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ABSTRACT

A spheromak problem was investigated taking into account the Hall-currents with the aim that in the first period of the heating it may be good approximation. For certain initial conditions the magnetic field components are calculated, they exhibit nonlinear oscillations. For a simplified model the field was also determined analytically, enabling treatment of the forced oscillations too. The modelling yields a qualitative similarity.

АННОТАЦИЯ

Исследовалась проблема сферомака в начальной стадии нагрева, когда Холловские токи играют существенную роль. При определенных начальных условиях были вычислены компоненты магнитного поля, которые оказались нелинейно осциллирующими. Для упрощенной модели поле было определено также и аналитически, далее рассматривались вынужденные колебания. Вторая модель показывала аналогичные свойства с первой.

KIVONAT

Megvizsgáltuk a szferomak problémát a felfütési szakasz kezdetén, amikor a Hall áramok jelentős szerepet játszanak. Bizonyos kezdeti feltételek mellett kiszámitottuk a mágneses tér komponenseit, melyek nemlineáris oszcillációkat mutatnak. Egy egyszerüsitett modellre a teret analitikusan is meghatároztuk, és a kényszerrezgéseket is tárgyaltuk. E modell az előzőhöz hasonló tulajdonságokat mutat.

Introduction

The spheromak as a "natural" plasma configuration was considered in astrophysics [1], by analogy with the Hill's configuration of the hydrodynamics. The field lines and the plasma arrangement of the classical spheromak is shown in Fig. 1. It is characterized by a force-free condition

$$\vec{j} = k\vec{B} \qquad (1)$$

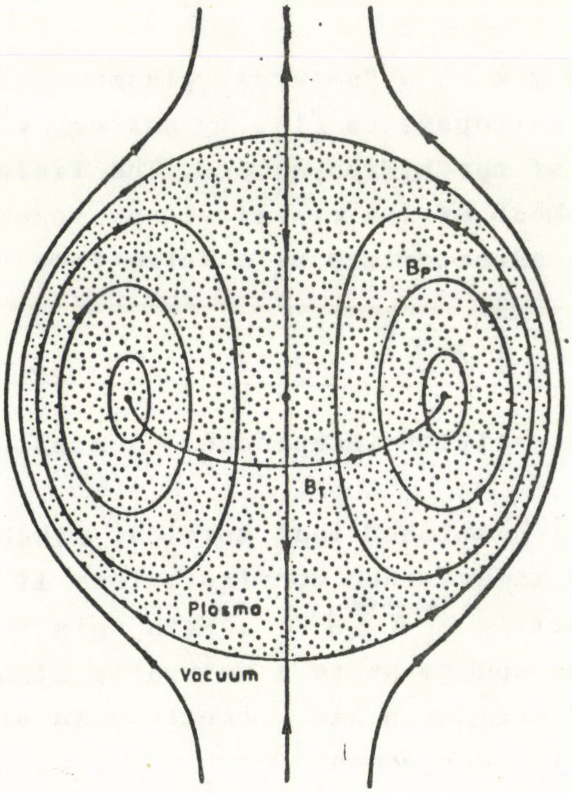
where k is a position-independent constant, that is $\vec{j}\times\vec{B}=0$

Taylor [2] mentioned that toroidal discharges spontaneously evolve toward such configurations if the plasma conserves the quantity $\int \vec{A}\cdot\vec{B}\,dV$. With this conservative restriction the spheromak is a naturally stable system.

Increased attention has recently been directed toward this "ideal" magnetic arrangements with a motivation that the stability of reversed pinches and tokamaks would be a consequence of the spheromak stability. Rosenbluth and Bussac [3] investigated the spheromak problem by the Mercier and the Taylor stability methods and demonstrated a very good resistance against MHD instabilities. Because of the small magnetic shear the β limit is low, but it can be improved by a slight modification in the geometry /oblimak/. In contrast, at the coil, the β-limit is very favourable and equivalent to the best tokamak arrangements [4].

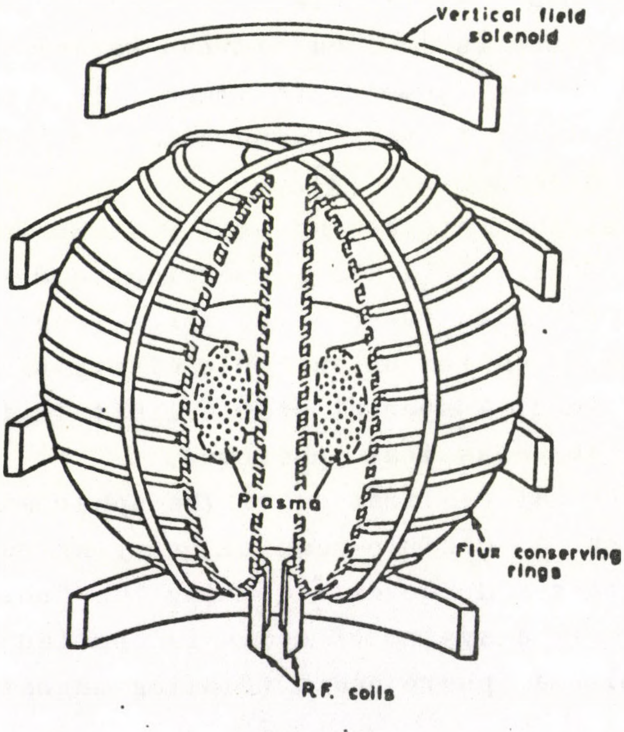
A general and interesting analysis has been carried out by G. and L. Vahala [6]. The geometrical and MHD /magnetic helicity/ constraints suggest the force-free solution of the MHD-equations. Then for an ensemble of eigensolutions an average is formed. A special stable limit is found in the ground mode, the classical spheromak.

The technical realization of the spheromak is a more difficult task. A possible mean of carrying out has been proposed by Hugrass and Tuczek [7] using the "rotamak concept" /Fig. 2./. There a system of coils is applied poloidally around the plasma sphere and a rotating magnetic field induces



B_P
B_T
Plasma
Vacuum

Fig. 1.



Vertical field solenoid
Plasma
Flux conserving rings
R.F. coils

Fig. 2.

the azimuthal current. However, the azimuthal asymmetry of the classical spheromak is violated. Normally because of the skin effect, the heating is not too effective but - first of all in a non ideal plasma - the Hall effect plays a role too. As a result of this the situation fundamentally changes, on a different scale the field penetrates the plasma. If the currents are strong enough the spheromak configuration may appear.

In the paper of Hugrass et alii [5] the problem of the rotating field in an approximation of cylindrical symmetry is discussed. Their computation shows that the field penetrates the plasma and the field lines exhibit a twisting, similar to the familiar solutions of the hydrodynamical equations where the nonlinearity plays an essential role.

In the present paper we should like to investigate a time dependent problem which can help us to understand the physics of the heating period in a spheromak plasma. In this period the motion of the plasma can be considered as negligible, and the collisions are important. Since the heating arises by applying external alternating currents we must consider three spatial dimensions and the time as variables.

In the nextchapter we derive the fundamental equations, and we give the mathematical results for the non-symmetric sheromak model. Thereafter we discuss the first termes of an infinite system of equations and their numerical solutions. The main result is the existence of nonlinear oscillations, which are not in the work [5]. An other approximation is given where we give an analytical solution which is in qualitative accordance with the numerical calculations.

Fundamental equations

The classical spheromak is shown in Fig. 1. The magnetic field lines have radial, poloidal and azimuthal components, satisfying the force-free condition (1). To solve the Ampère's low

$$\mathrm{rot}\,\vec{B} = \frac{4\pi}{c}\,\vec{j} \qquad (2)$$

this condition induces a solution in a series of Legendre polynomials P_n^m, angular functions $e^{im\varphi}$ and spherical Bessel functions j_n, with the help of a scaler function

$$\psi_{nmq} = j_n(k_{nq}r)\,P_n^m(\cos\vartheta)\,e^{im\varphi} \tag{3}$$

The magnetic field is given in the form

$$\vec{B}_{nmq} = \vec{r}\times\nabla\psi_{nmq} + \frac{1}{k_{nq0}}\nabla\times\left(\vec{r}\times\nabla\psi_{nmq}\right) \tag{4}$$

We shall use in this paper the Maxwell equations complemented by the Hall-currents

$$\left.\begin{aligned} \dot{\vec{B}} &= \frac{c^2}{4\pi\sigma}\nabla^2\vec{B} - \frac{c}{4\pi n_e e}\nabla\times\left[(\nabla\times\vec{B})\times\vec{B}\right] \\ \operatorname{div}\vec{B} &= 0 \end{aligned}\right\} \tag{5}$$

The $\vec{B}$ -field can have a prescribed component /it may be time -dependent too/ stemming from external coils. Let us denote it by $\vec{B}_o$, so for the plasma interior we have

$$\left.\begin{aligned} \frac{\partial}{\partial t}(\vec{B}+\vec{B}_o) &= a_1\nabla^2(\vec{B}+\vec{B}_o) - a_2\nabla\times\left[(\nabla\times\vec{B})\times(\vec{B}+\vec{B}_o)\right] \\ \nabla(\vec{B}+\vec{B}_o) &= 0 \end{aligned}\right\} \tag{5a}$$

Let the plasma radius be a. In a spheromak configuration B_r on the plasma surface vanishes or it has a prescribed value. According to the expansion into series (3), the solution has the form

$$\begin{aligned} \vec{B} = \sum_{nmq} e^{im\varphi} g_{nmq}(t)\Big\{ &\vec{e}_r \frac{n(n+1)}{k_{nq}r} j_n(k_{nq}r)P_n^m(\cos\vartheta) + \\ &+ \vec{e}_\vartheta\left[\frac{im}{\sin\vartheta} j_n(k_{nq}r)P_n^m(\cos\vartheta) - \frac{\sin\vartheta}{k_{nq}r}P_n'^m(\cos\vartheta)\frac{d}{dr}\left(r j_n(k_{nq}r)\right)\right] + \\ &+ \vec{e}_\varphi\left[\sin\vartheta\, j_n(k_{nq}r)P_n'^m(\cos\vartheta) + \frac{im}{\sin\vartheta}P_n^m(\cos\vartheta)\frac{1}{k_{nq}r}\cdot\frac{d}{dr}\left(r j_n(k_{nq}r)\right)\right]\Big\} \end{aligned} \tag{6}$$

