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BUDAPEST

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ABSTRACT

Using eight pseudoparticles the precise procedure of Gupta-Bleuler quantisation is given for the free massless spin 2 field.

АННОТАЦИЯ

Дана точная процедура квантования Гупта-Блейлера для свободного безмассового поля со спином два. Используется восемь псевдочастиц.

KIVONAT

Nyolc pszeudorészecskét felhasználva megadjuk a Gupta-Bleuler-féle kvantálási eljárás pontos formáját a szabad kettes spinü tömegnélküli mező esetére.

As far as it is known, the precise procedure of Gupta-Bleuler quantisation for the free massless spin 2 field was not given yet. Indeed, already Gupta formulated the procedure a long time ago [1], but he used ad hoc nine pseudoparticles. This is in fact an uncomprehensible choice. The potential of the spin 2 field is a symmetric $U^{ij}=U^{ji}$ tensor, and therefore there should obviously be two physical and eight unphysical polarisations. Of course, from the physical point of view the choice of nine or eight pseudoparticles is not essential, because in any case there are only two physical polarisations. Nevertheless, the precise formulation of the Gupta-Bleuler quantisation procedure is all the same useful. As it is well-known, in the case of self-interacting gauge fields the pseudoparticles determine the behaviour of ghost particles in Lorentz-gauge [2],[3].

This paper gives the precise procedure of quantisation. Of course, the questions that are identical to the case of spin 1 field - opposite sign of commutators, indefinite metric, etc... (for details see, e.g., [4]) - are not considered here.

In the Lorentz-gauge the equations of the free massless spin 2 field are given by [5]

$$\Box U^{ij} = 0 ; \qquad 2\, U^{ij}{}_{,j} = U^{,i} ; \qquad U^{i}_{i} \equiv U, \tag{1}$$

where an index after a comma denotes partial derivatives, $\Box \equiv \partial^i \partial_i$ and the Latin indices take the values 0,1,2,3. The indices are moved by the $\eta^{ij}=\eta_{ij} \equiv \mathrm{diag}(1,-1,-1,-1)$ Minkowskian metric tensor of flat space-time. We have still the following gauge freedom:

$$\tilde{U}^{ij} = U^{ij} + V^{(i,j)}; \qquad \Box V^{i} = 0, \tag{2}$$

where ( ) denotes symmetrisation without the factor $\frac{1}{2}$ and V^i are C^3 functions of x^i coordinates. In the momentum space we have

$$U^{ij} = (2\pi)^{-\frac{3}{2}} \int \frac{d\vec{k}}{\sqrt{2k_0}} \left(\bar{U}^{ij}(\vec{k}) e^{ik_n x^n} + \overset{+}{\bar{U}}{}^{ij}(\vec{k}) e^{-ik_n x^n} \right), \qquad (3)$$

where $k^i \equiv [k^0, \vec{k}]$ is the wave vector. Of course, all this is not new.

In order to introduce the eight pseudopolarisations we shall use the orthonormal e,f,n,m vectors, which are well-known [4]. They fulfil the following relations:

$$m^i m^j - e^i e^j - f^i f^j - n^i n^j = \eta^{ij} \; ; \quad e^i e_i = f^i f_i = n^i n_i = -m^i m_i = -1;$$

$$e^i f_i = e^i n_i = e^i m_i = f^i n_i = f^i m_i = n^i m_i = 0; \; k^i k_i = (k^0)^2 - |\vec{k}|^2 = 0;$$

$$k^i = k^0(n^i + m^i); \quad e^i \equiv [0, \vec{e}]; \quad f^i \equiv [0, \vec{f}]; \quad n^i \equiv [0, \frac{\vec{k}}{|\vec{k}|}];$$

$$m^i \equiv [1,0,0,0]. \qquad (4)$$

One may write:

$$\sqrt{2}\, \overset{+}{\bar{U}}{}^{ij}(\vec{k}) = \overset{+}{\bar{a}}(\vec{k})(e^i e^j + f^i f^j) + \overset{+}{\bar{b}}(\vec{k}) e^{(i} f^{j)} + \overset{+}{\bar{c}}(\vec{k}) e^{(i} n^{j)} +$$

$$+ \overset{+}{\bar{d}}(\vec{k}) e^{(i} m^{j)} + \overset{+}{\bar{g}}(\vec{k}) f^{(i} n^{j)} + \overset{+}{\bar{h}}(\vec{k}) f^{(i} m^{j)} + \overset{+}{\bar{p}}(\vec{k})(m^i m^j + n^i n^j) +$$

$$+ \overset{+}{\bar{r}}(\vec{k}) m^{(i} n^{j)} + \frac{\overset{+}{\bar{u}}(\vec{k}) + \overset{+}{\bar{v}}(\vec{k})}{\sqrt{2}} (m^i m^j - n^i n^j) +$$

$$+ \frac{\overset{+}{\bar{u}}(\vec{k}) - \overset{+}{\bar{v}}(\vec{k})}{\sqrt{2}} (e^i e^j + f^i f^j). \qquad (5)$$

