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ABSTRACT

It is shown that the source of any anomalies in the theory is the collective motion of the vacuum particles with arbitrarily large moments.

АННОТАЦИЯ

Показано, что причиной аномалий является коллективное движение частиц в вакууме, включающее частицы с произвольно большими импульсами.

KIVONAT

Megmutatjuk, hogy az anomáliák forrása a részecskék tetszőlegesen nagy impulzusu kollektiv mozgása a vákuumban.

1. GENERAL DESCRIPTION OF THE PHENOMENON

This paper has a mainly pedagogical character. Still, I feel, that it is not absolutely meaningless to rediscuss in another language one of the most beautiful and non-trivial phenomena in the modern field theory. What is the mystery of the phenomenon we call anomalies ? We have a theory in which the energy-momentum tensor and for example the axial charge are conserved on the classical level. We also have the quantum perturbation expansion in the theory which preserves conservations of energy-momentum of the particles and their helicities in any order. But, when we start to calculate the energy-momentum tensor and the axial current explicitely, for example in a given external field, we find out, that these quantities are not conserved even in second order on the external field.

This discovery [1] becomes very important, when we are outside the region where perturbation theory is valid [2]. In order to understand what is going on, let us consider the simplest case - free vacuum of massless fermions in external electromagnetic field. Even more, let us imagine this vacuum, as a gas of classical massless particles, moving independently in external field.

Such a movement can be described as a movement in the phase space $\vec{p}, \vec{x}$

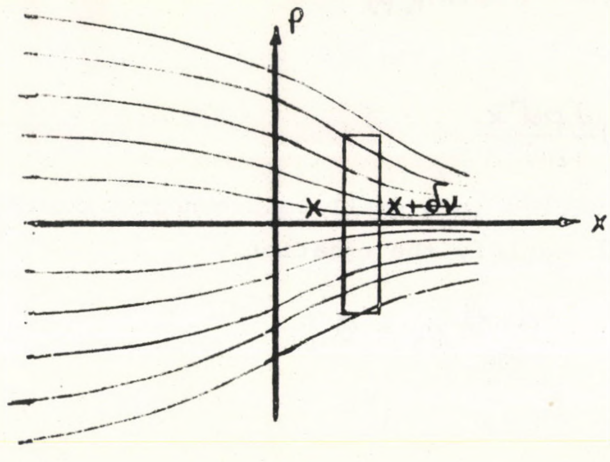

Fig.1.

where each particle has a trajectory p(t), x(t). In the relativistic notations we can introduce four-component vectors $p_\mu(\tau)$, $x_\mu(\tau)$ as functions of some invariant parameter τ . These vectors satisfy the equations of motion.

$$\dot{x}_\mu = p_\mu(\tau) \tag{1}$$

$$\dot{p}_\mu = F_{\mu\nu} p_\nu(\tau) \tag{2}$$

The current and energy-momentum of each particle are

$$j_\mu(x) = \int p_\mu(\tau)\delta[x_\mu - x_\mu(\tau)]\, d\tau \tag{3}$$

$$T_{\mu\nu}(x) = \int p_\mu(\tau) p_\nu(\tau)\delta[x_\mu - x_\mu(\tau)]\, d\tau \tag{4}$$

These quantities are conserved in a sense

$$\partial_\mu j_\mu = 0 \tag{5}$$

$$\partial_\nu T_{\mu\nu} = F_{\mu\nu}(x) j_\nu(x) \tag{6}$$

If we are not interested in looking for each particle, we can introduce the densities of j_μ and $T_{\mu\nu}$ in phase space

$$j_\mu(p,x) = \rho(p,x) p_\mu \tag{7}$$

$$T_{\mu\nu}(p,x) = \rho(p,x) p_\mu p_\nu \tag{8}$$

where

$$\rho(p,x)\frac{d^4p\, d^4x}{(2\pi)^4}$$

is the number of particles in a small volume near p,x. Due to the Liouville theorem $\rho(p,x)$ satisfy the equation

$$\frac{\partial \rho(p,x)}{\partial x_\mu}\dot{x}_\mu + \frac{\partial \rho(p,x)}{\partial p_\mu}\dot{p}_\mu = 0 \tag{9}$$

where $\dot{x}_\mu$ and $\dot{p}_\mu$ are defined by (1) and (2).
If we want to calculate the densities of j_μ and $T_{\mu\nu}$ in the normal space-time x_μ we must integrate (7) and (8) over the momenta p_μ and of course we get infinity. Does this infinity have physical significance ? The essence of Dirac 's idea to describe the vacuum, as a sea of the particles with negative energies is, that the particles with infinite momenta could not influence any physics, because they produce the homogeneous distribution of the current, which could not be observable. This means, that if we have an external field, the characteristic frequencies of which are much less than $\varkappa$, we could consider $j_\mu(p,x)$ and $T_{\mu\nu}(p,x)$ integrated over some volume V in the momentum space, which is large enough to contain any momenta of the order of $\varkappa$ and no physics would depend on this volume. Let us consider such quantities

$$j_\mu^V(x) = \int_V \frac{d^4p}{(2\pi)^4}\, \rho(p,x)\, p_\mu \qquad (10)$$

$$T_{\mu\nu}^V(x) = \int_V \frac{d^4p}{(2\pi)^4}\, \rho(p,x)\, p_\mu p_\nu \qquad (11)$$

and ask: are these quantities conserved or not ?
According to (9)

$$\partial_\mu j_\mu^V = -\int_V \frac{d^4p}{(2\pi)^4} \frac{\partial}{\partial p_\mu}(\rho \dot{p}_\mu) = -\int_S \frac{d^3\sigma}{(2\pi)^4}\, \rho\, \dot{p}_n \qquad (12)$$

$$\partial_\nu T_{\mu\nu}^V = -\int \frac{d^4p}{(2\pi)^4}\, p_\mu \frac{\partial}{\partial p_\nu}(\rho\, \dot{p}_\nu) = -\int \frac{d^3\sigma}{(2\pi)^4}\, \rho\, p_\mu \dot{p}_n + F_{\mu\nu} j_\nu^V \qquad (13)$$

These equations show that $\partial_\mu j_\nu^V$ and $\partial_\nu T_{\mu\nu}^V$ are defined by integrals over a large surface in momentum space of the flow of $j_\mu(p,x)$ and $T_{\mu\nu}(p,x)$ from the region of large momenta to the small one. It is obvious, that Dirac's idea can be literally true only, if there is no flow of any quantities like charge, energy etc. from the world of infinitely large momenta to our world of finite momenta.