In our calculations the expansion is performed with the real functions $\sin$ and $\cos$, so (6) is transformed to

$$\begin{aligned} \vec{B} = \sum_{nmq}\vec{B}_{nmq} = \sum_{nmq} &\vec{e}_r\left[\frac{n(n+1)}{k_{nq}r} j_n(k_{nq}r)P_n^m(\cos\vartheta)\cdot(-)\right] + \\ &+ \vec{e}_\vartheta\left[-\frac{m}{\sin\vartheta} j_n(k_{nq}r)P_n^m(\cos\vartheta)\cdot(+) - \frac{\sin\vartheta}{k_{nq}r}P_n'^m(\cos\vartheta)\frac{d}{dr}\left(r j_n(k_{nq}r)\right)\cdot(-)\right] + \\ &+ \vec{e}_\varphi\left[\sin\vartheta\, j_n(k_{nq}r)P_n'^m(\cos\vartheta)\cdot(-) - \frac{m}{\sin\vartheta}P_n^m(\cos\vartheta)\frac{1}{k_{nq}r}\frac{d}{dr}\left(r j_n(k_{nq}r)\right)\right] \end{aligned} \tag{7}$$

Here $\vec{e}_r, \vec{e}_\vartheta, \vec{e}_\varphi$ are the unit fectors, j_n are the spherical Bessel functions,

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+1/2}(x)$$

P_n^m are the Legendre polynomials. The symbols $(+)$ and $(-)$ represent the following expressions

$$\begin{aligned} (+) &\doteq \beta_{nmq}(t)\cos m\varphi + \alpha_{nmq}(t)\sin m\varphi \\ (-) &\doteq \alpha_{nmq}(t)\cos m\varphi - \beta_{nmq}(t)\sin m\varphi \end{aligned} \tag{8}$$

For the $\vec{B}_o$ -field we get the similar expressions, then we use

$$\begin{aligned} (+) &\to \delta_{nmq}(t)\cos m\varphi + \gamma_{nmq}(t)\sin m\varphi \\ (-) &\to \gamma_{nmq}(t)\cos m\varphi - \delta_{nmq}(t)\sin m\varphi \end{aligned} \tag{8a}$$

If we are able to determine the time dependent α and β coefficients the problem is solved. In accordance with the construction (7) the components satisfy

$$\nabla \times \vec{B}_{nmq} = k_{nq}\, \vec{B}_{nmq} \tag{9}$$

i.e. the force-free condition.

When $\vec{B}$ vanishes on the plasma surface, then $k_{nq} \cdot a$ must be a root /say the q-th / of the $j_n(x)$ function, see eq. (7). If we apply the same condition to the $\vec{B}_o$ -field too, we can take into account simultaneously the known coefficients γ and δ , which may have prescribed /e.g. time dependent/ values. In the case of non-vanishing condition the $k_{nq}a$ -values must be determined by a linear combination

$$c_n \dot{j}_n + d_n j_n = 0 \tag{10}$$

here

$$\dot{j}_n(x) \doteq \frac{1}{x}\frac{d}{dx}\left(x\, j_n(x)\right) \tag{10a}$$

We investigated the case of vanishing B_r if $r = a$.

The nonlinear term in eq. (5) or $(5a)$ is tedious to determine. For the radial component

$$\sum_{\substack{nn' \\ mm' \\ qq'}} \vec{e}_r \frac{k_{nq}}{r\sin\vartheta} \left\{ \left[-\frac{m' n(n+1)}{\sin\vartheta\; k_{nq} r} j_n j_{n'} \cdot (\sin\vartheta\, P_n^m)^{\cdot} P_{n'}^{m'} \cdot (-)(+') + \right.\right.$$

$$+\frac{n(n+1)}{k_{nq}r} j_n j_{n'} (\sin\vartheta\, P_n^m)^{\cdot} \dot{P}_{n'}^{m'} (-)(-')\Big] - \Big[\longrightarrow'\Big] +$$

$$+\Big[-\frac{n(n+1)n'(n'+1)}{k_{nq}r} j_n j_{n'} \sin\vartheta\, P_n^m P_{n'}^{m'} (-)(-') -$$

$$- m'\frac{n'(n'+1)}{k_{n'q'}r} j_n j_{n'} \dot{P}_n^m P_{n'}^{m'} (+')(-) - \tag{11}$$

$$- m m' \frac{n'(n'+1)}{\sin\vartheta\, k_{n'q'}r} \cdot j_{n'} j_n P_{n'}^{m'} P_n^m (+)(+')\Big] - \Big[\longrightarrow'\Big]$$

The same abbreviations are used as in (8) . The brackets $[\longrightarrow']$ symbolize the terms where the previous bracket is to be repeated but with interchanged indices; that is, n, m, q must be changed to n', m', q' and vice versa. The arguments in the functions j and j_n are chosen according to the indices,

$$j_n \doteq j_n(k_{nq}r)\,; \qquad j_{n'} \doteq j_{n'}(k_{n'q'}r)$$

and so on. The dots stand for the derivatives according to ϑ.

Now we have to determine the terms in the differential equation (5) or (5a) . Multiplying by

$$\begin{Bmatrix}\cos m''\varphi \\ \sin m''\varphi\end{Bmatrix} j_{n''}(k_{n''q''}r) P_{n''}^{m''}(\cos\vartheta)$$

and by a weighting function and integrating it upon the volume, we can isolate the linear terms, and we get a system of non-linear differential equations for the time-dependent coefficient. From the orthogonality of the Bessel functions it is easy to see that this weighting function is $r^3 \sin\vartheta$

The linear part of the equation is now

$$\left.\begin{matrix} -\frac{\partial}{\partial t}\beta_{n''m''q''} \\ \\ +\frac{\partial}{\partial t}\alpha_{n''m''q''} \end{matrix}\right\} = -a_1 k_{n''q''}^2 \left\{\begin{matrix} -\beta_{n''m''q''} - \delta_{n''m''q''} \\ \\ \alpha_{n''m''q''} + \gamma_{n''m''q''} \end{matrix}\right. \tag{12}$$

We used the time-independent $\vec{B}_0$ -field, but the generalization for the time dependent case is straightforward. The eq. (12) is the equivalent of

$$\frac{1}{a_1}\dot{\vec{B}}_{nmq} - \Delta\vec{B}_{nmq} = -k_{nq}^2\vec{B} \tag{13}$$

After some lengthy algebra the nonlinear part is

$$-4\pi a_2 \frac{1}{C_{n''m''q''}} \sum_{\substack{\{n n'\} \\ \{m m'\} \\ \{q q'\}}} \frac{1}{a^2} \lambda_{nq} \Big\{ \Big[\mathcal{F}_1 \Big\{ {n \atop q} \Big\} \Big(\frac{1}{\lambda_{nq}} Z(-+') K(1) +$$

$$+ \frac{1}{\lambda_{nq}} Z(+-') K(11) \Big) + \mathcal{F}_2 \Big\{ {n \atop q} \Big\} \Big(\frac{1}{\lambda_{nq}} K(2) Z(--') + \qquad (14)$$

$$+ \frac{n(n+1)}{\lambda_{nq}} Z(++') K(10) \Big) \Big] - \Big[\longrightarrow ' \Big] \Big\}$$

where $C_{n''m''q''}$ is a normalization factor,

$$C_{n''m''q''} \doteq$$

$$\doteq \frac{\pi^2}{4} \frac{n''(n''+1)}{n''+1/2} \left(\mathcal{F}'_{n''}(\lambda_{n''q''}) \right)^2 \frac{(n''+m'')!}{(n''-m'')!} \cdot \frac{1}{\lambda^2_{n''q''}} \qquad (15)$$

The functions $\mathcal{F}_1 \{ {n \atop .q} \}$ and $\mathcal{F}_2 \{ {n \atop q} \}$ are numerically determined up to the index orders of 5. We have given an extract in Table 1a, 1b, 1c. The bracket symbolizes the dependency on the index triplets n, n', n'' and q, q', q''. The functions $\mathcal{F}$ stem from the integration on the triple multiplication of the Bessel functions.

The $Z(\quad)$ symbols are the results of the integration on the angular functions $\cos(m, m', m''\varphi), \sin(m, m', m''\varphi)$, and their values for the non-vanishing cases are given in Table 2a, 2b. Non-vanishing values are obtained only obtained if the indices satisfy the following possibilities

$$m - m' = m'' \quad \text{or} \quad m' - m = m'' \quad \text{or} \quad m + m' = m''$$

$$\text{or} \quad m'' = 0 \quad \text{or} \quad m = 0 \quad \text{or} \quad m' = 0 \qquad (16)$$

For the sake of brevity, in Table 2 a,b

$$\alpha \doteq \alpha_{nmq} \qquad \alpha' \doteq \alpha_{n'm'q'} + \gamma_{n'm'q'}$$

$$\beta \doteq \beta_{nmq} \qquad \beta' \doteq \beta_{n'm'q'} + \delta_{n'm'q'} \qquad (17)$$

The functions $K(i)$ are tabulated for the first indices in Table 4, stemming from the integration of the Legendre polynomials. $K(i)$-s can be calculated exactly using the formulae of Table 3. Since all numbers m, m', m'' are positive, condition (16) can be satisfied only in two cases. In eq. (14) it is easy to see that either the functions $K(1)$, $K(11)$ or $K(2)$, $K(10)$ are non zero depending upon the parity of the index sums x and y /see Table 3/.

Table 4 contains the first $K(i)$, $K(i')$ coefficients. Empty places have the value o. It is easy to see that a change in the indices $n \to n'$, $m \to m'$ and vice versa means a change in the columns in Table 4. The function $K(10)$ is always symmetric.

We calculated numerically the function $K(i)$ to the order $\begin{pmatrix} 5 5 5 \\ 5 5 5 \end{pmatrix}$. It is possible to see that all terms increase. So we need to investigate carefully the convergence of (14). The factor $C_{n''m''q''}$ increases linearly or factorially depending upon $n''+m''$, $n''-m''$, see (15). Therefore roughly speaking we can say that for a well chosen initial condition eq. (14) gives a good approximation.

