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A. SÜTŐ
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Hungarian Academy of Sciences

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A. SÜTŐ, G.T. ZIMÁNYI

Central Research Institute for Physics
H-1525 Budapest 114, P.O.B.49, Hungary

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ABSTRACT

Classical spin models in the presence of random interactions or external fields are considered. A general argument shows that the normal (non-Parisi-type) replica trick is bound to yield the correct free energy provided that this latter is an analytic function of the strength ε of the random variables. As an illustration, the free energy of the one-dimensional Ising model in random external field is calculated up to sixth order in ε by direct computation and also by the replica method, and the coincidence of the two results is demonstrated.

АННОТАЦИЯ

Исследованы классические системы спинов в присутствии случайного взаимодействия или внешнего поля. Показано, что обыкновенный (то есть, не типа Паризи) метод реплики дает правильную свободную энергию, если она является аналитической силовой функцией ε случайной переменной. В качестве примера вычислена свободная энергия одномерной модели Изинга в случайном внешнем поле до шестого порядка от ε, сначала непосредственно, а потом используя метод реплики. Полученные результаты совпадают.

KIVONAT

Klasszikus spinrendszereket vizsgálunk véletlen kölcsönhatás, vagy külső tér jelenlétében. Megmutatjuk, hogy a szokásos /nem Parisi tipusu/ replika-módszer a helyes szabadenergiát adja, ha az utóbbi analitikus függvénye a véletlen változó ε erősségének. Példaként kiszámitjuk az egydimenziós Ising-lánc szabadenergiáját véletlen külső tér esetén ε-ban hatodrendig direkt módszerrel és a replika-eljárással, és a két eredményt egyezőnek találjuk.

1. INTRODUCTION

The replica trick (Edwards and Anderson 1975) is one of the few powerful tools to calculate the properties of disordered systems. Its use is nevertheless shadowed by the fact that the prescriptions for the computation are not well defined from a mathematical point of view. The trick exploits the identity

$$\overline{\ln Z} = \frac{d}{dx}\,\overline{Z^x}\Big|_{x=0} = \lim_{x\to 0}(\overline{Z^x}-1)/x \qquad (1)$$

where Z is the partition function and the bar indicates averaging over the randomness. The average of the x th power of Z for general x is by no means easier to calculate than the average of the logarithm; for $x=m$ positive integer, however, $\overline{Z^m}$ is the partition function of an m-replica system, coupled by the averaging. This - at least in principle - can be calculated for general m, so that one obtains $(\overline{Z^m})^{1/N} = \lambda(m)$ and the trick consists of continuing $(\lambda(m)-1)/m$ to $m=0$ and identifying the result with $\overline{\ln Z}/N$. This is a problematic procedure because the function $\lambda(m)$ may not be unique. Indeed, there are many ways to calculate a partition function; and though the sequence of numbers $\{\lambda_m = (\overline{Z^m})^{1/N}\}_{m=0}^{\infty}$ is the same for all cases, the function $\lambda(m)$ may depend on the particular method used for the calculation. This possible ambiguity can be avoided by using nothing else than the sequence of values λ_m and trying to reproduce $(\overline{Z^x})^{1/N}$ as a function of x out of these values.

Formulating the question in this way we arrive at an interpolation problem the solution of which would consist of the following steps:

(i) Establish the analytical properties (A) of $\lim_{N\to\infty} (\overline{Z^x})^{1/N}$ for complex values of x.

(ii) Show that there is a unique function $\lambda(x)$ with the properties (A) so that $\lambda(m) = \lambda_m$ for all non-negative integers m. Then

$$\lambda(x) = \lim_{N\to\infty} (\overline{Z^x})^{1/N} \qquad (2)$$

(iii) Construct $\lambda(x)$ by using the sequence of numbers λ_m.

This program, what we call the normal replica method, is rather difficult to carry through. (It was partly done for the model of Sherrington and Kirkpatrick (1975), see van Hemmen and Palmer (1979)). Rather than following the line (i)-(iii) we provide a sufficient condition for the replica method to yield the correct result. This hinges upon the existence of the limit in Eq. (2) and its analyticity *as a function of* ε, the strength of random fields or couplings. (The existence of $\lambda(x) \equiv \lambda(x,\varepsilon)$, as defined in (2), for real x and ε can be shown and will be presented elsewhere.) Our finding verifies the general expectation that the replica trick is correct at high temperatures where the analyticity in ε holds true. In case of lack of analyticity in ε, the replica method is still capable to yield the existing derivatives of the free energy with respect to ε.

In the second part of this paper the general result on the replica method is presented. In the third part we discuss the one-dimensional Ising model in a random magnetic field. We calculate the free energy up to sixth order in ε by direct computation and by applying the replica trick; then the agreement of the two results can be checked explicitely. We end up with a brief discussion of the singularity at $T=0$.

