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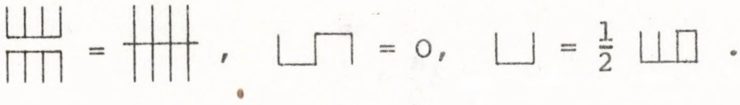

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ABSTRACT

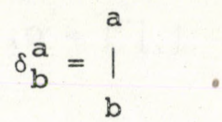
The simplest physical system to have a non-trivial intrinsic structure in Minkowski space-time is a three-twistor particle. We investigate this structure and the two pictures of the particle as an extended object in space-time and as a point in unitary space. We consider the effect of twistor translations on the mass triangle defined by the partial centre of mass points in space-time. Finally we consider the connections between twistor rotations and spin and we establish the spin deficiency formula.

АННОТАЦИЯ

В пространстве-времени Минковски простейшей физической системой является частица, состоящая из трёх твисторов. Нами изучается эта частица, являющаяся обширной системой в пространстве-времени и точкой в унитарном пространстве. В прострастве-времени массовый треугольник определяют парциальные массовые центры. Записав эффект твисторных трансляций на массовый треугольник, определив связь между спином и ротациями твисторов, нами выведена формула спинового дефекта.

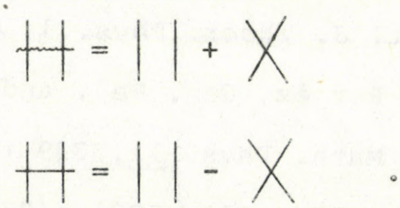
KIVONAT

Minkowski téridőben a legegyszerübb, nemtriviális belső szerkezetü fizikai rendszer a háromtvisztor-részecske. Megvizsgáljuk ezt a részecskét, amely a téridőben kiterjedt és az unitér térben pontszerü rendszer. A téridőben a parciális tömegközéppontok tömegháromszöget definiálnak. Felirjuk a tvisztor-transzlációk hatását a tömegháromszögre. Megállapitjuk a spin és a tvisztor-elforgatások kapcsolatát és levezetjük a spinhiány-formulát.

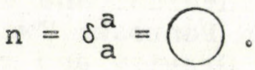


The notation converts identities of the kind $U^f_{gr}\ \delta^r_h = U^f_{gh}$ into trivial partitions of some index line.

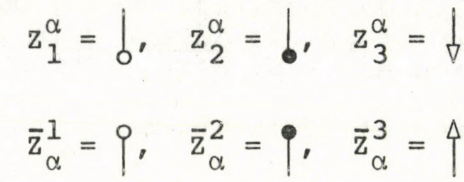
Symmetrization and skewing in like indices is denoted according to the scheme



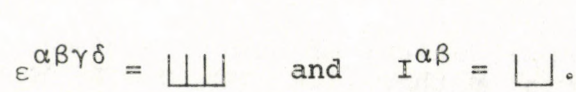
The dimension n of the tensor system is given by the loop



In twistor theory one is interested in dimension n=4. Taking twistor complex conjugates is an involution that has the effect of turning the symbols upside down. Thus the blobs of twistors Z^α_1, Z^α_2, Z^α_3 and of their complex conjugates are drawn



The skew unit twistor and the infinity twistor are denoted, respectively

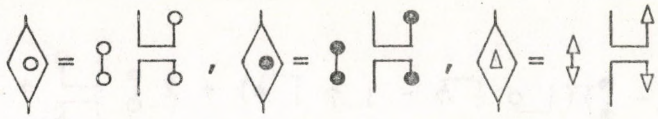


It is useful in computations to keep in mind some of their algebraic properties in the blob notation such as

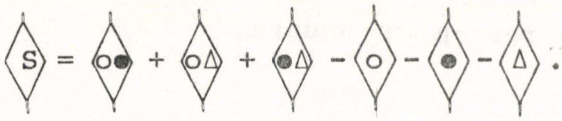
1. INTRODUCTION

In Penrose's theory of twistors, zero-mass objects are assigned a fundamental role. The zero-mass particles are represented (classically) by a single twistor while the massive particles are represented by several twistors[1-6]. The basic idea of twistor particle theory is that the kinematic variables, e.g. momentum and angular momentum, associated with the massive particles can be expressed in terms of two or more twistors. (This description is via the so-called kinematic twistor). On the other hand the internal structure of the particle does depend critically on the number of twistors used in the description. (Frequently in twistor theory[6-8] leptons are described by two-twistor systems while hadrons are described by three-twistor systems.) The linear transformations among the two or three (or more) twistors which preserve the kinematic twistor (or variables) are referred to as internal symmetry transformations[7], (IST). The two-twistor particles have the simplest space-time description. They can be thought of as either a real center of mass world-line with an associated momentum and spin or as a complex center of mass world-line.

Three-twistor particles are the simplest systems which <u>do</u> possess an extended structure in complex Minkowski-space. A three twistor particle with twistors X^α, Y^α, Z^α can be thought of as being (in some sense) composed of the three pairs (X^α, Y^α), (Y^α, Z^α) and (Z^α, X^α) of twistors[5] and thus it would have a substructure of three two-twistor massive sub-systems. These structure are however not disjoint since any pair of them has a twistor in common. Furthermore the world-lines, masses, spins, etc. of the parts make up those of the entire system. A point to be


(7.8)

of the zero-mass constituents, we obtain the spin definiency formula in the blob notation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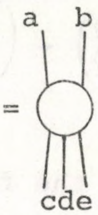
APPENDIX: THE BLOB NOTATION

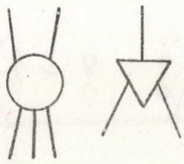
The blob notation of abstract tensor systems has been first introduced by Penrose[14]. Its advantages over the more conventional formalisms of Ricci, Levi-Civita and Einstein are best understood probably in terms of the physiology of human perception.

A tensor is drawn in the diagrammatic notation as a blob with arms and legs depicting the upper and lower indices:

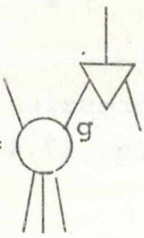
$T^{ab}_{cde} =$ 

$U^{f}_{gh} =$ 

The outer product of tensors T^{ab}_{cde} and U^{f}_{gh} is the juxtaposition of blobs

$T^{ab}_{cde}\, U^{f}_{gh} =$ 

To contract a pair of indices, one connects the corresponding arm and leg,

$T^{ag}_{cde}\, U^{f}_{gh} =$ 

A Kronecker symbol δ^{a}_{b} is represented by a line segment:

emphasized is that the subsystems and their kinematic properties are not invariant under the IST.