First non trivial terms

We summarize the first nonlinear terms in Table 5. We limit ourselves to the index sums 8, that is, $n+m+n'+m'+q+q' < 8$. We wish first to investigate the cylindrically non symmetric cases, and we neglect the higher order field inversions, so $q = q' = q'' = 1$.
In our opinion this is not a great restriction.

In Table 5 we can see the symmetry of terms. As we go from $\alpha \to \beta$ the coefficients are the same, e.g. see the coefficients of the second terms in α_{211} and β_{211}. The terms β and δ with $m'' = 0$ are always zero, therefore some terms are absent in β_{111}. For example for α_{211} and β_{211} if we transit from α to β, the nonlinear terms change regularly the higher order functions $\beta \to \alpha$ and in the linear term / γ and δ are given functions/ $\alpha \to \beta$ or $\gamma \to \delta$.

In Fig. 3 we give a numerical solution of the autonomous part of the system of equations (5), i.e. without external field. Then we give as initial condition as a model, the values for $t = 0$

$$\begin{array}{ll} \alpha_{101} = 20 & \beta_{111} = 0 \\ \alpha_{111} = 20 & \alpha_{211} = 0 \\ \alpha_{201} = 20 & \beta_{211} = 0 \end{array} \qquad (18)$$

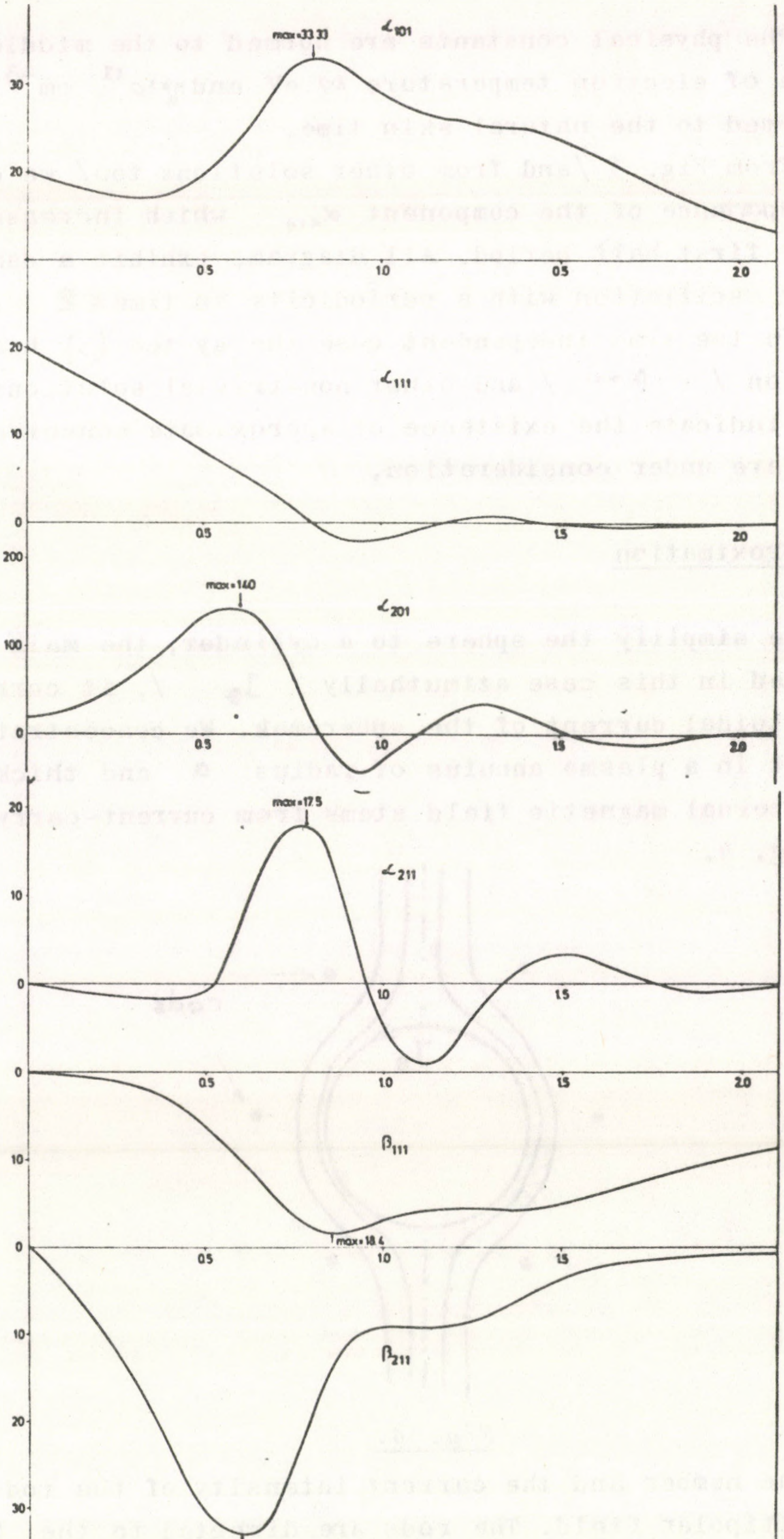

max=33.33
α101
α111
max=14.0
α201
max=17.5
α211
β111
max=18.4
β211
max=33

Fig. 3.

The physical constants are normed to the middle spheromak plasma of electron temperature 10 eV and $n_e \approx 10^{13}$ cm^{-3}, the time is normed to the natural skin time.

From Fig. 3 /and from other solutions too/ we can judge the importance of the component α_{111} which increases abruptly in the first half period. All diagrams exhibit a damped non-linear oscillation with a periodicity in time ≈ 2 .

In the time independent case the system (5) has a trivial solution / $\alpha = \beta = 0$ / and other non-trivial solutions too. These indicate the existence of approximate conservation laws, which are under consideration.

An approximation

We simplify the sphere to a cylinder, the main current is directed in this case azimuthally / J_θ /, it corresponds to the poloidal current of the spheromak. We concentrate this current in a plasma annulus of radius a and thickness δ . The external magnetic field stems from current-carrying rods, see Fig. 4.

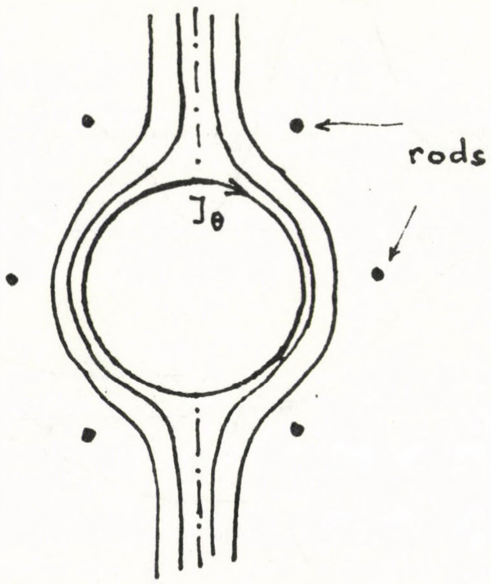


Fig. 4.

The number and the current intensity of the rods defines the multipolar field. The rods are directed to the z -axis.

The problem for a multipolar field, constant in time, was treated by Thonemann and Kolb [8], in a theta-pinch stability problem. If we apply the same approximation to our spheromak arrangement, we can obtain useful qualitative results.

The theta-pinch model supposes an azimuthal current in the plasma ring, it will be the poloidal current of the spheromak. The toroidal current of the spheromak will be the Hall-current of the Thonemann-Kolb model, it is directed parallel to the z -axis.

In the beginning the plasma is quiescent, $v \sim 0$. Thereafter if the density is not too low and the magnetic vield is not too high, the electrons - with the Hall-current - exert a rotation upon the magnetic field and in one period the internal magnetic field is separated from the external one, see e.g. [8],[9]. In this case the internal magnetic field exhibits a dipole or multipole structure. If the plasma is accelerated, the net current pushes out the field from the plasma interior.

We shall use this geometry for the rotating multipole problem. We use cylindrical coordinates / r, θ, z /, and the two-fluid MHD-equations are considered. For the ions the equation of motion yields, neglecting the convectional derivatives

$$n m_i \frac{d v_{i\theta}}{dt} = J_z (B_r + b_r) \tag{19}$$

where $v_i = v_\theta$, supposing $v_z = v_r \equiv 0$; $E_r = 0$

The electron equation is written in the form of the Ohm's low,

$$E_z = \eta J_z + \left(v_\theta - \frac{J_\theta}{ne}\right)(B_r + b_r)$$
$$E_\theta = \eta J_\theta + \frac{J_z}{ne}(B_r + b_r) \tag{20}$$

With B_r and B_θ we denote the given multipole field; b_r and b_θ stem from the Hall term. We suppose the multipole field having a given time dependence. For example, an ℓ -pole field

$$B_{r\ell} = B_{o\ell} \cos(\ell\theta - \omega_\ell t); \quad B_{\theta\ell} = -B_{o\ell} \sin(\ell\theta - \omega_\ell t) \tag{21}$$

For the sake of simplicity we assume a one-component ℓ -pole,

$$\omega_\ell = \omega, \quad B_{om} = 0, \text{ if } m \neq \ell \tag{22}$$

The plasma ring has a diameter a and a thickness δ . The azimuthal current J_θ causes a jump in the axial magnetic field

$$B_z^{int} + \mu_0 \delta J_\theta = B_z^{ext} \tag{23}$$

The induction law yields

$$\frac{1}{a}\frac{\partial E_z}{\partial \theta} = -\frac{\partial}{\partial t}(B_r + b_r) \tag{24}$$

We use same symbols as a ref. [8]. The axial magnetic field is assumed to be constant. From this relation the jump condition yields

$$E_\theta = -\frac{1}{2} a \frac{\partial B_z^{int}}{\partial t} = \frac{1}{2} a \delta \mu_0 \frac{\partial J_\theta}{\partial t} \tag{25}$$