2. THE ANALYSIS OF THE REPLICA METHOD BY A SERIES EXPANSION OF THE FREE ENERGY

We consider a classical spin system described by the Hamiltonian

$$-\beta \mathcal{H} = \mathcal{H}_0 + \varepsilon \cdot \mathcal{H}_1 \tag{3}$$

Here $\mathcal{H}_0$ is a non-random Hamiltonian and

$$\mathcal{H}_1 = \sum_b \xi_b \cdot \varphi_b(\underline{s}) \tag{4}$$

where the summation runs over certain sets b of lattice sites, $\varphi_b(\underline{s})$ is some function of the spin variables at the sites in b and ξ_b are independent random variables. The partition function of a system of N spins, described by the Hamiltonian (3), can be written as

$$Z = \sum_{\underline{s}} e^{\mathcal{H}_0} \cdot e^{\varepsilon \mathcal{H}_1} = Z_0 \cdot \langle e^{\varepsilon \mathcal{H}_1} \rangle_0 \tag{5}$$

where z_0 is the partition function of the unperturbed ($\varepsilon = 0$) system. Similarly,

$$\left(\overline{z^x}/z_0^x\right)^{1/N} = \overline{\langle \exp(\varepsilon \cdot \mathcal{H}_1)\rangle_0^x}^{1/N} \tag{6}$$

On the right-hand side one has to take first the thermal average with the distribution $\sim \exp \mathcal{H}_0$, then the x th power, then the average over the random variables ξ_l and finally the N th root. Without the thermal average, this expression would depend only on $\varepsilon \cdot x$; but even in the presence of thermal averaging the dependences on x and ε are intimately related. For N finite, the expression (6) is an analytic function of ε at $\varepsilon = 0$. Our main observation is that by taking its expansion in powers of ε, the l th order term in ε will be of the l th order also in x. Indeed,

$$\left(\overline{z^x}/z_0^x\right)^{1/N} \equiv \Pi(x,\varepsilon) =$$

$$= 1 + \sum_{l=1}^{\infty} \varepsilon^l \sum_{q=1} \binom{1/N}{q} \sum_{l_1 \geq 1} \cdots \sum_{l_q \geq 1} \sum_{k_1=1}^{l_1} \cdots \sum_{k_q=1}^{l_q} \binom{x}{k_1} \cdots \binom{x}{k_q} \cdot B_{l_1 \cdots l_q ; k_1 \cdots k_q}$$

$$l_1 + \cdots + l_q = l$$

$$= 1 + \sum_{l=1}^{\infty} a_l(x) \cdot \varepsilon^l \tag{7}$$

where a_l is defined by the last equality and

$$B_{l_1 \cdots l_q ; k_1 \cdots k_q} = \sum_{n_{11} \geq 1} \cdots \sum_{n_{1k_1} \geq 1} \cdots \sum_{n_{q1} \geq 1} \cdots \sum_{n_{qk_q} \geq 1} \prod_{i=1}^{q} \left(\overline{\prod_{j=1}^{k_i} \frac{1}{n_{ij}!} \langle \mathcal{H}_1^{n_{ij}} \rangle_0} \right) \tag{8}$$

$$n_{11} + \cdots + n_{1k_1} = l_1 \quad n_{q1} + \cdots + n_{qk_q} = l_q$$

As we see, $a_\ell(x)$ is an ℓ th order polynomial in x. Its relation to the quenched free energy can be seen by differentiating Eq. (7) with respect to x and setting $x=0$:

$$\left.\frac{\partial \Pi}{\partial x}\right|_{x=0} = \frac{1}{N}\overline{\ln Z} - \frac{1}{N}\ln Z_0 = \sum_{\ell=1}^{\infty} a'_\ell(0)\cdot\varepsilon^\ell \qquad (9)$$

Therefore, by knowing $a'_\ell(0)$ for all ℓ, the quenched free energy is also known (this latter being analytic at $\varepsilon = 0$ for N finite). The calculation of a_ℓ from Eqs. (7) and (8) would be rather difficult. Let us suppose, however, that one is able to calculate $\Pi(x,\varepsilon)$ for $x = m$ *general non-negative integer;* i.e. one obtains a function $\hat{\Pi}(m,\varepsilon)$ so that $\hat{\Pi}(m,\varepsilon) = \Pi(m,\varepsilon)$ identically in ε for any non-negative integer m. Though $\hat{\Pi}(x,\varepsilon) \neq \Pi(x,\varepsilon)$ may happen for non-integer x, the Taylor coefficients a_ℓ of Π can be computed from $\hat{\Pi}$. Indeed, let us observe that the $\ell+1$ numbers $a_\ell(0)=0$, $a_\ell(1), \ldots, a_\ell(\ell)$ completely determine the polynomial $a_\ell(x)$ by Lagrange interpolation. On the other hand, $\hat{\Pi}(m,\varepsilon)$ is analytic in ε at $\varepsilon=0$ and we have

$$\frac{1}{\ell!}\frac{\partial^\ell}{\partial\varepsilon^\ell}\hat{\Pi}(m,\varepsilon)\Big|_{\varepsilon=0} = \frac{1}{\ell!}\frac{\partial^\ell}{\partial\varepsilon^\ell}\Pi(m,\varepsilon) = a_\ell(m) \qquad (10)$$

for all ℓ and integer m. We can therefore draw our first conclusion:

