D n-component cubic model was presented in Paper I. At the same time we determined the different representations of the 1D quantum problem that serve as convenient calculational bases in the different phases (ferro-, para-, partially ordered). To investigate the duality properties of the model we use the strong coupling representation (Paper I, eqs. 2.1 and 2.2), and rewrite the Hamiltonian in the form:

$$H = - \frac{\lambda_1}{2n} \sum_i \sum_{k=1}^{n} \Omega_i^{2k-1} \Omega_{i+1}^{2n-2k+1} - \frac{\lambda_2}{2n} \sum_i \sum_{k=1}^{n-1} \Omega_i^{2k} \Omega_{i+1}^{2n-2k}$$

$$- h_1 \sum_i \sum_{k=1}^{n-1} (M_i^k + M_i^{n+k}) - h_2 \sum_i M_i^n$$

(2.1)

Here $\lambda_1$ and $\lambda_2$ denote the couplings between the neighbouring spins, while $h_1$ and $h_2$ are the strengths of the external fields appearing in the Hamiltonian version of the model. $\Omega$ and $M$ are 2nx2n matrices:

$$\Omega = \begin{bmatrix} 1 & 0 & & & & 0 \\ 0 & \omega & & & & \\ & & \omega^2 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & 0 \\ 0 & & \cdot\cdot & & 0 & \omega^{2n-1} \end{bmatrix} \qquad M = \begin{bmatrix} 0 & 0 & \cdots & & 0 & 1 \\ 1 & 0 & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & 0 & 1 & 0 \end{bmatrix}$$

and $\omega = \exp\left(\frac{2\pi i}{2n}\right)$. These matrices commute in different sites; on the same site they satisfy the $Z(2n)$-algebra:

$$\Omega_i^k \Omega_i^\ell = \Omega_i^{k+\ell}$$

$$M_i^k M_i^\ell = M_i^{k+\ell} \qquad (2.2)$$

$$M_i^k \Omega_i^\ell = \omega^{k\ell} \Omega_i^\ell M_i^k$$

Let ut now introduce a dual lattice and define the operators $\Omega_i^k$, $M_i^k$ on the sites of the dual lattice, i.e. on the links of the original lattice:

$$\overline{\Omega}_i^k = \prod_{j=i} M_j^k$$

$$\overline{M}_i^k = \Omega_i^{2n-k} \Omega_{i+1}^k$$

It is easy to see that these operators also satisfy the $Z(2n)$--algebra. The (2.1) Hamiltonian operator can be written in terms of the new operators as:

$$H = -\frac{\lambda_1}{2n} \sum_i \sum_{k=1}^{n} \overline{M}_i^{2k-1} - \frac{\lambda_2}{2n} \sum_i \sum_{k=1}^{n-1} \overline{M}_i^{2k}$$

$$\qquad (2.3)$$

$$- h_1 \sum_i \sum_{k=1}^{n-1} (\overline{\Omega}_i^k \overline{\Omega}_{i+1}^{2n-k} + \overline{\Omega}_i^{n+k} \overline{\Omega}_i^{n-k}) - h_2 \sum_i \overline{\Omega}_i^n \overline{\Omega}_{i+1}^n$$

Comparing (2.1) and (2.3), one can see that the model is not, in general, self-dual. However, by using the parametrization

$$\frac{\lambda_1}{2n} = 1$$

$$h = \frac{2nh_1}{\lambda_1} = \frac{2nh_2}{\lambda_2} \qquad (2.4)$$

$$\lambda = \frac{\lambda_2}{\lambda_1} = \frac{h_2}{h_1}$$

the Hamiltonian can be written as

$$H = H_o + H_p$$

$$H_o = -\sum_i \sum_{k=1}^{2n-1} \Omega_i^k \Omega_{i+1}^k - h\sum_i \sum_{k=1}^{2n-1} M_i^k - (\lambda-1)\left[\sum_i \sum_{k=1}^{n-1} \Omega_i^{2k} \Omega_{i+1}^{2n-2k}\right.$$

$$(2.5)$$

$$\left. + h \sum_i \sum_{k=1}^{n-1} M_i^{2k}\right]$$

$$H_p = (\lambda-1)\cdot h \sum_i \left(\sum_{k=1}^{n-1} M_i^{2k} - M_i^n\right)$$

$H_o$ obeys the duality relation:

$$H_o(h,\lambda) = h\, H_o(h^{-1},\lambda)$$

and its self-dual line is h=1 independently of $\lambda$. The original
model (described by H) is self-dual if $H_p$ = 0. This is true
for the Ashkin-Teller model, where n=2, and at the 2n-state
Potts point, i.e. at $\lambda$ = 1. The plane given by eq. (2.4) was
used by Kohmoto et al (1981) in their study on the Ashkin-
Teller model. It turned out that this subspace contains all

the interesting regions of the phase diagram (Iglói and Sólyom 1984). Finally we would mention that the (2.5) form of the Hamiltonian will be used in the §.4. to perform an anisotropy expansion around the 2n-state Potts point in powers of $\lambda-1$.