The vacuum region is current free, so

$$\nabla \cdot \vec{b} = 0 \; ; \quad \nabla \times \vec{b} = 0 \tag{26}$$

$\vec{b}$ can be deduced from the Poisson-equation of a scalar function ϕ ,

$$\phi = \sum_{m=1}^{\infty} (C_1 r^m + C_2 r^{-m})(C_3 \cos m\theta + C_4 \sin m\theta) \tag{27}$$

The coefficient are time-dependent. The finiteness of ϕ imposes restrictions. For $r<a$ the induced radial field is

$$b_r = \sum m C_1 r^{m-1} (C_3 \cos m\theta + C_4 \sin m\theta) \tag{28}$$

When $r>a$:

$$b_r = \sum m C_1 r^{-m-1} a^{2m} (C_3 \cos m\theta + C_4 \sin m\theta) \tag{29}$$

The azimuthal component for $r<a$ is

$$b_\theta = -\sum m C_1 r^{m-1} (C_3 \sin m\theta - C_4 \cos m\theta) ; \tag{30}$$

and for $r > a$

$$b_\theta = \sum m C_1 a^{2m} r^{-m-1} \left(C_3 \sin m\theta - C_4 \cos m\theta \right) \qquad (31)$$

If we define α and β coefficients, the jumping condition yields

$$b_\theta^{ext} - b_\theta^{int} = \mu_0 \delta J_z \; ; \qquad r = a \qquad (32)$$

$$\alpha_m \doteq m C_1 C_3 a^{m-1} \; ; \quad \beta_m \doteq m C_1 C_4 a^{m-1} \qquad (33)$$

Thus

$$J_z = \frac{2}{\mu_0 \delta} \sum \left(\alpha_m \sin m\theta - \beta_m \cos m\theta \right) \qquad (34)$$

The average velocity in the θ -direction is obtained by integrating upon θ , so the equation of motion in an averaged form is

$$\dot{v}_\theta = \frac{1}{2\pi n m_i} \int_0^{2\pi} J_z \left(B_r + b_r \right) d\theta = - \frac{1}{n m_i \mu_0 \delta} \sum_l B_c \left(\beta_l \cos \omega t - \alpha_l \sin \omega t \right) \qquad (35)$$

For our simplified problem we treat only one l -value, so the sum over l can be omitted. The sum over m reduces only for the case $m = l$. So for α_l and β_l we get two equations from the induction law,

$$\frac{1}{a} \left\{ \frac{2\eta}{\mu_0 \delta} l \beta_l - l \left(B_0 \cos \omega t + \alpha_l \right) \left(v_\theta - \frac{J_\theta}{ne} \right) \right\} = - \dot{\beta}_l - B_0 \omega \cos \omega t \qquad (36)$$

$$\frac{1}{a} \left\{ \frac{2\eta}{\mu_0 \delta} l \alpha_l + l \left(B_0 \sin \omega t + \beta_l \right) \left(v_\theta - \frac{J_\theta}{ne} \right) \right\} = - \dot{\alpha}_l + B_0 \omega \sin \omega t \qquad (37)$$

These equations differ from the Thonemann-Kolb equations by the periodic forcing terms. If $\omega \to 0$ they are identical.

By a transformation in time and in amplitudes the equations /36/,/37/,/34/,/35/ simplify, and yield, respectively

$$\frac{dA}{dT} = l \left[-A + \left(\sin \Omega T + B \right) \left(V + \mathfrak{F} \right) R_3 \right] + \Omega \sin \Omega T \qquad (38)$$

$$\frac{dB}{dT} = \ell\left[-B-(\cos\Omega T + A)(V+\mathcal{J})R_3\right] - \Omega\cos\Omega T \tag{39}$$

$$\frac{d\mathcal{J}}{dT} = -\mathcal{J} + R_1\frac{dV}{dT} \tag{40}$$

$$\frac{dV}{dT} = R_2\left(B\cos\Omega T - A\sin\Omega T\right) \tag{41}$$

Here the time is normed to the natural decay time of the plasma ring, T_d

$$T_d = a\delta\mu_0/2\eta \;;\quad T = t/T_d \;;\quad \Omega = \omega T_d \tag{42}$$

and the other normed quantities are, according to [8],

$$R_3 = T_d/T_v = T_d/(ne\,a/J_0) \tag{43}$$

J_0 being the electroniccurrent density at $t=0$

$$R_1 = \frac{2m_i}{ne^2 a\delta\mu_0} \;;\quad R_2 = \frac{B_0^2\, e\, a\, T_d}{m_i\mu_0\delta\, J_0} \tag{44}$$

The A and B variable are normed to B_0

$$A = \alpha_\ell/B_0 \;;\quad B = \beta_\ell/B_0 \tag{45}$$

and V, $\mathcal{J}$ to J_0

$$V = -\frac{v_0}{J_0/ne} \tag{46}$$

$$\mathcal{J} = J_\theta/J_0$$

The system is solved for the theta pinch problem if $\Omega = 0$, numerically and in the small time approximation analytically in [8]. Some interesting features can be ascertained, viz.

1. The integrated current density till time $\sim T_d$ behaves as $\exp(-t/T_d)$, practically without quick oscillations.

2. A and B oscillate quickly in the small time $/t \ll T_d/$ region, the frequency decreases more or less exponentially.

The small time behaviour leads to a linear equation which is simple to solve. Later the influence of the decaying current begins to dominate the solutions and so the frequency is no longer constant.

We wish to solve the system of equations (38 – 41) for a more general case:

1. We retain the external pumping,

2. We simulate the decay of the current by an exponential function $\exp(-t/T_d)$

1. We treat a quiescent plasma, $V = 0$ and for small times $\mathcal{J} \approx 1$. A system now yields for A and B

$$\dot{A} = -l A + l R_3 B + l R_3 \sin \Omega T + \Omega \sin \Omega T \tag{47}$$

$$\dot{B} = -l B - l R_3 B - l R_3 \cos \Omega T - \Omega \cos \Omega T \tag{48}$$

This easy to solve for forced oscillations

$$\begin{Bmatrix} A \\ B \end{Bmatrix} = e^{-lt/T_d}\left[\begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix} \cos \omega_0 T + \begin{Bmatrix} A_2 \\ B_2 \end{Bmatrix} \sin \omega_0 T\right] + \begin{Bmatrix} -b_2 \\ b_1 \end{Bmatrix} \sin \Omega T + \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} \cos \Omega T \tag{49}$$

b_1 and b_2 are

$$b_1 = -\frac{(l^2 R_3^2 - \Omega^2)\left[l^2(1+R_3^2) - \Omega^2\right] + 2 l^2 \Omega (l R_3 + \Omega)}{(l^2(1+R_3^2) - \Omega^2)^2 + (2 l \Omega)^2} \tag{50}$$

$$b_2 = \frac{-l(l R_3 + \Omega)\left[l^2(1+R_3^2) - \Omega^2\right] + 2 l \Omega (l^2 R_3^2 - \Omega^2)}{(l^2(1+R_3^2) - \Omega^2)^2 + (2 l \Omega)^2} \tag{51}$$

If $\Omega \to 0$ we obtain

$$b_1 = -1 + \frac{1}{1 + R_3^2}$$
$$b_2 = R_3/(1+R_3^2) \tag{52}$$

as in [8].

In the case of the theta pinch $R_3 \approx 200$ so $b_1 \approx -1$ $b_2 \approx 0$. For the spheromak $R_3 = O(10)$ and $\Omega = O(1,10)$. The oscillation frequency is

$$\omega_0 = \ell\sqrt{1+R_3^2} \tag{53}$$

which is a fair approximation.

2. If we make into account the decay of the current J_θ,

$$J = J_0 e^{-t/T_d} \tag{54}$$

this form is valid for small times and small R_1 values. We get the following equations

$$\dot{A} = \ell(-A + (\sin\Omega T + B)J_0 R_3 e^{-T}) + \Omega\sin\Omega T \doteq -\ell A + gBe^{-T} + S \tag{55}$$

$$\dot{B} = \ell(-B - (\cos\Omega T + A)J_0 R_3 e^{-T}) - \Omega\cos\Omega T \doteq -\ell B - gAe^{-T} - C \tag{56}$$

for $V \approx 0$. Here

$$g = \ell J_0 R_3 \tag{57}$$

$$C = \ell R_3 J_0 e^{-T}\cos\Omega T + \Omega\cos\Omega T \tag{58}$$

$$S = \ell R_3 J_0 e^{-T}\sin\Omega T + \Omega\sin\Omega T \tag{59}$$

By substitution and derivations it is easy to get

$$\ddot{B} + (2\ell+1)\dot{B} + [\ell(\ell+1) + g^2 e^{-2T}]B = -\ell(\ell+1)C + \dot{C} - Sge^{-T} \doteq R \tag{60}$$

and a similar equation for A .

The homogeneous equation is a type of Bessel equation. First by substitution

$$B = e^{lT} u \tag{60}$$

giving

$$\ddot{u} + u\left(-\frac{1}{4} + g^2 e^{-2T}\right) = e^{(l+1/2)T} R \tag{61}$$

and by $e^{-T} \doteq x$ yields

$$x^2 \frac{d^2u}{dx^2} + x\frac{du}{dx} + (q^2x^2 - 1/4)u = \frac{1}{x^{l+1/2}} R \tag{62}$$

The solutions of the eq. (62) for the homogeneous case are

$$A_h = e^{-lT}\left(D_1 \cos(qe^{-T}) - D_2 \sin(qe^{-T})\right) \tag{63}$$

$$B_h = e^{-lT}\left(D_1 \sin(qe^{-T}) + D_2 \cos(qe^{-T})\right) \tag{64}$$

The solutions of the inhomogeneous equations are to be obtained by the variation of the constants D_1, D_2,

$$D_1 = D_{10} - \int e^{lT} \frac{d}{dT}\left(\cos(\Omega T - qe^{-T})\right) dT \tag{65}$$

$$D_2 = D_{20} - \int e^{lT} \frac{d}{dT}\left(\sin(\Omega T - qe^{-T})\right) dT \tag{66}$$

So we get an integrated form for A and B .