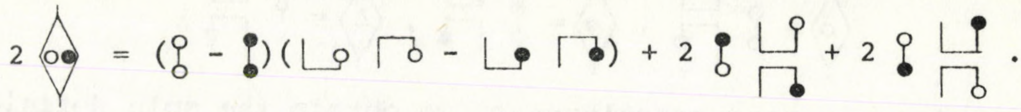
The present work is an attempt to study the geometry of this twistor decomposition of three-twistor systems. The two main questions investigated are (a) what are the changes in the subsystems caused by the action of the IST, and (b) what is the relationship of the total system variables to those of the related sub-system variables. For pedagogical reasons we have chosen a purely classical approach, most of the results easily surviving quantization.

In Section 2 the general theory[6-8] of massive n-twistor particles is reviewed. At the center of the formalism lies the kinematic twistor in terms of which the momentum, angular momentum and center of mass line of the particle are expressed. As mentioned before, the IST of the system leaves the kinematic twistor invariant.

In Section 3 we specialize to three-twistor systems and study the IST which turns out to be the inhomogeneous SU(3) group (ISU(3)). Three spaces play a critical role here (1) (twistor space T on which we choose three points $Z_i^\alpha = (X^\alpha, Y^\alpha, Z^\alpha)$, or alternately three copies of T, i.e. T x T x T, (2) complex Minkowski space which has the naturally chosen triple of points x^a, y^a, z^a, each being the intersections of the twistor pairs (Y^α, Z^α), (Z^α, X^α), and (X^α, Y^α) and (3) unitary space, a three-complex dimensional affine space on which the elementary representation of the ISU(3) group acts as the isometries. It can be viewed as the homogeneous space ISU(3)/SU(3).

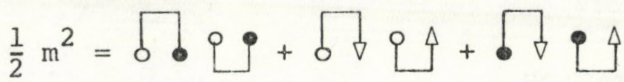
In Section 4 we investigate the translation subgroup of the ISU(3) group and show that a translation along a given axis in unitary space leaves the corresponding twistor unchanged while shifting the other two in complex space-time along their subsystem or partial center of mass line. We also find a unique correspondence between the time development of the system in Minkowski space and a special translation in unitary space.

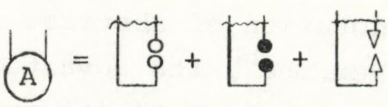
In Section 5 we study the SU(3) subgroup of ISU(3) and show how from its generators we can find a unique "complex internal center of mass" world line in unitary space. This is in analogy with the use of the homogeneous Lorentz group generators to find we obtain



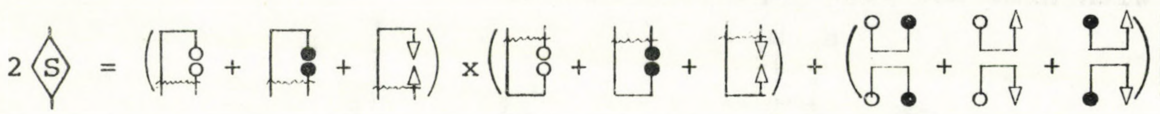
(7.4)

We now compute the spin of a three-twistor particle similarly. Inserting the rest-mass square,

 (7.5)

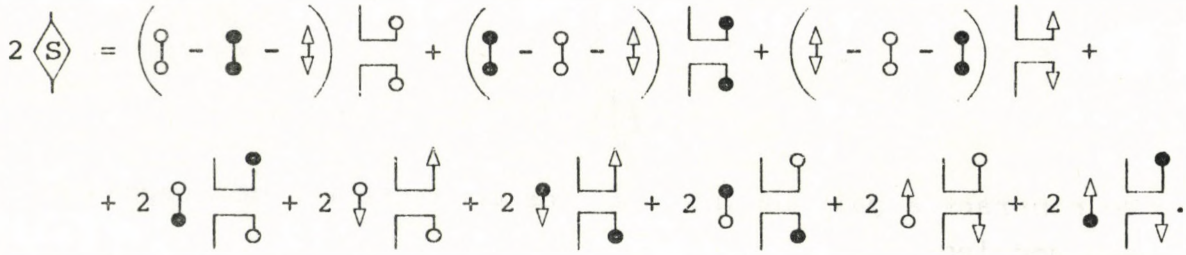
and the kinematical twistor 

in Eq. (7.2) we have



(7.6)

Using a judicious amount of identities of the form (7.3) for various subsystems, we obtain for the spin of the three--twistor particle:



(7.7)

When we compare this expression with the spin twistors of the massive subsystems (Cf. Eq. (7.4)) and with the spin twistors the center of mass[1]. We further discuss the internal or unitary spin (which is analogous to the Pauli-Lubanski vector) and show its relationship to the space-time spin. Section 6 deals with the mass triangle defined by the partial center of mass points while Section 7 presents the spin deficiency formula, i.e. the relationship between the total spin and the constituent spins. According to this formula the total spin is the sum of the massive subsystem spins minus the spins of the three twistors.

In the concluding sections we use the Penrose blob notation[14] to facilitate lengthy algebraic computation. An introduction to the blob notation is given in an Appendix.

2. THE TWISTOR CONSTITUENTS

Consider a massive particle in Minkowski space-time as a system of $n \geq 2$ massless constituent twistors Z_i^α, $i=1,\dots n$. The particle has the kinematical twistor[1]

$$A^{\alpha\beta} = 2Z_i^{(\alpha}I^{\beta)\gamma}\bar{Z}_\gamma^i \tag{2.1}$$

where $I^{\alpha\beta}$ is the infinity twistor which breaks the conformal invariance. The summation convention holds for Roman twistor "flavor" indices (Flavor indices share the property of Greek twistor indices that they are raised and lowered by complex conjugation, $(\overline{Z_i^\alpha}) = (\bar{Z}_\alpha^i)$. Each term on the right of Eq. (2.1) for a fixed value of i is the kinematical twistor of one of the massless constituents.