As we will show below this flow is not always absent. In QED it is absent for the vector current and not absent for the axial current and for the energy momentum tensor. The question arises, whether this situation is hopeless and we have no theory for our poor world of finite momenta. Is there any example of similar situation in physics ? The answer is "yes". We know that in thermodynamics, when we are considering a system which is the part of a large system and is in thermodynamical equilibrium with it, than the energy of our system is of course not conserved, but since the amount of energy change is uniquely defined in equilibrium one is able to introduce free energy which is conserved in a quasi-equilibrium process. We can use the same trick in our case. If we will calculate surface terms (12) (13) in an unique way and if they will have the structure of divergencies in the usual space-time, we will be able to introduce the conserved free axial current and the free energy-momentum tensor.

Fortunately this is the case. As we will show, this description of the vacuum of fermions could be extended in order to take into account the quantum nature of particle motion in the vacuum. Doing this one gets that the surface terms lead to well-known results.

$$\partial_\mu j^V_\mu = -\partial_\mu j^\delta_\mu \qquad \partial_\nu T^V_{\mu\nu} = -\partial_\nu T^\delta_{\mu\nu} + F_{\mu\nu} j^V_\nu \tag{14}$$

$$j^\delta_\mu = -\frac{\alpha}{4\pi} \varepsilon_{\mu\nu\rho\sigma} A_\nu F_{\rho\sigma} \tag{15}$$

$$T^\delta_{\mu\nu} = \frac{\alpha}{6\pi}\left[\frac{5}{6} F_{\mu\sigma} F_{\nu\sigma} - \frac{11}{24}\delta_{\mu\nu} F^2\right] \tag{16}$$

where α is the fine-structure constant.

These results give us the possibility to introduce the conserved free axial current j^f_μ and the conserved energy-momentum $T^f_{\mu\nu}$

$$j^f_\mu = j^V_\mu + j^\delta_\mu \tag{17}$$

$$T^f_{\mu\nu} = T^V_{\mu\nu} + T^\delta_{\mu\nu} \tag{18}$$

Conservation of these quantities leads in many cases to the same results as conservation of the original j_μ and $T_{\mu\nu}$ but in some cases there might be a difference.

The main difference can come from the fact, that T^f_{oo} is not positively defined. This could lead to a real instability of the theory, when the charges of the particles would be increased infinitely in spite of the conservation of $T^f_{\mu\nu}$.

The description of the anomalies we have presented shows that the source of anomalies is a collective motion of particles with arbitrarily large momenta in the vacuum. This collective motion transfers the axial charge and the energy-momentum from the world of particles with infinitely large momenta to our world of finite momenta. This description makes the main features of the phenomenon very obvious.

First of all it is clear, that if we represent our field $F_{\mu\nu}$ by a finite number of photons we will never see the phenomenon, because one photon can interact only with one particle and therefore a finite number of photons is not able to change the momenta of an infinite number of particles.

It does not mean, however, that Feyman's diagrams contain no such effects. Indeed, infrared divergencies of the theory are in fact the manifestation of the existence of the classical fields with infinite number of photons in the processes of particles interaction. Calculating probabilities of different infrared divergent processes, we are adding to the initial or to the final states an infinite amount of photons in order to take into account the contribution of the classical field to the structure of these states. The classical field is important not only for the diagrams which are really infrared divergent, but also for those which contain no infrared divergencies. The reason of this is the fact, that the absence of infrared divergencies of the diagrams is very often due to calculations of different photon corrections to the process, what means the existence of some sum rule, reflecting in a correct way the presence of classical fields during the intermedium stages of the process. For example, we could calculate the axial charge of a photon simply calculating the diagram of Fig.2.

The general picture of any process of particle scattering or creation looks as follows. A classical electromagnetic field created during the time of interaction produces that collective motion, and as a result of it particles get some axial charge and additional energy.

Fig.2.

After a sufficiently large time particles will be far a part from each other and states of the system become the product of independent stationary states. Any of these states will have a fixed value of axial charge and energy, as a function of other quantum numbers of the states. Any collective motions must be stopped at that time because these states are stationary. If we would look for the features of this stationary states and compare for example their axial charges before and after the interaction we would find that charge is not conserved. It is easy to calculate photon and fermion charges by means of the diagrams on Fig.2.

The result is

$$q_{ph} = \frac{\alpha}{4\pi} \xi_{ph} \tag{19}$$

where ξ_{ph} is the helicity of the photon.

If we calculate free axial charges for photon and fermion according to (17) we naturally find $q^f_{ph} = 0$ and $q^f_{ferm} = \xi_{ferm}$.

Up to now we discussed only the role of anomalies in the manifestation of the importance of classical fields and collective motions in the vacuum during the process of particles interactions. Let us discuss now some properties of j^f_μ and $T^f_{\mu\nu}$ which are of significance. One interesting feature, which is difficult to understand without our interpretation of anomalies, is the non-zero value of the trace of $T^f_{\mu\nu}$.

$$T^f_{\mu\mu} = T^{\delta}_{\mu\mu} = \frac{\alpha}{6\pi} F^2 \neq 0 . \tag{20}$$

This non-zero value of $T^f_{\mu\mu}$ is not so surprising, as it would be if $T^f_{\mu\nu}$ has been the energy-momentum tensor of any collective of massless particles, but not some useful effective quantity as it is.