In the small time limit we can expand the exponential into series. Taking into account only the linear terms we can execute the integration and it yields

$$A = e^{-lt/Td}\left(D_{10}\cos(qe^{-t/Td}) - D_{20}\sin(qe^{-t/Td})\right) - \tag{67}$$

$$- \frac{\Omega + q}{\sqrt{l^2 + (\Omega + q)^2}} \cos(\omega t - \varphi)$$

$$B = e^{-\ell t/T_d}\left(D_{10}\sin\left(g e^{-t/T_d}\right) + D_{20}\cos\left(g e^{-t/T_d}\right)\right) - \frac{\Omega + g}{\sqrt{\ell^2 + (\Omega^2+g^2)^2}}\sin(\omega t - \varphi) \tag{68}$$

$$\sin\varphi = \ell \Big/ \sqrt{\ell^2 + (\Omega^2+g)^2} \tag{69}$$

So the exponential term has little influence on the stationary amplitudes A and B.

The D_{10} and D_{20} constants are integration constants, the functions $\sin$ and $\cos$ vary with the frequency of

$$t^{-1} g e^{-t/T_d} \tag{70}$$

in accordance with the analysis of [8].

The results in the approximation $\Omega \to 0$ are given below. It is interesting to give it, because these are the results of [8] in the time interval $t \ll T_d$. Then $t/T_d \ll 1$, we can integrate the system

$$D_1 \cong D_{10} + \int g e^{(\ell-1)T} \sin g e^{-T} \, dT \tag{71}$$

$$D_2 \cong D_{20} + \int g e^{(\ell-1)T} \cos g e^{-T} \, dT \tag{72}$$

By the substitution $g e^{-T} \doteq x$

$$D_1 = D_{10} + \int g^{\ell} x^{-\ell} \sin x \, dx \tag{73}$$

$$D_2 = D_{20} + \int g^{\ell} x^{-\ell} \cos x \, dx \tag{74}$$

So we get a type of exponential integrals Si, Ci. The difficulty is in the variation of the arguments for $T \approx 0$, there the curves of [8] are not too clear. So we chose the time as x_1, the solutions are

$$A = \frac{1}{2}x^3\left\{\left(2\frac{B_1}{x_1^3} + \frac{1}{x_1^2}\right)\sin(x_1 - x) + \left(2\frac{A_1}{x_1^3} + \frac{1}{x_1}\right)\cos(x - x_1) - (Ci\,x_1 - Ci\,x)\sin x + (Si\,x_1 - Si\,x)\cos x - 1/x\right\} \tag{75}$$

$$B = \frac{1}{2} x^3 \left\{ \left(\frac{2B_1}{x_1^3} + \frac{1}{x_1^2} \right) \cos(x - x_1) + \left(2\frac{A_1}{x_1^3} + \frac{1}{x_1} \right) \sin(x - x_1) + \right.$$

$$\left. + (\mathrm{Si}\,x_1 - \mathrm{Si}\,x) \sin x + (\mathrm{Ci}\,x_1 - \mathrm{Ci}\,x) \cos x - \frac{1}{x^2} \right\} \qquad (76)$$

Here A_1 is A at time x_1, B_1 similarly.

Calculation of the integrals yields the curves in Fig. 5a and Fig. 5b. The tendency is the same as in the numerical results. The initial conditions are

$$T = 1 \qquad A = -1, \quad B = 0{,}05 \qquad (77)$$

We chose these so, that we could reproduce [8]. A and B decay with an average frequency 400 in T_d -units, for $R_2 = 170$ $l = 3$. The average of B tends to small negative values with increasing time /see Fig. 5b/.

By this control calculation we can say that in the small time limit the simplifying suppositions $V \sim 0$ and $J \sim \exp$ are valid, and the results are not too bad.

The results of the spheromak calculations and of the simplified model are in qualitative accordance. The coupling constant for the spheromak case is

$$g = O(10\,l) \qquad (78)$$

for a temperature 10^5 °K and a density 10^{13} cm^{-3}, using standard resistivity. So the average frequency is in order of unity. This frequency is produced by the numerical integration of the system (5), without external fields and the initial conditions (18). Naturally the geometry is different and the components A and B are different from α_{nmq} and β_{nmq} . But the same order of frequency motivates that the frequency first of all depends upon physical constants and not upon the initial conditions or geometry. On the other hand the complexity of the system (5) complicates the results too, so it seems necessary to solve the system for more cases.

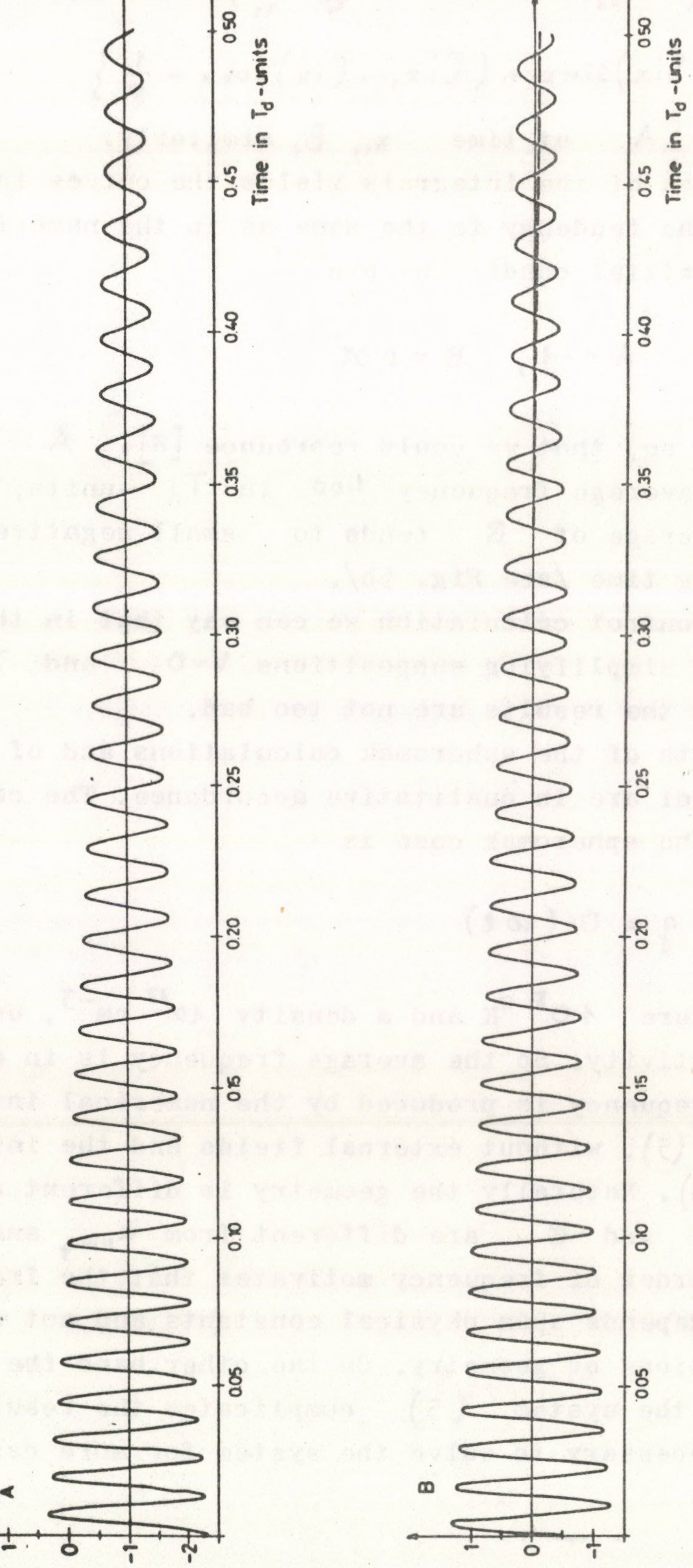


Fig. 5a.

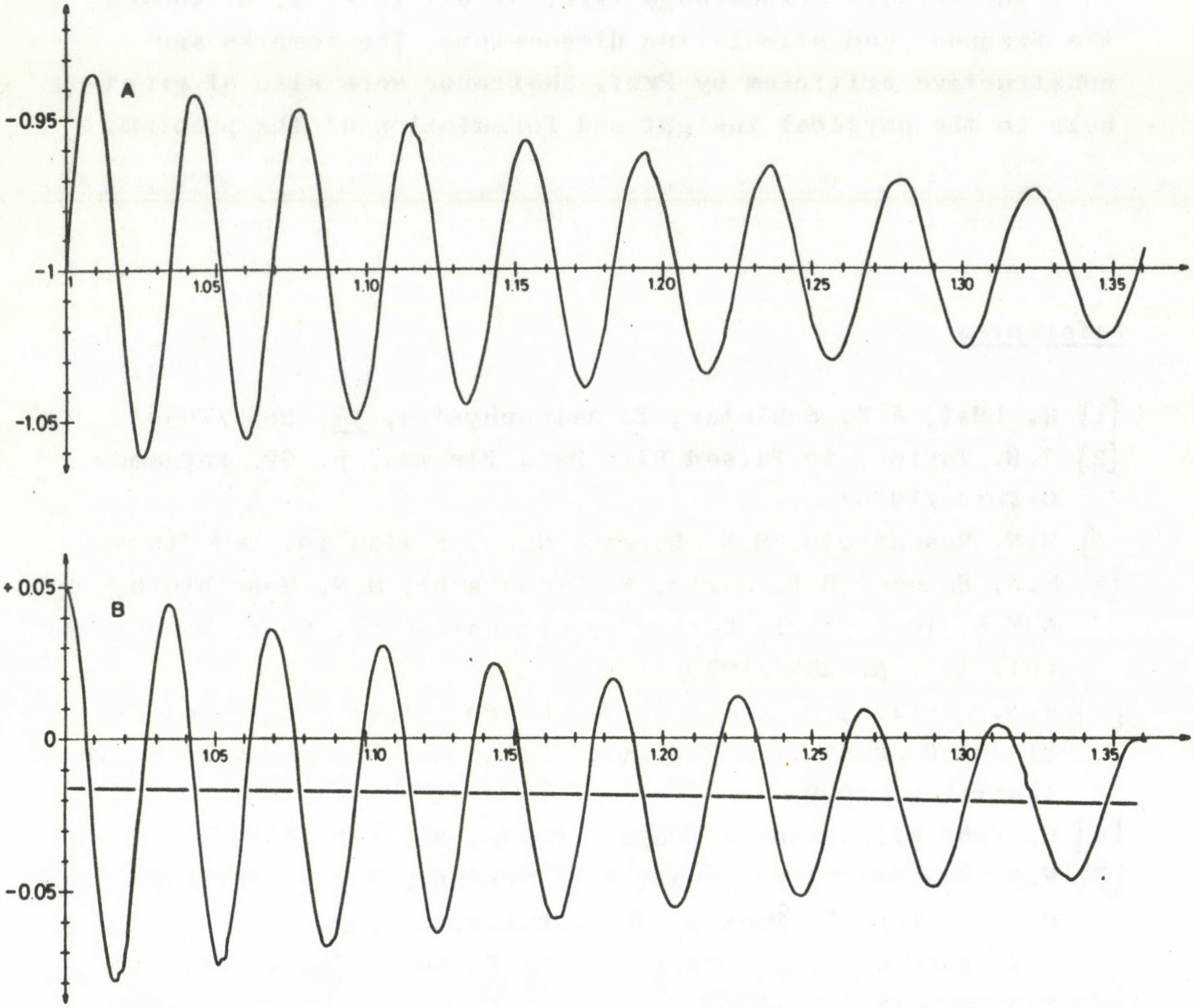

A
-0.95
-1
-1.05
1
1.05
1.10
1.15
1.20
1.25
1.30
1.35
B
+0.05
0
-0.05
1.05
1.10
1.15
1.20
1.25
1.30
1.35

Fig. 5b.