(C1) *For a finite system (N finite) the replica-partition functions $\overline{Z^m}$ always determine the quenched free energy, by determining its Taylor series about $\varepsilon=0$.*

This result does not necessarily imply that the quenched free energy can be obtained by the usual replica trick. The generally valid method is to determine $a_\ell(m)$ for $m = 1, 2, \ldots, \ell$ and for any ℓ through Eq. (10), then to build up the polynomial $a_\ell(x)$ by interpolation, to calculate $a'_\ell(0)$ and finally to construct the free energy from its Taylor expansion (9). The validity of the replica trick, as it is generally applied, can be checked in the following way:

(C2) *Consider the Taylor coefficients of* $\hat{\Pi}$,

$$\hat{a}_\ell(x) = \frac{1}{\ell!} \frac{\partial^\ell}{\partial \varepsilon^\ell} \hat{\Pi}(x, \varepsilon) \Big|_{\varepsilon = 0} \tag{11}$$

If $\hat{a}_\ell(x)$ *is a polynomial for all* ℓ , *then the usual replica trick holds true for the whole free energy and separately for any order in* ε , *namely*

$$\frac{1}{N} \overline{\ln Z} - \frac{1}{N} \ln Z_0 = \frac{\hat{\Pi}(m, \varepsilon) - 1}{m} \Big|_{m=0} \tag{12}$$

and

$$a'_\ell(0) = \frac{\hat{a}_\ell(m)}{m} \Big|_{m=0} \tag{13}$$

Indeed, if $\hat{a}_\ell(x)$ is a polynomial for all ℓ then due to (10) it coincides with the polynomial $a_\ell(x)$ for all ℓ and therefore $\hat{\Pi}(x, \varepsilon)$ coincides with $\Pi(x, \varepsilon)$. The result then follows from the first equality of Eq. (9) and from the

continuity of $\hat{\Pi}$ at $x=0$. An analogous observation was made by Parisi (1982), in a field theoretical model with random magnetic field, though that expansion proceeds in the powers of the coefficient of the ϕ^4 term.

So far, our considerations referred to finite systems: all the functions Π, α_ℓ, $\hat{\Pi}$, $\hat{\alpha}_\ell$ depended on N , the number of spins. Our real interest is in the validity of the replica trick at $N\to\infty$ where the trick may break down. Below we formulate a sufficient condition that this does not happen. We notice that the existence of the thermodynamic limit of $\Pi(x,\varepsilon)$ can be shown for real x and ε under rather general conditions.

(C3) *The conclusions (C1) and (C2) remain valid in the infinite system if the thermodynamic limit of* $\Pi(x,\varepsilon)$ *is an analytic function of* ε *at* $\varepsilon=0$.

What we need is the term-by-term convergence of the Taylor series (7) as N goes to infinity. In this case $\alpha_\ell(x)$ = $= \lim_N \alpha_\ell^{(N)}(x)$ is also an ℓ th order polynomial of x and the previous considerations can be repeated without any change. Now (C3) implies that the breakdown of the replica trick indicates a singularity of $\Pi(x,\varepsilon)$ at $\varepsilon=0$.

3. THE ONE-DIMENSIONAL ISING MODEL IN RANDOM EXTERNAL FIELD (1d-RFIM)

In this section we investigate the free energy of the 1d-RFIM firstly by means of a direct method and secondly by applying the replica trick. The Hamiltonian is defined by

$$\mathcal{H}_0 = K \sum_i S_i S_{i+1}$$

$$\varepsilon \cdot \mathcal{H}_1 = \sum H_i S_i = \varepsilon \sum h_i S_i \qquad (14)$$

where h_i are independent random variables with the same distribution g so that $g(h) = g(-h)$ and $\int h^2 \cdot g(h)\, dh = 1$. It is thought that the free energy is analytic in ε at $\varepsilon = 0$ for finite values of K, therefore we expect the coincidence of the two results.

3.1. DIRECT METHOD

The partition function is expressed as

$$Z = \mathrm{Tr} \prod_i \underset{\sim}{T}_i$$

where

$$\underset{\sim}{T}_i = \begin{pmatrix} e^{H_i} & 0 \\ 0 & e^{-H_i} \end{pmatrix} \cdot \begin{pmatrix} e^{K} & e^{-K} \\ e^{-K} & e^{K} \end{pmatrix} \qquad (15)$$

Using the invariance of the trace and applying

$$\underset{\sim}{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad (16)$$

for a term-by-term unitary transformation in Eq. (15), we

obtain

$$Z = C^N \cdot \left(\prod_i c_i\right) \cdot Tr\left\{\prod_i \left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + t_i \cdot \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right] \cdot \begin{pmatrix}1 & 0\\ 0 & T\end{pmatrix}\right\} \qquad (17)$$

where $C = \cosh K$, $S = \sinh K$, $T = \tanh K$, $c_i = \cosh H_i$ and $t_i = \tanh H_i$.