The kinematical twistor is decomposed into the spinor parts

$$\left[A^{\alpha\beta}\right] = \begin{bmatrix} -2i\mu^{AB} & p^A_{B'} \\ p_B^{A'} & 0 \end{bmatrix}. \tag{2.2}$$

Here μ^{AB} is the total angular momentum spinor (symmetric in its indices) and the Hermitian spinor $p^A_{B'}$ is the four-momentum. The center-of-mass line of the system consists of the points of real

Thus, typically, in tensor notation

$$c^2 = \frac{2}{(Z_1^\alpha Z_3^\beta I_{\alpha\beta})(\bar{Z}_\gamma^1 \bar{Z}_\delta^3 I^{\gamma\delta})(Z_2^\mu Z_3^\nu I_{\mu\nu})(\bar{Z}_\lambda^2 \bar{Z}_\rho^3 I^{\lambda\rho})} \times$$

$$\times \{((Z_3^\kappa Z_1^{[\eta} \; I_{\kappa\eta}) Z_2^{\zeta]} \bar{Z}^2_{[\zeta} (\bar{Z}^1_{\sigma]} \bar{Z}^3_\tau \; I^{\tau\sigma}) Z_3^\omega \bar{Z}_\omega^3 \; -$$

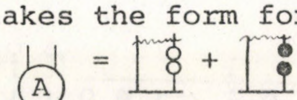
$$- \; (Z_3^\kappa Z_1^{[\eta} \; I_{\kappa\eta}) Z_2^{\zeta]} \bar{Z}^3_\zeta \bar{Z}^3_\tau I^{\tau\sigma} \bar{Z}^1_{[\sigma} \bar{Z}^2_{\omega]} Z_3^\omega \}$$

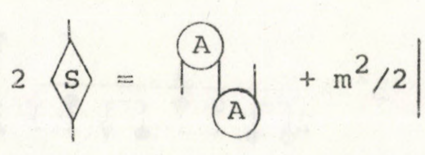
where antisymmetrization in indices in the brackets is understood.

7. THE SPIN DEFICIENCY FORMULA

In this section we prove the spin deficiency theorem. The content of this theorem is that joining three two-twistor particles by identifying their constituents pairwise gives a total spin which is the sum of the spins diminished by the spins of the twistor constituents,

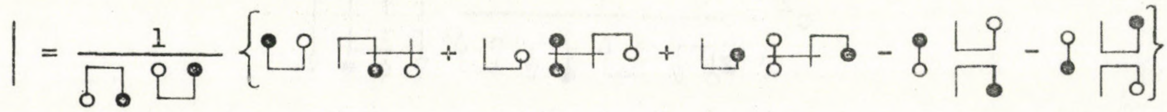
$$S_k^i = S_{12k}^i + S_{23k}^i + S_{31k}^i - S_{1k}^i - S_{2k}^i - S_{3k}^i. \qquad (7.1)$$

Equation (2.1) takes the form for a two-twistor particle in the blob notation:  . Inserting this in the spin twistor (2.11) written in terms of blobs as



(7.2)

and using the identity[5]



(7.3)

Minkowski space-time

$$x^{AA'}(\tau) \stackrel{\text{def}}{=\!=\!=} m^{-2}(\mu^{AB}p_B^{\;A'} + \bar{\mu}^{A'B'}p_{B'}^{\;A}) + \tau m^{-1}p^{AA'} \qquad (2.3)$$

where $m^2 \stackrel{\text{def}}{=\!=\!=} p_{B'}^{\;A}p_A^{\;B'}$ is the squared rest-mass and the real parameter τ is the proper time. It does not appear to be possible to express the center of mass in terms of the kinematical quantities in a manifestly twistor invariant form, due to the fact that the concept of the center of mass in not invariant with respect to translations. However, using the constituent twistors directly, one can define a center of mass point twistor[8]:

$$R^{\alpha\beta} \stackrel{\text{def}}{=\!=\!=} 2m^{-2}Z_i^{\alpha}\,Z_k^{\beta}\,\bar{M}^{ik}. \qquad (2.4)$$

The quantities

$$M_{ik} \stackrel{\text{def}}{=\!=\!=} Z_i^{\alpha}\,Z_k^{\beta}\,I_{\alpha\beta} \qquad (2.5)$$

are called mass amplitudes and for n > 2 are the partial mass amplitudes of the two-component subsystem labeled by i and k, and are such that the mass-squared of the system may be written as a sum of partial mass-squares:

$$m^2 = M_{ik}\,\bar{M}^{ik}. \qquad (2.6)$$

The point twistor (2.4) decomposes according to

$$R^{\alpha\beta} = \begin{bmatrix} -\frac{1}{2}\,r_{RR'}r^{RR'}\varepsilon^{AB} & ir_{B'}^{\;A} \\ -\,ir_{A'}^{\;B} & \varepsilon_{A'B'} \end{bmatrix} \qquad (2.7)$$

where $r^{AA'}$, the center of mass point, is a point of the complex Minkowski space-time. The point $r^{AA'}$ lies on the complex center of mass line of the system. The complex center of mass line is the set of points[9]

$$z^{AA'} = m^{-2}(\mu^{AB}p_B^{\;A'} + \bar{\mu}^{A'B'}p_{B'}^{\;A}) + iS^{AA'}m^{-2} + \lambda p^{AA'}m^{-1} \qquad (2.8)$$

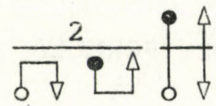
Taking the sum, and adding the lengths of imaginary parts [Eq. (6.6)],

$$c^2 = (x_{\underline{A}}-x_{\underline{B}})^2 = \frac{1}{4}[R^{\alpha\beta}_{\underline{A}}\,\bar{R}^{\underline{A}}_{\alpha\beta} + R^{\alpha\beta}_{\underline{B}}\,\bar{R}^{\underline{B}}_{\alpha\beta} - (R^{\alpha\beta}_{\underline{A}} + \bar{R}^{\underline{A}\alpha\beta})(R_{\underline{B}\alpha\beta} + \bar{R}^{\underline{B}}_{\alpha\beta})]. \tag{6.10}$$

For calculational convenience and to illustrate its usefulness, Penrose's blob notation will be used to write the lengths in terms of the constituent twistors /Cf. Appendix/.
Let us write

$Z^\alpha_1 =$ , $Z^\alpha_2 =$, $Z^\alpha_3 =$. (6.11)

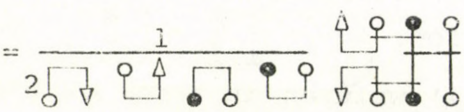
From equation (2.8), and $\underline{A} = (13)$, $\underline{B} = (23)$,

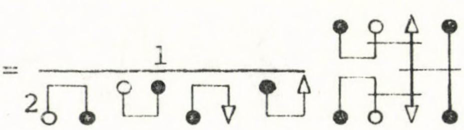
$R^{\alpha\beta}_{\underline{A}}\,\bar{R}^{\underline{B}}_{\alpha\beta} =$  (6.12)

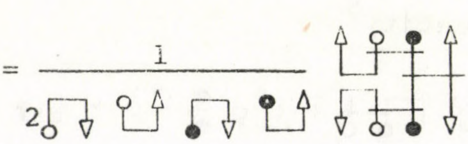
where the mass amplitudes are

$M_{\underline{A}} =$ , $\bar{M}^{\underline{B}} =$  (6.13)

Proceeding with the evaluation of terms in Eq. (6.10) we obtain the expressions for the sides of the mass triangle:

$a^2 =$  ,

$b^2 =$  , (6.14)

$c^2 =$  .