It is interesting to notice also, that from this point of view $T^f_{\mu\nu}$ is a quite improbable candidate to determine the interaction of a system with gravitational field. May be this interpretation gives some insight on the problem of the cosmological term in the theory of gravitation. One of the problems with the cosmological term is the following. Having a lot of different condensates in the vacuum / Higgs field condensate, strong interactions condensate/, why we have no cosmological term in the gravitation. The answer, may be, lies in the fact, that in all these condensation processes the energy only transfers from the region of large momenta to that of small ones, not changing the gravitation field of the vacuum.

The most important property of the j^f_μ which was widely discussed in the literature, is that it is not gauge invariant. What could mean that the amount of the axial current transferred from the region of large momenta is not gauge invariant? The answer is, that the amount of current transferred during some processes from the region of the large momenta depends not only on parameters of the initial and final states, but also on the details of these processes. In particular, if the system went from the initial state to the final one and back through different stages, the amount of transferred current would be different. It is like hysteresis in macrophysics. In QED, although j^f_μ is not gauge invariant, the free axial charge $q^f = \int j^f_0 d^3x$ is gauge invariant and therefore our previous discussion makes sense. In QCD even q^f is not gauge invariant for non-perturbative fields and these fields could lead to observable nonconservation of the helicity in a number of processes.

2. QUASI-CLASSICAL DESCRIPTION OF THE SOLUTIONS OF THE DIRAC EQUATION FOR MASSLESS PARTICLES IN ELECTROMAGNETIC FIELD

In this section we will show how to formulate the picture we have described for massless fermions, obeying not the classical equation of motion, but the Dirac equation. We are used to the fact, that in quantum mechanics particles cannot be characterized by coordinates and momenta simultaneously. This is, however, not in contradiction with the possibility of introducing the local momentum $p_\mu(x)$ in such a way, that the current density $\bar\psi_{p_0\lambda}(x)\gamma_\mu \psi_{p_0\lambda}(x)$ would be proportional $p_\mu(x)$.

$$j_\mu(p_0 x) = \bar\psi_{p_0\lambda}(x)\gamma_\mu \psi_{p_0\lambda} = \rho(x,p_0)p_\mu(x,p_0) \qquad (21)$$

If $\rho(x,p_0)$ is chosen in an appropriate way $p_\mu(x,p_0)$ is obviously a local momentum of the classical particles in the quasiclassical approximation for ψ . Outside the quasiclassical approximation we have the freedom in the definition of $p_\mu(x,p_0)$ up to a factor $\alpha(x,p_0)$, which could be included in $\rho(x, p_o)$. If we fix this factor in some way, we will know $p_\mu(x,p_o)$ solving the Dirac equation for $\psi_{p_o\lambda}(x)$. The interesting feature of the massless fermion is, that we can proceed also in the opposite way. We can find out $\psi_{p_o\lambda}$ up to the phase factor, if we know $p_\mu(x)$. It means that we must have some equation for $p_\mu(x)$, which is an equivalent of the Dirac equation for ψ . In the quasiclassical approximation this equation will be, of course, the classical equation (2). Anyhow, if we know $p_\mu(x, p_o)$, we can write down the vacuum current for right or left particles separately in the form (10) .

$$j_\mu^\lambda(x) = 2\int \frac{d^4p_o}{(2\pi)^3}\,\delta(p_o^2)\,\vartheta(-p_{oo})\,\rho^\lambda(p_o x)\,p_\mu^\lambda(x,p_o) \tag{22}$$

If we introduce in integral (23) a new variable $p_\mu = p_\mu^\lambda(x,p_o)$ instead of p_μ^o we get the expression for $j_\mu^\lambda(x)$

$$j_\mu^\lambda(x) = 2\int \frac{d^4p}{(2\pi)^4}\,\rho^\lambda(p,x)\,p_\mu \tag{23}$$

$$\rho^\lambda(p,x) = 2\pi\,\delta(p_o^2)\,\vartheta(-p_{oo})\,\rho^\lambda(p_o,x)\left\|\frac{\partial p_o}{\partial p}\right\| \tag{24}$$

In (24) $p_{o\mu}$ must be expressed through p_μ by means of the equations

$$p_\mu = p_\mu(x, p_o) \tag{25}$$

The expression (23 coincides with (10) for $V \to \infty$.
In the same way we can write down the energy momentum tensor

$$T_{\mu\nu}^\lambda(x,p_o) = \frac{i}{2}\left\{\bar\psi_{p_o\lambda}\gamma_\nu\nabla_\mu\psi_{p_o\lambda} + \text{Herm conj.}\right\} \equiv \rho^\lambda(p_o,x)\,T_{\mu\nu}^\lambda(p,x) \tag{26}$$

$$T_{\mu\nu}^\lambda(x) = \int\frac{d^4p^o}{(2\pi)^3}\,\delta(p_o^2)\,\vartheta(-p_{oo})\,T_{\mu\nu}^\lambda(x,p_o) = \int\frac{d^4p}{(2\pi)^3}\,\rho^\lambda(p,x)\,T_{\mu\nu}(p,x) \tag{27}$$

Here $\rho^\lambda(p,x)$ is defined by (24) . The expression for $T_{\mu\nu}(p,x)$ we will calculate below.

The conservation law for $j_\mu^\lambda(p_0, x)$ and for $T_{\mu\nu}^\lambda(p_0 x)$ immediately leads to the equations (12) (13), independently of the explicite form $p_\mu(x, p_0)$ or $T_{\mu\nu}(p, x)$. From the equation

$$\partial_\mu j_\mu^\lambda(x, p_0) = 0 \tag{28}$$

and from properties of the Jacobians it is easy to find the equality:

$$p_\mu \partial_\mu \rho^\lambda(p, x) + \frac{\partial}{\partial p_\nu} \rho^\lambda(p, x) p_\mu \partial_\mu p_\nu = 0 \tag{29}$$

which is equivalent to (9) and leads to (12) with $\dot{p}_\mu \equiv p_\nu \partial_\nu p_\mu$. Similarly from the equation

$$\partial_\nu T_{\mu\nu}^\lambda(p_0, x) = F_{\mu\nu} j_\nu^\lambda(p_0, x) \tag{30}$$

we have

$$\begin{aligned}\partial_\nu \rho^\lambda(p, x) T_{\mu\nu}(p, x) &= \\ &= F_{\mu\nu} \rho^\lambda(p, x) p_\nu(x) - \frac{\partial}{\partial p_\rho}\left(T_{\mu\nu}^\lambda \partial_\nu p_\rho\right)\end{aligned} \tag{31}$$

which leads to (13) with

$$p_\mu \dot{p}_\rho \to T_{\mu\nu} \partial_\nu p_\rho$$

Let us discuss now the concrete structure of equations for $p_\mu(x, p_0)$ and $\psi(x)$ as a function of $p_\mu(x)$.