Observing that

$$Tr\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^n = 0 \qquad \text{for } n \text{ odd}$$

and collecting the surviving terms we get

$$Z = (2C)^N \cdot \left(\prod_i c_i\right) \cdot \left\{1 + \sum_{n=1}^{N/2} \sum_{1 \le i_1 < \ldots < i_{2n} \le N} t_{i_1} \ldots t_{i_{2N}} \cdot T^{(i_2 - i_1 + \ldots + i_{2n} - i_{2n-1})}\right\} \qquad (18)$$

in which we recognize the usual high-temperature expansion of the partition function.

The free energy is determined by the average of the logarithm of the partition function. In order to obtain the series of $\overline{\ln Z}$ in powers of ε , one has to take the logarithm of Eq. (18), then expand the logarithm of the bracket. This is followed by the averaging over h_i and then by a resummation of the powers of T . Using the fact, that $\overline{t_i \cdot t_j} = \delta_{ij}\, \overline{t^2}$ and collecting carefully all the possible pairings of the t_i-s , we obtain, up to sixth order

$$-\frac{\beta F}{N} \equiv f = \ln 2G + \overline{\ln C} - \frac{1}{2} S^2 \overline{t^2}^2 + 2 \cdot S^4 \overline{t^2}^3 =$$

$$= \ln 2G + \frac{1}{2} \overline{h^2} \cdot \varepsilon^2 - \frac{1}{2}\left(\frac{1}{6} h^4 + S^2 \overline{h^2}^2\right) \cdot \varepsilon^4 + \qquad (19)$$

$$+ \left(\frac{1}{45} \overline{h^6} + \frac{2}{3} S^2 \cdot \overline{h^4} \cdot \overline{h^2} + 2 S^4 \overline{h^2}\right) \cdot \varepsilon^6 + \mathcal{O}(\varepsilon^8)$$

Here we dropped the site-index i . The free energy up to second order was obtained independently by Bruinsma and Aeppli (1983 a,b) and up to fourth order by Györgyi and Ruján(1984).

3.2. REPLICA METHOD

In this subsection we rederive the above series expansion for the free energy by the replica method. From the considerations of Section 2. one knows that in order to go up to the mth power in the series expansion, it is sufficient to calculate only the partition function of the $1,2,\ldots,m$ replica-systems. Nevertheless, in what follows, we provide the derivation for a general number of replicas so as to get a full comparison with the results of the previous section.

The partition function for the m -replica system is

$$Z^m = Tr \prod_i \underset{\sim}{T}_i^{\otimes m} \qquad (20)$$

where $\underset{\sim}{T}_i^{\otimes m}$ denotes the m th tensorian power of the transfer matrix $\underset{\sim}{T}_i$ (15).

Due to the independence of the random fields at different sites, averaging of Eq. (20) is equivalent to averaging the

elements of $\underset{\sim}{T}_i^{\otimes}$. This yields a $2^m \times 2^m$ site--independent transfer matrix $\underset{\sim}{T}$ and hence

$$\overline{Z^m} = \mathrm{Tr}\, \underset{\sim}{T}^N$$

The elements of $\underset{\sim}{T}$ are labelled by the configurations of m spins:

$$\underset{\sim}{T}_{\underset{\sim}{S},\underset{\sim}{\sigma}} = e^{K \cdot \underset{\sim}{S}\underset{\sim}{\sigma}} \overline{e^{H \sum_i^m S_i}} \tag{21}$$

where $\underset{\sim}{S} = (S^1, \ldots, S^m)$. Developing the average of the exponential and keeping the terms up to sixth order we find:

$$\underset{\sim}{T}_{\underset{\sim}{S}\underset{\sim}{\sigma}} = e^{K \underset{\sim}{S}\underset{\sim}{\sigma}} \cdot \Big(\alpha_m + \beta_m \cdot \sum_{i<j} S^i S^j + \gamma_m \cdot \sum_{i<j<k<l} S^i S^j S^k S^l + \delta_m \cdot \sum_{i<j<k<l<m<n} S^i S^j S^k S^l S^m S^n \Big) \tag{22a}$$

where

$$\begin{aligned}
\alpha_m &= 1 + \frac{m}{2}\,\overline{H^2} + \left(\frac{m^2}{8} - \frac{m}{12}\right)\overline{H^4} + \left(\frac{m^3}{48} - \frac{m^2}{24} + \frac{m}{45}\right)\overline{H^6} \\
\beta_m &= 1 + \left(\frac{m}{2} - \frac{2}{3}\right)\frac{\overline{H^4}}{\overline{H^2}} + \left(\frac{7m^2}{24} - \frac{5m}{4} + \frac{62}{45}\right)\frac{\overline{H^6}}{\overline{H^2}} \\
\gamma_m &= \frac{\overline{H^4}}{\overline{H^2}} + \left(\frac{m}{2} - \frac{4}{3}\right)\frac{\overline{H^6}}{\overline{H^2}} \\
\delta_m &= \frac{\overline{H^6}}{\overline{H^2}}
\end{aligned} \tag{22b}$$