## 3. 1/n expansion

For several spin systems, in which the number of components of the spin is a parameter and the $n \to \infty$ limit is exactly soluble, a well-converging 1/n expansion can be defined. The method was introduced by Kogut (1980) and was applied to several spin- and gauge-systems. The success of the method is due to the fact that even in first order infinite terms of different orders of the usual Brillouin-Wigner perturbation series have to be summed. In contrast to this procedure, we can mention another expansion, say 1/m, where m is the number of coupled spins (Iglói et al. 1984). Now the different terms in powers of 1/m corresponds to the different orders of the Brillouin-Wigner perturbation series. The convergence in the latter case is less pronounced. By comparing the two methods we can see that, in some sense, there is an obvious analogy with the different perturbation methods for dense gases or liquids. The Mayer expansion for dense gases used for treating the rapidly varying perturbations, is the analogue of the 1/n expansion. On the other hand, the usual high-temperature expansion used for treating slowly varying perturbation in liquids and gases are the analogues of the 1/m expansion.

The essence of the 1/n expansion may be summarized as follows. The ground state energy of the system as a function of the coupling has to be determined for different powers of 1/n, both in the weak-coupling and the strong-coupling

regime. The crossing point for the two expressions is identified with the phase transition point in the given order. Furthermore, the difference in the slopes of the two curves at the crossing point is proportional to the latent heat, also in the given order of 1/n. The latent heat defined in this way is positive for large values of n; however, it becomes negative with decreasing values. As was pointed out by Kogut (1980), the polynomial form of the finite series is not able to account for the essential singularity in the latent heat corresponding to the second to first order change in the transition. Therefore the latent heat obtained by 1/n expansion has a physical meaning only for its positive values. The crossover value of n is defined by the zero of the latent heat expression, while for its negative values the transition is assumed to be of second order. The estimate for the crossover value of n turned out to be fairly accurate even in first order for the Potts model in two and three dimensions (Kogut et al. 1980, Kogut and Sinclair 1981). The further terms give a slight improvement though, the convergence of the series is rather slow. For illustration, the approximate crossover values of n are summarized in Table 1 for the n-state Potts model in 2 dimensions. Here, the exact solution of Baxter was used both for the latent heat (Baxter 1973) and for the jump in the magnetization (Baxter 1982):

$$L = 1 - \frac{6}{n} + \frac{2}{n^2} + \frac{8}{n^3} + \frac{26}{n^4} + \frac{76}{n^5} + \ldots$$

$$\Delta M = 1 - \frac{1}{n} - \frac{3}{n^2} - \frac{9}{n^3} - \frac{27}{n^4} - \frac{82}{n^5} \ldots$$

From Table 1 it can be seen that the two series for $n_c$ go from different directions to the exact value of 4.

The 1/n expansion is easier to apply for systems where self-duality holds. In these cases the weak-coupling and strong-coupling series are connected by self-duality, and the approximate transition points coincide with the self-dual point in every order of the calculation.

In our model, where self-duality generally does not hold, the two expansions have to be carried out separately. The first few terms of the weak- and strong-coupling series are given in the Appendix. In the neighbourhood of the phase transition points the terms of the series can be arranged in powers of 1/n if the condition

$$2n\ h_1^2 \gg h_2^2 \qquad\qquad (3.1)$$

is fulfilled. This condition is true for the cubic transition line. Unfortunately, the multicritical point cannot be investigated by this method. In that region all terms of the series become of the same order of magnitude, a fact that clearly represents the role of competition in creating multicritical points. In the following the phase diagram and the latent heat of the cubic transition are given up to second order.

The ground state energy in the strong- and weak-coupling limit up to second order in 1/n is given in the Appendix. The phase-transition point is given as:

$$2h = 1+\lambda + \frac{1}{2n}\, \alpha_1(\lambda) + \frac{1}{(2n)^2}\, \alpha_2(\lambda) + \ldots \qquad (3.2)$$

Here

$$\alpha_1(\lambda) = -(\lambda-1)(2+\lambda + \frac{\lambda-1}{\lambda+3})$$

while the form of $\alpha_2(\lambda)$ is given in Fig. 1. The latent heat
is:

$$L = 2nh\,\{1 + \frac{1}{2n}\, \delta_1(\lambda) + \frac{1}{(2n)^2}\, \delta_2(\lambda) + \ldots\} \qquad (3.3)$$

where

$$\delta_1(\lambda) = -6 + (\lambda-1) - 2(\frac{\lambda-1}{\lambda+3})^2$$

while $\delta_2(\lambda)$ is sketched in Fig. 1.

