KFKI-1981-48

M. BANAI

ON A QUANTIZATION OF SPACE-TIME AND THE CORRESPONDING QUANTUM MECHANICS (PART I)

Hungarian Academy of Sciences

CENTRAL RESEARCH INSTITUTE FOR PHYSICS

BUDAPEST

2017

KFKI-1981-48

ON A QUANTIZATION OF SPACE-TIME AND THE CORRESPONDING QUANTUM MECHANICS (PART I)

Miklós Banai

Central Research Institute for Physics
H-1525 Budapest 114, P.O.B. 49, Hungary

and

Publishing House of the Hungarian Academy of Sciences
H-1363 Budapest, P.O.B. 24, Hungary

HU ISSN 0368 5330
ISBN 963 371 831 7

ABSTRACT

An axiomatic framework for describing general space-time models is outlined. Space-time models to which irreducible propositional systems belong as causal logics are quantum theoretically interpretable and their event spaces are Hilbert spaces. Such a quantum space-time is proposed via a "canonical" quantization of Minkowski space M^4. As a basic assumption the time t and the place r of an event satisfy the CCR $[t,r]=-i\hbar'$. In that case the event space is a complex Hilbert space of countable dimension. When $\hbar'\to 0$, M^4 is provided as the classical limit of this quantum space-time. Unitary symmetries consist of Poincaré-like symmetries: translations, rotations and inversion, and of gauge-like symmetries. Space inversion implies the time inversion, and vice versa. This quantum space-time reveals a confinement phenomenon: the test particle is "confined" in an $\hbar$ size region of M^4 at any time. In the one particle theory over this quantum space-time, the Klein--Gordon eq. and the Dirac eq. may be reinterpreted as bare mass eigenvalue eq.'s for a scalar and a spinor particle, respectively. This quantum mechanics reduces to the usual relativistic quantum mechanics when $\hbar'\to 0$. An example explains the potential model of the Ψ-particle. This comparison with the Ψ-particle gives $\hbar'\approx 1\ \frac{\text{fermi}}{\text{GeV}}$.

АННОТАЦИЯ

Описывается аксиоматический подход к описанию общих пространство-временных моделей. Те модели пространства-времени, которые имеют неприводимые системы пропозиции в качестве причинных логик, обладают теоретико-полевой интерпретацией, и их пространства событий являются Гильбертовыми. Задается такое квантовое пространство-время с помощью канонического квантования пространства Минковского M^4. Это квантовое пространство-время имеет свойство запирания. В одночастичной теории на этом пространстве-времени уравнения Клейна-Гордона и Дирака имеют интерпретацию уравнений на собственное значение голой массы для скалярной или спинорной частицы.

KIVONAT

Egy axiomatikus keretet vázolunk általános téridő-modellek leirására. Azok a téridő-modellek, amelyekhez irreducibilis propozició rendszerek tartoznak mint kauzális logikák, kvantumelméletileg interpretálhatók és az eseménytereik Hilbert-terek. Megadunk egy ilyen kvantumtéridőt az M^4 Minkowski--tér "kanonikus" kvantálásán keresztül. Alapvető feltevésként egy esemény t ideje és r helye teljesiti a $[t,r] = -i\hbar'$ felcserélési relációt. Ebben az esetben az eseménytér egy komplex szeparábilis Hilbert-tér. Ha $\hbar'\to 0$, M^4-t mint ennek a kvantumtéridőnek a klasszikus határesetét kapjuk vissza. Unitér szimmetriák Poincaré-szerü szimmetriákat: eltolások, forgatások és inverziók, és mértékszerü szimmetriákat tartalmaznak. A tértükrözés maga után vonja az időtükrözést és viszont. Ez a kvantumtéridő bezárási jelenséget fed föl: a próbarészecske bezáródik M^4 egy $\hbar$ méretü cellájába minden időpillanatban. Az egyrészecske-elméletben ezen a téridőn a Klein-Gordon-egyenlet és a Dirac-egyenlet ujra interpretálhatók mint csupasz tömeg-sajátérték egyenletek egy skalár ill. egy spinor részecskére. Ez a kvantummechanika a szokásos relativisztikus kvantummechanikára redukálódik ha $\hbar'\to 0$. Egy példa magyarázza a Ψ-részecske potenciál modelljét. Ez az összehasonlitás a Ψ-részecskével a $\hbar'\approx 1\ \frac{\text{fermi}}{\text{GeV}}$ értéket adja.

1. Introduction

The view is widely accepted that the difficulties of conventional local quantum (q) field theories arise from their paradoxical and semantically inconsistent nature, namly they are q theories over a classical(c) space-time, they are cq theories in the terminology of Finkelstein (1974). There exist in the literature many different approaches to resolve this inconsistency and to achieve a semantically consistent -- which must and do be a proper feature of a successful theory stressed oftenly by von Weizsäcker (1973, 1974)-- local q field theory and at the same time to explain the very nature of space-time or to arrive at a q space-time. Some of the most essential approaches are the space-time code theory of Finkelstein (1974) et al. (1974), the ur theory of von Weizsäcker (1974) et al. (1975, 1977, 1979, 1981), the twistor theory of Penrose (1975) et al.(1972) and more recently the attempt of Marlow (1981a, 1981b)[1]. However these approaches have not been completed and it is not too easy to see in their present stage whether they will really achieve the goal or not. Therefore we think there are still possibilities for other approaches.

Recently the present author proposed a generalization of q logic of the type of Piron (1976) and Gudder (1970) for local field theories (lft) using the new technique of lattice valued logics (Banai (1980, 1981a)). This q logical approach offers us a new possibility to approach the problem above and to develop a consequent q version of space-time. As a continuation of the investigation of the ideas in these papers mentioned we elaborate here the suggestions given in Banai (1981b) and propose a "canonical quantization" of Minkowski space $\mathbb{M}^4$ and formally develop a Hilbert space formalism for describing this q space-time and for q mechanics over this q substratum. Our guiding principles consist of two hypotheses:

(A) The space-time of a local q physical system should be q theoretically fully interpretable.

(B) The time and place of an event could not be measured, in principle, with orbitrary precision.

The first hypothesis is required by the semantical consistency and it determines the mathematical framework of the corresponding space-times. Following from the clearcut result of Cegła and Jadczyk (1977) about the causal logic of $\mathbb{M}^4$ (A) will be equivalent in mathematical terms with the requirement that the causal logic of the space-time should satisfy the covering law and thus the causal logics of the corresponding space-times become propositional systems of Piron (1976). The second hypothesis can be formulated mathematically in the Heisenberg-type uncertainty relation

$$\Delta t \, \Delta r \geq \frac{1}{2} \hbar', \qquad r^2 = x_1^2 + x_2^2 + x_3^2 \qquad (1)$$

where $\hbar'$ is a Planck-constant characteristic for space-time, and (1) will lead to a "canonically" quantized version of $\mathbb{M}^4$, to a concrete q space-time.

The main content of this paper is presented as follows. In sec. 2 an axiomatic framework for describing general space-time models, following from a q logical approach of lft (Banai (1981a)), is outlined. Space-time models to which irreducible propositional systems of Piron (1976) belong as causal logics are q theoretically fully interpretable and, if their causal logics contain at least four atoms, their event spaces are generalized Hilbert spaces. In sec. 3 such a q space-time model is proposed via a "canonical" quantization of $\mathbb{M}^4$. As a basic assumption following from (B) the time t and the place r of an event satisfy the CCR $[\hat{t}, \hat{r}] = -i\hbar'$ which implies (1). In that case the event space is a complex Hilbert space H of countable dimension, events are rays in H, observables are self-adjoint operators in H and symmetries are unitary or anti-unitary operators in H. In the formal limit $\hbar' \rightarrow 0$, $\mathbb{M}^4$ is provided as the c limit of this q space-time. In sec. 4 it is shown that the unitary symmetries of q space-time consist of Poincaré-like symmetries: translations, rotations and inversion, and of gauge-like symmetries. The space inversion implies the time inversion, and vice versa, in this q space-time. In the c limit the unitary symmetries are reduced to the Poincaré symmetries of $\mathbb{M}^4$. In sec. 5 some properties of q space-time is studied and it is seen that this q space-time reveals a confinement phenomenon: the test particle is "confined" in an $\hbar$ size region of $\mathbb{M}^4$ at any time. Sec. 6 deals with the one particle theory over this q substratum and the Klein-Gordon eq. and the Dirac eq. are reinterpreted as mass eigenvalue eq.'s for the mechanical (or bare) mass of a scalar and of a spinor particle, respectively, which particles are free or interact with an external field. This q mechanics is reduced to the usual relativistic (r) q mechanics on $\mathbb{M}^4$ in the formal limit $\hbar' \rightarrow 0$. In sec. 7 an example, a particle in a Coulomb potential, explains why the potential model of the Υ-particle describes so beautifully the spectrum of this particle in a non-relativistic way. This comparison with the Υ-particle gives $\hbar' \approx 1 \frac{\text{fermi}}{\text{GeV}}$ in natural units. In sec. 8 concluding remarks close this paper.

2. Space-time models from a q logical approach of LFT

(1) In Banai (1980) the local physical system $P(\Omega)$ is represented by a lattice-valued logic (L, ℓ, V); the value lattice ℓ have

to reflect the causal structure of the physical space Ω over which the system spreads. Thus the physical space of the system should be determined by ℓ together with its event structure, symmetries and observable aspects.

(2) Given ℓ abstractly in a concrete lattice-valued logic (L, ℓ, V) representing the system $P(\Omega)$, then a/ events are represented by the atoms (or more generally by the maximal filters) of ℓ, b/ symmetries are given by the automorphisms of ℓ ; the symmetry group of ℓ is $Aut(\ell)$, the geometrical symmetry group of the corresponding physical space is generated by $Aut(\ell)$, c/ observables are morphisms (σ-morphisms or c-morphisms) from Boolean lattices associated with the measuring apparatuses (classical systems) to ℓ.

(3) Causal relation: Definition. Two enents are causally disconnected (connected) whenever the two events are compatible (non-compatible); they are orthogonal, they belong to a distributive sublattice in ℓ. The elements of ℓ (two regions generated by the two elements) are causally disconnected if they are compatible.

We say that the "causal logic" ℓ is non-relativistic, respectively, relativistic if ℓ is distributive, respectively, non-distributive.

(4) Assumptions: We restrict ourselves, from now on, to CROC--valued logics representing a local $P(\Omega)$. Thus ℓ will be a CROC, complete orthomodular lattice. To simplify the event structure of ℓ, a further assumption is to be ℓ atomic CROC, and thus events are in one-to-one correspondence with the atoms of ℓ.

(5) Examples: a/ $\ell = \bigcup_{t \in \mathbb{R}} \mathcal{P}_t(\mathbb{R}^3)$, this is the causal logic of Galilean space-time $X = \mathbb{R} \times \mathbb{R}^3$. The Borel sets of X constitute a σ-complete lattice ℓ^B in ℓ. The Galilean group G on X acts as a group of automorphisms of ℓ and ℓ^B.

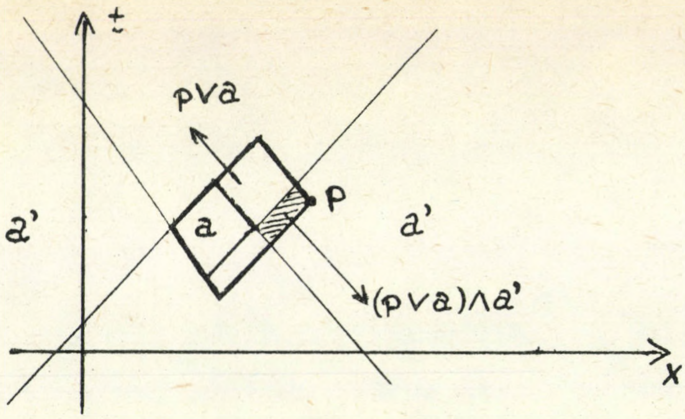
The events are the points of X. The observables in ℓ^B (σ--morphisms) generate Borel functions on X, and conversely (Cegła and Jadczyk (1976)).

b/ ℓ is the causal logic of $\mathbb{M}^4$, i.e. the elements of ℓ are given by double -orthogonal sets in $\mathbb{M}^4$ and these sets form a CROC as it was shown by Cegła and Jadczyk (1977). The events of ℓ are the points of $\mathbb{M}^4$. Maximal complete Boolean sublattices of ℓ correspond in one-to-one to spacelike hyperplanes in $\mathbb{M}^4$; the atoms of a maximal Boolean sublattice in ℓ are the points of a spacelike hyperplane in $\mathbb{M}^4$, all these events are causally disconnected. The subset ℓ^B of ℓ consisting of all Borel sets in ℓ is a σ-complete, orthomodular lattice. Every automorphism of ℓ is induced by a transformation of $\mathbb{M}^4$ preserving interiors of light cones, and so, by the result of Borchers and Hegerfeldt (1972), is a Poincaré transformation up to dilation. Thus the full group of symmetries, Aut(ℓ) consists of dilations and Poincaré transformations. An observable in ℓ^B (σ-morphism) generates a Borel function on a spacelike hyperplane in $\mathbb{M}^4$, and conversely.

(6) As we see the non-r causal logics are almost exhausted by the physically interesting and well studied example a/ though there are theoretically open questions in this case, too. Nevertheless we now concern the physically more interesting cases of r causal logics.

In example b/ ℓ is an atomic CROC, moreover it is an irreducible atomic CROC; the center of ℓ consists of the empty set and $\mathbb{M}^4$ only. But the covering law is not satisfied by ℓ (one can easily verify this considering the content of the law on a two dimensional figure:

$p \wedge a = \emptyset$, but $(p \vee a) \wedge a'$ is not an atom in general! as it should follow from the covering law if it is satisfied.)?



Thus ℓ is not a propositional system in the sense of Piron (1976) and so it does not be realizable via a (generalized) Hilbert space in accordance with the Piron's realization theorem about propositional systems.

(7) On the other hand if we suppose that the covering law is satisfied by ℓ (e.g., in the case of CROC-valued propositional systems (L, ℓ, V) representing rlft systems (Banai (1980))) then the r causal logic ℓ becomes an irreducible propositional system of Piron and thus, if ℓ containes at least four atoms, it is realizable with the lattice of the closed subspaces of a generalized Hilbert space H over a division ring K in accordance with the Piron's result. Furthermore the corresponding r space-time can be fully operationally defined (at least to the extent of the Piron's q physical approach). Now, by Piron (1976), we know that the covering law guarantees in a q system, knowing the response of the system undergo an ideal measurement of the first kind, to calculate the final pure state as a function of the initial pure state. Without this axiom we cannot completely determine the final state; and although the measurement may be ideal, the perturbation results in a loss of information, even if we take the response of the system into account.

A pure state is represented by an atom in the propositional system; in the causal logic, an atom represents an event (of a test particle moving in the corresponding space-time). Thus the covering law ensures us to be able to predict the subsequent event of a test particle observing with an ideal measurement of the first kind, as a function of the initial (previous) event. We may have in this way such space-time models which only employ operationally definable and observable concepts and which are <u>q theoretically interpretable</u>. So we call such a r space-time model to which an irreducible propositional system belongs as r causal logic a <u>q space-time</u>.

(8) Let us collect the main results following from q theory and concerning on q space-times.

<u>Theorem 1</u>. Let (L, ℓ, V) be an irreducible CROC-valued propositional system representing a pure rlft system. If the r causal logic ℓ is of rank at least equal to 4 then ℓ can be realized by the lattice $\mathcal{P}(H)$ of closed linear subspaces of a generalized Hilbert space H over a field K. (The vector space (H, K, ϕ) is a generalized Hilbert space iff $u + u^{\perp} = H$, $\forall u \in \mathcal{P}(H)$, $u^{\perp} = \{ f \in H \mid \phi(f,g) = 0, \forall g \in u\}$, where ϕ is a definite Hermitian form constructed over this space.)

<u>Proof</u>. See in Piron (1976).

<u>Theorem 2</u>. The events of the causal logic ℓ in Th.1. can be represented by the rays of H. Two distinct events are causally disconnected if the corresponding vectors make the definite Hermitian form vanish.