If we have for example a right-handed spinor $\gamma_5 \psi(x) = \psi(x)$, then for any ψ_e we can find $p_\mu(x)$ satisfying the condition

$$\hat{p}(x)\psi(x) = 0 \tag{32}$$

From (32)

$$p^2(x) = 0 \, , \qquad \bar{\psi}\gamma_\mu \psi \sim p_\mu(x) \, . \tag{33}$$

If $\psi(x)$ satisfies the Dirac equation

$$\hat{\nabla}\psi(x) = (\hat{\partial} + i\hat{A})\psi = 0 \tag{34}$$

with $A_\mu(x)$ vanishing at space infinity, it can be written in the form

$$\psi(x) = e^{-ip^0_\mu x + i\chi - \chi'} S(x)\varphi(p_0) \qquad (35)$$

where χ and χ' are real functions of x and $\varphi(p_0)$ is a free spinor with momentum $p_{0\mu}$ $\hat{p}_0\varphi(p_0) = 0$; $S(x)$ is a matrix of some Lorenz transformations,

$$\bar{S}S = 1 , \quad \bar{S} = \gamma_0 S^+ \gamma_0$$

From (32) and (35) we can define p_μ and $S(x)$ in such a way, that

$$\hat{p} = S\hat{p}_0\bar{S} \qquad (36)$$

One can propose the following idea to find $\psi(x)$ and $p_\mu(x)$ in quasi-classical manner.
Let us introduce

$$\mathcal{F}_{\mu\nu}(x) = \partial_\nu p_\mu - \partial_\mu p_\nu \qquad (37)$$

and suppose, for a moment, that we know $\mathcal{F}_{\mu\nu}(x)$. Then from (37), using the condition $p^2(x) = 0$ we have

$$p_\nu \partial_\nu p_\mu(x) = \mathcal{F}_{\mu\nu}(x) p_\nu(x) \qquad (38)$$

We can understand this equation, as the Newton equation, if we will determine the classical particle's trajectory in the field $\mathcal{F}_{\mu\nu}$ by the equations

$$\dot{x}_\mu = p_\mu \qquad (39)$$

$$\dot{p}_\mu = \mathcal{F}_{\mu\nu} p_\nu \qquad (40)$$

and will remember, that $\dot{p}_\mu$ — the derivative $p_\mu(x)$ along the trajectory — is equal to the derivative $p_\mu(x)$ along $p_\nu(x)$ $(p_\nu \partial_\nu p_\mu)$.
If we know $\mathcal{F}_{\mu\nu}(x)$ we can find $p_\mu(x)$, solving the classical equations (39) (40) with the condition $p_\mu(x) = p^0_\mu$ at $x \to \infty$.
We also could find $S(x)$, because from (40) and (36) we have

$$\dot{S} = \hat{\mathcal{F}} S$$
$$\hat{\mathcal{F}} = \frac{1}{4}\sigma_{\mu\nu}\mathcal{F}_{\mu\nu} \qquad (41)$$

From (41)

$$S = T e^{\int_{-\infty}^{0} d\tau \hat{\mathcal{F}}[x(\tau)]} \tag{42}$$

where integration in (42) is performed along the trojectory going from the point x_μ to infinity and having at infinity the momentum p^0_μ . Now we can try to define the effective force $\mathcal{F}_{\mu\nu}$ from the Dirac equation.

From (35)

$$\nabla_\mu \psi = \left\{ -i(p^0_\mu - \partial_\mu \chi - A_\mu) - \partial_\mu \chi' + \partial_\mu S\bar{S} \right\} \psi \tag{43}$$

substituting (43) into (34) and using the condition $\gamma_5 \psi = \psi$, we have

$$(\hat{p}_1 + i\hat{p}_2)\psi = 0 \tag{44}$$

where

$$p_{1\mu} = p^0_\mu - \partial_\mu \chi - A_\mu + h_\mu \tag{45}$$

$$p_{2\mu} = -\partial_\mu \chi' + g_\mu \tag{46}$$

$$h_\mu = \frac{i}{4} Sp(\bar{S}\gamma_5 \gamma_\mu \hat{\partial} S) \tag{47}$$

$$g_\mu = \frac{1}{4} Sp(\bar{S}\gamma_\mu \hat{\partial} S) \tag{48}$$

From (44)

$$p_1^2 = p_2^2 \qquad p_1 p_2 = 0 \tag{49}$$

From (32), (44)

$$p p_1 = 0 \qquad p p_2 = 0 \tag{50}$$

In order to determine a spinor completely we have to fix the system of coordinates. The three vectors $p_1 p_2 p$ are just the self-vectors which provide a natural coordinate system for spinor ψ . Really, if we multiply (44) by $\hat{p}_1$ we get the equation

$$[p_1^2 + i\sigma_{\mu\nu} p_{1\mu} p_{2\nu}]\psi = 0 \tag{51}$$

In a coordinate system where two space-like (42) vectors $p_{1\mu}, p_{2\mu}$ have only two components $p_{1\mu} = (0, p_x, 0, 0)$, $p_{2\mu} = (0, 0, p_y, 0)$ the (52) reads /in a representation of where is diagonal/

$$(1 - \sigma_z)\psi = 0 \tag{52}$$

This means, that in this coordinate system a two component spinor describes the particle with the momentum along the z axis. From (52) p_μ is just this momentum vector.