Let us notice, that in Eq. (22a) the sum subsequent to α_m is

of the order of ε^2 . Therefore we can write

$$\underset{\sim}{T} = \underset{\sim}{A} + \varepsilon^2 \underset{\sim}{B}$$

and look for the largest eigenvalue of $\underset{\sim}{T}$ in the form

$$\lambda(m) = \lambda_1 + \varepsilon^2 \cdot \lambda \tag{23}$$

where λ_1 is the largest eigenvalue of the matrix

$$\underset{\sim}{A} = \alpha_m \cdot e^{K \underset{\sim}{S} \underset{\sim}{G}}$$

It is assumed, that ε is sufficiently small so that no level-crossing occurs and the deviation of $\lambda(m)$ from λ_1 is small.

In Appendix A we derive a formula for the evaluation of the determinant of a perturbed matrix. We employ this for the secular equation to compute $\lambda(m)$. In the case when $\underset{\sim}{A}$ is diagonal, the equation for λ is

$$\lambda\left(1 - \varepsilon^2 \sum_{i=2}^{n} \frac{b_{11}+b_{ii}}{\lambda_1-\lambda_i}\right) - b_{11} + \varepsilon^2 \sum \frac{b_{11}b_{ii} - b_{1i}b_{i1}}{\lambda_1 - \lambda_i} -$$

$$- \varepsilon^4 \sum_{\substack{i,j=2 \\ i<j}}^{n} \frac{b_{ii}b_{ij}b_{j1} + b_{1j}b_{ji}b_{i1}}{(\lambda_1-\lambda_i)(\lambda_1-\lambda_j)} = 0 \qquad ; n = 2^m \tag{24}$$

Here b_{ij} are elements of the matrix $\underset{\sim}{B}$, after the transformation which diagonalizes $\underset{\sim}{A}$ is carried out on it. We shall see later, that b_{11} is equal to zero so that λ is of the order of ε^2 . For this case, Equation (24) contains all the terms up to ε^4 .

Now the matrix $\underset{\sim}{A}$ can be diagonalized with the orthogonal matrix $\underset{\sim}{U}^{\otimes m}$ where $\underset{\sim}{U}$ diagonalizes the 2 x 2 transfer matrix and was given explicitly in Eq. (16).

By inspection one can justify, that

$$\underset{\sim}{U}^{\otimes m}_{\underset{\sim}{s},\underset{\sim}{\sigma}} = 2^{-m/2} \cdot \operatorname{sgn} \prod_{i=1}^{m} (s^i + \sigma^i + 1) \tag{25}$$

A direct evaluation proves indeed that with (25) one gets

$$\underset{\sim}{U}^{\otimes m} \cdot \underset{\sim}{U}^{\otimes m} = \underset{\sim}{1}^{\otimes m} \tag{26}$$

and

$$\underset{\sim}{U}^{\otimes}_{\underset{\sim}{s}\underset{\sim}{s}'} e^{K \underset{\sim}{s}'\underset{\sim}{\sigma}'} \underset{\sim}{U}^{\otimes m}_{\underset{\sigma}{\sim}'\underset{\sim}{\sigma}} = \delta_{\underset{\sim}{s}\underset{\sim}{\sigma}} \cdot 2^m \cdot S^{\frac{1}{2}(m - \sum_{i=1}^{m} \sigma_i)} \cdot C^{\frac{1}{2}(m + \sum_{i=1}^{m} \sigma_i)} \tag{27}$$

For the transformation of $\underset{\sim}{B}$ one also needs

$$\underset{\sim}{U}^{\otimes m}_{\underset{\sim}{s},\underset{\sim}{s}'} \cdot \Big(\sum_{i_1 < \ldots < i_{2p}} s'_{i_1} \cdots s'_{i_{2p}} \cdot e^{K \underset{\sim}{s}'\underset{\sim}{\sigma}'} \Big) \underset{\sim}{U}^{\otimes m}_{\underset{\sigma}{\sim}'\underset{\sim}{\sigma}} =$$

$$= \begin{cases} 2^m \cdot S^{\frac{1}{2}(m - \sum_{i=1}^{m} \sigma_i)} \cdot C^{\frac{1}{2}(m + \sum_{i=1}^{m} \sigma_i)} \\ \qquad \text{for all configurations where } \underset{\sim}{s} \text{ and } \underset{\sim}{\sigma} \\ \qquad \text{differ precisely in } 2p \text{ sites} \\ 0 \qquad \text{otherwise} \end{cases} \tag{28}$$

For completeness, we verify this last statement in Appendix B. It also implies the vanishing of b_{11} and, in general, of b_{ii} .