<u>Proof</u>. See in Piron (1976).

So we see that the q substitute for the c event space $\mathbb{M}^4$ is a Hilbert space H corresponding to the rlft system represented by the irreducible CROC-valued propositional system (L, ℓ, V), similarly to q mechanics where the q mechanical substitute for c phase space is the Hilbert space.

<u>Theorem 3</u>. (Wigner) Let H be a generalized Hilbert space of dimension at least equal to 3, realizing a r causal logic ℓ. Every

isomorphism of $\mathcal{P}(H)$ onto itself is induced by a semilinear transformation of (H, K) onto itself. A semilinear transformation (σ, σ') of (H, K) onto itself induces an isomorphism of $\mathcal{P}(H)$ onto itself iff there exists $\alpha \in K$ such that $\sigma'^{-1}\phi(\sigma f, \sigma g) = \phi(f,g)\alpha$, $\forall f,g \in H$.

Proof: Th.(3.28) in Piron(1976).

Corollary. If H is a complex Hilbert space of at least dimension 3, every symmetry is induced by a transformation u which is linear or antilinear. In the linear case $\phi(uf,ug) = \phi(f,g)$, $\forall f,g\in H$, and in the antilinear case $\phi(uf,ug) = \phi(g,f)$, $\forall f,g\in H$.

But the transformation u is not entirely determined by the specification of the symmetry. Two u's which differ by a complex factor of unit modulus induce the same symmetry.

Thus the symmetry group Aut(ℓ) of the r causal logic in Th.1. generates, roughly speaking, the unitary group of the corresponding (generalized) Hilbert space H, that is to say the geometrical symmetry group of the corresponding q space-time represented by H is the unitary group of H.

Now an observable is a c-morphism of a Boolean CROC associated with a measuring apparatus into the r causal logic ℓ. When the field K is isomorphic to one of three fields the reals, the complexes, or the quaternions one can state in the Hilbert realization (Piron(1976) Th.(3.53)):

Theorem 4. Each observable of a r causal logic which is an irreducible propositional system $\mathcal{P}(H)$ defines an Abelian von Neumann algebra over H. If H is of countable dimension, this algebra is generated by a self-adjoint operator. If H is finite dimensional, every observable has a purely discrete spectrum.

A state w can be defined on a q space-time as a generalized probability measure on ℓ. The main result along this line the

the Gleason's theorem (Gleason (1957)).

Theorem 5. Let H be a complex Hilbert space representing the event space of a q space-time. Every generalized probability measure defined onto $\mathcal{P}(H)$ is of the form $w(Q) = tr(Q\varrho)$, $\forall Q \in \mathcal{P}(H)$, where ϱ is a von Neumann density operator.

This theorem was proved by Gleason for countable dimensional case and by Eirels and Horst (1975) for uncountable dimensional case with the assumption of Continuum Hypothesis. Using this theorem and the properties of von Neumann density operators, the mean value of an observable $\hat{A}$ which is a self-adjoint operator in H has the form

$$\langle\hat{A}\rangle = tr(\hat{A}\varrho) = \sum_{i=1}^{\infty}\mu_i tr(\hat{A}P_i) = \sum_{i=1}^{\infty}\mu_i \phi(x_i, \hat{A}x_i) = \\ = \sum_{i=1}^{\infty}\mu_i\langle x_i|\hat{A}|x_i\rangle \qquad (2.1)$$

where $\mu_i \in \mathbb{R}_+$, $\sum\mu_i = 1$, P_i's are mutually orthogonal projectors of rank 1 and x_i's constitute an orthonormal basis in H. In particular the mean value of $\hat{A}$ in a pure state which is represented by a one dimensional projector or by a ray, is

$$\langle\hat{A}\rangle_P = tr(\hat{A}P) = \langle x|\hat{A}|x\rangle \qquad (2.2)$$

Because of the definition of an event we can say that the expectation value of an observable $\hat{A}$ at the event $P = |x\rangle\langle x|$ is given by (2.2).

Remark: Conserved curents define states on the causal logic of $\mathbb{M}^4$ in example b/ as was shown by Cegła and Jadczyk (1979).

(9) Maximal Boolean subalgebras in ℓ of example b/ correspond to spacelike hyperplanes in $\mathbb{M}^4$, similarly, maximal Boolean sublattices in ℓ (in Th.1.) generate spacelike hyperplanes in the corresponding q space-time. For, let B be a maximal Boolean sublattice in ℓ, then $B = \mathcal{P}(\Omega)$ where Ω is the set of atoms of B. Every event (the points of Ω) in B is causally disconnected.

Now Ω with the discrete topology is a copletely regular space and thus its Stone-Cech compactification Γ exists, which is a compact Hausdorff space, extremely disconnected. On the other hand, let A be the Abelian von Neumann algebra generated by B, then A is a commutative C^{*}-algebra (moreover a W^{*}-algebra) and so it is representable as a function algebra $C(\bar{\Gamma})$ where $\bar{\Gamma}$ is a compact Hausdorff space, extremely disconnected; $\bar{\Gamma}$ is the spectrum space of A. It is clear that $\Gamma = \bar{\Gamma}$ up to topological isomorphism. This completely disconnected compact Hausdorff space Γ is what we can (and do) call a spacelike hyperplane in the q space-time representable by a complex Hilbert space H. Following from Th.5., all probability measures, states, on Γ are determined as convex combinations of pure states and a pure state is represented by a Dirac measure on Γ concentrated on a point of Γ . To summarize we can state:

Theorem 6. Let ℓ be a r causal logic realizable with a $\mathcal{P}(H)$ where H is a complex Hilbert space of dimension at least equal to 4. Every maximal Boolean sublattice B of ℓ determines a spacelike hyperplane Γ in the corresponding q space-time represented by H. Γ is a completely disconnected compact Hausdorff space and can be identified with the spectrum space of the Abelian von Neumann algebra generated by B. Every state on B can be represented on Γ as a probability measure of the form $\mu = \sum_{x\in\Gamma} \lambda_x \mu_x$ where $\lambda_x \geq 0$, $\sum \lambda_x = 1$ and μ_x is the Dirac measure concentrated on $x \in \Gamma$.

So we see that a q space-time has a much more discrete inner topological structure compared with the space-time $\mathbb{M}^4$; a spacelike hyperplane in $\mathbb{M}^4$ is a connected locally compact Hausdorff space in its usual topology.

(10) We saw that the observables of a q space-time representable by a complex Hilbert space of countable dimension are self-adjoint operators. Thus the observable time and space coordinates of an event (which are supposed, in c space-time, that they are observables) become self-adjoint operators in such a q space-time. The space-time coordinate 4-vector plaies a distinguished role in $\mathbb{M}^4$; all other observables on $\mathbb{M}^4$ can be expressed as functions of this 4-vector. Thus the determination of the commutation property of the coordinate time and space operators is decisive for us. It will be done this in the following sec. using a heuristic argument.

3. A "canonical" quantization of Minkowski space

(11) From now on we restrict our attention to such q space-time models which are represented by complex Hilbert spaces of countable dimension. In these cases the whole well-known mathematical apparatus of q mechanics can be exploited to build up a sensible and, probably, satisfactory q version of space-time.

We note that any two such q space-time models, i.e. represented by two complex Hilbert spaces of countable dimension, are unitarily equivalent because of any two such Hilbert spaces are unitarily equivalent.

Let H be a complex Hilbert space of countable dimension, representing the event space of a r space-time model. Let $\hat{A}$ and $\hat{B}$ be two observables in this q space-time, i.e. two self-adjoint operators in H and let ϕ be a unit vector defining an event in both of their domains, and such that $\hat{A}\phi$ is in the domain of $\hat{B}$ and vice versa. Denote $\delta(\hat{A},\phi)$ the dispersion of $\hat{A}$ in the event ϕ, i.e. $\delta(\hat{A},\phi)$ is a quantitative measure of the degree of "spreadoutness" of an observable in a given event (pure state):

$$\delta(\hat{A},\phi) = \langle [\hat{A} - \langle \hat{A}\phi | \phi \rangle]^2 \phi | \phi \rangle^{\frac{1}{2}} = \langle \hat{A}^2\phi|\phi\rangle - \langle \hat{A}\phi|\phi\rangle^2 .$$

Then, as it is well-known from q mechanics (see, e.g., in Mackey(1963)), the product of the two dispersions $\delta(\hat{A},\phi)$ and $\delta(\hat{B},\phi)$ is bounded below by $|\langle(\hat{B}\hat{A} - \hat{A}\hat{B})\phi|\phi\rangle|/2$:

$$\delta(\hat{A},\phi) \cdot \delta(\hat{B},\phi) \geq \frac{1}{2} |\langle[\hat{B}, \hat{A}]\phi|\phi\rangle| \qquad (3.1)$$

When $\hat{A}$ and $\hat{B}$ do not commute this is a limitation to the degree to which the probability distributions of the corresponding observables may be independently concentrated near to single points. (12) Now let $\hat{x}_o = c\hat{t}$, $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$ be the self-adjoint operators corresponding to the coordinate time and space observables (of an event of a test particle)[3]. These observables play a role in $\mathbb{M}^4$ similar to the role of the conjugate momentum and coordinate observables in c mechanics; all other observables on $\mathbb{M}^4$ are the functions of these observables. The non-commuting property of the conjugate momentum and coordinate observables has a central role in q mechanics. Thus we suspect that, similarly, there is a corresponding relation between the coordinate time and space observables in q space-time.

In q mechanics, following from the CCR, arbitrary small cells of phase space built up from p and q do not correspond to physically observable reality. A similar statement in q space-time that the arbitrary small cells of $\mathbb{M}^4$ - this is the analogous of phase space - do not correspond to physically observable reality, i.e. with physical measurements with vanishing dispersions. In other terms we are not able to distinguish, by measurements with zero dispersions, two events arbitrarily close in $\mathbb{M}^4$ from each other. On the other hand we may expect - taking into acount the great empirical success of non-r q mechanics which presupposes the Euclidean structure of space and that any particle is localizable in space to a point also in q mechanics, and the conservation of angular momentum (Segal (1965)) - that the spacelike coordinates $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$ of a test particle are measurable without dispersion, i.e. $\hat{x}_1$, $\hat{x}_2$ and $\hat{x}_3$ are commutable among themselves.

Let $\Delta A = \delta(\hat{A}, \phi)$ then we formulate this heuristic argument in the following Heisenberg-type uncertainty relation[4]

$$\Delta t \Delta r \geq \frac{1}{2} \hbar', \qquad r = \sqrt{x_1^2 + x_2^2 + x_3^2} \qquad (3.2a)$$

or

$$\Delta x_0 \Delta r \geq \frac{1}{2} \hbar, \qquad \hbar = c\hbar' \qquad (3.2b)$$

where $\hbar'$ is a constant characteristic for space-time. This uncertainty relation means that the time and place of an event cannot be measured with arbitrary precision, in principle, anyhow the measuring apparatuses are refined. We can derive this relation, applying (3.1), from the following Heisenberg-type commutation relation (CR):

$$[\hat{t}, \hat{r}] = -i\hbar' \cdot 1 \qquad (3.3a)$$

or $$\hat{r}^2 = \hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2$$

$$[\hat{x}_0, \hat{r}] = -i\hbar \cdot 1 \qquad (3.3b)$$

where 1 is the identity operator on H and equality are understood on the common domain of the both side (and this remark will be valid for all formal equalities between unbounded operators in H they extensively appear in what follows!). We choose the CR (3.3) our (second) <u>basic assumption</u> to set up an operationally defined and phenomenologically allowable concrete q space-time.

(13) We can easily determine a concrete realization of (3.3). Let $(H, \phi) = (L^2(\mathbb{R}), \langle f_1 | f_2 \rangle = \int_{\mathbb{R}} dq\, \bar{f}_1(q) f_2(q))$. Then the following self-adjoint unbounded operators satisfy (3.3) (when they are suitably restricted):

$$\hat{t}\,\varphi(q) = -i\hbar' \frac{d}{dq}\varphi(q), \qquad \hat{x}_0 \varphi(q) = -i\hbar \frac{d}{dq}\varphi(q)$$
$$\hat{r}\,\varphi(q) = q \cdot \varphi(q) \qquad (3.4)$$

But the realization (3.4) is unique up to unitary equivalence in the sense of the Stone-von Neumann theorem (1932).

Note that the pair $(q, -i\frac{d}{dq})$ of operators constitutes an irreducible system of operators in $L^2(\mathbb{R})$ in the sense that only the scalar multiplies of the identity operator commute with both of them.

Now let $(H, \phi) = \left(L^2(\mathbb{R}^3), \langle f_1, f_2\rangle = \int_{\mathbb{R}^3} d^3\underline{x}\ \bar{f}_1(\underline{x})f_2(\underline{x})\right)$ then

$$\hat{x}_i\,\phi(\underline{x}) = x_i\cdot\phi(\underline{x}) \qquad (3.5a)$$

To determine the self-adjoint representation of (3.3) in this case, let us use the following trick.[5] Let us consider the map $V: L^2(\mathbb{R}) \longrightarrow L^2_{\varrho}(\mathbb{R});\ \varphi(q) \mapsto \frac{1}{q}\varphi(q)$, where $d\varrho(q) = q^2dq$. This is a unitary map. Denote h the multiplication operator and $-i\frac{d}{dh}$ the differentiation operator in $L^2_{\varrho}(\mathbb{R})$. Then $VqV^{-1} = h$ and $V(-i\frac{d}{dq})V^{-1} = -i\frac{1}{h}\frac{d}{dh}h = -i(\frac{d}{dh} + \frac{1}{h}) := f$. Furthermore let $\xi : \mathbb{R}^3 \longrightarrow \mathbb{R}\times(0, \frac{\pi}{2})\times(0, 2\pi)\ ;\ \underline{x} \longmapsto \left(|x|\,\mathrm{sign}\,x_3,\ \arccos\frac{|x_3|}{|x|},\ \arcsin\frac{x_1}{\sqrt{x_1^2+x_2^2}}\right)$, $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2} = r$, then $\xi^{-1}(h, \vartheta, \chi) = (|h|\sin\vartheta\cos\chi,\ |h|\sin\vartheta\sin\chi,\ h\cos\vartheta)$, and the mapping $U : L^2(\mathbb{R}^3) \longrightarrow L^2_{\varrho}(\mathbb{R})\otimes L^2_{\nu}(0, \frac{\pi}{2})\otimes L^2(0, 2\pi);\ \varphi \longmapsto \varphi\circ\xi^{-1}$ is unitary, where $d\nu(\vartheta) = \sin\vartheta\, d\vartheta$. Then we can write

$$U\left(h\otimes \mathrm{id}_{L^2_\nu(0,\frac{\pi}{2})}\otimes \mathrm{id}_{L^2(0,2\pi)}\right)U^{-1}\phi(h,\vartheta,\chi) = |x|\,\mathrm{sign}\,x_3\cdot\phi(\underline{x}) :=$$

$$:= \underline{r}\cdot\phi(\underline{x}),$$

$$U\left(f\otimes \mathrm{id}_{L^2_\nu(0,\frac{\pi}{2})}\otimes \mathrm{id}_{L^2(0,2\pi)}\right)U^{-1}\phi(h,\vartheta,\chi) =$$

$$= -i\,\frac{1}{|x|\,\mathrm{sign}\,x_3}\left(\sum_1^3 x_i\frac{\partial}{\partial x_i} + \mathrm{id}_{L^2(\mathbb{R}^3)}\right)\phi(\underline{x}) =$$

$$= -i\,\frac{1}{\underline{r}}\left(x_i\frac{\partial}{\partial x_i} + 1\right)\phi(\underline{x}) := -i\,\frac{d}{d\underline{r}}\phi(\underline{x})$$

and these self-adjoint operators clearly satisfy (3.3) if we put the coefficient $\hbar$ or $\hbar'$ on the appropriate place. Thus we have

$$\hat{x}_0\,\phi(\underline{x}) = -i\hbar\,\frac{d}{d\underline{r}}\,\phi(\underline{x}) \qquad (3.5b)$$

$$\hat{r}\ \phi(\underline{x}) = \underline{r}\cdot\phi(\underline{x}), \qquad (3.5c)$$

$$\hat{t}\ \phi(\underline{x}) = -i\hbar'\ \frac{d}{d\underline{r}}\ \phi(\underline{x}) \qquad (3.5d)$$

The operator $\hat{r}$ in (3.5c) clearly satisfies the condition $\hat{r}^2 = \hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2$.