We also give the results in the original space of
couplings up to first order, when the formulae are relatively
simple:

$$4nh_1 = \lambda_1+\lambda_2 + \frac{1}{2n}\left\{ -4nh_2+2\lambda_1 - \frac{(\lambda_2-\lambda_1)^2}{\lambda_2+3\lambda_1} \right\} + \sigma(\frac{1}{n^2}) \qquad (3.4)$$

$$L = 2nh_1\left\{ 1 - \frac{1}{2n}\left[ 7 - \frac{h_2}{h_1} + 2\left(\frac{\lambda_2-\lambda_1}{\lambda_2+3\lambda_1}\right)^2 \right] + \sigma(\frac{1}{n^2}) \right\} \qquad (3.5)$$

In the following let us discuss some consequences of
these formulae.

i) In zeroth order, i.e. in the $n \to \infty$ limit, the phase-
transition points (3.2) and (3.4) are the same as in the
mean-field calculation (Section 3 Paper I). Therefore the
mean-field phase diagram is exact for the cubic transition
line in the $n \to \infty$ limit.

ii) The position of the multicritical point for large values of n can be obtained from condition (3.1). It is

$$2nh_1^2 \approx h_2^2$$

This expression is also in agreement with the results of mean-field calculation.

iii) The latent heat depends on the values of the couplings, even in first order of $1/n$. Therefore there is no universality for the first order transition.

iv) In order to investigate the convergence of the series (3.2) for the phase transition points, we compare it with the results of other methods. At $\lambda = 1$ (at the 2n-state Potts point) eq. (3.2) is exact. In another part of the phase-diagram: at $\lambda = 0$, we compare it with the result of the self-dual RG calculation (Section 4, Paper I). Fig. 2 shows that the results of the two methods are quite close to each other for $n \gtrsim 3$.

v) As already mentioned, the latent heat expression gives the possibility to estimate the crossover values of $n_c$, where the phase transition changes from first order to second order. The estimated $n_c$ values for different values of $\lambda$ are given in Fig. 3. It is generally supposed (Nienhuis et al. 1983) that the exact value of $n_c$ is 2 independently of the value of $\lambda$. In our case, even in this low-order of the calculation, the estimated values of $n_c$ are in the range 2.8-3.6, and there is no strong dependence of $n_c$ on $\lambda$. So this picture can be considered to be consistent with the conjectured one.

vi) By expanding eqs. (3.2) and (3.3) around the 2n-state Potts point in powers of $(\lambda-1)$, the following series can be obtained:

$$2h = 2+(\lambda-1)\left[1 - \frac{3}{2n} - \frac{3}{(2n)^2} \ldots\right] +$$

$$+ (\lambda-1)^2\left[-\frac{5}{4} \cdot \frac{1}{2n} + \frac{27}{8} \cdot \frac{1}{(2n)^2} \ldots\right] + \ldots \tag{3.6}$$

$$L = 2nh\left\{1 - \frac{6}{2n} + \frac{2}{(2n)^2} +\ldots+ (\lambda-1)\left[\frac{1}{2n} - \frac{5}{(2n)^2} \ldots\right]+\ldots\right\} \tag{3.7}$$

It will be shown in the next section that in first order of $(\lambda-1)$ the coefficients can be summed. This exercise will be done by using anisotropy expansion around the 2n-state Potts point.

## 4. $(\lambda-1)$ expansion around the 2n-state Potts point

Let us turn back to expression (2.5), where the Hamiltonian of the system is split into a self-dual part $(H_o)$ and a perturbation $(H_p)$. Let us suppose that n>2. The phase transition is of first order so the transition is accompained by a crossing of energy levels. Let us denote the difference between the two lowest levels by $F(h,\lambda-1)$, and the same quantities for $H_o$ and $H_p$ are denoted by $F_o$ and $F_p$, respectively. The gap vanishes at the phase transition point, so:

$$F(h^*, \lambda-1) = 0 \qquad (4.1)$$

In the 2n-state Potts point, and for the system described by the Hamiltonian $H_o$ the transition point is just $h^*=1$, therefore

$$F(1,0) = 0, \quad F_o(1,\lambda-1) = 0$$

By expanding (4.1) in first order in $(\lambda-1)$:

$$F(h^*,\lambda-1) = F_o(1,\lambda-1) + \left.\frac{\partial F_o}{\partial h}\right|_{\substack{\lambda=1 \\ h=1}} (h^*-1) + F_p(1,\lambda-1) + \ldots = 0$$

Thus the change in the phase transition point is

$$h^*-1 = - \frac{F_p(1,0)}{\frac{\partial}{\partial h} F_o(h,0)} \qquad (4.2)$$

The denominator on the right-hand side of (4.2) is proportional to the latent heat of the 2n-state Potts model, that can be written by using the Hellman-Feynman theorem as:

$$-\frac{\partial}{\partial h} F_o(h,0) = <1|\sum_i \sum_{k=1}^{2n-1} M_i^k|1> - <0|\sum_i \sum_{k=1}^{2n-1} M_i^k|0> \qquad (4.3)$$

where $|0>$ and $|1>$ denote the ground state and the first excited states of the 2n-state Potts model, respectively. The numerator of (4.2) has a form similar to (4.3), viz.

$$F_p(1,0) = (\lambda-1)[<1|\sum_i \left(\sum_{k=1}^{n-1} M_i^{2k} - M_i^n\right)|1> -$$

$$-<0|\sum_i \left(\sum_{k=1}^{n-1} M_i^{2k} - M_i^n\right)|0>]$$

The two expressions are proportional to each other and their ratio in (4.2) is just $(n-2)/(2n-1) \cdot (\lambda-1)$.