If the initial spinor $\varphi(p_0)$ is determined by vector p^0_μ and two space-like vectors $\xi^0_{1\mu}, \xi^0_{2\mu}$ $\xi^{02}_1 = \xi^{02}_2 = -1$ in such a way, that $\frac{1}{2}(1+\gamma_5)\psi = \sqrt{p_{00}}\binom{1}{0}$ in the coordinate system of vectors $p^0_\mu, \xi^0_{1\mu}, \xi^0_{2\mu}$ then the matrix S which transforms $p^0_\mu \to p_\mu$ $(\hat{p} = S\hat{p}_0\bar{S})$ will transform $\xi^0_{1\mu}, \xi^0_{2\mu}$ into $\xi_{1\mu}, \xi_{2\mu}$

$$\hat{\xi}_1 = S\xi^0_1\bar{S} \tag{53}$$

$$\hat{\xi}_2 = S\xi^0_2\bar{S}$$

Now, we can say, that

$$p_{1\mu} = \alpha_1 p_\mu + \beta_{11}\xi_{1\mu} + \beta_{12}\xi_{2\mu} \tag{54}$$

$$p_{2\mu} = \alpha_2 p_\mu + \beta_{21}\xi_{1\mu} + \beta_{22}\xi_{2\mu} \tag{55}$$

$$\beta_{11} = \beta_{22} \qquad \beta_{12} = -\beta_{12} \quad , \tag{56}$$

because the self-vectors $p_{1\mu}$ and $p_{2\mu}$ can differ from the self-vectors $\xi_{1\mu}, \xi_{2\mu}$ by a scale factor, by rotations in the plane $\xi_{1\mu}, \xi_{2\mu}$ and by addition of the vector proportional to p_μ.These conditions are enough to determine $\mathcal{F}_{\mu\nu}(x)$ if we fix the scale of p_μ what was not done up to now. We will do this putting $\alpha_1 = 1$. As a result we have

$$p_\mu = p_{1\mu} + \beta_{11}\xi_{1\mu} + \beta_{12}\xi_{2\mu} \tag{57}$$

and according to (45)

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} + h_{\mu\nu} + \xi_{\mu\nu} \tag{58}$$

$$h_{\mu\nu} = \partial_\nu h_\mu - \partial_\mu h_\nu \tag{59}$$

$$\xi_{\mu\nu} = \partial_\nu(\beta_{11}\xi_{1\mu} + \beta_{12}\xi_{2\mu}) - (\mu \to \nu) \tag{60}$$

The equation (55) is the equation for $\mathcal{F}_{\mu\nu}$ because as we will see, $h_{\mu\nu}$ and $\xi_{\mu\nu}$ is determined of $\mathcal{F}_{\mu\nu}$ as integrals over classical trajectories.
The $h_{\mu\nu}$ is expressed explicitely by (47) and (42).
The calculation of $\xi_{\mu\nu}$ is a little bit more complicated. Before doing this, let us calculate ψ i.e. χ and χ'.
From (50) and (45), (46) we have

$$\dot{\chi} = -p_\mu A_\mu + h_\mu p_\mu + p_\mu \overset{\circ}{p}_\mu \tag{61}$$

$$\dot{\chi}' = p_\mu g_\mu \tag{62}$$

If we will write down

$$\overset{\circ}{p}_\mu = p_\mu - \int_{-\infty}^{x} \mathcal{F}_{\mu\nu}\, dx'_\nu \tag{63}$$

and notice, that from (48)

$$p_\mu g_\mu = \frac{1}{2}\partial_\mu p_\mu \tag{64}$$

we can write

$$\chi = -\underline{\int} A_\nu(x')dx'_\nu + \underline{\int} h_\nu(x')dx'_\nu - \underset{x'>x''}{\underline{\int}} dx'_\nu dx''_\mu \mathcal{F}_{\mu\nu}(x'') \tag{65}$$

$$\chi' = \frac{1}{2}\int_{-\infty}^{0} d\tau\, \partial_\nu p_\nu(x') \equiv \frac{1}{2}\,\underline{\partial_\nu p_\nu} \tag{66}$$

By $\underline{\varphi}$ we denote the integration of any quantity φ along the trajectory going from point x to $-\infty$.
Using these expressions for χ and χ' we can write $p_{1\mu}$ and $p_{2\mu}$ in the form

$$\begin{aligned} p_{1\mu} = \overset{\circ}{p}_\mu + \underline{\int} dx'_\nu \frac{\partial x'_\varrho}{\partial x_\mu}\left[F_{\varrho\nu}(x') + h_{\varrho\nu}(x')\right] + \\ + \partial_\mu \underset{x'>x''}{\underline{\int}} dx'_\varrho dx''_\sigma \mathcal{F}_{\varrho\sigma}(x'') \end{aligned} \tag{67}$$

$$p_{2\mu} = \int dx'_\nu \frac{\partial x'_\varrho}{\partial x_\mu} q_{\varrho\nu}(x') \tag{68}$$

These formulae give us a possibility to find β_{11}, β_{12} and consequently $\xi_{\mu\nu}$.

$$\beta_{11} = \beta_{22} = \int dx'_\nu \, \xi_{2\mu}(x) \frac{\partial x'_\varrho}{\partial x_\mu} g_{\varrho\nu}(x') \tag{69}$$

$$\beta_{12} = -\beta_{21} = -\int dx'_\nu \, \xi_{1\mu}(x) \frac{\partial x'_\varrho}{\partial x_\mu} g_{\varrho\nu}(x') \tag{70}$$

$$\xi_{\mu\nu} = \partial_\mu(\xi_{1\nu}\xi_{2\varrho} - \xi_{2\nu}\xi_{1\varrho}) \int dx'_\alpha \frac{\partial x'_\beta}{\partial x_\varrho} g_{\alpha\beta}(x') - (\mu \to \nu) \tag{71}$$

This expression for $\xi_{\mu\nu}$ shows, that $\xi_{\mu\nu}$ is invariant under rotation of ξ_1 and ξ_2 in their plane.
In order to make it explicite it is natural to introduce the fourth vector ζ_μ of the coordinate system defining the spinor.