Note that the system $(x_i, -i\frac{\partial}{\partial x_i})$, i=1,2,3, of operators constitutes an irreducible system of operators in $L^2(\mathbb{R}^3)$, and, while the solution in (3.4) for $(\hat{t}, \hat{r})$ is irreducible, the solution in (3.5c,d) is not irreducible in $L^2(\mathbb{R}^3)$.[6] Furthermore observe that, in this representation, $L^2(\mathbb{R}^3)$ is orthogonally decomposed into the direct sum $L^2(\mathbb{R}^3) = H_1 \oplus H_2 = L^2(\mathbb{R}^2 \times \mathbb{R}_+) \oplus L^2(\mathbb{R}^2 \times \mathbb{R}_-)$, and the pair $(\hat{t}, \hat{r})$ acts in H_1 as $\left(-i\hbar'\ \frac{1}{r}\frac{\partial}{\partial r}\ r\ ,\ r\right)$ and in H_2 as $\left(+i\hbar'\ \frac{1}{r}\frac{\partial}{\partial r}\ r, -r\right)$; $(\hat{t}, \hat{r})$ has purely positive spectrum in H_1 and purely negative spectrum in H_2.

With the aid of this representation of $\hat{x}_0$ we obtain, after a formal calculation, the CR's between the components of the coordinate 4-vector $\hat{x}_\mu$:

$$[\hat{x}_0, \hat{x}_i] = -i\hbar\ \frac{\hat{x}_i}{\underline{r}}, \quad i=1,2,3 \qquad (3.6a)$$

or

$$[\hat{x}_\mu\ , \hat{x}_\nu] = -i\hbar\ \hat{A}_{\mu\nu}\ , \qquad \mu,\nu = 0,1,2,3 \qquad (3.6b)$$

where

$$[\hat{A}_{\mu\nu}] = \frac{1}{\underline{r}}\begin{pmatrix} 0, & \hat{x}_1, & \hat{x}_2, & \hat{x}_3 \\ -\hat{x}_1, & 0, & 0, & 0 \\ -\hat{x}_2, & 0, & 0, & 0 \\ -\hat{x}_3, & 0, & 0, & 0 \end{pmatrix} = [-\hat{A}_{\nu\mu}] \qquad (3.7)$$

By means of (3.1) we get the uncertainty relations for the components of $\hat{x}_\mu$:

$$\Delta x_0 \Delta x_i \geq \frac{1}{2}\hbar \left|\overline{\left(\frac{x_i}{\underline{r}}\right)}\right|, \qquad \overline{\left(\frac{x_i}{\underline{r}}\right)} = \langle\phi|\frac{\hat{x}_i}{\underline{r}}|\phi\rangle \qquad (3.8a)$$

or

$$\Delta x_\mu\ \Delta x_\nu \geq \frac{1}{2}\hbar\ |\overline{A}_{\mu\nu}|\ , \qquad \overline{A}_{\mu\nu} = \langle\phi|\hat{A}_{\mu\nu}|\phi\rangle \qquad (3.8b)$$

We can write for the expectation values of the coordinates of an event realized by a unit vector $\phi \in H$:

$$\overline{x}_\mu = \mathrm{tr}(P_\phi\ \hat{x}_\mu) = \langle\phi|\hat{x}_\mu|\phi\rangle = \int_{\mathbb{R}^3} \bar{\phi}\ \hat{x}_\mu\ \phi\ \ d^3\underline{x}\ , \qquad (3.9)$$

$$\overline{x}_0 = -i\hbar \int_{\mathbb{R}} dq \int_0^{\frac{\pi}{2}} d\vartheta \;\sin\vartheta \int_0^{2\pi} d\chi \;\overline{\phi}(q,\vartheta,\chi)\frac{\partial}{\partial q}\phi(q,\vartheta,\chi) =$$

$$= -i\hbar \int_{\mathbb{R}^3} d^3\underline{x}\;\phi(\underline{x})\frac{d}{d\underline{r}}\phi(\underline{x}) \tag{3.9a}$$

$$\overline{x}_i = \int_{\mathbb{R}^3}\phi(\underline{x})x_i\,\phi(\underline{x})\;d^3\underline{x} \tag{3.9b}$$

(14) When $H = L^2(\mathbb{R}^3)$ so that the $\hat{x}_\mu$ is as indicated above in (3.5), each event vector ϕ may be written in the form $\phi = \sqrt{\varsigma}\,e^{is}$ where $\varsigma = |\phi|^2$ and s is a real-valued function on $\mathbb{R}^3$ which is determined only up to an additive constant. ς and s together uniquely determine ϕ, and ς has an obvious physical significance. If E is any Borel subset of $\mathbb{R}^3$ then $\int_E \varsigma(x_1,x_2,x_3)d^3\underline{x}$ is the probability that measurements of the x_i of the event will give a value for the 3-tuple x_1, x_2, x_3 lying in E. Therefore we can interprete $\mathbb{R}^3$ as the <u>classical physical space</u> in which the measuring apparatuses (c objects) behaving stationary take place (cloks, measuring lines, ect.; cf. below sec.4.).

s also has a simple physical meaning. Assuming that ϕ is suitably differentiable, let us compute the expected value of the observable $\hat{t}$. It is

$$\langle\phi|\hat{t}\phi\rangle = -i\hbar' \int_{\mathbb{R}} dq \int_0^{\frac{\pi}{2}} d\vartheta\sin\vartheta \int_0^{2\pi} d\chi \;(\sqrt{\varsigma}\,e^{-is})\frac{\partial}{\partial q}(\sqrt{\varsigma}\,e^{is}) =$$

$$= -i\hbar' \int_{\mathbb{R}} dq \int_0^{\frac{\pi}{2}} d\vartheta\sin\vartheta \int_0^{2\pi} d\chi \;\frac{1}{2}\frac{\partial\varsigma}{\partial q} + \hbar' \int_{\mathbb{R}} dq \int_0^{\frac{\pi}{2}} d\vartheta\sin\vartheta \;\times$$

$$\times \int_0^{2\pi} d\chi \;\;\varsigma\frac{\partial s}{\partial q} = \frac{\hbar'}{2i}\int_0^{\frac{\pi}{2}} d\vartheta\sin\vartheta \int_0^{2\pi} d\chi \;\left[\varsigma\right]_{-\infty}^{+\infty} + \hbar' \int_{\mathbb{R}} dq \int_0^{\frac{\pi}{2}} d\vartheta \;\times$$

$$\times \sin\vartheta \int_0^{2\pi} d\chi \;\;\varsigma\frac{\partial s}{\partial q} = \hbar' \int_{\mathbb{R}^3} d^3\underline{x}\;\;\varsigma\frac{ds}{d\underline{r}}$$

If we have an event, i.e. a state of a test particle, in which ς is highly concentrated, i.e. in which x_1, x_2, x_3 is almost sure to be very near to x_1^o, x_2^o, x_3^o, then the time coordinate t will have an expected value very near to $\hbar'\frac{ds}{d\underline{r}}(x_1^o,x_2^o,x_3^o)$ (cf. below). In any case $\underline{x} \longmapsto \hbar'\frac{ds}{d\underline{r}}(\underline{x})$ gives a map that associates a time coordinate value to every set of space coordinate values.

The mean of this time coordinate value with respect to ϱ is the expected value of the time coordinate for the event $\sqrt{\varrho}\,e^{is}$. In this sense $\hbar' \frac{ds}{dr}$ describes the coordinate time of the event.

We note that the operators $\hat{x}_1, \hat{x}_2, \hat{x}_3$ form a complete commuting family, thus an event is completely determined by the measurement of the observables $\hat{x}_1, \hat{x}_2, \hat{x}_3$, while the measurement of the observable $\hat{t}$ does not determine completely the event.

(15) Let us consider the c limit of the model. According to q mechanics the c limit is provided by the set of events for which the dispersions of the time observable $\hat{t}$ and of the place observable $\hat{r}$ are minimal, i.e. for which

$$\Delta t \, \Delta r = \frac{1}{2}\hbar' \tag{3.10}$$

This condition is equivalent with the following equation for the corresponding events (see in von Neumann(1955)):

$$(\hat{t} - \bar{t})\phi = i\gamma(\hat{r} - \bar{r})\phi\ , \quad \phi \in H, \quad \gamma \in (0, +\infty) \tag{3.11}$$

where $\bar{t} = \langle \phi | \hat{t}\,\phi \rangle$ and $\bar{r} = \langle \phi | \hat{r}\,\phi \rangle$. Let $H = L^2(\mathbb{R})$ then (3.11) can be written as follows

$$(-i\hbar' \frac{d}{dq} - \bar{t})\,\phi(q) = i\,\gamma\,(q - \bar{r})\,\phi(q)$$

then

$$\frac{d\phi}{dq} = \left\{ -\frac{\gamma}{\hbar'} q - \frac{\gamma}{\hbar'}\bar{r} + \frac{i\bar{t}}{\hbar'} \right\} \phi\ ,$$

$$\phi(q) = C \exp\left\{ \int_{-\infty}^{+\infty} dq \left(-\frac{\gamma}{\hbar'} q - \frac{\gamma}{\hbar'}\bar{r} + \frac{i\bar{t}}{\hbar'} \right) \right\} = C \exp\left\{ -\frac{\gamma}{2\hbar'} q^2 + \right.$$

$$\left. + \frac{\gamma}{\hbar'}\bar{r}q + \frac{i\bar{t}}{\hbar'} q \right\} = C' \exp\left\{ -\frac{\gamma}{2\hbar'}(q - \bar{r})^2 + \frac{i\bar{t}}{\hbar'} q \right\}$$

Because of $\gamma > 0$, $\|\phi\|^2 = \int_{-\infty}^{+\infty} |\phi(q)|^2\, dq < \infty$, so $\phi \in L^2(\mathbb{R})$. The constant C' can be determined from the condition $\|\phi\|^2 = 1$. It is

$$1 = \|\phi\|^2 = |C'|^2 \int_{-\infty}^{+\infty} dq \exp\left\{ -\frac{\gamma}{\hbar'}(q - \bar{r})^2 \right\} = |C'|^2 \int_{-\infty}^{+\infty} dx\ \times$$

$$\times\, e^{-\frac{\gamma}{\hbar'}x^2} = |C'|^2 \sqrt{\frac{\pi\hbar'}{\gamma}}\ , \text{ then } |C'| = \left(\frac{\gamma}{\pi\hbar'} \right)^{\frac{1}{4}}.$$

Thus the events we have been looking for have the form

$$\phi(q) = \left(\frac{\gamma}{\pi \hbar'}\right)^{\frac{1}{4}} \exp\left\{-\frac{\gamma}{2\hbar'}(q-\bar{r})^2 + \frac{i\bar{t}}{\hbar'} q\right\} \qquad (3.12)$$

where $\gamma \in (0, +\infty)$, $\bar{r} \in (-\infty, +\infty)$, $\bar{t} \in (-\infty, +\infty)$, and for these events

$$\Delta t = \sqrt{\frac{\hbar'\gamma}{2}}, \qquad \Delta r = \sqrt{\frac{\hbar'}{2\gamma}} \qquad (3.13)$$

$\phi(q)$ in (3.12) describes a wave packet around the point $\bar{r}$, with the width $\sqrt{\frac{4\hbar'}{\gamma}}$. If we take the formal limit "$\hbar' \to 0$" then $\phi(q)$ concentrates at the point $\bar{r}$, i.e.

$$\lim_{"\hbar' \to 0"} \phi(q) = \lim_{"\hbar' \to 0"} \left(\frac{\gamma}{\pi \hbar'}\right)^{\frac{1}{4}} e^{-\frac{\gamma}{2\hbar'}(q-\bar{r})^2} e^{\frac{i\bar{t}}{\hbar'} q} =$$

$$= \delta(q-\bar{r})\, e^{\frac{i\bar{t}}{\hbar'} q} = |\bar{r}, \bar{t}\rangle \qquad (3.14)$$

It is clear that the operators $\hat{t}$ and $\hat{r}$ commute on the events $|\bar{r}, \bar{t}\rangle$. In this way we can approximate the events of $\mathbb{M}^4$ with the events $\phi(q)|\bar{\vartheta}, \bar{\chi}\rangle$ where $|\bar{\vartheta}, \bar{\chi}\rangle$ is a common eigenfunction of the angle operators $\hat{\vartheta}$ and $\hat{\chi}$, and in the formal limit "$\hbar' \to 0$"

$$\lim_{"\hbar' \to 0"} \phi(q)|\bar{\vartheta}, \bar{\chi}\rangle = |\bar{r}, \bar{\vartheta}, \bar{\chi}, \bar{t}\rangle \longleftrightarrow (x_0, x_1, x_2, x_3) \in \mathbb{M}^4,$$

i.e. we have a one-to-one mapping.

We note that the operators $\hat{t}$ and $\hat{r}$ have continuous spectrum with eigenfunctions $|\bar{t}\rangle = \exp(\frac{i\bar{t}}{\hbar'} q)$ and $|\bar{r}\rangle = \delta(q-\bar{r})$, respectively, where $\bar{t}, \bar{r} \in (-\infty, +\infty)$.

4. The symmetries of q space-time

(16) The symmetries of q space-time introduced in the foregoing section are generated by unitary or antiunitary operators U in H according to Th.3. and Corollary, and two U's which differ by a complex factor of unit modulus induce the same symmetry. Let us determine first those symmetries of q space-time what we call Poincaré-like symmetries, i.e. translations, rotations and in-

versions in q space-time.