By applying Eq. (28) we find that after the unitary transformation $\underset{\sim}{B}$ has the form

$$\underset{\sim}{B}_{\underset{\sim}{S}\underset{\sim}{\sigma}} = 2^m \cdot S^{\frac{1}{2}(m-\sum_{i=1}^{m}\sigma_i)} \cdot C^{\frac{1}{2}(m+\sum_{i=1}^{m}\sigma_i)} \cdot \begin{cases} \beta_m & \text{if } \underset{\sim}{S} \text{ and } \underset{\sim}{\sigma} \text{ differ in two sites} \\ \gamma_m & \text{if } \underset{\sim}{S} \text{ and } \underset{\sim}{\sigma} \text{ differ in four sites} \\ \delta_m & \text{if } \underset{\sim}{S} \text{ and } \underset{\sim}{\sigma} \text{ differ in six sites} \\ \vdots & \\ 0 & \text{otherwise} \end{cases}$$

Moreover, the Equation (24) for λ simplifies further to

$$\lambda = \varepsilon^2 \sum_{i=2}^{n} \frac{b_{1i} b_{i1}}{\lambda_1 - \lambda_i} + \varepsilon^4 \sum_{i,j=2}^{n} \frac{b_{1i} b_{ij} b_{j1}}{(\lambda_1 - \lambda_i)(\lambda_1 - \lambda_j)} + \mathcal{O}(\varepsilon^6) \qquad (29)$$

where the numbering of the configurations goes in the decreasing order of their magnetization. In the case of degeneracy the order is arbitrary.

The correction to λ_1 (c.f. Eq.(23)) was defined as $\varepsilon^2 \cdot \lambda$, it is therefore sufficient to compute the matrix-elements b_{ij} up to $\mathcal{O}(\varepsilon^2)$ so as to get the result precise up to $\mathcal{O}(\varepsilon^6)$. Selecting out the non-vanishing elements we obtain finally:

$$\lambda(m)/(2C)^m = \hat{\Pi}(m,\varepsilon) = 1 + \frac{m}{2}\,\overline{h^2}\cdot\varepsilon^2 + \Big[\Big(\frac{m^2}{8} - \frac{m}{12}\Big)\overline{h^4} + \frac{1}{2}m(m-1)\,S^2\,\overline{h^2}^2\Big]\cdot\varepsilon^4 + \Big[\Big(\frac{m^3}{48} - \frac{m^2}{24} + \frac{m}{45}\Big)\overline{h^6} + m\cdot(m-1)\cdot\Big(\frac{m}{2} - \frac{2}{3}\Big)\cdot S^2\cdot\overline{h^2}\cdot\overline{h^4} + m\cdot(m-1)(m-2)\,S^4\,\overline{h^2}^3\Big]\cdot\varepsilon^6 + \mathcal{O}(\varepsilon^8)$$

$$= 1 + \hat{a}_2(m)\cdot\varepsilon^2 + \hat{a}_4(m)\cdot\varepsilon^4 + \hat{a}_6(m)\cdot\varepsilon^6 + \mathcal{O}(\varepsilon^8) \qquad (30)$$

where the notations $\hat{\Pi}$ and $\hat{a}_\ell$ correspond to those introduced in the previous part. Now $\hat{a}_2$, $\hat{a}_4$ and $\hat{a}_6$ are polynomials, hence we can deduce from (C2) that they must coincide respectively with a_2, a_4 and a_6 (c.f. Eq. (7)). Then by Eq. (13) we get

$$a_2'(0) = \frac{1}{2}\,\overline{h^2} \quad , \quad a_4'(0) = -\frac{1}{12}\,\overline{h^4} - \frac{1}{2}\,S^2\,\overline{h^2}^2$$

$$a_6'(0) = \frac{1}{45}\,\overline{h^6} + \frac{2}{3}\,S^2\,\overline{h^2}\cdot\overline{h^4} + 2\cdot S^4\,\overline{h^2}^3 \qquad (31)$$

in a complete agreement with Eq. (19).

Let us notice, that the actual order of the polynomial $a_\ell(x)$ is only $\ell/2$. This can be understood, by inspecting Eqs. (7), (8) and (14), as the consequence of the occurrence of a vanishing factor $\langle S_i\rangle_0$ in all terms proportional to x^k for $k > \ell/2$.

4. DISCUSSION

In this paper we have studied the possibility to reproduce the free energy of general random models from the free energies of the corresponding m-replicated systems, where m is positive integer. As a sufficient condition (which may prove to be also necessary) we found the analyticity of the free energy as a function of the strength (or variance) ε of the random variables at $\varepsilon=0$. In models with a phase transition the analyticity in ε is likely to break down below the critical temperature, therefore one may generally expect the necessity of some procedure *à la Parisi* (1979) - or the choice of other methods - in this temperature region.

An interesting feature of the explicitely discussed example (1d-RFIM) is the divergence of the fourth and higher-order terms of the free energy as K goes to infinity (c.f. Eq. (19)). This of course does not imply that f diverges exponentially with increasing K; it rather implies that the radius of convergence of the ε-expansion goes to zero exponentially fast as $K\to\infty$. The non-analyticity at $\varepsilon=0$ is somewhat analogous to the singularity at $p=0$ found in the model of Grinstein and Mukamel (1983), though the expansion (19) is meaningless for their case.