Thus, the phase transition points in first order are given by:

$$h^* = 1 + \frac{n-2}{2n-1}(\lambda-1) + \ldots \qquad (4.4)$$

This result coincides with the result of the $\frac{1}{n}$ expansion in eq. (3.6). It is pointed out that eq. (4.4) holds for the single phase transition part of the phase diagram, otherwhere it may be considered as a "duality line".

The latent heat can be calculated from the slope of the $F(h,\lambda-1)$ curve at $h^*$:

$$L = -\left.\frac{\partial F(h,\lambda)}{\partial h}\right|_{h^*}$$

By expanding L in first order of $\lambda-1$:

$$L = -\left.\frac{\partial F_o(h,\lambda)}{\partial h}\right|_{h=h^*} - \left.\frac{\partial F_p}{\partial h}\right|_{h=1} \tag{4.5}$$

The first term can be expanded as

$$-\left.\frac{\partial F_o(h,\lambda)}{\partial h}\right|_{h=h^*} = -\left.\frac{\partial F_o(h,1)}{\partial h}\right|_{h=1} - \left.\frac{\partial^2 F_o(h,1)}{\partial h^2}\right|_{h=1} (h^*-1) \tag{4.6}$$

$$-\left.\frac{\partial^2 F_o(h,\lambda)}{\partial h \partial \lambda}\right|_{\substack{h=1 \\ \lambda=1}} (\lambda-1) + \ldots$$

The first term is just the latent heat of the 2n-state Potts model; the second is zero due to the duality properties of the Potts model. The third term of (4.6) as well as the second term in (4.5) are proportional to the latent heat of the 2n-state Potts model. Therefore we can write up to first order:

$$L = L_{Potts}\left[1 + \frac{1}{2n-1}(\lambda-1) + \ldots\right] \tag{4.7}$$

This formula is also in accordance with the result of the 1/n expansion (3.7). According to eq. (4.7), in the n-component cubic model on the cubic transition line for $n \leq 2$ the latent heat vanishes whereas for $n > 2$ there is finite latent heat. Thus at n=2, the phase transition changes its nature from second order to first order. Furthermore the latent heat shows essential singularity at n=2 along the cubic transition line, similarly to the 4 state Potts model

(Baxter 1973). Finally it is mentioned that the higher order terms of the anisotropy expansion should be exposed by the derivatives of the free energy of the Potts model at the critical point; however, these are not yet known.

## 5. Summary

In this paper, which represents the second part of our work on the (1+1)D n-component cubic model, the properties of the system were investigated along the cubic transition line, for first order transition. We used 1/n expansion and anisotropy expansion, which supply complementary information. The results show the cubic transition to be of first order for n>2, independently of the coupling. The latent heat depends on the coupling (there is no universality), and has essential singularity at n=2. At the multicritical point all terms of the series are of the same order of magnitude, which prevents the application of a perturbational expansion.

## Acknowledgment

## Appendix

### 1. Strong-coupling expansion

The strong-coupling representation (Paper I, eqs. 2.1 and 2.2) is used in the calculation. The unperturbed part of the Hamiltonian is

$$H_O = H_\lambda + \sum_i h_2$$

while the perturbation

$$V = H_h - \sum_i h_2$$

The ground state of $H_O$ is given by

$$\psi_O^{S,O} = |1\ 1...1>$$

with the energy $E_O^{S,O} = -N \frac{\lambda_1}{2} + Nh_2$.

The lower lying excitations of $H_O$ contain one flipped spin, and they are of two kinds:

$$\psi_{1,k}^{S,O} = |1\ 1...1\ k\ 1...1>$$

with energy $E_{1,k}^{S,O} = E_O^{S,O} + \lambda_1 + \lambda_2$, and

$$\psi_{1,n+1}^{S,O} = |1\ 1...1\ (n+1)1...1>$$

with energy: $E_{1,n+1}^{s,o} = E_o^{s,o} + 2\lambda_1$. The two excitations are
$N(2n-2)$ and $N$-fold degenerate, respectively. The higher
lying excitations of $H_o$ contain more flipped spins.