where λ is complex and $S_{AA'}$ denotes the Pauli-Lubanski spinor

$$S_{AA'} = i(\mu_{AB}\, p^{B}_{A'} - \bar{\mu}_{A'B'}\, p^{B'}_{A}). \tag{2.9}$$

In the rest-frame of the particle defined by the form of the four-momentum $[p^{AA'}] = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, the Pauli-Lubanski spinor becomes proportional to the nonrelativistic spin. Thus the spin $j_{AA'}$ is given

$$j_{AA'} = m^{-1}\, S_{AA'}. \tag{2.10}$$

The Pauli-Lubanski spinor is the sole nonvanishing part of the spin twistor[5]

$$S^{\beta}_{\alpha} = \frac{1}{2}(\bar{A}_{\alpha\rho}\, A^{\rho\beta} + \frac{1}{2}\, m^{2}\, \delta^{\beta}_{\alpha}) \tag{2.11}$$

according to

$$S^{\beta}_{\alpha} = \begin{bmatrix} 0 & 0 \\ -S^{A'B} & 0 \end{bmatrix}. \tag{2.12}$$

The internal symmetry transformations of the kinematical twistor are[10]

$$\begin{aligned} \hat{Z}^{\alpha}_{i} &= U^{k}_{i}(Z^{\alpha}_{k} + \Lambda_{k\ell}\, I^{\alpha\beta}\, \bar{Z}^{\ell}_{\beta}) \\ \hat{\bar{Z}}^{i}_{\alpha} &= \bar{U}^{i}_{k}(\bar{Z}^{k}_{\alpha} + \bar{\Lambda}^{k\ell}\, I_{\alpha\beta}\, Z^{\beta}_{\ell}) \end{aligned} \tag{2.13}$$

where $[U^{i}_{k}]$ is an nxn unitary matrix and $[\Lambda_{k\ell}]$ is skew. The transition $Z^{\alpha}_{i} \to \hat{Z}^{\alpha}_{i}$ amounts to selecting a new set of massless constituents for which the kinematical twistor of the system remains unchanged. Thus the kinematical variables discussed remain unchanged under internal symmetry transformations, with the exception of the center-of-mass twistor. The center-of-mass point is defined directly in terms of constituents and it has been shown

$$R_1^{\alpha\beta}\, R_{2\alpha\beta} = -\frac{1}{2} R_1^{\alpha\beta}\, e_{\alpha\beta\gamma\delta}\, R_2^{\gamma\delta} = (r_1^{AA'} - r_2^{AA'})(r^1_{AA'} - r^2_{AA'}). \tag{6.5}$$

The length of the imaginary part of vector $r^{AA'}$ is

$$R^{\alpha\beta}\, \bar{R}_{\alpha\beta} = 4y^{AA'}\, y_{AA'}. \tag{6.6}$$

The partial mass centers are null separated in complex Minkowski space-time since any pair of them lie on a common twistor. Let us however consider the real mass triangle. From Eq. (6.5), the condition of null separation for the arbitrary subsystems $\underline{A}$ and $\underline{B}$ is

$$-R_{\underline{A}}^{\alpha\beta}\, R_{\underline{B}\alpha\beta} = (x_{\underline{A}} - x_{\underline{B}})^2 - (y_{\underline{A}} - y_{\underline{B}})^2 + 2i(x_{\underline{A}} - x_{\underline{B}}).(y_{\underline{A}} - y_{\underline{B}}) = 0. \tag{6.7}$$

Hence using Eq. (2.8), we can express the side c connecting the mass points $\underline{A}$ and $\underline{B}$ in terms of the spin vectors $S^{AA'} = m^2 y^{AA'}$ as

$$c^2 \overset{\text{def}}{=\!=} (x_{\underline{A}} - x_{\underline{B}})^2 = (m_{\underline{A}}^{-2} S_{\underline{A}} - m_{\underline{B}}^{-2} S_{\underline{B}})^2. \tag{6.8}$$

It is quite surprising that the spin difference appears in a side length of the mass triangle. From Eq. (6.7) we further have that each side of the mass triangle is orthogonal to the difference of the spins at the endpoints of the side.

To complete the analysis of the mass triangle, we now ask how the lengths of the sides of the mass triangle depend directly on the constituent twistors. A direct substitution into Eq. (6.8) yields unwieldy results. Instead we consider the variants of Eq. (6.7)

$$\begin{aligned}
-\bar{R}^{\underline{A}}_{\alpha\beta}\, \bar{R}^{\underline{B}\alpha\beta} &= (x_{\underline{A}} - x_{\underline{B}})^2 - (y_{\underline{A}} - y_{\underline{B}})^2 - 2i(x_{\underline{A}} - x_{\underline{B}}).(y_{\underline{A}} - y_{\underline{B}}) \\
-R_{\underline{A}}^{\alpha\beta}\, \bar{R}^{\underline{B}}_{\alpha\beta} &= (x_{\underline{A}} - x_{\underline{B}})^2 - (y_{\underline{A}} + y_{\underline{B}})^2 + 2i(x_{\underline{A}} - x_{\underline{B}}).(y_{\underline{A}} + y_{\underline{B}}) \qquad (6.9) \\
-\bar{R}^{\underline{A}\alpha\beta}\, R_{\underline{B}\alpha\beta} &= (x_{\underline{A}} - x_{\underline{B}})^2 - (y_{\underline{A}} + y_{\underline{B}})^2 - 2i(x_{\underline{A}} - x_{\underline{B}}).(y_{\underline{A}} + y_{\underline{B}}).
\end{aligned}$$

by Hughston[8] that internal transformations (2.13) move the complex center of mass point over the entirety of the complex center-of-mass line.

The central dogma of twistor particle theory asserts that the state of the system at any instant of time is completely described by the values of the constituent twistors Z^{α}_{i} and by their complex conjugates. The space of n-twistors TxTx...xT admits a naturally defined symplectic form $dZ^{\alpha}_{i} \wedge d\bar{Z}^{i}_{\alpha}$. In the sense of Hamiltonian dynamics Z^{α}_{i} and $\bar{Z}^{i}_{\alpha}$ together play the role of canonically conjugate variables. Accordingly, any function of the form $f(Z^{\alpha}_{i}, \bar{Z}^{i}_{\alpha})$ is called a <u>dynamical quantity</u>[11].