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Appendix A

We consider the matrix $\underset{\sim}{T} = \underset{\sim}{A} + \varepsilon^2 \underset{\sim}{B}$ and look for its largest eigenvalue in the form $\lambda(m) = \lambda_1 + \varepsilon^2 \lambda$. The secular equation is obtained as

$$|\underset{\sim}{T} - \lambda(m)\cdot\underset{\sim}{1}| = |\underset{\sim}{A} - \lambda_1 \underset{\sim}{1}| + \varepsilon^2 \cdot \left\{ \begin{vmatrix} b_{11}-\lambda & b_{12} & \cdots \\ a_{21} & a_{22}-\lambda_1 & \cdots \\ \vdots & & \\ a_{n1} & a_{n2} & \cdots \end{vmatrix} + \ldots + \begin{vmatrix} a_{11}-\lambda_1 & a_{12} & \cdots \\ \vdots & & \\ a_{n-1,1} & a_{n-1,2} & \cdots \\ b_{n1} & b_{n2} & \cdots b_{nn}-\lambda \end{vmatrix} \right\}$$

$$+ \varepsilon^4 \cdot \left\{ \begin{vmatrix} b_{11}-\lambda & b_{12} & b_{13} & \cdots \\ b_{21} & b_{22}-\lambda & b_{23} & \cdots \\ a_{31} & a_{32} & a_{33}-\lambda_1 & \\ \vdots & \vdots & & \end{vmatrix} + \ldots \right\} + \mathcal{O}(\varepsilon^6)$$

The treatment simplifies considerably, if $\underset{\sim}{A}$ is diagonal. Then, dividing by $\prod_{i=2}^{n} (\lambda_1 - \lambda_i)$ one obtains

$$b_{11} - \lambda + \varepsilon^2 \cdot \left\{ \frac{1}{\lambda_2 - \lambda_1} \begin{vmatrix} b_{11}-\lambda & b_{12} \\ b_{21} & b_{22}-\lambda \end{vmatrix} + \frac{1}{\lambda_3 - \lambda_1} \begin{vmatrix} b_{11}-\lambda & b_{13} \\ b_{31} & b_{33}-\lambda \end{vmatrix} + \ldots \right\} +$$

$$+ \varepsilon^4 \cdot \left\{ \frac{1}{(\lambda_2 - \lambda_1)\cdot(\lambda_3 - \lambda_1)} \begin{vmatrix} b_{11}-\lambda & b_{12} & b_{13} \\ b_{21} & b_{33}-\lambda & b_{23} \\ b_{31} & b_{32} & b_{33}-\lambda \end{vmatrix} + \ldots \right\} + \mathcal{O}(\varepsilon^6) = 0$$

Collecting the terms up to $\mathcal{O}(\varepsilon^4)$ we immediately arrive at Eq. (24).

Appendix B

Here we show the equality (28) for $p=1$; the generalization is straightforward.

One has to compute

$$\underset{\sim}{U}^{\otimes m}_{\underset{\sim}{s}\underset{\sim}{s}'} \cdot \Big(\sum_{i<j} s_i' s_j'\Big) e^{K \underset{\sim}{s}'\underset{\sim}{\sigma}'} \underset{\sim}{U}^{\otimes m}_{\underset{\sim}{\sigma}'\underset{\sim}{\sigma}} \equiv \underset{\sim}{\Omega}_{\underset{\sim}{s},\underset{\sim}{\sigma}}$$

Using the well-known decomposition of the exponential and Eq. (25),

$$\underset{\sim}{\Omega}_{\underset{\sim}{s},\underset{\sim}{\sigma}} = 2^{-m} \sum_{\underset{\sim}{s}',\underset{\sim}{\sigma}'} \sum_{i<j} s_i' s_j' \sum_{I \subset M} \Big(\prod_{k\in I} s_k' \cdot \operatorname{sgn}(s_k + s_k' + 1)\Big) \Big(\prod_{k\in I} \sigma_k' (\sigma_k' + \sigma_k + 1)\Big) \cdot$$

$$\cdot \Big(\prod_{k \in M\setminus I} \operatorname{sgn}(s_k + s_k' + 1)\Big) \cdot \Big(\prod_{k\in M\setminus I} \operatorname{sgn}(\sigma_k' + \sigma_k + 1)\Big) \cdot S'^{|I|} \cdot C^{m-|I|}$$

where $M = \{1, 2, \ldots, m\}$ is the set of replica-indices and $|I|$ denotes the number of elements of I .

Let us consider a particular term and transform it, as follows:

$$\sum_{\underset{\sim}{s}'_I} \prod_{k\in I} s_k' \cdot \operatorname{sgn}(s_k + s_k' + 1) = \sum_{\underset{\sim}{s}'_{I_+^s}} \prod_{k\in I_+^s} s_k' \cdot \sum_{\underset{\sim}{s}'_{I_-^s}} \prod_{k\in I_-^s} 1 =$$

$$= 2^{|I_-^s|} \cdot \sum_{\underset{\sim}{s}'_{I_+^s}} \prod_{k\in I_+^s} s_k' = 2^{|I|} \cdot \delta_{I_+^s, 0}$$

where I_+^s is the subset of I containing all the cases, when s_k is an up spin and I_-^s is its complement.