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I	J	K	L	M	N	J1(N,Q)	J2(N,Q)	J3(N,Q)
1	1	1	1	1	1	0.15996E-01	-0.29474E-02	-0.20474E-02
1	1	1	1	1	2	0.42476E-02	0.18454E-02	0.18454E-02
1	1	1	1	1	3	-0.18824E-03	0.37953E-03	0.37953E-03
1	1	1	1	2	1	0.42476E-02	-0.69016E-02	0.18454E-02
1	1	1	1	2	2	0.64186E-02	-0.23972E-02	-0.24979E-03
1	1	1	1	2	3	0.19431E-02	0.10462E-02	0.96306E-03
1	1	1	1	3	1	-0.18824E-03	-0.18930E-02	0.37953E-03
1	1	1	1	3	2	0.19431E-02	-0.36855E-02	0.76306E-03
1	1	1	1	3	3	0.33507E-02	-0.12702E-02	-0.61028E-05
1	1	1	2	1	1	0.42476E-02	0.18454E-02	-0.69016E-02
1	1	1	2	1	2	0.64186E-02	-0.24979E-03	-0.23972E-02
1	1	1	2	1	3	0.19431E-02	0.96306E-03	0.10462E-02
1	1	1	2	2	1	0.64186E-02	-0.23972E-02	-0.23972E-02
1	1	1	2	2	2	0.47587E-02	-0.20284E-02	-0.20284E-02
1	1	1	2	2	3	0.34289E-02	-0.54844E-03	-0.54844E-03
1	1	1	2	3	1	0.19431E-02	-0.3685[illegible]E-02	0.10462E-02
1	1	1	2	3	2	0.34289E-02	-0.26811E-02	-0.54844E-03
1	1	1	2	3	3	0.28120E-02	-0.14567E-02	-0.84810E-03
1	1	1	3	1	1	-0.18824E-03	0.37953E-03	-0.18930E-02
1	1	1	3	1	2	0.19431E-02	0.96306E-03	-0.36855E-02
1	1	1	3	1	3	0.33507E-02	-0.61028E-05	-0.12702E-02
1	1	1	3	2	1	0.19431E-02	0.10462E-02	-0.36855E-02
1	1	1	3	2	2	0.34299E-02	-0.54844E-03	-0.26811E-02
1	1	1	3	2	3	0.28120E-02	-0.84810E-03	-0.14567E-02
1	1	1	3	3	1	0.33507E-02	-0.12702E-02	-0.12702E-02
1	1	1	3	3	2	0.28120E-02	-0.14567E-02	-0.14567E-02
1	1	1	3	3	3	0.25227E-02	-0.11932E-02	-0.11932E-02

I	J	K	L	M	N	J1(N,Q)	J2(N,Q)	J3(N,Q)
1	1	2	1	1	1	0.10891E-01	-0.26812E-02	-0.26812E-02
1	1	2	1	1	2	0.48675E-02	0.10728E-02	0.10728E-02
1	1	2	1	1	3	0.84343E-03	0.50492E-03	0.50492E-03
1	1	2	1	2	1	0.14877E-02	-0.45993E-02	0.14013E-02
1	1	2	1	2	2	0.48439E-02	-0.28822E-02	-0.35953E-03
1	1	2	1	2	3	0.26494E-02	0.20532E-03	0.69402E-03
1	1	2	1	3	1	-0.77489E-03	-0.45376E-03	0.30516E-04
1	1	2	1	3	2	0.36656E-03	-0.27283E-02	0.76105E-03
1	1	2	1	3	3	0.25622E-02	-0.18946E-02	-0.76637E-04
1	1	2	2	1	1	0.14877E-02	0.14013E-02	-0.45993E-02
1	1	2	2	1	2	0.48439E-02	-0.35953E-03	-0.28822E-02
1	1	2	2	1	3	0.26404E-02	0.69402E-03	0.20532E-03
1	1	2	2	2	1	0.34527E-02	-0.12734E-02	-0.12734E-02
1	1	2	2	2	2	0.31751E-02	-0.16908E-02	-0.16908E-02
1	1	2	2	2	3	0.32409E-02	-0.10581E-02	-0.10581E-02
1	1	2	2	3	1	0.44364E-03	-0.20248E-02	0.95114E-03
1	1	2	2	3	2	0.18394E-02	-0.19839E-02	-0.66222E-04
1	1	2	2	3	3	0.20129E-02	-0.16050E-02	-0.70179E-03
1	1	2	3	1	1	-0.77489E-03	0.30516E-04	-0.45376E-03
1	1	2	3	1	2	0.36656E-03	0.76105E-03	-0.27283E-02
1	1	2	3	1	3	0.25622E-02	-0.76637E-04	-0.18946E-02
1	1	2	3	2	1	0.44364E-03	0.95114E-03	-0.20248E-02
1	1	2	3	2	2	0.18394E-02	-0.66222E-04	-0.19839E-02
1	1	2	3	2	3	0.20129E-02	-0.70179E-03	-0.16050E-02
1	1	2	3	3	1	0.16567E-02	-0.52759E-03	-0.52759E-03
1	1	2	3	3	2	0.15555E-02	-0.84421E-03	-0.84421E-03
1	1	2	3	3	3	0.17052E-02	-0.96148E-03	-0.96148E-03

Table 1a

I	J	K	L	M	N	J1(N,Q)	J2(N,Q)	J3(N,Q)
1	2	1	1	1	1	0.10891E-01	-0.10216E-02	-0.26812E-02
1	2	1	1	1	2	0.14877E-02	0.13715E-02	0.14013E-02
1	2	1	1	1	3	0.77489E-03	-0.41072E-04	0.30516E-04
1	2	1	1	2	1	0.48675E-02	-0.49721E-02	0.10728E-02
1	2	1	1	2	2	0.48439E-02	-0.85410E-03	-0.35953E-03
1	2	1	1	2	3	0.36656E-03	0.11080E-02	0.76105E-03
1	2	1	1	3	1	0.84343E-03	-0.21517E-02	0.50492E-03
1	2	1	1	3	2	0.26494E-02	-0.28025E-02	0.69402E-03
1	2	1	1	3	3	0.25622E-02	-0.22006E-03	-0.76637E-04
1	2	1	2	1	1	0.14877E-02	0.13715E-02	-0.45993E-02
1	2	1	2	1	2	0.34527E-02	-0.70184E-04	-0.12734E-02
1	2	1	2	1	3	0.44364E-03	0.55318E-03	0.95114E-03
1	2	1	2	2	1	0.48439E-02	-0.85410E-03	-0.28822E-02
1	2	1	2	2	2	0.31751E-02	-0.93450E-03	-0.16908E-02
1	2	1	2	2	3	0.18394E-02	-0.44780E-04	-0.66222E-04
1	2	1	2	3	1	0.26494E-02	-0.28025E-02	0.20532E-03
1	2	1	2	3	2	0.32409E-02	-0.16117E-02	-0.10581E-02
1	2	1	2	3	3	0.20129E-02	-0.62432E-03	-0.70179E-03
1	2	1	3	1	1	0.77489E-03	-0.41072E-04	-0.45376E-03
1	2	1	3	1	2	0.44364E-03	0.55318E-03	-0.20248E-02
1	2	1	3	1	3	0.16567E-02	-0.27955E-04	-0.52759E-03
1	2	1	3	2	1	0.36656E-03	0.11080E-02	-0.27283E-02
1	2	1	3	2	2	0.18394E-02	-0.44780E-04	-0.19839E-02
1	2	1	3	2	3	0.15555E-02	-0.39696E-03	-0.84421E-03
1	2	1	3	3	1	0.25622E-02	-0.22006E-03	-0.18946E-02
1	2	1	3	3	2	0.20129E-02	-0.62432E-03	-0.16050E-02
1	2	1	3	3	3	0.17052E-02	-0.59481E-03	-0.96148E-03