(17) <u>q space-time translations</u>: A space-time translation means a translation on the spectrum of the "4-position" operator $\hat{x}_\mu$, i.e. in mathematical terms:

$$\hat{x}'_\mu = U_a \hat{x}_\mu U_a^{-1} = \hat{x}_\mu + a_\mu \cdot 1 \ , \qquad a_\mu \in \mathbb{R} \qquad (4.1)$$

where $U_a = U(a_0, a_1, a_2, a_3)$ is a 4-parameter <u>unitary</u> group in H. The events transform under translations according to

$$\phi' = U_a \phi \quad , \qquad \phi \in H \qquad (4.2)$$

If $a = (a_0, a_1, a_2, a_3)$ is infinitesimal we can write

$$U_a = 1 - \frac{i}{\hbar} a^\mu \hat{p}_\mu \qquad (4.3)$$

where $\hat{p}_\mu$ is the self-adjoint generator of the translations in the "μ-direction". One get from (4.1)

$$U_a \hat{x}_\mu U_a^{-1} = (1 - \frac{i}{\hbar} a^\nu \hat{p}_\nu) \hat{x}_\mu (1 + \frac{i}{\hbar} a^\nu \hat{p}_\nu) = \hat{x}_\mu + \frac{i}{\hbar} [\hat{x}_\mu , p_\nu] a^\nu =$$
$$= \hat{x}_\mu + a_\mu \cdot 1$$

Then we have the CCR's

$$[\hat{p}_\mu , \hat{x}_\nu] = i\hbar g_{\mu\nu} \ , \quad g^{oo} = - g^{ii} = 1 \qquad (4.4)$$

The solution of these CCR's in $H = L^2(\mathbb{R}^3)$ is

$$\hat{x}_i \phi(\underline{x}) = x_i \phi(\underline{x}) \ , \qquad \hat{p}_i \phi(\underline{x}) = -i\hbar \frac{\partial}{\partial x_i} \phi(\underline{x}) \ , \qquad (4.5a)$$

$$\hat{x}_o \phi(\underline{x}) = -i\bar{\hbar} \frac{d}{d\underline{r}} \phi(\underline{x}) \ , \quad \hat{p}_o \phi(\underline{x}) = \frac{\hbar}{\bar{\hbar}} \underline{r} \phi(\underline{x}) \qquad (4.5b)$$

namely $[\hat{p}_o, \hat{x}_o] \phi(\underline{x}) = \frac{\hbar}{\bar{\hbar}} \frac{\bar{\hbar}}{i} \left(\underline{r} \frac{d\phi}{d\underline{r}} - \frac{d}{d\underline{r}}(\underline{r}\phi)\right) = i\hbar \ \phi(\underline{x})$ where (3.5) was used. This solution also is unique up to unitary equivalence in the sense mentioned in sec.3.. By means of this representation, one can determine the CR's between the components of $\hat{p}_\mu$. They are

$$[\hat{p}_i, \hat{p}_j] \phi(\underline{x}) = 0 \ ,$$
$$[\hat{p}_o, \hat{p}_i] \phi(\underline{x}) = \frac{\hbar}{\bar{\hbar}} \frac{\hbar}{i} \left(\underline{r} \frac{\partial}{\partial x_i} \phi(\underline{x}) - \frac{\partial}{\partial x_i} (\underline{r} \ \phi(\underline{x}))\right) = i \frac{\hbar^2}{\bar{\hbar}} \frac{\partial \underline{r}}{\partial x_i} \phi(\underline{x}) =$$
$$= i \frac{\hbar^2}{\bar{\hbar}} \frac{x_i}{\underline{r}} \phi(\underline{x})$$

Thus we obtained

$$[\hat{p}_i, \hat{p}_j] = 0, \qquad [\hat{p}_o, \hat{p}_i] = i\hbar\bar{\hbar}^2 \, \frac{\hat{x}_i}{r} \tag{4.6a}$$

or

$$[\hat{p}_\mu , \hat{p}_\nu] = i\hbar\bar{\hbar}^2 \, \hat{A}_{\mu\nu} \tag{4.6b}$$

where $\hat{A}_{\mu\nu}$ is given by (3.7) and the abbreviation $\frac{\hbar}{\bar{\hbar}} = \bar{\hbar}$ was introduced. By (3.1) one gets the following uncertainty relations between the components of $\hat{p}_\mu$:

$$\Delta p_o \Delta p_i \geq \frac{1}{2} \hbar\bar{\hbar}^2 \left(\overline{\frac{x_i}{r}} \right) \tag{4.7a}$$

or

$$\Delta p_\mu \Delta p_\nu \geq \frac{1}{2} \hbar\bar{\hbar}^2 \, \overline{|A_{\mu\nu}|} \tag{4.7b}$$

The $\hat{p}_\mu$ is a self-adjoint operator thus it corresponds to an observable in q space-time and clearly we can identify it with the <u>"4-momentum" observable</u> of a test particle; $\hat{E} = c\hat{p}_o$ is the energy observable, $\hat{p}_i$'s are the components of the 3-momentum observable.

<u>Note</u>: Let $\hat{p}^2 = \hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2$ then one gets, with an argument similar to that used in the case of the observables $\hat{t}$ and $\hat{r}$ above, the CR

$$[\hat{p}_o, \hat{p}] = i\hbar\bar{\hbar}^2 \cdot 1 \tag{4.8a}$$

and the uncertainty relation $\Delta p_o \Delta p \geq \frac{1}{2} \hbar\bar{\hbar}^2$. In momentum representation we have

$$\hat{p}_i \, \phi(\underline{p}) = p_i \cdot \phi(\underline{p})$$

$$\hat{p} \, \phi(\underline{p}) = \underline{p} \cdot \phi(\underline{p}) = |p| \text{ sign } p_3 \, \phi(\underline{p})$$

$$\hat{p}_o \, \phi(\underline{p}) = i\hbar\bar{\hbar}^2 \frac{d}{d\underline{p}} \, \phi(\underline{p}) = i\hbar\bar{\hbar}^2 \frac{1}{\underline{p}} \left(p_i \frac{\partial}{\partial p_i} + 1 \right) \phi(\underline{p})$$

Now the energy observable $\hat{E} = c\hat{p}_o$ of a test particle commutes with the momentum observable $\hat{p}$ according to

$$[\hat{E}, \hat{p}] = ic\hbar\bar{\hbar}^2 = ic \, \frac{\hbar^2}{\bar{\hbar}} = i\hbar'\bar{\hbar}'^2 \tag{4.8b}$$

where $\bar{\hbar}' = \frac{\hbar}{\hbar'}$, and thus it follows the uncertainty relation

$$\Delta E \, \Delta p \geq \frac{1}{2} \hbar'\bar{\hbar}'^2 \tag{4.9}$$

for the energy and momentum of a test particle, which means that the energy and momentum of a test particle cannot be measurable in an event of the test particle, with vanishing dispersions. We can consider the c limit of the "4-momentum space" in a way similar to above in the case $(\hat{t}, \hat{r})$. The c 4-momentum space is approximated to the greatest extent in this model by the set of the following wave packet-like events

$$\phi(q) = \left(\frac{|\gamma|}{\pi \hbar' \hbar'^2}\right)^{\frac{1}{4}} \exp\left\{-\frac{|\gamma|}{2\hbar'\hbar'^2}(q - \bar{p})^2 + \frac{i\bar{E}}{\hbar'\hbar'^2} q\right\}$$

where $\gamma < 0$ and

$$\Delta E = \sqrt{\frac{\hbar' \hbar'^2 |\gamma|}{2}}, \quad \Delta p = \sqrt{\frac{\hbar' \hbar'^2}{2|\gamma|}}, \quad \Delta E \Delta p = \frac{1}{2} \hbar' \hbar'^2 .$$

Now consider the transformation of an event $\phi \in H = L^2(\mathbb{R}^3)$ under an infinitesimal q space-time translation $U_a = 1 - \frac{i}{\hbar} a^\mu p_\mu$. It is

$$\phi' = U_a \phi = \phi(\underline{x}) - \frac{i}{\hbar} a^\mu p_\mu \phi(\underline{x}) \qquad (4.10)$$

Then for a finite translation in space when $a_0 = 0,\ a_i \neq 0$:

$$\phi'(\underline{x}) = U(a_1,a_2,a_3)\phi(\underline{x}) = \left[\exp\left(\frac{i}{\hbar} a_i \hat{p}_i\right)\right]\phi(\underline{x}) = \phi(\underline{x} + \underline{a}), \quad (4.11)$$

and for a finite translation in time when $a_0 \neq 0,\ a_i = 0$:

$$\phi'(\underline{x}) = U_{a_0}\phi(\underline{x}) = \left[\exp\left(-\frac{i}{\hbar} a_0 \hat{p}_0\right)\right]\phi(\underline{x}) = \phi(\underline{x}) \exp\left(-\frac{i}{\hbar} a_0 \underline{r}\right) \quad (4.12)$$

Remarks: 1/ The Minkowski space $\mathbb{M}^4$ is isomorphic to the parameter space of the translation group of q space-time. Now this is a noncommutative group and all of its irreducible representations are infinite dimensional and unitary equivalent as this follows from (4.6) and (4.8). $\mathbb{M}^4$ is given by the following set

$$\mathbb{M}^4 := \left\{ \underline{y} \mid \underline{y} = a_\mu e^\mu ,\ a_\mu \in \mathbb{R} ,\ \langle e^\mu , e_\nu \rangle = \delta^\mu{}_\nu \right\}$$

The corresponding set of self-adjoint operators in L(H) (the set of linear operators in H) is given by

$$QM^4 := \left\{ \hat{\underline{y}} \mid \hat{\underline{y}} = (\hat{x}_\mu + a_\mu \cdot 1)e^\mu ,\ a_\mu \in \mathbb{R} ,\ [\hat{x}_\mu , \hat{x}_\nu] = -i\hbar \hat{A}_{\mu\nu} , \langle e^\mu , e_\nu \rangle = \delta^\mu{}_\nu \right\}$$

where QM^4 is also endowed with a vector space structure. The metric in $\mathbb{M}^4$ is formulated by the expression $s^2 = g_{\mu\nu}(y^\mu - x^\mu)(y^\nu - x^\nu) = g_{\mu\nu} a^\mu a^\nu$, the corresponding quantity in QM^4 is $\hat{s}^2 = g_{\mu\nu}(\hat{y}^\mu - \hat{x}^\mu)(\hat{y}^\nu - \hat{x}^\nu) =$

$= g_{\mu\nu} a^{\mu} a^{\nu} \cdot 1$ and its average value is $\bar{s}^2 = \langle \phi | \hat{s}^2 \phi \rangle = g_{\mu\nu} a^{\mu} a^{\nu} = s^2$, $\forall \phi \in H$. Therefore $\hat{s}^2$ is translation invariant in q space-time.

2/ What is the relation between $\langle \phi | U_a \phi \rangle$, $\phi \in H$, and $s^2 = a_{\mu} a^{\mu}$? The latter describes the causal relation of two points in $\mathbb{M}^4$, while the former describes the same for two events ϕ and $U_a \phi$ in q space-time (Th.2.). For infinitesimal translations $\langle \phi | U_a \phi \rangle = \langle \phi | \phi \rangle - \frac{i}{\hbar} a^{\mu} \langle \phi | \hat{p}_{\mu} \phi \rangle = 1 - \frac{i}{\hbar} a^{\mu} \bar{p}_{\mu}$. This is not null in general, but for localized events in space, i.e. for $\phi(\underline{x}) = | \bar{\underline{x}} \rangle = \delta(\underline{x} - \bar{\underline{x}}) (\notin H)$ and for spacelike translations of these "events", $a_o = 0$, $- a_i a_i < 0$, $s^2 < 0$; $\langle \bar{\underline{x}} | U_{\underline{a}} | \bar{\underline{x}} \rangle = \langle \bar{\underline{x}} | \bar{\underline{x}} + \underline{a} \rangle = 0$, i.e. $| \bar{\underline{x}} \rangle$ and $| \bar{\underline{x}} + \underline{a} \rangle$ are causally disconnected. Now for a time-like translation of the event $\phi \in H$, we have: $a_o \neq 0$, $\underline{a} = 0$, $s^2 = a_o^2 > 0$; $\langle \phi | U_{a_o} \phi \rangle = \int_{\mathbb{R}} \bar{\phi}(q) \phi(q) \exp(-\frac{i}{\hbar} a_o q) \, dq \neq 0$, i.e. ϕ and $U_{a_o} \phi$ are causally connected. Generally we can say that the null cone structure become "smeared out" in q space-time.

(18) <u>q space-time rotations</u>: We introduce the rotations via the action of a 6-parameter unitary group $\omega_{\mu\nu} \mapsto U_{\omega_{\mu\nu}}$ in H, where $\omega_{\mu\nu} = - \omega_{\nu\mu}$, and for a $\phi \in H$

$$\phi' = U_{\omega_{\mu\nu}} \phi = e^{-\frac{i}{\hbar} \hat{M}_{\mu\nu} \omega_{\mu\nu}} \phi \, , \tag{4.13a}$$

$$\hat{x}'_{\varrho} = U_{\omega_{\mu\nu}} \, \hat{x}_{\varrho} \, U^{-1}_{\omega_{\mu\nu}} \tag{4.13b}$$

where $\hat{M}^{+}_{\mu\nu} = \hat{M}_{\mu\nu}$ and $\hat{M}_{\mu\nu} = -\hat{M}_{\nu\mu}$. To determine completely the rotations we have to give the concrete form of the self-adjoint generators $\hat{M}_{\mu\nu}$.[7] Because $\hat{M}_{\mu\nu}$ is a self-adjoint operator it corresponds to an observable of the test particle; we identify $\hat{M}_{\mu\nu}$ with the $\mu\nu$-component of the angular momentum observable of the test particle. Spin degrees of freedom have been not attached to the test particle so the total angular momentum observable is equal to the orbital angular momentum observable, i.e. $\hat{M}_{\mu\nu} = \hat{L}_{\mu\nu}$

and thus

$$\hat{M}_{\mu\nu} = \hat{L}_{\mu\nu} = \hat{x}_\mu \hat{p}_\nu - \hat{x}_\nu \hat{p}_\mu \qquad (4.14)$$

according to the c form of $\hat{L}_{\mu\nu}$. We shall call the component $\hat{M}_{oi} = -\hat{M}_{io} = \hat{K}_i$ the generator of a boost in the i-direction in space, while the component $\hat{M}_{ij} = -\hat{M}_{ji}$ is the generator of the usual space rotations in the i-j plane. Let us compute the CR's $[\hat{M}_{\mu\nu}, \hat{x}_\varrho]$, $[\hat{M}_{\mu\nu}, \hat{p}_\varrho]$ and $[\hat{M}_{\mu\nu}, \hat{M}_{\varrho\sigma}]$ with the aid of the CR's (3.6b), (4.6b) and (4.4). A formal noncommutative algebraic calculation yields , e.g., for $[\hat{M}_{\mu\nu}, \hat{x}_\varrho]$

$$[\hat{M}_{\mu\nu}, \hat{x}_\varrho] = [\hat{x}_\mu\hat{p}_\nu - \hat{x}_\nu\hat{p}_\mu, \hat{x}_\varrho] = [\hat{x}_\mu\hat{p}_\nu, \hat{x}_\varrho] - [\hat{x}_\nu\hat{p}_\mu, \hat{x}_\varrho] =$$
$$= \hat{x}_\mu[\hat{p}_\nu, \hat{x}_\varrho] + [\hat{x}_\mu, \hat{x}_\varrho]\hat{p}_\nu - \hat{x}_\nu[\hat{p}_\mu, \hat{x}_\varrho] - [\hat{x}_\nu, \hat{x}_\varrho]\hat{p}_\mu = i\hbar\,(g_{\nu\varrho}\hat{x}_\mu -$$
$$- g_{\mu\varrho}\hat{x}_\nu) - i\bar{\hbar}\,(\hat{A}_{\mu\varrho}\hat{p}_\nu - \hat{A}_{\nu\varrho}\hat{p}_\mu)\,,$$

and similarly

$$[\hat{M}_{\mu\nu}, \hat{p}_\varrho] = i\hbar\,(g_{\varrho\nu}\hat{p}_\mu - g_{\varrho\mu}\hat{p}_\nu) + i\hbar\bar{\hbar}^2\,(\hat{x}_\mu\hat{A}_{\nu\varrho} - \hat{x}_\nu\hat{A}_{\mu\varrho})\,,$$
$$[\hat{M}_{\mu\nu}, \hat{M}_{\varrho\sigma}] = i\hbar\,(g_{\nu\varrho}\hat{M}_{\mu\sigma} + g_{\mu\sigma}\hat{M}_{\nu\varrho} + g_{\varrho\mu}\hat{M}_{\sigma\nu} + g_{\sigma\nu}\hat{M}_{\varrho\mu}) + i\bar{\hbar}\big[(\hat{A}_{\sigma\nu}\hat{p}_\varrho\hat{p}_\mu -$$
$$- \bar{\hbar}^2\hat{x}_\mu\hat{x}_\varrho\hat{A}_{\sigma\nu}) + (\hat{A}_{\varrho\mu}\hat{p}_\sigma\hat{p}_\nu - \bar{\hbar}^2\hat{x}_\nu\hat{x}_\sigma\hat{A}_{\varrho\mu}) + (\hat{A}_{\mu\sigma}\hat{p}_\varrho\hat{p}_\nu - \bar{\hbar}^2\hat{x}_\nu\hat{x}_\varrho\hat{A}_{\mu\sigma}) +$$
$$+ (\hat{A}_{\nu\varrho}\hat{p}_\sigma\hat{p}_\mu - \bar{\hbar}^2\hat{x}_\mu\hat{x}_\sigma\hat{A}_{\nu\varrho})\big]$$

Note that $[\hat{A}_{\mu\nu}, \hat{x}_\varrho] = 0$ but $[\hat{A}_{\mu\nu}, \hat{p}_\varrho] \neq 0$, for, $[\hat{A}_{\mu\nu}, \hat{p}_o] = 0$, $[\hat{A}_{ij}, \hat{p}_\varrho] = 0$, $[\hat{A}_{oo}, \hat{p}_\varrho] = 0$, $[\hat{A}_{oi}, \hat{p}_j] = i\hbar\,\frac{1}{\underline{r}}\,(\delta_{ij} - \frac{1}{r^2}\,\hat{x}_i\hat{x}_j)$.