We introduce the Poisson bracket of dynamical quantities $f(Z,\bar{Z})$ and $g(Z,\bar{Z})$

$$[f,g] = -\left(\frac{\partial f}{\partial Z^{\alpha}_{r}} \frac{\partial g}{\partial \bar{Z}^{r}_{\alpha}} - \frac{\partial f}{\partial \bar{Z}^{r}_{\alpha}} \frac{\partial g}{\partial Z^{\alpha}_{r}}\right). \tag{2.14}$$

The Poisson bracket is antisymmetric in f and g and is imaginary when both f and g are real. From (2.14) we identify the general coordinates q^{i}_{α} and canonically conjugate momenta p^{α}_{i} as follows,

$$q^{\alpha}_{i} = -Z^{\alpha}_{i}, \qquad p^{i}_{\alpha} = \bar{Z}^{i}_{\alpha}. \tag{2.15}$$

The twistor variables have the Poisson brackets

$$[Z^{\alpha}_{i}, Z^{\beta}_{k}] = 0, \quad [Z^{\alpha}_{i}, \bar{Z}^{k}_{\beta}] = \delta^{k}_{i}\, \delta^{\alpha}_{\beta}. \tag{2.16}$$

A transformation generated by a dynamical quantity f is given

$$\delta Z^{\alpha}_{i} = i[Z^{\alpha}_{i}, f]$$
$$\delta \bar{Z}^{i}_{\alpha} = i[\bar{Z}^{i}_{\alpha}, f]. \tag{2.17}$$

The unitary transformations $\hat{Z}^{\alpha}_{i} = U^{k}_{i} Z^{\alpha}_{k}$ are generated by the functions

$$B^{i}_{k} = Z^{\alpha}_{k} \bar{Z}^{i}_{\alpha} \tag{2.18}$$

6. THE MASS TRIANGLE

The subsystem of twistors Z_1^α and Z_2^α has the squared rest mass

$$m_{12}^2 = 2M_{12}\bar{M}^{12} = 2d^3\bar{d}_3.$$

The center of mass point twistor (2.4) of the subsystem may be written

$$R_{12}^{\alpha\beta} = \frac{1}{M_{12}}(Z_1^\alpha Z_2^\beta - Z_2^\alpha Z_1^\beta). \qquad (6.1)$$

Using Eq. (6.1) and the center of mass twistors of the remaining massive subsystems, we obtain the center of mass of the three-twistor particle as the linear combination

$$R^{\alpha\beta} = \frac{2}{m^2}(M_{12}\bar{M}^{12}R_{12}^{\alpha\beta} + M_{23}\bar{M}^{23}R_{23}^{\alpha\beta} + M_{31}\bar{M}^{31}R_{31}^{\alpha\beta}). \qquad (6.2)$$

The non-diagonal spinor part of (6.2) which is linear in the position vectors [Cf. Eq. (2.7)] is

$$r^{AA'} = m^{-2}(m_{12}^2 r_{12}^{AA'} + m_{23}^2 r_{23}^{AA'} + m_{31}^2 r_{31}^{AA'}). \qquad (6.3)$$

Thus the center-of-mass point $r^{AA'}$ of the particle is a weighted mean of the partial center of masses. Hence the four mass center points lie in a plane. The center of mass $r^{AA'}$ is in the barycenter of the triangle formed by the partial mass centers. Note however that the weights are mass-squares rather than masses. We now compute the sides of the triangle.

The invariant distance of two complex points in Minkowski space-time,

$$r_1^{AA'} = x_1^{AA'} + iy_1^{AA'}$$
$$r_2^{AA'} = x_2^{AA'} + iy_2^{AA'} \qquad (6.4)$$

can be expressed in terms of the point twistors $R_1^{\alpha\beta}$ and $R_2^{\alpha\beta}$ as[8] forming a Hermitian matrix, and the transformations $\tilde{Z}^{\alpha}_{i} = Z^{\alpha}_{i} + \Lambda_{ik}\, I^{\alpha\beta}\, \bar{Z}^{k}_{\beta}$ are generated by the mass amplitudes

$$M_{ik} = Z^{\alpha}_{i} Z^{\beta}_{l} I_{\alpha\beta} \quad \text{and} \quad \bar{M}^{ik} = \bar{Z}^{i}_{\alpha} \bar{Z}^{k}_{\beta} I^{\alpha\beta}. \tag{2.19}$$

The Λ-transformations commute and are called internal translations.

As long as we are concerned with a massive particle in free motion, the decomposition into allowed twistor constituents is immaterial for the motion of the particle. This reflects the invariance of the kinematical twistor with respect to the internal transformations (2.13). The idea is, however, that the behaviour of the particle in interactions should depend on the substructures present in a twistor decomposition.

The n-twistor particle where $n \geq 3$, possesses massive parts. Such a substructure consists of two or more null constituents. Clearly, a two-twistor particle has no massive subsystems. The simplest place to study massive subsystems is a three-twistor particle. In the next section we discuss some features unique to three-twistor particles.

3. STRUCTURE OF THREE-TWISTOR PARTICLES

The internal structure of a massive particle described by three twistors can be examined in terms of the three two-twistor subsystems obtained by considering the three twistors pairwise. Each such two-twistor subsystem defines a massive particle in space-time with well-defined (real and complex) center-of-mass line, spin and center-of-mass point. These physical properties of the subsystems combine to yield the properties of the entire system in unexpected and interesting ways, given an ordered triple of twistors $(Z^{\alpha}_{1}, Z^{\alpha}_{2}, Z^{\alpha}_{3}) \in T$. (It is sometimes preferable to think of the triple as a point in TxTxT.) Any one of these, Z^{α}_{1}, has its kinematical twistor, $A^{\alpha\beta}_{i}$, and any pair of these, $(Z^{\alpha}_{i}, Z^{\alpha}_{j})$, $i < j$, has its associated kinematical twistor $A^{\alpha\beta}_{ij}$. While a single twistor describes a massless system in Minkowski where we introduce the real parameter j which can take any real value for a classical system (and it will take the values 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, ... after quantization). This common magnitude is a Casimir invariant of both the Poincaré and of the inhomogeneous SU(3) groups, the second common Casimir invariant being the mass square:

$$d^i \bar{d}_i = p_{AA'} p^{AA'} = \frac{m^2}{2} .$$

There is a further property that connects the space-time spin and (the negative of) unitary spin: <u>The projections of the spin twistor (2.11) onto the constituent twistors are the negative of the components of the unitary spin,</u>

$$S^\alpha_\beta Z^\beta_i \bar{Z}^k_\alpha = - S^k_i . \qquad (5.12)$$

This result (which can be proved by direct computation) generalizes a relation holding for two-twistor particle spins[5]. For a two-twistor particle, however, the unitary spin is replaced by the conformally invariant quadratic expressions

$$Z^\alpha_i \sigma^i_k \bar{Z}^k_\alpha, \ i, k = 1, 2,$$

where $[\sigma^i_k]$ are the Pauli matrices.