Deriving a few analogous formulas we arrive at

$$\Omega_{\underset{\sim}{S},\underset{\sim}{\sigma}} = \sum_{i<j} \sum_{\underset{\sim}{S}'} S_i' S_j' \sum_{I \subset M} \delta_{I^{\sigma}_{+},0} \cdot \delta_{(M\setminus I)^{\sigma}_{-},0} \cdot S^{|I|} \cdot C^{m-|I|} \cdot$$

$$\cdot \left(\prod_{k \in I} S_k' \cdot \mathrm{sgn}\,(S_k + S_k' + 1) \right) \cdot \left(\prod_{k \in M\setminus I} \mathrm{sgn}\,(S_k + S_k' + 1) \right)$$

Visibly only one set survives from the "I-summation" and that is M^{σ}_{-}, containing all the replicas, where the $\underset{\sim}{\sigma}$ spin is downward. Doing the $\underset{\sim}{S}'$ summation we must take care for the two additional spin $S_i' \cdot S_j'$. We employ a further transformation formula, to be checked directly

$$\sum_{S_k'} S_k' \cdot \mathrm{sgn}\,(S_k + S_k' + 1) = (1 - S_k)$$

Denoting the set of the two fixed points by $\{i,j\}$ we obtain

$$\Omega_{\underset{\sim}{S},\underset{\sim}{\sigma}} = S^{|M^{\sigma}_{-}|} C^{|M^{\sigma}_{+}|} \cdot \prod_{k \in M^{\sigma}_{-}} (1 - S_k) \cdot \prod_{\ell \in M^{\sigma}_{+}} (1 + S_\ell) \cdot$$

$$\cdot \left\{ \sum_{i,j \in M^{\sigma}_{-}} \frac{(1+S_i)(1+S_j)}{(1-S_i)(1-S_j)} + \sum_{i,j \in M^{\sigma}_{+}} \frac{(1-S_i)(1-S_j)}{(1+S_i)(1+S_j)} + \sum_{i \in M^{\sigma}_{-}} \sum_{j \in M^{\sigma}_{+}} \frac{(1+S_i)(1-S_j)}{(1-S_i)(1+S_j)} \right.$$

$$\left. + \sum_{i \in M^{\sigma}_{+}} \sum_{j \in M^{\sigma}_{-}} \frac{(1-S_i)(1+S_j)}{(1+S_i)(1-S_j)} \right\}$$

The two products in the first row would compel the $\underset{\sim}{S}$ configuration to coincide with the $\underset{\sim}{\sigma}$ configuration. However, the terms in the curly bracket force them to differ precisely in two sites. As we noted the generalization of the above procedure is quite obvious and this completes the proof of Eq. (28).

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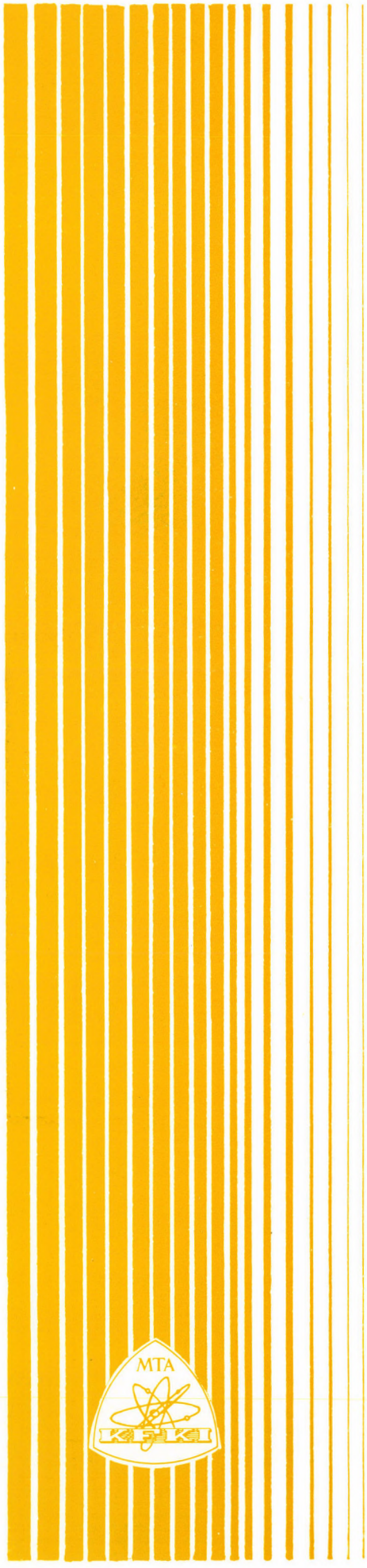

Kiadja a Központi Fizikai Kutató Intézet
Felelős kiadó: Kroó Norbert
Szakmai lektor: Dr. Forgács Gábor
Nyelvi lektor: Dr. Forgács Gábor
Példányszám: 125 Törzsszám: 84-299
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