The first few terms of the Brillouin-Wigner perturbational series for the ground state energy are the following:

$$E_o^{s,o} = -N(\frac{\lambda_1}{2} + h_2)$$

$$E_o^{s,1} = 0$$

$$E_o^{s,2} = -N\left[(2n-2)\frac{h_1^2}{\lambda_1+\lambda_2} + \frac{h_2^2}{2\lambda_1}\right]$$

$$E_o^{s,3} = -N\left[(2n-2)(2n-4)\frac{h_1^3}{(\lambda_1+\lambda_2)^2} + 2(2n-2)\frac{h_1^2 h_2}{(\lambda_1+\lambda_2)2\lambda_1} + \right.$$

(A1)

$$\left.(2n-2)\cdot\frac{h_1^2 h_2}{(\lambda_1+\lambda_2)^2}\right]$$

$$E_o^{s,4} = -N\left[(2n-2)(2n-4)^2\frac{h_1^4}{(\lambda_1+\lambda_2)^3} +...\right]$$

$$\vdots$$

A well defined $1/n$ expansion exists if condition (4.1) is
fulfilled. Then the ground state energy can be written as:

$$\frac{E_o^s}{N} = \varepsilon_o^{s,o} + \frac{1}{2n}\varepsilon_o^{s,1} + \frac{1}{(2n)^2}\varepsilon_o^{s,2} +... \qquad (A2)$$

The zeroth order term is just:

$$\varepsilon_o^{s,o} = -\frac{\lambda_1}{2}$$

The first order term is the sum of a geometric series, which stands in the first column in the right hand sides of (A1). These elements have the property that in the $\langle O|VgV...gV|O\rangle$ perturbational expression the V perturbation always acts on the same spin. The sum of these terms is

$$\varepsilon_o^{s,1} = -2nh_1 \frac{\frac{2nh_1}{\lambda_1+\lambda_2}}{1 - \frac{2nh_1}{\lambda_1+\lambda_2}} + 2nh_2$$

The second-order term in (A2) is more complicated and consists of different parts, which can be summarized as follows:

1. V always acts on one spin

   i)   There is no intermediate state of $|n+1\rangle$, there is no flip with n.

   ii)  There is no intermediate state of $|n+1\rangle$, there is one flip with n.

   iii) There is one intermediate state of $|n+1\rangle$, there is one flip with n.

        (This must be the first or the last spin flip.)

   iv)  There is one intermediate state of $|n+1\rangle$, there is no flip with n.

2. V acts on two spins

    i)  The two spins are not nearest neighbours.

    ii) The two spins are nearest neighbours.

The higher order terms in (A2) contain expressions with more flipped spins and the terms with fewer flipped spins have to be evaluated in the given order of $1/n$.

## Appendix

### 2. Weak-coupling expansion

The weak-coupling form of the Hamiltonian (Paper I, eqs. 2.5 and 2.6) is used in this calculation. The unperturbed part of the Hamiltonian is

$$H_O = H_h + \sum_i (n-1) \frac{\lambda_2}{2n}$$

while the perturbation is

$$V = H_\lambda - \sum_i (n-1) \frac{\lambda_2}{2n}$$

The ground state of $H_O$ is given by

$$\psi_O^{W,O} = |1'1'...1'\rangle$$

with the energy $E_O^{W,O} = -N[2(n-2)h_1 - (n-1)\frac{\lambda_2}{2n}]$

The lower lying excitations contain a flipped pair of spins, and they are of two kinds:

$$\psi_{1,2k}^{W,O} = |1'1'...1'(2k)'(2n+2-2k)'1'...1'\rangle \quad k=1,2...n$$

with energy $E_{1,2k}^{W,O} = E_O^{W,O} + 4nh_1 + 4(h_2-h_1)$, and

$$\psi_{1,2k+1}^{W,O} = |1'1'...1'(2k+1)'(2n+1-2k)'1'...1'\rangle$$

$$k=1,2,...,n-1$$

with energy $E_{1,2k+1}^{W,O} = E_O^{W,O} + 4nh_1$. The two excitations are $N \cdot n$ and $N(n-1)$-fold degenerate, respectively. The higher lying excitations of $H_O$ contain more flipped spins.

The first few terms of the perturbational series for the ground state energy are:

$$E_O^{W,O} = -N[(2n-2)h_1 - (n-1)\frac{\lambda_2}{2n}], \qquad E_O^{W,1} = 0$$

$$E_O^{W,2} = -N\left[n(\frac{\lambda_1}{2n})^2 \frac{1}{4nh_1 + 4(h_2-h_1)} + (n-1)(\frac{\lambda_2}{2n})^2 \frac{1}{4nh_1}\right]$$

$$E_O^{W,3} = -N\left[n(n-1)\frac{(\frac{\lambda_1}{2n})^2 \cdot \frac{\lambda_2}{2n}}{[4nh_1 + 4(h_2-h_1)]^2} + \right.$$

$$+ 2n(n-1)\frac{(\frac{\lambda_1}{2n})^2 \frac{\lambda_2}{2n}}{[4nh_1 + 4(h_2-h_1)]4nh_1} +$$

$$\left. + (n-1)(n-2)\frac{(\frac{\lambda_2}{2n})^3}{(4nh_1)^2}\right]$$

The $1/n$-expansion can be written in the region (4.1) as

$$\frac{E_O^W}{N} = \varepsilon_O^{W,O} + \frac{1}{2n}\varepsilon_O^{W,1} + \frac{1}{(2n)^2}\varepsilon_O^{W,2} + \dots \tag{A3}$$