space-time, a pair of twistors describes a massive system. The internal symmetry transformations change the kinematical twistors of the one and two twistor subsystems while the kinematical twistor of the entire system is unchanged and in fact this constrains the changes in $A_i^{\alpha\beta}$ and $A_{ij}^{\alpha\beta}$ since

$$A^{\alpha\beta} = \frac{1}{2} \sum_{\substack{i,j \\ i<j}} A_{ij}^{\alpha\beta} = \sum_i A_i^{\alpha\beta} = \sum_{i,j} A_{ij}^{\alpha\beta} - \sum_i A_i^{\alpha\beta}$$

The internal states will be used in the description of interactions and the manner in which the various concepts are linked is of importance. To proceed further the internal symmetry group must be examined more closely. The internal translations are generated by the partial mass amplitudes which can be given equivalently by

$$d^i = \frac{1}{2} \varepsilon^{ijk} M_{jk} \quad \text{and} \quad \bar{d}_i = \frac{1}{2} \varepsilon_{ijk} \bar{M}^{jk}. \tag{3.1}$$

The mass squared is easily given by $m^2 = 2d^i\bar{d}_i$ and is positive. The unitary transformations are generated by B^i_k which satisfy

$$\bar{d}_i \, B^i_k \, d^k = 0. \tag{3.2}$$

Writing B^i_k as a trace-free part, A^i_k, plus a trace results in

$$B^i_k = A^i_k + \delta^i_k \, B \tag{3.3}$$

where $B = \frac{1}{3} B^r_r$. From (3.2), the trace may be written as

$$B = 2m^{-2} \, \bar{d}_i \, A^i_k \, d^k. \tag{3.4}$$

Thus the trace of the generators of the unitary transformations can be written in terms of the remaining 14 internal symmetry generators and without loss of generality the unitary transformations will be restricted to SU(3) in the remainder of the paper.

As in (3.1) introducing alternative translation parameters

$$[S^i_j, S^k_\ell] = (d^k \bar{d}_j - \delta^k_j d^r \bar{d}_r) S^i_\ell - (d^i \bar{d}_\ell - \delta^i_\ell d^r \bar{d}_r) S^k_j \tag{5.9}$$

which are the SU(2) Poisson brackets. When the transformations generated by S^i_j are referred to arbitrary three-twistor systems, then they do not belong to ISU(3) since S^i_j is not a linear combination of the A^i_j's and d^i's. We can consider, however, the restriction of these transformations to systems with a fixed momentum $d^i = D^i$. Then, S^i_j are ISU(3) generators. We thus have the theorem:

<u>The unitary spin S^i_j generates the SU(2) subgroup of ISU(3) leaving the mass amplitudes invariant</u> [Cf. Eq. (5.7)].

The mass amplitudes determine the scalar products of the momentum parts $\pi_{iA'}$ of constituent twistors by $M_{ik} = \pi_{iA'} \pi^{A'}_k$. Hence the effect of transformations generated by S^i_j is a rigid rotation of the momentum spinors $\pi_{iA'}$ (together with the frame defined by any pair of them) about the total four-momentum $p^A{}_{A'}$. For systems with a fixed value of the unitary momentum, $d^i = D^i$, we may choose coordinates $D^i = \frac{m}{\sqrt{2}} \delta^i_3$. In this coordinate system, the unitary spin has the component form, with $J_{\underline{a}}$ real,

$$[S^i_k] = \frac{1}{2} m^2 \begin{bmatrix} J_3 & J_1 + iJ_2 & 0 \\ J_1 - iJ_2 & J_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

Thus the Poisson brackets (5.9), restricted to systems with a fixed momentum, may be written

$$[J_{\underline{a}}, J_{\underline{b}}] = i\varepsilon_{\underline{abc}} J_{\underline{c}} \qquad \underline{a}, \underline{b}, \underline{c}, = 1, 2, 3. \tag{5.10}$$

It has been known for some time[7] that the magnitudes of the space-time spin and of the unitary spin are equal:

$$\frac{1}{2} m^2 S^i_k S^k_i = - S_{AA'} S^{AA'} \equiv j(j + 1) m^{-2} \tag{5.11}$$

$$t^i = \frac{1}{2} \varepsilon^{ijk} \Lambda_{jk}, \tag{3.5}$$

the internal symmetry transformations take the form

$$\hat{Z}_i^\alpha = U_i^k (Z_k^\alpha - \varepsilon_{k\ell m} t^\ell I^{\alpha\beta} \bar{Z}_\beta^m) \tag{3.6}$$

with U_i^k an element of SU(3) and t^ℓ an element of $\mathbb{C}^3$. This transformation is represented by the pair $(\underline{\underline{U}},\underline{t})$ and the group product structure follows from the composition of two successive transformations. Thus $(\underline{\underline{U}},\underline{t})$ followed by $(\underline{\underline{U}}',\underline{t}')$ gives after a short calculation

$$(\underline{\underline{U}}',\underline{t}') \cdot (\underline{\underline{U}},\underline{t}) = (\underline{\underline{U}}'\underline{\underline{U}},\underline{t} + \underline{\underline{U}}^+ \underline{t}') \tag{3.7}$$

where $\underline{\underline{U}}^+$ is the Hermitian adjoint of $\underline{\underline{U}}$. Equation (3.7) defines the 14-parameter group denoted by ISU(3) and called the inhomogeneous SU(3) group of internal symmetry transformations (IST). This group acts on $\mathbb{C}^3$ with coordinates z^i, $i = 1, 2, 3$ as a group of point transformations where $(\underline{\underline{U}},\underline{t})$ gives

$$\hat{z}^i = U_k^i (z^k + t^k). \tag{3.8}$$

That is to say the ISU(3) is realized as the isometry group on $\mathbb{C}^3$ of the Hermitian line element $dz^i\, d\bar{z}_i$ and this gives $\mathbb{C}^3$ the structure of a unitary space, U^3. An alternate point of view is to consider U^3 as the homogeneous space ISU(3)/SU(3). We will give later yet another method of obtaining U^3.

The transformations in (3.6) which act on T^3 constitute the twistor realization of ISU(3). The same elements of ISU(3) act on U^3 via (3.8). We need not bother to compute the generators of the ISU(3) group in the isometry representation since they are already available in the twistor realization [cf. eqs. (2.18), (3.3), and (3.1)]:

$$S^i_j = S(X^i \bar{X}_j - Y^i \bar{Y}_j) \tag{5.8}$$

with

$$S^i_j X^j = SX^i, \; S^i_j Y^j = -SY^i, \; S \geq 0$$

$$X^i \bar{d}_i = Y^i \bar{d}_i = 0, \quad X^i \bar{X}_i = Y^i \bar{Y}_i = 1.$$

The vector $S^i = \sqrt{S}\, X^i$ contains all the information in S^i_j. From (5.4) and the first term in (5.6) we have

$$\hat{B} = B + i\frac{m^2}{3} (\tau - \bar{\tau}).$$

Up to the choice of origin $\hat{B}$ can be identified with the imaginary part of τ. There exists a real line which is imbedded in the complex line and parametrized by the real part of τ defined by $\hat{B} = 0$.