Introduce the following notation

$$\hat{N}_{\mu\nu} = \hat{p}_\mu \hat{p}_\nu - \bar{\hbar}^2\, \hat{x}_\nu \hat{x}_\mu \qquad (4.15)$$

and observe that $\hat{p}_\mu\hat{p}_\nu - \bar{\hbar}^2\hat{x}_\nu\hat{x}_\mu = \hat{p}_\nu\hat{p}_\mu + i\hbar\bar{\hbar}^2\,\hat{A}_{\mu\nu} - i\hbar\bar{\hbar}^2\,\hat{A}_{\mu\nu} - \bar{\hbar}^2\hat{x}_\mu\hat{x}_\nu = \hat{p}_\nu\hat{p}_\mu - \bar{\hbar}^2\hat{x}_\mu\hat{x}_\nu$, so $\hat{N}_{\mu\nu} = \hat{N}_{\nu\mu}$. With the use of these facts and notation let us summarize the CR's above

$$[\hat{M}_{\mu\nu}, \hat{x}_\varrho] = i\hbar\,(g_{\nu\varrho}\hat{x}_\mu - g_{\mu\varrho}\hat{x}_\nu) + i\bar{\hbar}\,(\hat{A}_{\nu\varrho}\hat{p}_\mu - \hat{A}_{\mu\varrho}\hat{p}_\nu)\,, \qquad (4.16)$$
$$[\hat{M}_{\mu\nu}, \hat{p}_\varrho] = i\hbar\,(g_{\varrho\nu}\hat{p}_\mu - g_{\varrho\mu}\hat{p}_\nu) + i\hbar\bar{\hbar}^2\,(\hat{A}_{\nu\varrho}\hat{x}_\mu - \hat{A}_{\mu\varrho}\hat{x}_\nu)\,, \qquad (4.17)$$
$$[\hat{M}_{\mu\nu}, \hat{M}_{\varrho\sigma}] = i\hbar\,(g_{\nu\varrho}\hat{M}_{\mu\sigma} + g_{\mu\sigma}\hat{M}_{\nu\varrho} + g_{\varrho\mu}\hat{M}_{\sigma\nu} + g_{\sigma\nu}\hat{M}_{\varrho\mu}) +$$
$$+ i\bar{\hbar}\,(\hat{A}_{\nu\varrho}\hat{N}_{\mu\sigma} + \hat{A}_{\mu\sigma}\hat{N}_{\nu\varrho} + \hat{A}_{\varrho\mu}\hat{N}_{\sigma\nu} + \hat{A}_{\sigma\nu}\hat{N}_{\varrho\mu}) \qquad (4.18)$$

We see that the CR's of the Hermitian generators of the translations and rotations in q space-time agree with the CR's of the infinitesimal generators of the Poincaré group in the order of $\hbar$, and for $\mu = i$, $\nu = j$, $\varrho = k$ and $\sigma = \ell$ they completely are equal to the CR's of the infinitesimal generators of the rotation and translation group in 3-space. Thus we can say that an observable $\hat{F} = \hat{F}(p_\mu, \hat{x}_\nu) = \hat{F}(\hat{\underline{p}}, \hat{\underline{x}})$ or a set of such observables is a "scalar" or a "4-vector" or a "4-tenzor" if it is in order a scalar or a 4-vector or a 4-tenzor in the order of $\hbar$ according to the usual definition of these objects in rq mechanics. So, e.g., $\hat{p}_\mu$ and $\hat{x}_\mu$ are 4-vectors, an other example, $\hat{W}^\mu = \frac{1}{2}\varepsilon^{\mu\nu\varrho\sigma}\hat{M}_{\nu\varrho}\hat{p}_\sigma$ (the Pauli-Lubanski 4-vector) then an easy calculation produces, using (4.17) and (4.18), that $[\hat{M}_{\mu\nu}, \hat{W}_\varrho] = i\hbar\,(g_{\nu\varrho}\hat{W}_\mu - g_{\mu\varrho}\hat{W}_\nu) + i\hbar\,\mathcal{W}_{\mu\nu\varrho}$, thus $\hat{W}_\mu$ is a 4-vector, too.

We note that the usual Casimir operators $\hat{P}^2 = \hat{p}_\mu\hat{p}^\mu$ and $\hat{W}^2 = -\hat{W}_\mu\hat{W}^\mu$ of the Poincaré group clearly do not remain invariant operators!

We can conclude that the restricted Poincaré-like transformations (translations and rotations) in q space-time are induced by the elements of a 10-parameter unitary group $(a,\omega) \mapsto U(a,\omega)$ in H, and the events and the observables transform under such an element according to

$$\phi' = U(a,\omega)\phi \quad , \qquad \phi \in H \tag{4.19}$$

and

$$\hat{F}' = U(a,\omega)\,\hat{F}\,U(a,\omega)^{-1} \tag{4.20}$$

For infinitesimal transformations we can write

$$U(a,\omega) = 1 - \frac{i}{\hbar}\,a^\mu\hat{p}_\mu - \frac{i}{2\hbar}\,\omega^{\mu\nu}\hat{M}_{\mu\nu} \tag{4.21}$$

where $\hat{p}_\mu$ and $\hat{M}_{\mu\nu}$ satisfy the CR's (4.6), (4.17) and (4.18).

(19) <u>q space-time inversions</u>: Let P be the space inversion operator then by definition $P\phi(\underline{x}) = \phi(-\underline{x})$, $P^2\phi(\underline{x}) = \phi(\underline{x})$, $\phi(\underline{x}) \in L^2(\mathbb{R}^3)$, and $\langle P\phi | P\phi\rangle = \langle\phi|\phi\rangle$ thus P is a unitary operator and $P^2 = 1$, $P^+ = P = P^{-1}$, as in the old q mechanics. Furthermore we get from these relations

$$P\,\hat{\underline{x}}\,P = -\,\hat{\underline{x}} \tag{4.22}$$

and it follows from the definition of $\hat{r}$

$$P\,\hat{r}\,P = -\hat{r}\;, \qquad P(\hat{r}\,\phi(\underline{x})) = -\underline{r}\,\phi(-\underline{x}) \qquad (4.23)$$

Then the CR (3.3) provides $P(\hat{t}\hat{r} - \hat{r}\hat{t})P = P\hat{t}PP\hat{r}P - P\hat{r}PP\hat{t}P =$ $= -P\hat{t}P\hat{r} + \hat{r}P\hat{t}P = -i\hbar' P^2 = -i\hbar'$, therefore

$$P\,\hat{t}\,P = -\,\hat{t}\;, \qquad (4.24a)$$

and

$$P\,\hat{x}_o P = -\,\hat{x}_o \qquad (4.24b)$$

This means that the space inversion <u>implies the time inversion</u> in q space-time and vice versa, and thus it implies the space-time inversion, too. We obtained that the time inversion and space inversion are not independent symmetry transformations of this q space-time model.

(20) <u>Other unitary symmetries of q space-time</u>: We can write for a one-parameter unitary group $a \mapsto U_a$ in H

$$U_a = \exp\{-i\hat{F}a\} \qquad (4.25)$$

where F is a self-adjoint operator in H and thus it is a function of the members of the irreducible system of operators in H, or, taking into acount (3.5) and (3.7), we could say that $\hat{F}$ is a function of the 4-position operatior $\hat{x}_\mu$, i.e. $\hat{F} = F(\hat{x}_o, \hat{\underline{x}})$, that is to say, $\hat{F}$ is a "q space-time dependent function".
Now let U(1) be the local gauge group of a clft locally invariant under U(1) (e.g., c electrodynamics) then the elements of U(1) have the form

$$U(f) = e^{-if(x)e} \qquad (4.26)$$

where f(x) is a function on M^4. Let us make the formal correspondence $f = f(x) \mapsto \hat{F} = f(\hat{x})$ between the elements of the set of functions in (4.26) and the elements of the set of operators in (4.25), then, for a fixed a, a unitary operator of the form (4.25) corresponds to each element of U(1) and the collection of such one-parameter unitary groups in H corresponds to U(1). This analogy suggests that we call the unitary transformations of the form (4.25) in H gauge-like transformations and the corresponding symmetries in q space-time gauge-like symmetries.

(21) We can close this section with the observation that the unitary symmetries of q space-time consist of Poincaré-like symmetries; translations, rotations and inversion in q space-time, and of gauge-like symmetries. The space inversion implies the time inversion, and vice versa, in q space-time. In the c limit of the model, in the formal limit $\hbar \to 0$, the unitary symmetries of q space-time reduce to the Poincaré symmetries of $\mathbb{M}^4$.

5. Some properties of q space-time

(22) Now we mean under a coordinate system of a given observer a coordinate system in 3-space spanned by three rectangular measuring lines, and a collection of clocks placed densely in this coordinate system in 3-space, as in c theory. These macroscopic measuring lines and clocks are the measuring apparatuses associated with the observables $\hat{r}$ and $\hat{t}$, respectively. The coordinate systems of different observers transform among themselves according to the law of special relativity in a good approximation, i.e. two such macroscopic coordinate systems are connected by Poincaré transformations (in a good approximation). The measuring apparatuses in a given coordinate system, associated with different observables, are c objects governed by the laws of c r theory. Such an arrangement of things are guaranteed by the existence of the c limit of q space-time under consideration, which means that the c description provides a good approximation in large space-time regions relative to $\hbar$ and these space-time regions are those in which the coordinate systems of different observers operationally are available for the observers. New effects due to the q nature of space-time should be expected in

$\hbar$ size regions of space-time.

(23) The general transformation laws of the observables and the events under Poincaré-like transformations are given by (4.19) and (4.20). Let us consider these transformations for infinitezimal Poincaré-like transformations.

1/ Infinitezimal translations: $U_a = 1 - \frac{i}{\hbar} a^\mu \hat{p}_\mu$; The transformation of events is

$$\phi' = \phi - \frac{i}{\hbar} a^\mu \hat{p}_\mu \phi , \qquad \phi \in H \tag{5.1}$$

and the transformation of observables is

$$\hat{F}' = (1 - \frac{i}{\hbar} a^\mu \hat{p}_\mu)\hat{F}(1 + \frac{i}{\hbar} a^\mu \hat{p}_\mu) = \hat{F} + \frac{i}{\hbar}[\hat{F}, \hat{p}_\mu] a^\mu \tag{5.2}$$

Thus the change of the observable $\hat{F}$ is

$$\delta \hat{F} = \frac{i}{\hbar} a^\mu [\hat{F}, \hat{p}_\mu] \tag{5.3}$$

and this allows us to define formally the "partial derivatives" of an observable $\hat{F} = F(\hat{x}_o, \underline{\hat{x}})$ with respect to the $\hat{x}_\mu$'s. They are

$$\frac{\partial F(\hat{x})}{\partial \hat{x}^\mu} := \lim_{a_\mu \to 0} \frac{\delta \hat{F}}{a^\mu} = \frac{i}{\hbar}[\hat{F}, \hat{p}_\mu] \tag{5.4}$$

2/ Infinitesimal Poincaré-like transformations: $U(a,\omega) = 1 - \frac{i}{\hbar} a^\mu \hat{p}_\mu - \frac{i}{2\hbar}\omega^{\mu\nu}\hat{M}_{\mu\nu}$; The transformation of events and of observables are

$$\phi' = \phi - \frac{i}{\hbar} a^\mu \hat{p}_\mu \phi - \frac{i}{2\hbar}\omega^{\mu\nu}\hat{M}_{\mu\nu}\phi , \tag{5.5}$$

$$\hat{F}' = (1 - \frac{i}{\hbar} a^\mu \hat{p}_\mu - \frac{i}{2\hbar}\omega^{\mu\nu}\hat{M}_{\mu\nu})\hat{F}(1 + \frac{i}{\hbar} a^\mu \hat{p}_\mu + \frac{i}{2\hbar}\omega^{\mu\nu}\hat{M}_{\mu\nu}) =$$

$$= \hat{F} + \frac{i}{\hbar}[\hat{F}, \hat{p}_\mu] a^\mu + \frac{i}{2\hbar}[\hat{F}, \hat{M}_{\mu\nu}]\omega^{\mu\nu} \tag{5.6}$$

and the change of $\hat{F}$ is

$$\delta \hat{F} = \hat{F}' - \hat{F} = \frac{i}{\hbar}\left([\hat{F}, \hat{p}_\mu] a^\mu + \frac{1}{2}[\hat{F}, \hat{M}_{\mu\nu}]\omega^{\mu\nu}\right) \tag{5.7}$$

<u>Examples</u>: a/ Let $\hat{F} = \hat{x}_\mu$ then

$$\hat{x}'_\mu = \hat{x}_\mu + \frac{i}{\hbar}[\hat{x}_\mu, \hat{p}_\nu] a^\nu + \frac{i}{2\hbar}[\hat{x}_\mu, \hat{M}_{\nu\rho}]\omega^{\nu\rho} = \hat{x}_\mu + a_\mu \cdot 1 +$$
$$+ \frac{1}{2}(g_{\rho\mu}\hat{x}_\nu - g_{\nu\mu}\hat{x}_\rho)\omega^{\nu\rho} + \frac{1}{2}\frac{\hbar\!\!\!/}{\hbar}(\hat{A}_{\rho\mu}\hat{p}_\nu - \hat{A}_{\nu\mu}\hat{p}_\rho)\omega^{\nu\rho} = \hat{x}_\mu + a_\mu \cdot 1 -$$
$$- \omega_\mu{}^\nu \hat{x}_\nu + \hbar\!\!\!/^{-1}\hat{A}_{\mu\rho}\omega^{\rho\nu}\hat{p}_\nu = (\delta_\mu{}^\nu - \omega_\mu{}^\nu)\hat{x}_\nu + a_\mu \cdot 1 + \hbar\!\!\!/^{-1}\hat{A}_{\mu\rho}\omega^{\rho\nu}\hat{p}_\nu \tag{5.8}$$

where (4.4) and (4.16) were used. If we compare (5.8) with the usual Poincaré transformation of x_μ then we see that (5.8) differs from that in the third term of the order $\hbar^{-1}$.

b/ Let $\hat{F} = \hat{p}_\mu$ then

$$\hat{p}'_\mu = \hat{p}_\mu + \frac{i}{\hbar}[\hat{p}_\mu, \hat{p}_\nu]a^\nu + \frac{i}{2\hbar}[\hat{p}_\mu, \hat{M}_{\nu\varrho}]\omega^{\nu\varrho} = \hat{p}_\mu - \hbar\hat{A}_{\mu\nu}a^\nu +$$
$$+ \frac{1}{2}(g_{\mu\varrho}\hat{p}_\nu - g_{\mu\nu}\hat{p}_\varrho)\omega^{\nu\varrho} + \frac{1}{2}\hbar(\hat{A}_{\varrho\mu}\hat{x}_\nu - \hat{A}_{\nu\mu}\hat{x}_\varrho)\omega^{\nu\varrho} = \hat{p}_\mu - \omega_\mu{}^\nu\hat{p}_\nu -$$
$$- \hbar\hat{A}_{\mu\nu}a^\nu + \hbar\hat{A}_{\mu\nu}\omega^{\nu\varrho}\hat{x}_\varrho = (\delta_\mu{}^\nu - \omega_\mu{}^\nu)\hat{p}_\nu - \hbar\hat{A}_{\mu\nu}(a^\nu - \omega^{\nu\varrho}\hat{x}_\varrho) \qquad (5.9)$$

where (4.6b) and (4.17) were used. Comparing (5.9) with the usual Poincaré transformation of p_μ we observe that (5.9) differs from that in the second term of the order $\hbar$, and that $\hat{p}_\mu$ is not translation invariant!