The zeroth order term is:

$$\varepsilon_O^{W,O} = -2nh_1 + \frac{\lambda_2}{2}$$

The first order term is the sum of elements, where the V perturbation acts on the same pair of spins:

$$\varepsilon_o^{w,1} = 4nh_1 - \lambda_2 - \frac{1}{2} \frac{\frac{(\lambda_2+\lambda_1)^2}{8nh_1}}{1 - \frac{\lambda_2+\lambda_1}{8nh_1}} - \frac{1}{2} \frac{\frac{(\lambda_2-\lambda_1)^2}{8nh_1}}{1 - \frac{\lambda_2-\lambda_1}{8nh_1}}$$

The second order term in (A3) consists of four parts:

1. V acts on one pair of spins

   i) Taking it into account that the energy denominators may be $4nh_1$ and $4nh_1 + 4(h_2-h_1)$

   ii) Taking it into account that the number of spin-flips are n and n-1 for $\frac{\lambda_1}{2n}$ and $\frac{\lambda_2}{2n}$ , respectively.

2. V acts on two pairs of spins

   i) The two pairs have no common spin.

   ii) The two pairs have one common spin.

The determination of the second order term in this case is somewhat more complicated than for the strong-coupling series. The reason for this is that the factor $\frac{\lambda_1}{2n}$ is associated with an odd number of spin flips therefore only the even powers of $\frac{\lambda_1}{2n}$ should appear in the expansion.

The higher order terms in (A3) contain more pairs of flipped spins.

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| Order | Crossover value of n | |
| --- | --- | --- |
| | latent heat | magnetization |
| 1 | 6 | 1 |
| 2 | 5.646 | 2.303 |
| 3 | 5.346 | 3 |
| 4 | 5.106 | 3.378 |
| 5 | 4.911 | 3.598 |

## Table 1

Zeros of the latent heat and the magnetiza-
tion series in powers of 1/n for the 2D
n-state Potts model.

Figure captions

Fig. 1 Second order expansion parameters of the phase transition point $(\alpha_2)$ and of the latent heat $(\delta_2)$.

Fig. 2 Phase transition point at $\lambda=0$, for different values of n. —— RG calculation, - - - 1/n expansion in second order, · · · 1/n expansion in first order.