To reiterate the material of this section, we have shown that a point in three twistor space selects an origin in unitary space. $A^i_j(Z)$ represent the eight generators of SU(3) rotations about this point while d^i are the generators of the three complex translations. The tensor field $\hat{A}^i_j(Z^\alpha_i, t^i)$ on the unitary space represents the SU(3) generators about the point $z^i = t^i$. Assuming that the three twistors Z^α_i are held fixed (i.e. we have a given internal structure) then simply from the algebraic structure of A^i_j and $\hat{A}^i_j$ one is led to the complex line and internal spin-tensor. At the present we make no attempt at a physical interpretation of the "origin", the internal center of mass line and internal spin tensor other than to say that they are to represent the internal structure of the three twistor particle. A different choice of the three twistors obtained from the IST would represent a different particle having a different origin, world-line and spin tensor but with the same kinematic values.

For a fixed numerical value of t^i, $\hat{A}^i_j$ are SU(3) generators. This is true in particular on the center of mass world-line. However an examination of (5.5) shows that t^i is a function of the Z^α_i and $\bar{Z}^i_\alpha$ and the functional dependence of $\hat{A}^i_j$ on the Z^α's is changed. From (5.7) we see that the S^i_j generate transformations which keep the mass line fixed and by direct calculation we have

$$A^i_k = Z^\alpha_k \bar{Z}^i_\alpha - \frac{1}{3}\,\delta^i_k\, Z^\alpha_r \bar{Z}^r_\alpha$$

$$d^i = \frac{1}{2}\,\varepsilon^{ijk}\, Z^\alpha_j\, I_{\alpha\beta}\, Z^\beta_k \qquad (3.9)$$

$$\bar{d}_i = \frac{1}{2}\,\varepsilon_{ijk}\, \bar{Z}^j_\alpha\, I^{\alpha\beta}\, \bar{Z}^k_\beta .$$

We identify the translation generators d^i and $\bar{d}_i$ as the components of the <u>complex internal momentum</u> of a particle in the unitary space. The generators A^i_k of SU(3) rotations constitute the <u>total internal angular momentum</u> of the particle. Under a translation $\hat{z}^i = z^i + t^i$, the total angular momentum and B changes as [Cf. Eq. (3.7)]

$$\hat{A}^i_k = A^i_k + d^i\,\bar{t}_k + t^i\,\bar{d}_k - \frac{1}{3}(d^r\,\bar{t}_r + t^r\,\bar{d}_r)\,\delta^i_k, \qquad (3.10)$$

$$\hat{B} = B + \frac{2}{3}(d^i\,\bar{t}_i + t^i\,\bar{d}_i) \qquad (3.11)$$

and the complex momentum remains invariant. The behaviour of dynamical quantities with respect to SU(3) rotations is implicit in our tensor notation.

For future use we wish to spell out the meaning of the transformation (3.10); the A^i_k are the generating functions of the isometries in unitary space with the origin as a fixed point while the $\hat{A}^i_k$ are the generating functions of isometries keeping the point $z^i = t^i$ fixed. In this manner $\hat{A}^i_k$ can be thought of as a tensor field on unitary space with $t^i = z^i$.

To summarize this section, there are three spaces which play fundamental roles here. The first is twistor space T on which we take three points $(Z^\alpha_1, Z^\alpha_2, Z^\alpha_3)$ to specify our massive system. (An alternative and sometimes necessary point of view would be to choose a twistor from each of three different copies of T). The IST, i.e. ISU(3), acts on these three points preserving the kinematic twistor. Since pairs of twistors define points of complex Minkowski space, the three twistors define three points in complex Minkowski space which are moved about by the IST. The third space is the unitary space U^3 having the isometry group ISU(3). The generators of ISU(3) define a

$$\hat{C}^i_j = C^i_j + d^i \bar{t}_j + t^i \bar{d}_j . \tag{5.2}$$

C^i_j can be decomposed into the four parts

$$C^i_j = \alpha d^i \bar{d}_j + \alpha^i \bar{d}_j + d^i \bar{\alpha}_j - \frac{2}{m^2} S^i_j \tag{5.3}$$

where

$$S^i_j d^j = S^i_j \bar{d}_i = S^i_i = 0, \quad S^i_k = \bar{S}^i_k ,$$

$$\alpha^i \bar{d}_i = d^i \bar{\alpha}_i = 0, \tag{5.4}$$

$$\alpha = (\frac{2}{m^2})^2 C^i_j d^j \bar{d}_i = \frac{3}{m^2} B, \quad \alpha^i = \frac{2}{m^2} C^i_j d^j - \alpha d^i ,$$

$$\bar{\alpha}_j = \frac{2}{m^2} C^i_j \bar{d}_i - \alpha \bar{d}_j .$$

(Note that the Hermitian adjoint is defined by $\bar{S}^i_k = \overline{S^k_i}$).

If we now insert (5.3) into (5.2) with

$$t^i = -\alpha^i + i\tau d^i , \quad \bar{t}_i = -\bar{\alpha}_i - i\bar{\tau} \bar{d}_i \tag{5.5}$$

we obtain

$$\hat{C}^i_j = (\alpha + i(\tau - \bar{\tau}))d^i \bar{d}_j - \frac{2}{m^2} S^i_j \quad . \tag{5.6}$$

Thus along the <u>internal center of mass line</u> defined by (5.5), $\hat{C}^i_j$ has only the first and last terms of its canonical decomposition. The Hermitian tensor S^i_j called the <u>internal spin-tensor</u> can be explicitly solved for and written $S^i_j = \varepsilon^{iab} \varepsilon_{jcd} (A^c_a + \frac{1}{2} B \delta^c_a) \bar{d}_b d^d$. From its derivation or by direct calculation it is seen that is (essentially) the invariant part of C^i_j under translations i.e.

$$[S^i_k , d^j] = 0, \qquad [S^i_k , \bar{d}_j] = 0 \tag{5.7}$$

and that it has a canonical decomposition vector field d^i and tensor field A^i_j on the unitary space in a manner analogous to the way the Lorentz group defines the momentum and angular momentum fields on Minkowski space. In section 5 we will show how A^i_j and d^i define an internal center of mass line and internal spin in analogy to the way angular momentum and momentum determine a center of mass and spin.