One has the conditions

$$(2\, \bar{U}^{ij}(\vec{k}) k_j - k^i\, \bar{U}(\vec{k})) |\phi\rangle = 0; \quad \langle\phi| (2 \overset{+}{\bar{U}}{}^{ij}(\vec{k}) k_j - \overset{+}{\bar{U}}(\vec{k}) k^i) = 0, \; (6)$$

where $|\phi>$ is the state vector of Fock-space. Hence

$$(\bar{d}(\vec{k}) - \bar{c}(\vec{k}))|\phi> = (\bar{h}(\vec{k}) - \bar{g}(\vec{k}))|\phi> = (\bar{p}(\vec{k}) - \bar{r}(\vec{k}))|\phi> =$$

$$= (\bar{u}(\vec{k}) - \bar{v}(\vec{k}))|\phi> = 0 \tag{7}$$

and

$$\sqrt{2}\, \bar{U}^{ij}(\vec{k})|\phi> = (\bar{a}(\vec{k})(e^i e^j - f^i f^j) + \bar{b}(\vec{k})e^{(i}f^{j)} + \frac{\bar{c}(\vec{k})}{k^0} e^{(i}k^{j)} +$$

$$+ \frac{\bar{g}(\vec{k})}{k^0} f^{(i}k^{j)} + \frac{\bar{p}(\vec{k})}{(k^0)^2} k^i k^j + \frac{\bar{u}(\vec{k})}{\sqrt{2}\, k^0}((m^i - n^i)k^j + k^i(m^j - n^j)))|\phi> \tag{8}$$

follows. The obvious relations for $<\phi|(\overset{+}{d}(\vec{k}) - \overset{+}{c}(\vec{k}))\ldots$ and $<\phi|\overset{+}{U}^{ij}(\vec{k})$ were not written down. Denoting

$$k_0 \sqrt{2}\, \overset{+}{\bar{V}}(\vec{k}) = \overset{+}{\bar{c}}(\vec{k})e^i + \overset{+}{\bar{g}}(\vec{k})f^i + \overset{+}{\bar{p}}(\vec{k})(m^i + n^i) + \frac{\overset{+}{\bar{u}}(\vec{k})}{\sqrt{2}}(m^i - n^i) \tag{9}$$

one immediately sees that only the polarisations given by operators $\overset{+}{\bar{a}}(\vec{k})$, $\overset{+}{\bar{b}}(\vec{k})$ have physical meaning; compare with (2). The commutators are the following:

$$[\bar{a}(\vec{k}), \overset{+}{a}(\vec{q})] = [\bar{b}(\vec{k}), \overset{+}{b}(\vec{q})] = [\bar{c}(\vec{k}), \overset{+}{c}(\vec{q})] = -[\bar{d}(\vec{k}), \overset{+}{d}(\vec{q})] =$$

$$= [\bar{g}(\vec{k}), \overset{+}{g}(\vec{q})] = -[\bar{h}(\vec{k}), \overset{+}{h}(\vec{q})] = [\bar{p}(\vec{k}), \overset{+}{p}(\vec{q})] = -[\bar{r}(\vec{k}), \overset{+}{r}(\vec{q})] =$$

$$= [\bar{u}(\vec{k}), \overset{+}{u}(\vec{q})] = -[\bar{v}(\vec{k}), \overset{+}{v}(\vec{q})] = \delta(\vec{k} - \vec{q}). \tag{10}$$

A long but straightforward calculation leads to the relation:

$$2[\bar{U}^{ij}(\vec{k}), \overset{+}{U}^{pr}(\vec{q})] = \delta(\vec{k} - \vec{q})(\eta^{ip}\eta^{jr} + \eta^{ir}\eta^{jp} - \eta^{ij}\eta^{pr}). \tag{11}$$

In order to write down the dynamical invariants we proceed as follows. As Lagrangian one may use

$$L = \frac{1}{2} :U^{ij,k}U_{ij,k}: - \frac{z}{2} :U^{,i}U_{,i}:, \tag{12}$$

where z is an arbitrary real number; $z \neq \frac{1}{4}$. For any z one obtains $\Box U^{ij} = 0$. In standard way from (12) one obtains the four-momentum:

$$P^i = \int d\vec{k}.k^i (\overset{+}{U}{}^{jm}(\vec{k})\, \bar{U}_{jm}(\vec{k}) - z\, \overset{+}{U}(\vec{k})\bar{U}(\vec{k})) \tag{13}$$

and hence

$$\langle\phi|P^i|\phi\rangle = \langle\phi|\int d\vec{k}.k^i(\overset{+}{a}(\vec{k})\bar{a}(\vec{k}) + \overset{+}{b}(\vec{k})\bar{b}(\vec{k}) + (1 - 2z)\overset{+}{v}(\vec{k})\bar{v}(\vec{k}))|\phi\rangle. \tag{14}$$

Thus obviously $z = \frac{1}{2}$ is the right choice in (12).

Note still that (11) is not the only possible commutator, because as potential one may use $(U^{ij} + y\eta^{ij} U)$ too, where y is real and $y \neq -\frac{1}{4}$. Because $\overset{+}{U}(\vec{k}) = 2\, \overset{+}{v}(\vec{k})$ holds, (11) may be substituted by

$$2[\bar{U}^{ij}(\vec{k}), \overset{+}{U}{}^{pr}(\vec{q})] = \delta(\vec{k}-\vec{q})(\eta^{ip}\eta^{jr} + \eta^{ir}\eta^{jp} - t\, \eta^{ij}\eta^{pr}), \tag{15}$$

where

$$t = 8y^2 + 4y + 1 > \frac{1}{2}. \tag{16}$$

We have seen: if one uses the vectors e,f,n,m, then the quantisation is straightforward but not trivial.

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