(24) Let us introduce the notion of the time derivative of an observable. We can write for the change of the expected value of an observable $\hat{F}$ under a translation in time with a in positive direction that

$$\Delta\bar{F} = \bar{F}_a - \bar{F}_o = \langle\Phi_a|\hat{F}\,\Phi_a\rangle - \langle\Phi|\hat{F}\,\Phi\rangle = \langle\Phi|U_a^{-1}\hat{F}U_a - \hat{F}|\Phi\rangle =$$
$$= a\langle\Phi|\frac{i}{\hbar}[\hat{p}_o, \hat{F}]|\Phi\rangle$$

then formally

$$\frac{d\hat{F}}{dt} := \lim_{a\to o}\frac{\Delta\bar{F}}{a} = c\langle\Phi|\frac{i}{\hbar}[\hat{p}_o, \hat{F}]|\Phi\rangle , \quad \forall\Phi\in H$$

and thus, also formally,

$$\frac{d\hat{F}}{dt} := i\frac{c}{\hbar}[\hat{p}_o, \hat{F}] = \frac{i}{\hbar}[\hat{E}, \hat{F}] \qquad (5.10)$$

<u>Notes</u>: a/ One can define the 3-velocity observable of a test particle as the time derivative of $\hat{r}$ and the i-th component of this observable as the time derivative of $\hat{x}_i$, i.e. $\hat{v} := \frac{d\hat{r}}{dt} = \frac{i}{\hbar}[\hat{E}, \hat{r}] = 0$, $\hat{v}_i := \frac{d\hat{x}_i}{dt} = \frac{ic}{\hbar}[\hat{p}_o, \hat{x}_i] = 0$, taking into acount (4.5b) and (4.4). Then we can interpret $\hat{r} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ as the 3-position observable of the test particle in <u>its</u> <u>rest</u> <u>frame</u> and thus $\hat{x}_o = c\hat{t}$ is <u>the</u> <u>proper</u> <u>time</u> observable of the test particle. Also $\hat{x}_\mu$ is the 4-position observable of the test particle in its rest frame. The 4-velocity observables of a test particle can be also defined as follows

$$\hat{u}_\mu := \frac{d\hat{x}_\mu}{dx^o} = \frac{i}{\hbar}[\hat{p}_o, \hat{x}_\mu] = -g_{\mu o} \qquad (5.11)$$

where (4.4) was applied. Let us execute an infinitesimal boost in the i-direction, then

$$\hat{x}'_i = U_{\omega_{oi}} \hat{x}_i U^{-1}_{\omega_{oi}} = \hat{x}_i + \frac{i}{\hbar}[\hat{x}_i, \hat{M}_{oi}]\omega_{oi} = \hat{x}_i - (\hat{x}_o + \hbar^{-1}\frac{\hat{x}_i}{\underline{r}}\hat{p}_i)\omega_{oi}$$

where (3.7) and (4.16) were applied. Thus the velocity of the test particle boosted infinitesimally in the i-direction is

$$\hat{v}'_i = \frac{d\hat{x}'_i}{d\hat{t}} = \frac{ic}{\hbar}[\hat{p}_o, \hat{x}'_i] = c\,\omega_{oi} - \frac{ic}{\hbar\hbar}\frac{\hat{x}_i}{\underline{r}}[\hat{p}_o, \hat{p}_i]\omega_{oi} - \frac{ic}{\hbar\hbar}[\hat{p}_o, \frac{\hat{x}_i}{\underline{r}}]\hat{p}_i\omega_{oi} = v_i\left[1 + \left(\frac{\hat{x}_i}{\underline{r}}\right)^2\right]$$

where $\omega_{oi} = \frac{v_i}{c}$ and we used (4.6a). The expectation value of this expression differs from the c one in the term $\left(\frac{\overline{x_i}}{\underline{r}}\right)^2$ which term is minimal if $\overline{x}_i = 0$ and is maximal if $\overline{x}_i = \overline{\underline{r}}$, in the latter case $\overline{v}'_i = 2v_i$, i.e. it is two times larger than in the c case. Now the square of the velocity of a test particle after boosted separately in all direction is

$$\hat{v}'^2 = \hat{v}'_i\hat{v}'_i = v_1^2\left[1 + \left(\frac{\hat{x}_1}{\underline{r}}\right)^2\right] + v_2^2\left[1 + \left(\frac{\hat{x}_2}{\underline{r}}\right)^2\right] + v_3^2\left[1 + \left(\frac{\hat{x}_3}{\underline{r}}\right)^2\right],$$

while the velocity of the test particle after a (non-separated) infinitesimal rotation is

$$\hat{v}' = \frac{d\hat{\underline{r}}'}{d\hat{t}} = \frac{i}{\hbar}[\hat{E}, \hat{\underline{r}}'] = \frac{2}{\underline{r}}(v_1\hat{x}_1 + v_2\hat{x}_2 + v_3\hat{x}_3)$$

where $\hat{\underline{r}}' = \hat{\underline{r}} + \frac{i}{2\hbar}[\hat{\underline{r}}, \hat{M}_{\mu\nu}]\omega^{\mu\nu}$, as one can verify this easily with a formal calculation.

b/ $\hat{t}$ transforms under an infinitesimal boost in the i-direction as follows

$$\hat{t}' = \hat{t} + \frac{i}{\hbar}[\hat{t}, \hat{M}_{oi}]\omega_{oi} = \hat{t} - \frac{2}{c}\hat{x}_i\omega_{oi} = \hat{t} - \frac{2}{c^2}\hat{x}_i v_i$$

where (5.8) was applied for $\mu = 0$ and $a_\mu = 0$. Then the change of $\hat{t}$ is $\delta\hat{t} = \hat{t}' - \hat{t} = -\frac{2}{c^2}\hat{x}_i v_i$ and the infinitesimal change of the expected value of $\hat{t}'$ observing this from the original frame is

$$d\bar{t}' = \bar{t}'_2 - \bar{t}'_1 = \bar{t}_2 - \bar{t}_1 + \delta\bar{t} = d\bar{t} - \frac{2}{c^2}v_i\overline{x}_i$$

which differs from the c expression in the factor 2. Now the transformation of $\hat{t}$ under an infitesimal rotation is

$$\hat{t}' = \hat{t} + \frac{i}{2\hbar}[\hat{t}, \hat{M}_{\mu\nu}]\omega^{\mu\nu} = \hat{t} - \frac{1}{c}(\omega_o{}^\mu\hat{x}_\mu - \hbar^{-1}\hat{A}_{o\mu}\omega^{\mu\nu}\hat{p}_\nu)$$

where (5.8) was used, and then

$$d\bar{t}' = d\bar{t} + \delta\bar{t} = d\bar{t} - \frac{1}{c}(\omega_o^{\ \mu}\bar{x}_\mu - \hbar^{-1}\hat{A}_{o\mu}\bar{p}_\nu\omega^{\mu\nu})$$

which differs from the c expression in the third term of the order $\hbar^{-1}$.

(25) We can say that an observable is conserved in time if and only if its time derivative vanishes, i.e. iff

$$\frac{d\hat{F}}{d\bar{t}} = \frac{ic}{\hbar}[\hat{p}_o, \hat{F}] = \frac{i}{\hbar}[\hat{E}, \hat{F}] = 0 \qquad (5.12)$$

<u>Examples</u>: 1/ $\hat{F} = \hat{p}_\mu$, the 4-momentum observable of a test particle moving freely in q space-time. Then from (4.6b)

$$\frac{d\hat{p}_\mu}{d\bar{t}} = i\frac{c}{\hbar}[\hat{p}_o, \hat{p}_\mu] = -c\hbar\,\hat{A}_{o\mu} \qquad (5.13)$$

For $\mu = 0$

$$\frac{d\hat{p}_o}{d\bar{t}} = 0 \qquad (5.13a)$$

so $\hat{p}_o$ is conserved in time and thus $\hat{E} = c\hat{p}_o$, the energy observable of the test particle is also conserved in time. For $\mu = i$

$$\frac{d\hat{p}_i}{d\bar{t}} = -c\hbar\frac{\hat{x}_i}{\underline{r}} \qquad (5.13b)$$

(see (3.7)), $\hat{p}_i$ is <u>not</u> conserved in time ! We know from c theory that $\frac{dp_i}{dt}$ is the i-th component of the force acting on the c particle, thus we can interpret $\hat{f}_i = \frac{d\hat{p}_i}{d\bar{t}}$ as the i-th component of the <u>force</u> acting on the <u>free</u> test particle in q space-time, and (5.13b) (or (5.13)) provides formally the equation of motion of the free test particle moving in q space-time. Then the <u>force</u> is given by

$$\hat{f} = \frac{d\hat{p}}{d\bar{t}} = i\frac{c}{\hbar}[\hat{p}_o, \hat{p}] = -\hbar' \qquad (5.14)$$

where (4.8) was used. Taking into acount this interpretation $\frac{d\hat{p}_\mu}{d\bar{x}_o}$ defines the 4-force observable $\hat{f}_\mu$ of a test particle in q space-time.

<u>Notes</u>: a/ We see that a constant force acts on the free test particle in q space-time, forcing it to the origin of its rest frame. It is an <u>atractive</u> <u>force</u>! Note that the expectation value of the energy $\hat{E} = c\hat{p}_o = \hbar'\underline{r}$ of the particle raises linearly with the radial distance $\bar{r}$ for positive $\bar{r}$ and $\bar{t}$ values (in the subspace H_1 of the event space H (cf. sec.3. paragraph (13))). Classically $\bar{E}$ is equiv-

alent with a linear potential $V(r) = \hbar' \cdot r$ in which the c point particle moves; the force is given in this potential by $\underline{F} = -\text{grad}\, V(r)$ and $F_i = -\frac{\partial V(r)}{\partial x_i} = -\hbar' \frac{x_i}{r}$. If we compare this phenomenon with the current quark models of hadrons then we can interpret this as a <u>confinement phenomenon in q space-time</u>!; the free test particle is forced to the origin of its rest frame in q space-time. We can give an estimation for the magnitude of this force if we choose the characteristic distance magnitude 10^{-15}m and time duration 10^{-24}sec in hadronic events as characteristic place and time uncertainty values, i.e. $\Delta r \sim 10^{-15}$m, $\Delta t \sim 10^{-24}$sec, then $\hbar' \sim \Delta r \Delta t \sim 10^{-39}$msec, $\hbar = c\hbar' \sim 3.10^{-31}\text{m}^2$, $\hbar = \frac{\hbar}{\hbar} \sim 3{,}33.10^{-4}\ \frac{\text{kg}}{\text{sec}}$ and $\hbar' = \frac{\hbar}{\hbar'} \sim 10^5$N. This provides an <u>enormous force</u> confining the particle inside an $\hbar' \sim \Delta r\, \Delta t \sim 10^{39}$msec size space-time bubble at any time. (Note that only the cells of $\mathbb{M}^4$, having at least a size $\hbar \sim 3.10^{-31}\text{m}^2$ correspond to physically observable reality and these minimal cells realize in $\mathbb{M}^4$ the set of events in q space-time which provides the c limit of the model, i.e. if "$\hbar \to 0$" these events concentrate on the points of $\mathbb{M}^4$. Redefining the translations on this set of events (bubbles) the generators of these translations will commute among themselves and thus they will be translation invariant as in the usual theory. Another note that the time translation invariance remain valid in this model.)

b/ Consider the change of the expectation value of an observable $\hat{F}$ in time. It is

$$\Delta \overline{F} = \langle U_{a_o}\phi\,|\hat{F}U_{a_o}\phi\rangle - \langle\phi|\hat{F}\phi\rangle = \frac{a_o}{c}\langle\phi|\frac{i}{\hbar}[\hat{p}_o, \hat{F}]|\phi\rangle =$$

$$= \langle\phi|\frac{d\hat{F}}{d\bar{t}}|\phi\rangle d\bar{t} = \overline{\left(\frac{d\hat{F}}{d\bar{t}}\right)} d\bar{t} \ , \quad d\bar{t} = \frac{a_o}{c}$$

then, formally,

$$\overline{F}(\bar{t}) = \int_o^{\bar{t}} d\bar{t}\, \langle\phi|\frac{d\hat{F}}{d\bar{t}}|\phi\rangle \tag{5.15}$$

Thus $\overline{E}(\bar{t}) = \int_o^{\bar{t}} d\bar{t}\ \langle\phi|\frac{d\hat{E}}{d\bar{t}}|\phi\rangle = 0$, $\overline{p}(\bar{t}) = \int_o^{\bar{t}} d\bar{t}\ \langle\phi|\frac{d\hat{p}}{d\bar{t}}|\phi\rangle = -\hbar'\bar{t}$,

$$\overline{p}_i(\overline{t}) = \int_0^{\overline{t}} d\overline{t}\ \langle\Phi|\frac{d\hat{p}_i}{d\overline{t}}|\Phi\rangle = -\hbar' \int_0^{\overline{t}} \langle\Phi|\frac{\hat{x}_i}{r}|\Phi\rangle = -\hbar' \overline{\left(\frac{x_i}{r}\right)}\cdot\overline{t}\ ,$$

note that $\frac{d^2\hat{p}_i}{d\overline{t}^2} = 0$, furthermore that the time avarage of $\overline{p}(\overline{t})$ and $\overline{p}_i(\overline{t})$ between $-\overline{t}$ and $+\overline{t}$ vanish , i.e.

$$\overline{p} = \lim_{\overline{t}\to\infty} \frac{1}{2\overline{t}} \int_{-\overline{t}}^{+\overline{t}} d\overline{t}\ \overline{p}(\overline{t}) = 0, \quad \overline{p}_i = \lim_{\overline{t}\to\infty} \frac{1}{2\overline{t}} \int_{-\overline{t}}^{+\overline{t}} d\overline{t}\ \overline{p}_i(\overline{t}) = 0.$$

2/ a/ $\hat{F} = \hat{M}_{ij}$ or $\hat{F} = \hat{J}_i = \varepsilon_{ijk}\hat{M}_{jk}$, the i-th component of the 3-angular momentum observable of a test particle. Then

$$\frac{d\hat{M}_{ij}}{d\overline{t}} = \frac{ic}{\hbar}[\hat{p}_o, \hat{M}_{ij}] = c\left\{(g_{oj}\hat{p}_i - g_{oi}\hat{p}_j) + \hbar(\hat{A}_{jo}\hat{x}_i - \hat{A}_{io}\hat{x}_j)\right\} =$$

$$= c\hbar\left(\frac{1}{r}\hat{x}_i\hat{x}_j - \frac{1}{r}\hat{x}_j\hat{x}_i\right) = 0 \qquad (5.16)$$

where (4.17) was applied. Then $\hat{J}_i$'s are conserved in time as we expected.

b/ $\hat{F} = \hat{J}_i\hat{J}_i = \hat{J}^2$. Then

$$\frac{d\hat{J}^2}{d\overline{t}} = i\frac{c}{\hbar}[\hat{p}_o, \hat{J}_i\hat{J}_i] = 0, \qquad (5.17)$$

thus $\hat{J}^2$ is conserved in time.