I	J	K	L	M	N	J1(N,Q)	J2(N,Q)	J3(N,Q)
1	2	2	1	1	1	0.77937E-02	-0.11054E-02	-0.23558E-02
1	2	2	1	1	2	0.24156E-02	0.10135E-02	0.86419E-03
1	2	2	1	1	3	-0.13040E-03	0.17601E-03	0.19847E-03
1	2	2	1	2	1	0.24156E-02	-0.35425E-02	0.86419E-03
1	2	2	1	2	2	0.41963E-02	-0.14864E-02	-0.43306E-03
1	2	2	1	2	3	0.13509E-02	0.63536E-03	0.59550E-03
1	2	2	1	3	1	-0.13040E-03	-0.90950E-03	0.19847E-03
1	2	2	1	3	2	0.13509E-02	-0.23961E-02	0.59550E-03
1	2	2	1	3	3	0.24810E-02	-0.96360E-03	-0.10755E-03
1	2	2	2	1	1	0.23451E-03	0.10228E-02	-0.31533E-02
1	2	2	2	1	2	0.27550E-02	-0.10816E-03	-0.17369E-02
1	2	2	2	1	3	0.10742E-02	0.48571E-03	0.44099E-03
1	2	2	2	2	1	0.27550E-02	-0.41056E-03	-0.17369E-02
1	2	2	2	2	2	0.23669E-02	-0.83512E-03	-0.16171E-02
1	2	2	2	2	3	0.21015E-02	-0.39392E-03	-0.60981E-03
1	2	2	2	3	1	0.10742E-02	-0.16718E-02	0.44099E-03
1	2	2	2	3	2	0.21015E-02	-0.13502E-02	-0.60981E-03
1	2	2	2	3	3	0.18150E-02	-0.91018E-03	-0.78859E-03
1	2	2	3	1	1	-0.75441E-03	-0.17338E-03	0.17894E-03
1	2	2	3	1	2	-0.26179E-03	0.42776E-03	-0.15137E-02
1	2	2	3	1	3	0.13229E-02	-0.45976E-04	-0.97375E-03
1	2	2	3	2	1	-0.26179E-03	0.85179E-03	-0.15137E-02
1	2	2	3	2	2	0.93410E-03	0.18507E-03	-0.15644E-02
1	2	2	3	2	3	0.11972E-02	-0.32435E-03	-0.11222E-02
1	2	2	3	3	1	0.13229E-02	0.17089E-04	-0.97375E-03
1	2	2	3	3	2	0.11972E-02	-0.31319E-03	-0.11222E-02
1	2	2	3	3	3	0.12996E-02	-0.49984E-03	-0.97092E-03

Table 1b

I	J	K	L	M	N	J1(N,Q)	J2(N,Q)	J3(N,Q)
2	1	1	1	1	1	0.10891E-01	-0.26812E-02	-0.10216E-02
2	1	1	1	1	2	0.14877E-02	0.14017E-02	0.13715E-02
2	1	1	1	1	3	-0.77489E-03	0.30516E-04	-0.41072E-04
2	1	1	1	2	1	0.14877E-02	-0.45993E-02	0.13715E-02
2	1	1	1	2	2	0.34527E-02	-0.12734E-02	-0.70185E-04
2	1	1	1	2	3	0.44364E-03	0.95114E-03	0.55319E-03
2	1	1	1	3	1	-0.77489E-03	-0.45376E-03	-0.41072E-04
2	1	1	1	3	2	0.44364E-03	-0.20248E-02	0.55318E-03
2	1	1	1	3	3	0.16567E-02	-0.52759E-03	-0.27955E-04
2	1	1	2	1	1	0.48675E-02	0.10728E-02	-0.49721E-02
2	1	1	2	1	2	0.48439E-02	-0.35953E-03	-0.85410E-03
2	1	1	2	1	3	0.36656E-03	0.76105E-03	0.11080E-02
2	1	1	2	2	1	0.45489E-02	-0.28822E-02	-0.85410E-03
2	1	1	2	2	2	0.31751E-02	-0.16908E-02	-0.93450E-03
2	1	1	2	2	3	0.18394E-02	-0.66222E-04	-0.44780E-04
2	1	1	2	3	1	0.36656E-03	-0.27283E-02	0.11080E-02
2	1	1	2	3	2	0.18394E-02	-0.19839E-02	-0.44780E-04
2	1	1	2	3	3	0.15555E-02	-0.84421E-03	-0.39696E-03
2	1	1	3	1	1	0.84343E-03	0.50492E-03	-0.21517E-02
2	1	1	3	1	2	0.26494E-02	0.69402E-03	-0.28025E-02
2	1	1	3	1	3	0.25622E-02	-0.76637E-04	-0.22006E-03
2	1	1	3	2	1	0.26494E-02	0.20532E-03	-0.28025E-02
2	1	1	3	2	2	0.32409E-02	-0.10581E-02	-0.16117E-02
2	1	1	3	2	3	0.20129E-02	-0.70179E-03	-0.62432E-03
2	1	1	3	3	1	0.25622E-02	-0.18946E-02	-0.22006E-03
2	1	1	3	3	2	0.20129E-02	-0.16050E-02	-0.62432E-03
2	1	1	3	3	3	0.17052E-02	-0.96148E-03	-0.59481E-03

I	J	K	L	M	N	J1(N,Q)	J2(N,Q)	J3(N,Q)
2	1	2	1	1	1	0.77937E-02	-0.23558E-02	-0.11054E-02
2	1	2	1	1	2	0.24156E-02	0.86419E-03	0.10135E-02
2	1	2	1	1	3	-0.13040E-03	0.19847E-03	0.17601E-03
2	1	2	1	2	1	0.23451E-03	-0.31533E-02	0.10228E-02
2	1	2	1	2	2	0.27550E-02	-0.17369E-02	-0.10816E-03
2	1	2	1	2	3	0.10742E-02	0.44099E-03	0.48571E-03
2	1	2	1	3	1	-0.75441E-03	0.17894E-03	-0.17338E-03
2	1	2	1	3	2	-0.26179E-03	-0.15137E-02	0.42776E-03
2	1	2	1	3	3	0.13229E-02	-0.97375E-03	-0.45976E-04
2	1	2	2	1	1	0.24156E-02	0.86419E-03	-0.35423E-02
2	1	2	2	1	2	0.41963E-02	-0.43306E-03	-0.14864E-02
2	1	2	2	1	3	0.13509E-02	0.59550E-03	0.63536E-03
2	1	2	2	2	1	0.27550E-02	-0.17369E-02	-0.41056E-03
2	1	2	2	2	2	0.23669E-02	-0.16171E-02	-0.83512E-03
2	1	2	2	2	3	0.21015E-02	-0.60981E-03	-0.39392E-03
2	1	2	2	3	1	-0.26179E-03	-0.15137E-02	0.85179E-03
2	1	2	2	3	2	0.93410E-03	-0.15644E-02	0.18507E-03
2	1	2	2	3	3	0.11972E-02	-0.11222E-02	-0.32435E-03
2	1	2	3	1	1	-0.13040E-03	0.19847E-03	-0.90950E-03
2	1	2	3	1	2	0.13509E-02	0.59550E-03	-0.23961E-02
2	1	2	3	1	3	0.24810E-02	-0.10755E-03	-0.96360E-03
2	1	2	3	2	1	0.10742E-02	0.44099E-03	-0.16718E-02
2	1	2	3	2	2	0.21015E-02	-0.60981E-03	-0.13502E-02
2	1	2	3	2	3	0.18150E-02	-0.78859E-03	-0.91018E-03
2	1	2	3	3	1	0.13229E-02	-0.97375E-03	0.17089E-04
2	1	2	3	3	2	0.11972E-02	-0.11222E-02	-0.31319E-03
2	1	2	3	3	3	0.12996E-02	-0.97097E-03	-0.49984E-03

Table 1c

$$\frac{\pi}{2}\cdot Z(\ \ )$$

$$\left[\int_0^{2\pi} d\varphi \cos\varphi\right] \times$$

	1 $m-m'=m''$	2 $m'-m=m''$	3 $m+m'=m''$
$(-)(+')$	$\alpha\beta'\frac{\pi}{2}-\beta\alpha'\frac{\pi}{2}$	$\alpha\beta'\frac{\pi}{2}-\beta\alpha'\frac{\pi}{2}$	$\alpha\beta'\frac{\pi}{2}+\beta\alpha'\frac{\pi}{2}$
$(-)(-')$	$\alpha\alpha'\frac{\pi}{2}+\beta\beta'\frac{\pi}{2}$	$\alpha\alpha'\frac{\pi}{2}+\beta\beta'\frac{\pi}{2}$	$\alpha\alpha'\frac{\pi}{2}-\beta\beta'\frac{\pi}{2}$
$(+)(+')$	$\beta\beta'\frac{\pi}{2}+\alpha\alpha'\frac{\pi}{2}$	$\beta\beta'\frac{\pi}{2}+\alpha\alpha'\frac{\pi}{2}$	$\beta\beta'\frac{\pi}{2}-\alpha\alpha'\frac{\pi}{2}$
$(+)(-')$	$\beta\alpha'\frac{\pi}{2}-\alpha\beta'\frac{\pi}{2}$	$\beta\alpha'\frac{\pi}{2}-\alpha\beta'\frac{\pi}{2}$	$\beta\alpha'\frac{\pi}{2}+\alpha\beta'\frac{\pi}{2}$
$(-')(+)$	$\beta\alpha'\frac{\pi}{2}-\alpha\beta'\frac{\pi}{2}$	$\beta\alpha'\frac{\pi}{2}-\alpha\beta'\frac{\pi}{2}$	$\beta\alpha'\frac{\pi}{2}+\alpha\beta'\frac{\pi}{2}$
$(-')(-)$	$\alpha\alpha'\frac{\pi}{2}+\beta\beta'\frac{\pi}{2}$	$\alpha\alpha'\frac{\pi}{2}+\beta\beta'\frac{\pi}{2}$	$\alpha\alpha'\frac{\pi}{2}-\beta\beta'\frac{\pi}{2}$
$(+')(+)$	$\beta\beta'\frac{\pi}{2}+\alpha\alpha'\frac{\pi}{2}$	$\beta\beta'\frac{\pi}{2}+\alpha\alpha'\frac{\pi}{2}$	$\beta\beta'\frac{\pi}{2}-\alpha\alpha'\frac{\pi}{2}$
$(+')(-)$	$\beta\alpha'\frac{\pi}{2}-\alpha\beta'\frac{\pi}{2}$	$\beta\alpha'\frac{\pi}{2}-\alpha\beta'\frac{\pi}{2}$	$\beta\alpha'\frac{\pi}{2}+\alpha\beta'\frac{\pi}{2}$