Fig. 3 The critical values of $n_c$ where the nature of the transition changes. - - - second order calculation, · · · first order calculation.
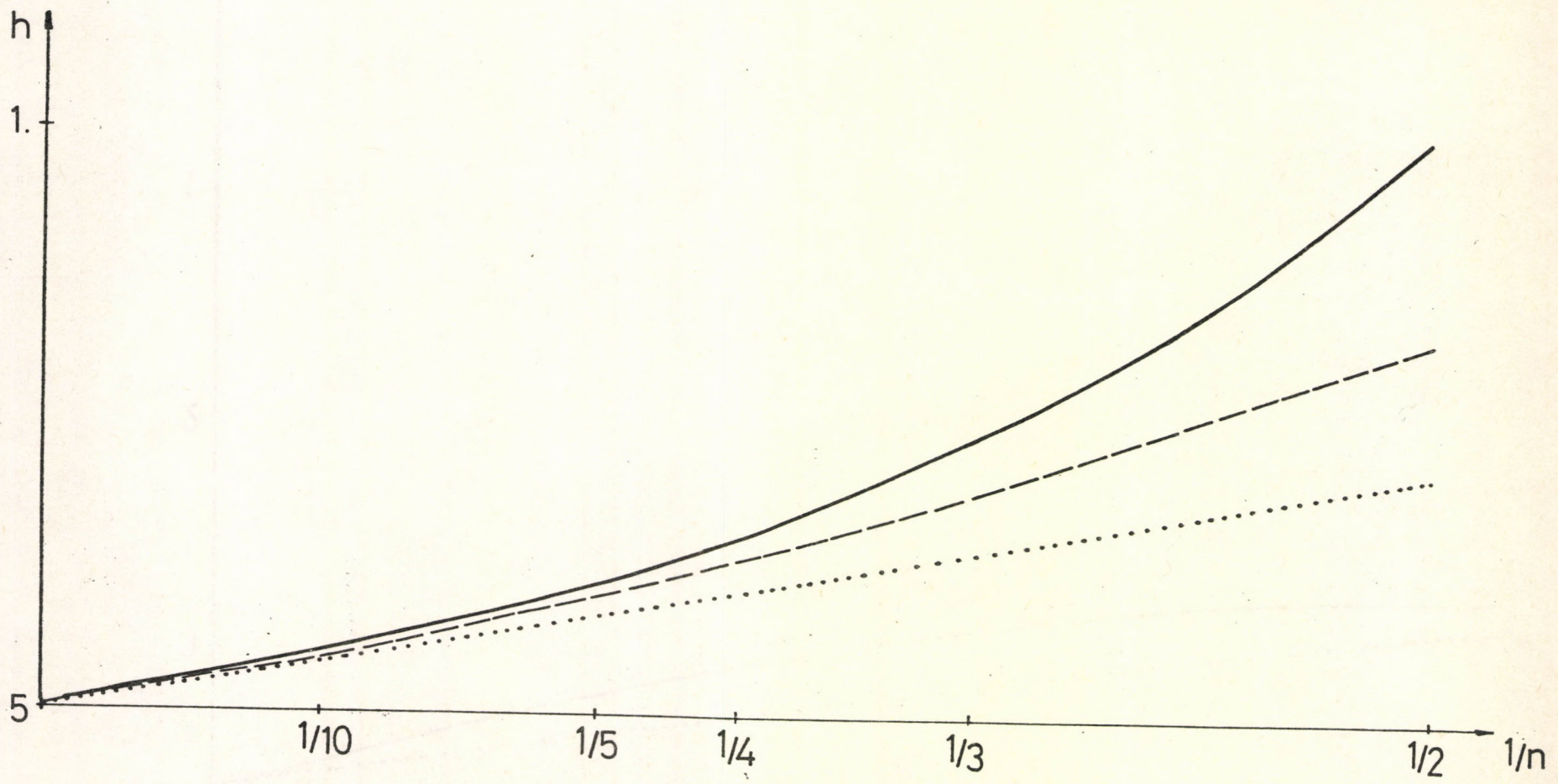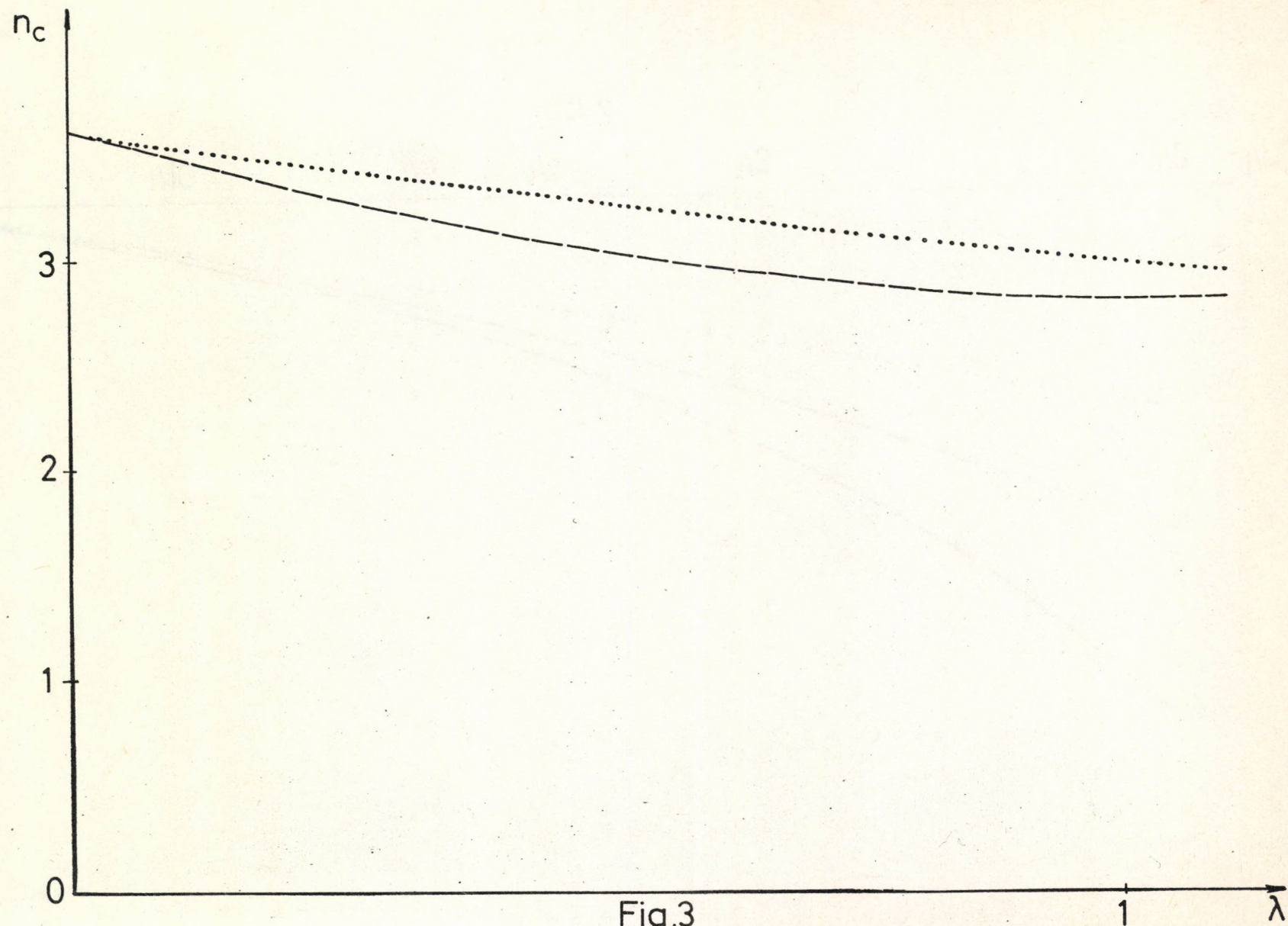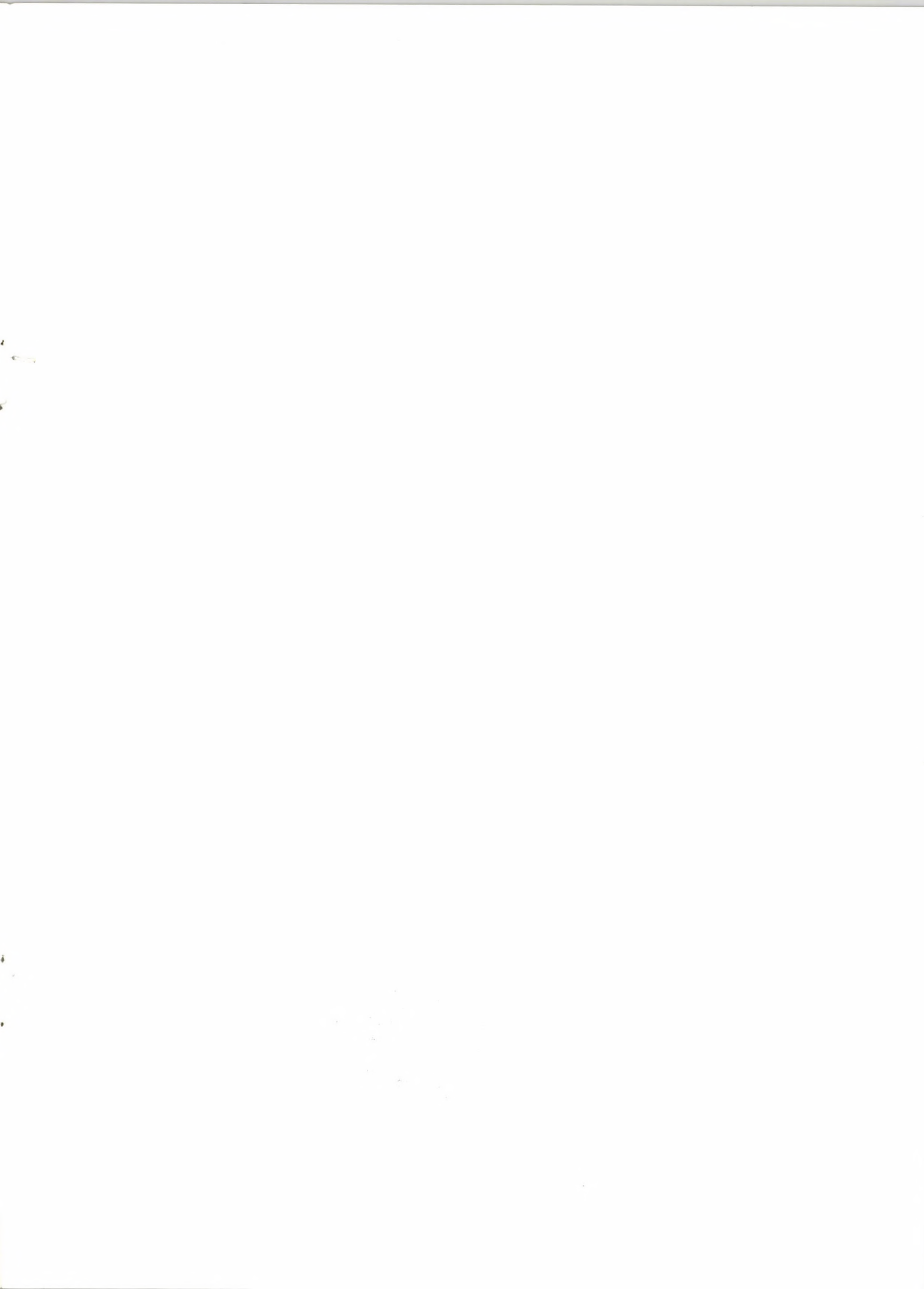
Fig.1.

Fig. 2.

Fig.3