3/ $\hat{F} = \hat{m}^2 = \frac{1}{c}\hat{p}_\mu\hat{p}^\mu$, the square of the mechanical (or bare) mass observable of a test particle in q space-time. (Note that $\hat{m}^2$ is clearly a scalar in the sense metioned above in sec.4, paragraph (18)). Then

$$\frac{d\hat{m}^2}{d\overline{t}} = \frac{i}{\hbar c}[\hat{p}_o, \hat{p}_\mu\hat{p}^\mu] = \frac{i}{\hbar c}\left([\hat{p}_o, \hat{p}_\mu]\hat{p}^\mu + \hat{p}^\mu[\hat{p}_o, \hat{p}_\mu]\right) = -\frac{\hbar}{c}(\hat{A}_{o\mu}\hat{p}^\mu +$$

$$+ \hat{p}^\mu\hat{A}_{o\mu}) = \frac{\hbar}{c}\left(\hat{p}_i\hat{A}_{oi} + \hat{A}_{oi}\hat{p}_i\right) = \frac{\hbar}{c}\left(2\hat{A}_{oi}\hat{p}_i + \frac{\hbar}{i}\frac{2}{r}\right) = \frac{2\hbar^2}{c}\left(\hat{x}_o - i\hbar\frac{\hat{1}}{r}\right) =$$

$$= 2\hbar^2\left(\hat{t} - i\hbar'\frac{\hat{1}}{r}\right) \qquad (5.18)$$

where (4.4) and (3.5b) were used. Thus $\hat{m}^2$ is not conserved in time. Observe that $\frac{d\hat{m}^2}{d\overline{t}}$ is not a self-adjoint operator, while $\hat{m}^2$ is! (The Lie bracket operation does not preserve the self-adjointness property.) We get for the real part $\mathrm{Re}\,\frac{d\hat{m}^2}{d\overline{t}} = 2\hbar^2\hat{t}$ that

$$\mathrm{Re}\ \overline{m}^2(\overline{t}) = \int_0^{\overline{t}} d\overline{t}\ \langle\Phi|2\hbar^2\hat{t}|\Phi\rangle = 2\hbar^2\frac{1}{2}\overline{t}^2 = \hbar^2\overline{t}^2 \quad \text{and}$$

$$\mathrm{Re}\ \overline{m}(\overline{t}) = \pm\hbar\ \overline{t} \qquad (5.19)$$

We can attach the negative root in (5.19) to the anti-particle and thus q space-time inversion + charge conjugation provides positive bare mass for anti-particles (cf. below). The time av-

erages of the two roots are also zero, i.e. Re $\bar{m} = \lim_{\bar{t}\to\infty} \frac{1}{2\bar{t}} \times$
$\times \int_{-\bar{t}}^{+\bar{t}} d\bar{t}\ \mathrm{Re}\ \bar{m}(\bar{t}) = 0.$

4/ $\hat{F} = \hat{m} = \frac{1}{c}\gamma^{\mu}\hat{p}_{\mu}$, the bare mass observable of a test particle with spin $\frac{1}{2}$, where the γ^{μ}'s are the Dirac matrices. Then

$$\frac{d\hat{m}}{d\bar{t}} = \frac{i}{\hbar}[\hat{p}_o, \gamma^{\mu}\hat{p}_{\mu}] = \frac{i}{\hbar}\gamma^{\mu}[\hat{p}_o, \hat{p}_{\mu}] = -\hbar\gamma^{\mu}\hat{A}_{o\mu} = \frac{\hbar}{r}\gamma_i\hat{x}_i \qquad (5.20)$$

So $\hat{m}$ does not be conserved in time. Futhermore

$$\underline{\underline{\bar{m}}}(\bar{t}) = \int_0^{\bar{t}} d\bar{t}\ \langle\phi|\frac{d\hat{m}}{d\bar{t}}|\phi\rangle = \hbar\,\overline{(\frac{1}{r}\ \gamma_i x_i)}\cdot\bar{t}$$

where $\underline{\underline{\bar{m}}}(\bar{t})$ is a 4x4 matrix. Let $\gamma_i = \begin{pmatrix} o & \sigma_i \\ -\sigma_i & o \end{pmatrix}$ where σ_i's are the Pauli matrices, then $\gamma_i\bar{x}_i = \begin{pmatrix} o & \sigma_i\bar{x}_i \\ -\sigma_i\bar{x}_i & o \end{pmatrix}$. Diagonalize $\sigma_i\bar{x}_i$ then $\sigma_i\bar{x}_i = \begin{pmatrix} \bar{r} & o \\ o & -\bar{r} \end{pmatrix}$ and thus

$$\overline{\frac{1}{r}\gamma_i x_i} = \begin{pmatrix} o & o & 1 & o \\ o & o & o & -1 \\ -1 & o & o & o \\ o & 1 & o & o \end{pmatrix} = \underline{\underline{\Gamma}}$$

We obtained

$$\underline{\underline{\bar{m}}}(\bar{t}) = \hbar\,\underline{\underline{\Gamma}}\cdot t \qquad (5.21)$$

We see that $\underline{\underline{\bar{m}}}(\bar{t})$ is purely real and implies negative bare mass expectation values for anti-particle, but q space-time inversion + charge conjugation provides again pozitive bare mass for anti-particles. In this q space-time model, only the <u>CPT theorem</u> would remain an exact symmetry of nature: as it is well-known the parity inversion implies the charge conjugation in nature, but we saw that the parity inversion implies the time inversion, too, in this q space-time background. We can identify the subspace H_1 of the event space H with the event space of the test particle, while the subspace H_2 belongs to the anti (test) particle.

The time average of $\underline{\underline{\bar{m}}}(\bar{t})$ vanishes, too, i.e. $\underline{\underline{\bar{m}}} = \lim_{\bar{t}\to\infty} \frac{1}{2\bar{t}} \times$
$\times \int_{-\bar{t}}^{+\bar{t}} d\bar{t}\ \underline{\underline{\bar{m}}}(\bar{t}) = 0.$

6. The one particle theory in q space-time

(26) Free case: The 4-position observable of the particle in its rest frame is $\hat{x}_\mu$ satisfying (3.6) ; $\hat{x}_o = c\hat{t}$ is the proper time observable of the particle. The 4-momentum observable of the particle is $\hat{p}_\mu$ satisfying (4.6). The 4-velocity observable of the particle in its rest frame is given by (5.11). The 4-force observable acting on the particle is $\hat{f}_\mu = \frac{d\hat{p}_\mu}{d\hat{x}_o} = \frac{i}{\hbar}[\hat{p}_o, \hat{p}_\mu] = -\hbar\hat{A}_{o\mu}$, and the equation of motion of the particle is given by (5.13).

a/ scalar particle: The state functions are the elements of H ; $\phi = \phi(\underline{x})$ if $H = L^2(\mathbb{R}^3)$, pure states are the events of q space-time. Let ϕ_o be a pure state of the particle then the particle is at the point $\bar{x}^o_\mu = \langle\phi_o|\hat{x}_\mu\phi_o\rangle$; the world line of the particle is given by the mapping $a_o \longmapsto U_{a_o}\phi_o = [\exp(-\frac{i}{\hbar}a_o\hat{p}_o)]\phi_o$: $\bar{x}_\mu(a_o) = \langle U_{a_o}\phi_o|\hat{x}_\mu U_{a_o}\phi_o\rangle$ with $\bar{x}_\mu(0) = \langle\phi_o|\hat{x}_\mu\phi_o\rangle$. The angular momentum observable is given by (4.14). The square of the bare mass observable is

$$\hat{m}^2 = \frac{1}{c^2}\hat{p}_\mu\hat{p}^\mu \tag{6.1}$$

Observe that

$$[\hat{M}_{ij}, \hat{p}_\mu\hat{p}^\mu] = \hat{p}^\mu[\hat{M}_{ij}, \hat{p}_\mu] + [\hat{M}_{ij}, \hat{p}_\mu]\hat{p}^\mu = i\hbar[(\hat{p}_j\hat{p}_i - \hat{p}_i\hat{p}_j) + (\hat{p}_i\hat{p}_j - \hat{p}_j\hat{p}_i)] + i\hbar\hbar^2[\hat{p}_o(\hat{A}_{jo}\hat{x}_i - \hat{A}_{io}\hat{x}_j) + (\hat{A}_{jo}\hat{x}_i - \hat{A}_{io}\hat{x}_j)\hat{p}_o] = 0$$

where (4.17) and (3.7) were applied. Then $[\hat{M}_{ij}, \hat{m}^2] = 0$, $[\hat{J}_i, \hat{m}^2] = 0$ and $[\hat{J}^2, \hat{m}^2] = 0$, thus $\hat{m}^2$, $\hat{J}_3$ and $\hat{J}^2$ constitute a complete commuting set of operators in H, therefore we can use they for labeling of the state vectors, i.e. if ϕ is a common eigenstate of these operators then we can write $\phi = |m^2, \mu, \ell\rangle$ where $\hat{m}^2|m^2\rangle = m^2|m^2\rangle$, $\hat{J}_3|\mu\rangle = \mu|\mu\rangle$ and $\hat{J}^2|\ell\rangle = \ell(\ell+1)|\ell\rangle$. The bare mass eigenvalue eq. of the particle is given by the Klein--Gordon eq.

$$\hat{p}_\mu\hat{p}^\mu\phi = c^2m^2\phi , \qquad \phi\in H \tag{6.2a}$$

or with (4.5)

$$\hbar^2 r^2 \Phi + \hbar^2 \Delta \Phi = c^2 m^2 \Phi , \qquad \Phi \in L^2(\mathbb{R}^3) \qquad (6.2b)$$

In polar system, one can write $\Phi = \Phi(r, \vartheta, \chi) = \frac{1}{r} R(r) Y_{\ell}^{\mu}(\vartheta, \chi)$, then the radial eq. of (6.2b), after a simple calculation, is

$$\frac{d^2 R(r)}{dr^2} + \left(\frac{1}{\hbar} r^2 - \frac{m^2 c^2}{\hbar^2} - \frac{\ell(\ell+1)}{r^2} \right) R(r) = 0 \qquad (6.2c)$$

This eq. gives a continuous spectrum for m^2.

b/ <u>Spinor particle with spin $\frac{1}{2}$</u>: Our basic assumtion (3.3) does not affect the internal degrees of freedom, so the state functions are Dirac spinors $\Psi = \Psi(\underline{x}, s) = \{\Phi_\alpha(\underline{x})\}$, $\alpha = 1,2,3,4$, $\Phi_\alpha \in H$, and the Ψ's are the elements of the Hilbert space $\mathcal{H}$ with the scalar product $\langle \Psi_1, \Psi_2 \rangle = \int_{\mathbb{R}^3} d^3\underline{x}\ \tilde{\Psi}_1 \gamma_0 \Psi_2 = \int_{\mathbb{R}^3} d^3\underline{x}\ \Psi_1^+(\underline{x}) \Psi_2(\underline{x}) = \sum_{\alpha=1}^4 \int_{\mathbb{R}^3} d^3\underline{x}\ \bar{\Phi}_1^\alpha(\underline{x}) \Phi_2^\alpha(\underline{x})$. The total angular momentum observable is $\hat{M}_{\mu\nu} = \hat{L}_{\mu\nu} + \hat{S}_{\mu\nu} = \hat{x}_\mu \hat{p}_\nu - \hat{x}_\nu \hat{p}_\mu + \frac{\hbar}{2} \sigma_{\mu\nu}$, $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. It is clear that $\hat{M}_{\mu\nu}$ satisfies (4.18). We identify the bare mass observable with

$$\hat{m} = \frac{1}{c} \gamma^\mu \hat{p}_\mu \qquad (6.3)$$

Observe that

$$[\hat{M}_{ij}, \gamma^\mu \hat{p}_\mu] = [\hat{L}_{ij} + \hat{S}_{ij}, \gamma^\mu \hat{p}_\mu] = \gamma^\mu [\hat{L}_{ij}, \hat{p}_\mu] + [\hat{S}_{ij}, \gamma^\mu] \hat{p}_\mu =$$
$$= i\hbar \gamma^\mu (g_{\mu j} \hat{p}_i - g_{\mu i} \hat{p}_j) + i\hbar \hbar^2 \gamma^\mu (\hat{A}_{j\mu} \hat{x}_i - \hat{A}_{i\mu} \hat{x}_j) + \frac{i\hbar}{4} \left([\gamma_i \gamma_j, \gamma_\mu] - [\gamma_j \gamma_i, \gamma_\mu] \right) \hat{p}^\mu = i\hbar (\gamma_j \hat{p}_i - \gamma_i \hat{p}_j) + i\hbar (\gamma_i \hat{p}_j - \gamma_j \hat{p}_i) = 0$$

where (4.17) and the relation $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}$ were used. Then $[\hat{M}_{ij}, \hat{m}] = 0$, $[\hat{J}_i, \hat{m}] = 0$ and $[\hat{J}^2, \hat{m}] = 0$, thus $\hat{m}$, $\hat{J}_3$ and $\hat{J}^2$ constitute a complete commuting set of operators in $\mathcal{H}$. One can use they for labeling of the state vectors, i.e. for a common eigenstate one can write $\Psi = |m, \mu, s, \ell, j\rangle$ where $\hat{m}|m\rangle = m|m\rangle$, $\hat{J}_3 |\mu, s\rangle = (\mu + s)|\mu, s\rangle$ and $\hat{J}^2 |\ell, j\rangle = [\ell(\ell+1) + j(j+1)] \times |\ell, j\rangle$. The bare mass eigenvalue eq. of the particle in its rest frame is given by the Dirac eq.

$$\gamma^\mu \hat{p}_\mu \Psi = cm \Psi , \qquad \Psi \in \mathcal{H} \qquad (6.4a)$$

or with (4.5)

$$\hbar \cdot \gamma_o \, \underline{r} \, \Psi(\underline{x}, s) - \frac{\hbar}{i} \gamma_i \frac{\partial}{\partial x_i} \Psi(\underline{x}, s) = cm \, \Psi(\underline{x}, s) \quad (6.4b)$$

Let us consider the corresponding second order eq. in a formal way. Let $\Psi = \frac{1}{mc}(\gamma^\nu \hat{p}_\nu + mc)\chi$ then $\gamma^\mu \gamma^\nu \hat{p}_\mu \hat{p}_\nu \chi = (g^{\mu\nu} - i\sigma^{\mu\nu}) \times \hat{p}_\mu \hat{p}_\nu \chi = (\hat{p}_\mu \hat{p}^\mu - \frac{i}{2}\sigma^{\mu\nu}[\hat{p}_\mu, \hat{p}_\nu])\chi$. Thus from (4.6b)

$$(\hat{p}_\mu \hat{p}^\mu + \frac{1}{2} \hbar\hbar^2 \sigma^{\mu\nu} \hat{A}_{\mu\nu})\chi = m^2c^2\chi \qquad (6.5a)$$

or, with $\sigma^{\mu\nu} = (i\underline{\alpha}, -\underline{\Sigma})$ and $\hat{A}_{\mu\nu} = \frac{1}{\underline{r}}(\underline{\hat{x}}, 0)$,

$$(\hat{p}_\mu \hat{p}^\mu - i\hbar\hbar^2 \frac{1}{\underline{r}} \underline{\alpha} \, \underline{\hat{x}})\chi = m^2c^2\chi \qquad (6.5b)$$

The extra term in this eq. indicates the interaction of "an electric moment $\hbar\hbar^2 \underline{\alpha}$ with the electric field $\frac{1}{\underline{r}} \underline{\hat{x}}$" in q space-time (from the analogy of cq Dirac eq. in an external field (see, e.g., in Schweber(1962))), in the rest frame of the free spinor particle.

Note that the second term in the lhs of (6.5b) is not self-adjoint, it is purely imaginary. This is the consequence of the fact that, while $\gamma^\mu \hat{p}_\mu$ is self-adjoint, $\gamma^\mu \gamma^\nu \hat{p}_\mu \hat{p}_\nu$ is not necessarily self-adjoint.