Table 2a

$$\frac{\pi}{2} \cdot Z(\ )$$

$$\cdot \left[\int_0^{2\pi} d\varphi \sin\varphi\right] \times$$

	1 $m - m' = m''$	2 $m' - m = m''$	3 $m + m' = m''$
$(-)(+')$	$-\alpha\alpha'\frac{\pi}{2} - \beta\beta'\frac{\pi}{2}$	$\alpha\alpha'\frac{\pi}{2} + \beta\beta'\frac{\pi}{2}$	$\alpha\alpha'\frac{\pi}{2} - \beta\beta'\frac{\pi}{2}$
$(-)(-')$	$+\alpha\beta'\frac{\pi}{2} - \beta\alpha'\frac{\pi}{2}$	$-\alpha\beta'\frac{\pi}{2} + \beta\alpha'\frac{\pi}{2}$	$-\alpha\beta\frac{\pi}{2} - \beta\alpha'\frac{\pi}{2}$
$(+)(+')$	$+\alpha\beta'\frac{\pi}{2} - \beta\alpha'\frac{\pi}{2}$	$-\alpha\beta'\frac{\pi}{2} + \beta\alpha'\frac{\pi}{2}$	$\alpha\beta'\frac{\pi}{2} + \beta\alpha'\frac{\pi}{2}$
$(+)(-')$	$\alpha\alpha'\frac{\pi}{2} + \beta\beta'\frac{\pi}{2}$	$-\alpha\alpha'\frac{\pi}{2} - \beta\beta'\frac{\pi}{2}$	$\alpha\alpha'\frac{\pi}{2} - \beta\beta'\frac{\pi}{2}$
$(-')(+)$	$+\alpha\alpha\frac{\pi}{2} + \beta\beta'\frac{\pi}{2}$	$-\alpha\alpha'\frac{\pi}{2} - \beta\beta'\frac{\pi}{2}$	$\alpha\alpha'\frac{\pi}{2} - \beta\beta'\frac{\pi}{2}$
$(-')(-)$	$\alpha\beta'\frac{\pi}{2} - \beta\alpha'\frac{\pi}{2}$	$-\alpha\beta'\frac{\pi}{2} + \alpha'\beta\frac{\pi}{2}$	$-\alpha\beta'\frac{\pi}{2} - \alpha'\beta\frac{\pi}{2}$
$(+')(+)$	$-\alpha'\beta\frac{\pi}{2} + \alpha\beta'\frac{\pi}{2}$	$\alpha'\beta\frac{\pi}{2} - \alpha\beta'\frac{\pi}{2}$	$\alpha'\beta\frac{\pi}{2} + \alpha\beta'\frac{\pi}{2}$
$(+')(-)$	$-\alpha\alpha'\frac{\pi}{2} - \beta\beta'\frac{\pi}{2}$	$\alpha\alpha'\frac{\pi}{2} + \beta\beta'\frac{\pi}{2}$	$\alpha\alpha'\frac{\pi}{2} - \beta\beta'\frac{\pi}{2}$

Table 2b

$$K(1) = -\sum_{k,k',k''} F\left(\left\{ {}^{n}{}_{k}{}^{m} \right\}\right) \cdot \left[(m+1)\mathring{\Pi}(x-1,y+1) - (n-m-2k)\mathring{\Pi}(x+1,y-1) \right] \cdot n(n+1) \cdot m'$$

$$K(2) = -\sum_{k,k',k''} F\left(\left\{ {}^{n}{}_{k}{}^{m} \right\}\right) \left[(m+1)m'\mathring{\Pi}(x-1,y+2) - m'(n-m-2k)\mathring{\Pi}(x+1,y) - \right.$$
$$\left. -(m+1)(n'-m'-2k')\mathring{\Pi}(x+1,y) + (n-m-2k)(n'-m'-2k')\mathring{\Pi}(x+3,y-2) \right] n(n+1) -$$
$$- n(n+1)n'(n'+1)\left[\mathring{\Pi}(x+1,y) \right]$$

$$K(10) = -\sum_{k,k',k''} F\left(\left\{ {}^{n}{}_{k}{}^{m} \right\}\right) \cdot \left[\mathring{\Pi}(x-1,y) \right] m\, m'$$

$$K(11) = -\sum_{k,k',k''} F\left(\left\{ {}^{n}{}_{k}{}^{m} \right\}\right) \cdot \left[m'\mathring{\Pi}(x-1,y+1) - (n'-m'-2k')\mathring{\Pi}(x+1,y-1) \right] m\, n(n+1)$$

$K(i) \longrightarrow K(i')$ i.e. indices $(n,m,k) \longrightarrow (n',m',k')$

$$F\left(\left\{ {}^{nm}_{k} \right\}\right) \doteq \binom{nm}{k} \cdot \binom{n'm'}{k'} \binom{n''m''}{k''} \; ; \qquad \binom{nm}{k} \doteq (-1)^{m+k} \frac{(2n-(2k+1))!!}{(2k)!\,(n-m-2k)!}$$

$$\text{Summation: } \sum_{k,k',k''} \doteq \sum_{k=0}^{E\left(\frac{n-m}{2}\right)} \sum_{k'=0}^{E\left(\frac{n'-m'}{2}\right)} \sum_{k''=0}^{E\left(\frac{n''-m''}{2}\right)} \; ; \quad E(x) \doteq \text{integer part of } x$$

$$\mathring{\Pi}(x,y) \doteq \frac{2^{\frac{x+1}{2}} \left(\frac{x-1}{2}\right)!\,(y-1)!!}{(x+y)!!} \; ; \quad \text{if } x \text{ is odd and } y \text{ is even, } x,y>0$$
$$\text{otherwise } \mathring{\Pi}(x,y) = 0$$

$$x \doteq m+m'+m'' \; , \; y \doteq n+n'+n''-(m+m'+m'')-2(k+k'+k'')$$

Table 3

$n\,n'\,n''$ $m\,m'\,m''$	K(1)	K(1')	K(2)	K(2')	K(10)	K(11)	K(11')
111 000							
111 110	$-\frac{8}{3}$	$-\frac{8}{3}$				$\frac{4}{3}$	$\frac{4}{3}$
111 011	$-\frac{4}{3}$						
111 101		$-\frac{4}{3}$					
211 000				-4			
121 000			-4				
211 010				-8	2		
121 110			-8		2		
211 011			-8	-8			
121 101			-8	-8			
121 011			$-\frac{52}{5}$	$-\frac{96}{5}$			
211 101			$-\frac{96}{5}$	$-\frac{52}{5}$			
221 000							
112 011			$-\frac{12}{5}$	$-\frac{32}{5}$			
112 101			$-\frac{32}{5}$	$-\frac{12}{5}$			
122 101		$\frac{48}{5}$				$-\frac{24}{5}$	
122 011	$-\frac{12}{5}$						$-\frac{72}{5}$
212 011	$\frac{48}{5}$						$-\frac{24}{5}$
212 101		$-\frac{12}{5}$				$-\frac{72}{5}$	

Table 4

α_{101}	$1.45450\,(\beta_{111}\gamma_{111} - \alpha_{111}\delta_{111}) - 1.31281\times 10^{-2}\,\alpha_{101}\alpha_{201} + 4.64465\times 10^{-2}\,\gamma_{201}\alpha_{101} - 5.95748\,\alpha_{201}\gamma_{101} -$ $- 4.25273\times 10^{-1}(\alpha_{211}\gamma_{111} + \beta_{211}\delta_{111}) - 9.37145\times 10^{-2}(\alpha_{111}\alpha_{211} + \beta_{111}\beta_{211}) + 3.31558\times 10^{-1}(\alpha_{111}\gamma_{211} + \beta_{111}\delta_{211})$
α_{201}	$1.11721\,(\beta_{211}\gamma_{111} - \alpha_{211}\delta_{111}) + 2.45633\,(\alpha_{111}\beta_{211} - \alpha_{211}\beta_{111}) - 8.71012\times 10^{-1}(\alpha_{111}\delta_{211} - \beta_{111}\gamma_{211})$
α_{111}	$-1.21208\times 10^{-1}(\alpha_{101}\delta_{111} - \beta_{111}\gamma_{101}) + 6.23246\times 10^{-2}\,\alpha_{201}(\alpha_{111} + \gamma_{111}) - 4.85905\times 10^{-2}\,\alpha_{111}(\alpha_{201} + \gamma_{201}) +$ $+ 2.15111\times 10^{-1}\,\alpha_{211}(\alpha_{101} + \gamma_{101}) - 1.67708\times 10^{-1}\,\alpha_{101}(\alpha_{211} + \gamma_{211})$
β_{111}	$-6.23246\times 10^{-2}\,\alpha_{201}(\beta_{111} + \delta_{111}) + 4.859049\times 10^{-2}\,\beta_{111}(\alpha_{201} + \gamma_{201}) - 2.151101\times 10^{-1}\,\beta_{211}(\alpha_{101} + \gamma_{101}) +$ $+ 1.67708\times 10^{-1}\,\alpha_{101}(\beta_{211} + \delta_{211})$
α_{211}	$-4.985756\times 10^{-2}(\alpha_{101}\gamma_{111} - \alpha_{111}\gamma_{101}) + 3.198065\times 10^{-1}\,\alpha_{101}(\beta_{211} + \delta_{211}) - 4.101999\times 10^{-1}\,\beta_{211}(\alpha_{101} + \gamma_{101}) -$ $- 4.450845\times 10^{-1}\,\beta_{111}(\alpha_{201} + \gamma_{201}) + 5.70888\times 10^{-1}\,\alpha_{201}(\beta_{111} + \delta_{111})$
β_{211}	$4.985756\times 10^{-2}(\alpha_{101}\delta_{111} - \beta_{111}\gamma_{101}) + 3.198065\times 10^{-1}\,\alpha_{101}(\alpha_{211} + \gamma_{211}) - 4.101999\times 10^{-1}\,\alpha_{211}(\alpha_{101} + \gamma_{101})$ $- 4.450845\times 10^{-1}\,\alpha_{111}(\alpha_{201} + \gamma_{201}) + 5.70888\times 10^{-1}\,\alpha_{201}(\alpha_{111} + \gamma_{111})$

Table 5

Kiadja a Központi Fizikai Kutató Intézet
Felelős kiadó: Szegő Károly
Szakmai lektor: Hrehuss Gyula
Nyelvi lektor: Harvey Shenker
Példányszám: 265 Törzsszám: 81-472
Készült a KFKI sokszorosító üzemében
Felelős vezető: Nagy Károly
Budapest, 1981. augusztus hó