$$\zeta^2 = 0, \quad p_1\zeta = 0, \quad p_2\zeta = 0, \quad p\zeta = 1 \tag{72}$$

$$\hat{\zeta} = S\hat{\zeta}_0\bar{S} \tag{73}$$

$\zeta_{0\mu} = \left(\frac{1}{p}, 0, 0, \frac{1}{p}\right)\frac{1}{2}$ in the coordinate system, where $p_{0\mu} = (-p^0, 0, 0, p)$

$$\xi_{1\nu}\xi_{2\varrho} - \xi_{1\varrho}\xi_{2\nu} = \varepsilon_{\nu\varrho\gamma\delta} p_\gamma \zeta_\delta \tag{74}$$

$$\xi_{\mu\nu} = \varepsilon_{\mu\nu\sigma\tau} \partial_\varrho f_{\varrho\sigma\tau} \tag{75}$$

$$f_{\varrho\sigma\tau} = \int dx'_\alpha q_{\beta\alpha}(x') \left[\frac{\partial x'_\beta}{\partial x_\varrho} p_\sigma \zeta_\tau + \frac{\partial x'_\beta}{\partial x_\sigma} p_\tau \zeta_\varrho + \frac{\partial x'_\beta}{\partial x_\tau} p_\varrho \zeta_\sigma\right] \tag{76}$$

3. CURRENT AND ENERGY-MOMENTUM TENSOR AT LARGE MOMENTA

Now we can calculate the current in terms of integrals on the classical trajectories.

According to (35) and (66)

$$j_\mu(x, p_0) = e^{-\partial_\rho p_\rho} p_\mu(x) \tag{77}$$

In order to calculate the vacuum current (24) we have to know the Jacobian (25). This Jacobian is the determinant of the matrix $C_{\mu\nu} = \frac{\partial p_\mu}{\partial p^0_\nu}$. Due to the equation of motion (40), $C_{\mu\nu}$ satisfies the equation

$$\dot{C}_{\mu\nu} = \left[-\partial_\mu p_\rho + \frac{\partial \mathcal{F}_{\mu\sigma}}{\partial p_\rho} p_\sigma \right] C_{\rho\nu} \tag{78}$$

In (78) we consider $\mathcal{F}_{\mu\nu}$ not as a function $\mathcal{F}_{\mu\nu}(x, p_0)$, but as a function $\mathcal{F}_{\mu\nu}(x,p) \equiv \mathcal{F}_{\mu\nu}(x, p_0(p,x))$. Denoting the expression in the brackets as a matrix $\varphi_{\mu\rho}$ we have

$$C = T e^{\varphi} \tag{79}$$

and

$$\| C \| = e^{Sp\,\varphi} = e^{-\partial_\rho p_\rho + \frac{\partial \mathcal{F}_{\mu\sigma} p_\sigma}{\partial p_\mu}} \tag{80}$$

Therefore the vacuum current (24) is equal

$$j_\mu(x) = 2 \int \frac{d^4 p}{(2\pi)^3} \delta(p^2)\, \vartheta(-p_{00})\, p_\mu\, e^{-\frac{\partial w_\rho}{\partial p_\rho}} \quad ; \tag{81}$$

$w_\rho = (\mathcal{F}_{\rho\sigma} - F_{\rho\sigma}) p_\sigma$ is the non-classical part of the accelerations along the trajectory. It is interesting, that the classical motion produces no current in the vacuum in the way classical electrons produce no diamagnetism in the metals.

According to (58) and (59)

$$w_\mu = \varepsilon_{\mu\nu\sigma\tau} P_\nu \zeta_\sigma P_\alpha q_{\alpha\tau} + h_{\mu\nu} p_\nu - \varepsilon_{\mu\nu\sigma\tau} (\partial_\rho p_\nu) f_{\rho\sigma\tau} \tag{82}$$

The first term in (82) is of the order of $\frac{1}{p}$, and could lead to linear divergency of the current, but this does not occure by symmetry reasons. Other terms give the usual logarithmically divergent expression for the current.

Let us consider now the divergence of the current

$$\partial_\mu j_\mu = -2\int \frac{d^4p}{(2\pi)^3}\,\delta(p^2)\frac{\partial}{\partial p_\mu}\left[w_\mu\,\vartheta(-p_{00})\,e^{-\frac{\partial w_\varrho}{\partial p_\varrho}}\right] \qquad (83)$$

If we introduce j^V_μ which is defined as the integral on a finite volume in the momentum space, then $\partial_\mu j^V_\mu$ will reduce to the integral over a large surface in this space

$$\partial_\mu j^V_\mu = -2\int_S \frac{d^3\sigma}{(2\pi)^3}\,w_n\,e^{-\frac{\partial w_\varrho}{\partial p_\varrho}}\,\delta(p^2) \qquad (84)$$

Unfortunately this integral depends on the form of the surface. This is natural, because the integral (81) is defined not well enough to differentiate it before the integration. In order to define j^V_μ correctly we could proceed in the usual way introducing a displacement of the point in the current. We could write, instead of (21)

$$j^V_\mu(x,p_0) = \bar{\psi}_{p_0}\left(x-\frac{\varepsilon}{2}\right)\gamma_\mu\,\psi_{p_0}\left(x+\frac{\varepsilon}{2}\right)e^{i\int_{x-\varepsilon/2}^{x+\varepsilon/2}A_\nu dx_\nu} \qquad (85)$$

Introducing (35) into (85) we have

$$j^V_\nu(p_0,x) = \frac{1}{4}Sp\left[\bar{S}\left(x-\frac{\varepsilon}{2}\right)\gamma_\nu\,S\left(x+\frac{\varepsilon}{2}\right)\hat{p}(1+\gamma_5)\right]e^{-\chi'(x+\frac{\varepsilon}{2})-\chi'(x-\frac{\varepsilon}{2})-ip^0_\mu\varepsilon_\mu-iF} \qquad (86)$$

$$F = \int_C A_\nu dx_\nu = \left[1+\frac{1}{24}(\varepsilon\partial)^2\right]\int dx'_\nu F_{\mu\nu}\varepsilon_\mu + O(\varepsilon^5) \qquad (87)$$

The conture C is defined by Fig.3.