4. INTERNAL TRANSLATIONS

We now explore the effect of internal translations on the structure of the three-twistor particle. Consider first a translation,

$$\hat{z}^i = z^i + t^i$$

with $t^i = (0,0,\Lambda)$, by the complex amount Λ in the z^3 direction of the unitary space. In the twistor realization, Eq. (3.7), the momentum parts of the constituent twistors [$\pi_{iA'}$, where $Z^\alpha_i = (\omega^A_i, \pi_{iA'})$] remain uneffected,

$$\hat{\pi}_{iA'} = \pi_{iA'} \tag{4.1}$$

and

$$\begin{aligned} \hat{\omega}^A_1 &= \omega^A_1 + \Lambda\bar{\pi}^{2A} \\ \hat{\omega}^A_2 &= \omega^A_2 - \Lambda\bar{\pi}^{1A} \\ \hat{\omega}^A_3 &= \omega^A_3 . \end{aligned} \tag{4.2}$$

We compare the change in the ω parts with the effect of a translation in Minkowski space-time

$$\tilde{x}^{AA'} = x^{AA'} + a^{AA'} . \tag{4.3}$$

This latter gives

$$\tilde{\omega}^A = \omega^A + ia^{AA'}\pi_{A'}, \quad \tilde{\pi}_{A'} = \pi_{A'} \tag{4.4}$$

5. SPIN AND ROTATION

In addition to its Minkowski space structure (momentum, mass, angular momentum, center of mass, etc) a three twistor particle has an associated unitary space structure, namely a point (or origin) in unitary space and a complex "internal center of mass" world-line also in unitary space with a related internal spin tensor (which is the unitary space analogue of the Pauli-Lubanski spin vector).

To see the point structure we note that three twistor space has 24 real dimensions (3x4x2) while the kinematic twistor $A^{\alpha\beta}$ has ten real components (momentum and angular momentum) and thus the kinematic subspace defined by $A^{\alpha\beta}$ constant is 14 real dimensional. The equivalence classes of points in this space (eight dimensional) defined as those points connected by SU(3) transformations, i.e. $Z'^{\alpha}_{i} = U^{j}_{i} Z^{\alpha}_{j}$, $U^{j}_{i} \in SU(3)$, can be identified with the points of unitary space. The equivalence classes can be parametrized by points in $\mathbb{C}^3$ (6 real dimensions) i.e. by the translations $Z'^{\alpha}_{i} = Z^{\alpha}_{i} + \Lambda_{ij} I^{\alpha\beta} \bar{Z}^{j}_{\beta}$, from some arbitrarily chosen "origin" Z^{α}_{i}.

(Note that by associating this arbitrarily chosen origin with the group identity element, the kinematic subspace can be considered as the ISU(3) group space. Note further that if we had considered originally the group IU(3), the U(1) part would have an action on the ISU(3) manifold which would not be the action of an ISU(3) element. Nevertheless locally one could duplicate the U(1) action by an ISU(3) element. This explains from a group theoretical point of view the relationship (3.4) between the U(1) generator and the ISU(3) generator).

In order to understand and see the internal center of mass line and internal spin tensor we define

$$C^{i}_{j} = A^{i}_{j} + \frac{1}{2} B\delta^{i}_{j} \tag{5.1}$$

and obtain from (3.10) and (3.11) the transformation law under translations $\hat{z}^{i} = z^{i} + t^{i}$

for any twistor. Consider, in particular, the pair Z_1^α and Z_2^α. Choose

$$a^{AA'} = \lambda(\bar{\pi}_1^{A}\pi^{1A'} + \bar{\pi}_2^{A}\pi^{2A'}) \tag{4.5}$$

where λ is complex. From Eqs. (4.4),

$$\tilde{\omega}_1^A = \omega_1^A + i\lambda\,\bar{\pi}^{2A}\pi_2^{A'}\pi_{1A'}$$

$$\tilde{\omega}_2^A = \omega_2^A - i\lambda\,\bar{\pi}^{1A}\pi_1^{A'}\pi_{2A'}$$

or using (2.5),

$$\begin{aligned}\tilde{\omega}_1^A &= \omega_1^A + i\lambda\, M_{12}\,\bar{\pi}_2^A \\ \tilde{\omega}_1^A &= \omega_1^A - i\lambda\, M_{12}\,\bar{\pi}_1^A .\end{aligned} \tag{4.6}$$

Choosing the parameter λ to be

$$\lambda = -i\Lambda/M_{12} \tag{4.7}$$

we have the following result:

The special internal translation with $(t^i) = (0,0,\Lambda)$ shifts the twistors Z_1^α and Z_2^α parallelly along the time-like center of mass line of the massive two-twistor subsystem they represent and leaves the twistor Z_3^α invariant. A similar result is obtained for translations along the other two axes.

Consider next the space-time translation in the direction of the total four-momentum of the system:

$$a^{AA'} = \tau(\bar{\pi}^{1A}\pi_1^{A'} + \bar{\pi}^{2A}\pi_2^{A'} + \bar{\pi}^{3A}\pi_3^{A'}). \tag{4.8}$$

Is this possibly an internal translation? From Eqs. (4.4) we obtain

$$\tilde{\omega}_1^A = \omega_1^A + i\tau(M_{12}\,\bar{\pi}^{2A} + M_{13}\,\bar{\pi}^{3A})$$

$$\tilde{\omega}_2^A = \omega_2^A + i\tau(M_{21}\,\bar{\pi}^{1A} + M_{23}\,\bar{\pi}^{3A}) \tag{4.9}$$

$$\tilde{\omega}_3^A = \omega_3^A + i\tau(M_{31}\,\bar{\pi}^{1A} + M_{32}\,\bar{\pi}^{2A})$$

or, in matrix form using (3.1),

$$\begin{pmatrix} \tilde{\omega}_1^A \\ \tilde{\omega}_2^A \\ \tilde{\omega}_3^A \end{pmatrix} = \begin{pmatrix} \omega_1^A \\ \omega_2^A \\ \omega_3^A \end{pmatrix} + i\tau \begin{pmatrix} 0 & d^3 & -d^2 \\ -d^3 & 0 & d^1 \\ d^2 & -d^1 & 0 \end{pmatrix} \begin{pmatrix} \bar{\pi}^{1A} \\ \bar{\pi}^{2A} \\ \bar{\pi}^{3A} \end{pmatrix} . \tag{4.10}$$

This defines an internal translation with

$$(t^i) = i\tau(d^1, d^2, d^3). \tag{4.11}$$

What we have is a translation in the direction of the unitary momentum d^i by the amount τ. The significance of this result lies in the fact that it establishes a map from the time development of the system in space-time to the development in unitary space parallel to the unitary momentum.

To conclude this section we observe that translations of the form (4.11) exhaust the unitary translations which can be pictured equivalently as space-time translations. The reason for this is that space-time translations not along the centre-of-mass line of the system alter the angular momentum. However the angular momentum is preserved by all internal transformations since these preserve the kinematic twistor.

63.201

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