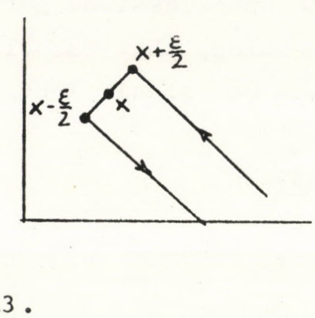

Fig.3.

In order to calculate $\partial_\nu j^V_\nu$ and $\partial_\mu T^V_{\mu\nu}$ we will need the explicite expression for the singular part of j^V_ν linear in the field. If we would multiply (86) by $e^{ip\varepsilon}$ we could expand the obtained expression in the power of ε. Doing this up to the third order in ε and linearly in the field, we have

$$e^{ip\varepsilon} j^V_\nu(p_0,x) = e^{-\partial_\rho p_\rho}\Big\{p_\nu\big(1+i\varepsilon W - \tfrac{i}{24}(\varepsilon\partial)^2\varepsilon_\mu \underline{F_{\mu\sigma}p_\sigma}\big) + \\ + \frac{i\varepsilon_\rho}{2}\partial_\rho \underline{\tilde{\mathcal{F}}_{\nu\sigma}p_\sigma} + \frac{1}{8}(\varepsilon\partial)^2\big[\underline{\mathcal{F}_{\nu\sigma}p_\sigma} - p_\nu\partial_\gamma \underline{\underline{\mathcal{F}_{\gamma\sigma}p_\sigma}}\big]\Big\} \tag{87}$$

$$\tilde{\mathcal{F}}_{\nu\sigma} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\mathcal{F}_{\alpha\beta} \tag{88}$$

Linearly in the field

$$\tilde{\mathcal{F}}_{\nu\sigma}p_\sigma = \tilde{F}_{\nu\sigma}p_\sigma + \tilde{\xi}_{\nu\sigma}p_\sigma \tag{89}$$

because, as it is easy to show from (47), $h_{\mu\nu}$ is quadratic in the field. From (48) and (75) (76)

$$\tilde{\xi}_{\nu\sigma}p_\sigma = \frac{1}{2}\partial^2\big[\underline{\mathcal{F}_{\nu\sigma}p_\sigma} - p_\nu\partial_\gamma \underline{\underline{\mathcal{F}_{\gamma\sigma}p_\sigma}}\big] = \frac{1}{2}\partial^2 \underline{\underline{f_\nu}} \tag{90}$$

$$f_\nu = \partial_\gamma\big(p_\gamma \mathcal{F}_{\nu\tau}p_\tau - p_\nu \mathcal{F}_{\gamma\tau}p_\tau\big) \tag{91}$$

The quantity $(\varepsilon\partial)^2\varepsilon_\mu F_{\mu\sigma}p_\sigma$ which entered in (82) could be written in the form

$$p_\nu(\varepsilon\partial)^2\varepsilon_\mu \underline{F_{\mu\sigma}p_\sigma} = -\frac{1}{\partial^2}(\varepsilon\partial)^3\underline{f_\nu} + i f'_\nu \tag{92}$$

$$f'_\nu = \frac{i}{\partial^2}(\varepsilon\partial)^2\partial_\gamma\big[p_\nu F_{\gamma\rho}\varepsilon_\rho - \varepsilon_\gamma F_{\nu\tau}p_\tau\big] \tag{93}$$

In (90) (91) and in the last term in (87) we can replace $\mathcal{F}_{\mu\nu}$ by $F_{\mu\nu}$. Using (87) – (93) we can write the vacuum current in the form

$$j^V_\nu = 2\int\frac{d^4p}{(2\pi)^3}e^{-ip\varepsilon}\delta(p^2)e^{-\frac{\partial W_\rho}{\partial p_\rho}}\vartheta(-p_0)\Big\{p_\nu(1-i\underline{\varepsilon W}) + \frac{i\varepsilon_\rho}{2}\partial_\rho\underline{\tilde{F}_{\nu\sigma}p_\sigma} + \\ + \frac{i(\varepsilon\partial)}{4}\partial^2\underline{\underline{\underline{f_\nu}}} + \frac{1}{8}(\varepsilon\partial)^2\underline{\underline{f_\nu}} + \frac{i}{24\partial^2}(\varepsilon\partial)^3\underline{f_\nu} + \frac{1}{24}f'_\nu\Big\} \tag{94}$$

In order to calculate the singular part of the current coming from the region of large momenta we can replace $e^{-\frac{\partial \underline{w_\rho}}{\partial p_\rho}}$ by $1-\underline{\frac{\partial w_\rho}{\partial p_\rho}}$.

Using the relations

$$\frac{\partial}{\partial p_\rho}\underline{w_\rho} = \underline{\frac{\partial}{\partial p_\rho} w_\rho} - \underline{\underline{\partial_\rho w_\rho}} \tag{95}$$

$$w_\nu - p_\nu \underline{\partial_\rho w_\rho} = \frac{1}{2}\partial^2 \underline{\tilde{F}_{\nu\sigma} p_\sigma} = \frac{1}{2}\partial^2 \underline{\tilde{F}_{\nu\sigma} p_\sigma} + \frac{1}{4}\partial^4 \underline{\underline{\underline{f_\nu}}} \tag{96}$$

which are consequences of the definition φ and (75), (76), (48) , we find out

$$j_\nu^V = 2\int \frac{d^4p}{(2\pi)^3} e^{-ip\varepsilon}\vartheta(-p_{00})\delta(p^2)\Big\{p_\nu + \frac{1}{2}\big[\partial^2 \underline{\underline{\tilde{F}_{\nu\sigma} p_\sigma}} + i(\varepsilon\partial)\underline{\tilde{F}_{\nu\sigma} p_\sigma}\big] +$$

$$+\frac{1}{4}\Big[\partial^4 \underline{\underline{\underline{f_\nu}}} + i(\varepsilon\partial)\partial^2 \underline{\underline{f_\nu}} + \frac{1}{2}(\varepsilon\partial)^2 \underline{\underline{f_\nu}} + \frac{i}{6\partial^2}(\varepsilon\partial)^3 \underline{f_\nu} + \frac{1}{6} f_\nu'\Big]\Big\} \tag{97}$$

Using the relations similar to (95) we can write the singular current of the right-hand particles in a compact form

$$j_\nu^V = 2\int \frac{d^4p}{(2\pi)^3} e^{-ip\varepsilon}\vartheta(-p_{00})\Big\{p_\nu\delta(p^2) + \underline{\frac{1}{2}\partial_\rho\frac{\partial}{\partial p_\rho}\tilde{F}_{\nu\sigma}p_\sigma\delta(p^2)} -$$

$$- \underline{\frac{1}{24\partial^2}\Big(\partial_\rho\frac{\partial}{\partial p_\rho}\Big)^3 f_\nu\delta(p^2)} + \frac{1}{24} f_\nu'\delta(p^2)\Big\} \tag{98}$$

This expression can be calculated explicitely.

$$\underline{\partial_\rho \frac{\partial}{\partial p_\rho}\tilde{F}_{\nu\sigma}p_\sigma\delta(p^2)} = 2\underline{p_\rho\partial_\rho\tilde{F}_{\nu\sigma}p_\sigma\delta'(p^2)} = 2\tilde{F}_{\nu\sigma}p_\sigma\delta'(p^2) \tag{99}$$

$$\underline{\Big(\partial_\rho\frac{\partial}{\partial p_\rho}\Big)^3} f_\nu = -8\partial^2\big[2\partial_\gamma F_{\gamma\nu}\delta'(p^2) - f_\nu\delta''(p^2)\big] + \partial^2 f_\nu' \tag{100}$$

Finally we have

$$j_\nu^V = \frac{1}{8\pi^2}\Big\{\tilde{F}_{\nu\sigma}\varepsilon_\sigma\frac{1}{\varepsilon^2} + \frac{1}{6}\partial_\gamma F_{\gamma\nu}\ln\varepsilon^2 + \frac{1}{6\varepsilon^2}\partial_\gamma\big[\varepsilon_\gamma F_{\nu\tau}\varepsilon_\tau - \varepsilon_\nu F_{\gamma\tau}\varepsilon_\tau\big]\Big\} \tag{101}$$

If we will average this expression over the directions of ε_μ the linearly divergent part of the current will be equal to zero, and the logarithmically divergent part will have the usual form $j_\nu^V = \frac{1}{12\pi^2} \ln\varepsilon\, \partial_\gamma F_{\gamma\nu}$.
The current (101) is conserved for any ε_μ. Using (101) we will find $\partial_\mu j_\mu$ and $\partial_\nu T_{\mu\nu}$ very easily.
According to (85)

$$\partial_\nu j_\nu^V = i F_{\nu\rho} \varepsilon_\rho j_\nu^V \tag{102}$$

On the right-hand side we can use the (101) and only the first term of it will contribute.
Therefore

$$\partial_\mu j_\mu^V = -\frac{1}{8\pi^2} F_{\nu\rho} \tilde{F}_{\nu\sigma} \frac{\varepsilon_\rho \varepsilon_\sigma}{\varepsilon^2} \tag{103}$$

This expression does not depend on ε_μ because the matrix $F_{\nu\rho} F_{\nu\sigma}$ is proportional to the unit matrix.

$$\partial_\mu j_\mu^V = \frac{1}{32\pi^2} F\tilde{F} \tag{104}$$

In order to calculate $\partial_\nu T_{\mu\nu}$ and to find $T_{\mu\nu}^f$ we have to define $T_{\mu\nu}^V$.
The simplest definition is the following:

$$T_{\mu\nu}^V = i \frac{\partial}{\partial\varepsilon_\mu} j_\nu^V(\varepsilon, x) \tag{105}$$

$$\partial_\nu T_{\mu\nu}^V = -F_{\nu\rho} \frac{\partial}{\partial\varepsilon_\mu} \varepsilon_\rho j_\nu^V = F_{\mu\nu} j_\nu + \varepsilon_\rho F_{\rho\nu} \frac{\partial j_\nu}{\partial\varepsilon_\mu} \tag{106}$$

In the last term of (106) we can again use (101). The first term of (93) will not contribute because it is odd in ε_μ and we have

$$\partial(T_{\mu\nu}^V + T_{\mu\nu}^\delta) = F_{\mu\nu} j_\nu \tag{107}$$

$$T_{\mu\nu}^\delta = \frac{1}{48\pi^2}\Big\{ + \frac{\varepsilon_\mu\varepsilon_\nu}{\varepsilon^2} F^2 + \Big(\delta_{\mu\nu} - \frac{\varepsilon_\mu\varepsilon_\nu}{\varepsilon^2}\Big)\frac{1}{\varepsilon^2}(\varepsilon F)^2 - \tag{108}$$

$$- \frac{2\varepsilon_\mu\varepsilon_\rho}{\varepsilon^2} F_{\rho\sigma} F_{\nu\sigma} - \frac{\varepsilon_\rho\varepsilon_\tau}{\varepsilon^2} F_{\rho\mu} F_{\tau\nu} \Big\}$$

As we see, $T^{\delta}_{\mu\nu}$ is finite, but it depends on the direction of ε_μ . $T^{f}_{\mu\nu}$ is conserved independently of the form of the cut-off and probably could be made independent of it.

If we average $T^{\delta}_{\mu\nu}$ over directions of ε_μ we find out

$$T^{d}_{\mu\nu} = -\frac{1}{48\pi^2}\left[\frac{5}{6}F_{\mu\sigma}F_{\nu\sigma} - \frac{11}{24}\delta_{\mu\nu}F^2\right] \qquad (109)$$

This expression was written in the first section. The important feature of the expression (108) for $T^{\delta}_{\mu\nu}$ is that the trace $T^{\delta}_{\mu\nu}$ does not depend on ε_μ and therefore

$$T^{f}_{\nu\nu} = T^{d}_{\nu\nu} = \frac{1}{48\pi^2}F^2 \qquad (110)$$

independently of any averaging.

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