(27) Interaction with an external field: The particle is interacting with an external field given classically by a 4-vector potential $A_\mu = A_\mu(x_o, \underline{x})$. Then formally $\hat{A}_\mu = A_\mu(\hat{x}_o, \underline{\hat{x}})$. The canonical 4-momentum observable of the system from its c counterpart is $\hat{P}_\mu = \hat{p}_\mu + \frac{e}{c}\hat{A}_\mu$ and the $\hat{P}_\mu$'s satisfy (4.4) and (4.6). Then $\hat{p}_\mu = \hat{P}_\mu - \frac{e}{c}\hat{A}_\mu$ (the 4-momentum observable of the particle). Let us consider (formally) the eq. of motion of the particle. The time derivative of $\hat{p}_\mu$ is

$$\frac{d\hat{p}_\mu}{dt} = \frac{ic}{\hbar}[\hat{P}_o, \hat{p}_\mu] = \frac{ic}{\hbar}[\hat{P}_o, \hat{P}_\mu] - \frac{ic}{\hbar}[\hat{P}_o, \hat{A}_\mu] = -\hbar' \hat{A}_{o\mu} - \frac{ie}{\hbar}[\hat{P}_o, \hat{A}_\mu]$$

and, using the notations $\frac{d\hat{x}_\mu}{d\hat{x}_o} = \hat{u}_\mu = -g_{o\mu}$ and $\frac{\partial \hat{A}_\mu}{\partial x^\nu} = \frac{i}{\hbar}[\hat{A}_\mu, \hat{P}_\nu]$ (cf.(5.4)), we obtain

$$\frac{d\hat{p}_\mu}{d\hat{x}_o} = -\hbar \, \hat{A}_{o\mu} - \frac{e}{c}\frac{\partial \hat{A}_\mu}{\partial \hat{x}^\nu}\hat{u}^\nu \qquad (6.6)$$

Note that the c counterpart of (6.6) is $\frac{dp^{\mu}}{ds} = \frac{e}{c} F^{\mu\nu} u_{\nu}$ where $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ (cf. below Note 2.).

a/ Scalar particle: We identify the square of the bare mass observable of the interacting particle with

$$\hat{m}^2 = \frac{1}{c^2}(\hat{P}_{\mu} - \frac{e}{c}\hat{A}_{\mu})(\hat{P}^{\mu} - \frac{e}{c}\hat{A}^{\mu}) \qquad (6.7)$$

then the corresponding bare mass eigenvalue eq. is given by the Klein-Gordon eq.

$$(\hat{P}_{\mu} - \frac{e}{c}\hat{A}_{\mu})(\hat{P}^{\mu} - \frac{e}{c}\hat{A}^{\mu})\phi = c^2 m^2 \phi\ , \qquad \phi \in H \qquad (6.8)$$

for the particle in the external field $\hat{A}_{\mu}$ in its rest frame.

b/ Spinor particle: The bare mass observable of the interacting particle is identified with

$$\hat{m} = \frac{1}{c}\gamma^{\mu}(\hat{P}_{\mu} - \frac{e}{c}\hat{A}_{\mu}) \qquad (6.9)$$

then the bare mass eigenvalue eq. is given by the Dirac eq.

$$\gamma^{\mu}(\hat{P}_{\mu} - \frac{e}{c}\hat{A}_{\mu})\Psi = cm\,\Psi\ , \qquad \Psi \in \mathcal{H} \qquad (6.10)$$

for the particle in the external field $\hat{A}_{\mu}$ in its rest frame.

Let us consider the corresponding second order eq. in a formal manner. Let $\Psi = \frac{1}{mc}(\gamma^{\mu}\hat{P}_{\mu} - \frac{e}{c}\hat{A}_{\mu} + mc)\chi$ then

$$\gamma^{\mu}\gamma^{\nu}(\hat{P}_{\mu} - \frac{e}{c}\hat{A}_{\mu})(\hat{P}_{\nu} - \frac{e}{c}\hat{A}_{\nu}) = (g^{\mu\nu} - i\sigma^{\mu\nu})(\hat{P}_{\mu} - \frac{e}{c}\hat{A}_{\mu})(\hat{P}_{\nu} - \frac{e}{c}\hat{A}_{\nu}) =$$
$$= (\hat{P}_{\mu} - \frac{e}{c}\hat{A}_{\mu})(\hat{P}^{\mu} - \frac{e}{c}\hat{A}^{\mu}) - \frac{i}{2}\sigma^{\mu\nu}[\hat{P}_{\mu} - \frac{e}{c}\hat{A}_{\mu},\ \hat{P}_{\nu} - \frac{e}{c}\hat{A}_{\nu}] =$$
$$= (\hat{P}_{\mu} - \frac{e}{c}\hat{A}_{\mu})(\hat{P}^{\mu} - \frac{e}{c}\hat{A}^{\mu}) - \frac{i}{2}\sigma^{\mu\nu}([\hat{P}_{\mu}, \hat{P}_{\nu}] - \frac{e}{c}[\hat{P}_{\mu}, \hat{A}_{\nu}] - \frac{e}{c}[\hat{A}_{\mu}, \hat{P}_{\nu}] + \frac{e^2}{c^2}[\hat{A}_{\mu}, \hat{A}_{\nu}]) = (\hat{P}_{\mu} - \frac{e}{c}\hat{A}_{\mu})(\hat{P}^{\mu} - \frac{e}{c}\hat{A}^{\mu}) + \frac{1}{2}\hbar^2\sigma^{\mu\nu}\hat{A}_{\mu\nu} +$$
$$+ \frac{1}{2}\frac{\hbar e}{c}\sigma^{\mu\nu}\hat{F}_{\mu\nu} - \frac{i}{2}(\frac{e}{c})^2\sigma^{\mu\nu}[\hat{A}_{\mu}, \hat{A}_{\nu}]$$

where the notation $\hat{F}_{\mu\nu} = \hat{\partial}_{\mu}\hat{A}_{\nu} - \hat{\partial}_{\nu}\hat{A}_{\mu}$ [see (5.4)] and (4.6b) was applied. Thus

$$\{(\hat{P}_{\mu} - \frac{e}{c}\hat{A}_{\mu})(\hat{P}^{\mu} - \frac{e}{c}\hat{A}^{\mu}) + \frac{1}{2}\sigma^{\mu\nu}(\hbar^2\hat{A}_{\mu\nu} + \frac{\hbar e}{c}\hat{F}_{\mu\nu}) - \frac{i}{2}(\frac{e}{c})^2 \times$$
$$\times \sigma^{\mu\nu}[\hat{A}_{\mu}, \hat{A}_{\nu}]\}\chi = c^2 m^2 \chi \qquad (6.11a)$$

or with $\hat{F}_{\mu\nu} = (-\underline{\hat{\mathcal{E}}}, \underline{\hat{\mathcal{H}}})$, $\sigma^{\mu\nu} = (i\,\underline{\alpha}, -\underline{\Sigma})$ and $\hat{A}_{\mu\nu} = \frac{1}{r}(\underline{\hat{x}}, 0)$

$$\{(\hat{P}_{\mu} - \frac{e}{c}\hat{A}_{\mu})(\hat{P}^{\mu} - \frac{e}{c}\hat{A}^{\mu}) + \frac{\hbar e}{c}\underline{\Sigma}\cdot\underline{\hat{\mathcal{H}}} - i\,\underline{\alpha}(\hbar^2\frac{\underline{\hat{x}}}{r} + \frac{\hbar e}{c}\underline{\hat{\mathcal{E}}}) -$$
$$- \frac{i}{2}(\frac{e}{c})^2\sigma^{\mu\nu}[\hat{A}_{\mu}, \hat{A}_{\nu}]\}\chi = c^2 m^2 \chi \qquad (6.11b)$$

This eq. differs from the cq one in two extra terms, following from the quantized nature of space-time. The first term is added to the electromagnetic field tenzor $\hat{F}_{\mu\nu}$ and it increases the electric moment - electric field interaction. The second term predicts further spin moment - electric moment interaction with the external field $\hat{A}_\mu$ in $\hbar$ size regions of $\mathbb{M}^4$. $[\hat{A}_\mu, \hat{A}_\nu] = 0$ if $\hat{A}_\mu$ describes a time independent electromagnetic field, a static field in the rest frame of the particle. When A_μ depends on the time in this rest frame then $[\hat{A}_\mu, \hat{A}_\nu]$ is, in general, not equal to zero.

Notes: 1/ The replacement m = 0 in the Dirac eq. (6.4) or (6.10) now has no meaning because this would have the consequence that $\gamma^\mu \hat{p}_\mu \Psi = 0, \ \forall \Psi \in \mathcal{H}$, or $\gamma^\mu(\hat{P}_\mu - \frac{e}{c}\hat{A}_\mu)\Psi = 0, \ \forall \Psi \in \mathcal{H}$, i.e. $\gamma^\mu \hat{p}_\mu \equiv 0$, or $\gamma^\mu(\hat{P}_\mu - \frac{e}{c}\hat{A}_\mu) \equiv 0$. Thus the zero rest mass Dirac eq. for neutrinos has no meaning in this model. The neutrino now can have eigenstates with zero (bare) mass but it must possess not zero bare mass states, too, or not vanishing expected values for its bare mass. A similar statement is valid for the photon in this model, too.

2/ The eq. (6.10) is invariant under the following gauge-like transformations

$$\Psi' = U_a \Psi = \left[\exp\left\{-\frac{ie}{\hbar c}\hat{F}a\right\}\right]\Psi, \quad c\hat{m}' = U_a \gamma^\mu(\hat{P}_\mu - \frac{e}{c}\hat{A}_\mu)\, U_a^{-1}, \quad a \in \mathbb{R} \qquad (6.12)$$

where $\hat{F} = F(\hat{x}_0, \hat{\underline{x}})$ is a self-adjoint operator in H (a "q space-time dependent function"). For infinitesimal a's one can write

$$c\hat{m}' = (1 - \frac{ie}{\hbar c}\hat{F}a)\gamma^\mu(\hat{P}_\mu - \frac{e}{c}\hat{A}_\mu)(1 + \frac{ie}{\hbar c}\hat{F}a) = \gamma^\mu\Big(\hat{P}_\mu - \frac{ie}{\hbar c}a[\hat{F}, \hat{P}_\mu] -$$
$$- \frac{e}{c}\hat{A}_\mu + \frac{ie^2}{c^2}[\hat{F}, \hat{A}_\mu]a\Big) = \gamma^\mu(\hat{P}_\mu - \frac{e}{c}\hat{A}_\mu - \frac{e}{c}\frac{\partial \hat{F}}{\partial \hat{x}^\mu}a + i(\frac{e}{c})^2[\hat{F}, \hat{A}_\mu]a)$$

In the formal limit "$\hbar \to 0$" these transformations clearly turn into the c gauge transformations and in this limit the transformations (6.12) mean the gauge transformations of the cq Dirac eq. in an external electromagnetic field given by A_μ. But it is obvious that the classically gauge invariant combination $\partial_\mu A_\nu - \partial_\nu A_\mu$ does not remain a gauge-like invariant combination in this model. (Note that A_μ classically is not observable but in q mechanics $\hat{A}_\mu$ is an observable if $\hat{A}_\mu$ is a self-adjoint operator in H.)

7. An example and the cq limit of the one particle theory

(28) As an example for a particle in an external field let us consider briefly the case of the Coulomb potential, i.e. $A_o = -\frac{e}{r}$ and $A_i = 0$. It is clear that this choice is only applicable when the source of this potential is in rest in the rest frame of the particle. The energy observable of the interacting particle is $\hat{E} = c\hat{p}_o = c\hat{P}_o - e\,\hat{A}_o = \bar{h}'\underline{r} + e^2\frac{1}{r}$ and the expectation value of the energy for positive time $\bar{t}$ is $\bar{E} = \bar{h}'\bar{r} + e^2\frac{1}{\bar{r}}$. Then we can consider this case classically that the particle is an external potential $V(r) = \bar{h}'r - \frac{e^2}{r}$ with the source in the origin of the 3-coordinate system of its rest frame. But we recognize in this example the very popular potential model of the Ψ-particle, in which the Ψ-meson consisting of a quark c (charm) and of an antiquark $\bar{c}$, is described by the Schrödinger eq. with the potential $V(r) = \lambda r + \beta\frac{1}{r}$ as a non-r two body system. Thus we can consider the Ψ-particle as the "hydrogen atom" of this q mechanics over q space-time. The two quarks are in their common rest frame, one of these quarks ($\bar{c}$) provides the external potential $U(r) = -\frac{e^2}{r} = \beta\frac{1}{r}$. The bare mass eigenvalue eq. of the second quark (c) is given by the Klein-Gordon eq.

$$\left(\bar{h}\,\underline{r} + \frac{e^2}{c}\,\frac{1}{r}\right)^2\phi - \hat{p}_i\hat{p}_i\phi = c^2m^2\phi\ , \qquad \phi\in H \tag{7.1}$$

or by the Dirac eq.

$$\gamma_o\left(\bar{h}\,\underline{r} + \frac{e^2}{c}\,\frac{1}{r}\right)\Psi - \gamma_i\hat{p}_i\Psi = cm\Psi\ , \quad \Psi\in\mathcal{H} \tag{7.2}$$

depending on the fact that the quark has a spin 0 or $\frac{1}{2}$, respectively. Furthermore the two quarks are confined in their common rest frame, in an $\bar{\hbar}$ size region in $\mathbb{M}^4$ at any time (cf. with sec.5. paragraph(25) Note a.). In this picture the two quarks are in rest with respect to each other, thus one can consider this system, in cq theory, in a <u>non-relativistic</u> <u>way</u> as two body system, calculating the energy spectrum of the bound states.

Now $\lambda = \bar{\bar{\hbar}}' = \frac{\hbar}{\bar{\hbar}}$, but $\lambda \approx 0,2\ \mathrm{GeV}^2$ (see e.g., in Tryon(1976)) then $\bar{\hbar}' = \frac{\hbar}{\lambda} \approx 5\ \mathrm{GeV}^{-2} = 1\ \frac{\mathrm{fermi}}{\mathrm{GeV}}$ in natural units, or in usual units: $\bar{\hbar} \approx 1{,}8.10^{-31}\mathrm{m}^2$, $\bar{\hbar}' = \frac{\bar{\hbar}}{c} \approx 6.10^{-40}\mathrm{msec}$, $\frac{\hbar}{\bar{\hbar}} = \bar{\bar{\hbar}} \approx 5{,}5.10^{-4}\mathrm{kgsec}^{-1}$ and $\bar{\bar{\hbar}}' = \frac{\hbar}{\bar{\hbar}'} \approx 1{,}66.10^5\mathrm{N}$. These values are in a good agreement with those which were expected heuristically in sec.5. paragraph (25) Note a.

<u>Notes</u>: a/ The solutions of the eq.'s (6.2), (6.4), (7.1) and (7.2) with their physical implications will be discussed elsewhere.
b/ The expectation value of the energy of the particle change for a repulsive potential $\frac{e^2}{r}$ in the same way as above for an attractive potential, and the linear term dominates beyond the turning point $r_o = e\sqrt{\frac{\bar{\hbar}'}{\bar{\hbar}}}$ now, too. Thus the two particles are confined again in spite of the repulsive Coulomb potential acting between them.

(29) Let us consider the formal limit $\bar{\hbar} \to 0$ in the one particle theory over q space-time. As we saw in sec.3. paragraph (15) the c limit of this q space-time model is realized by the events $\phi \in H$ for which $\Delta r \Delta t$ is minimal, i.e. $\Delta r \Delta t = \frac{1}{2}\bar{\hbar}'$. These events have the form (3.12). Let M be the set of such events then we can write formally: $M \ni \phi \xrightarrow{"\bar{\hbar} \to 0"} |\bar{x}_o, \underline{\bar{x}}\rangle \longleftrightarrow (x_o, \underline{x}) \in \mathbb{M}^4$ then $M \xrightarrow{\bar{\hbar} \to 0} \mathbb{M}^4$. Furthermore:

$$[\hat{x}_\mu, \hat{x}_\nu]\phi = -i\bar{\hbar}\,\hat{A}_{\mu\nu}\phi \xrightarrow{\bar{\hbar} \to 0} [\hat{x}_\mu, \hat{x}_\nu]\,|\bar{x}_o, \underline{\bar{x}}\rangle = 0,$$

$$[\hat{p}_\mu, \hat{p}_\nu]\phi = i\bar{\hbar}\bar{\bar{\hbar}}^2\hat{A}_{\mu\nu}\phi \xrightarrow{\bar{\hbar} \to 0} [\hat{p}_\mu, \hat{p}_\nu]\,|\bar{x}_o, \underline{\bar{x}}\rangle = 0$$

but

$$[\hat{p}_\mu, \hat{x}_\nu]\phi = i\hbar g_{\mu\nu}\phi \xrightarrow{\bar{\hbar} \to 0} [\hat{p}_\mu, \hat{x}_\nu]\,|\bar{x}_o, \underline{\bar{x}}\rangle = i\hbar g_{\mu\nu}|\bar{x}_o, \underline{\bar{x}}\rangle$$

i.e. these CCR's remain valid in the formal limit $\bar{\hbar} \to 0$. Thus we can say that, in the formal limit $\hbar \to 0$, the q space-time we proposed turns into the flat space-time $\mathbb{M}^4$, while q mechanics over this q space-time is reduced to the cq mechanics; the bare mass eigenvalue eq.'s (6.2), (6.8) and (6.4), (6.10) are reduced to the cq Klein-Gordon eq.'s and to the cq Dirac eq.'s, respectively. The situation can be represented formally as follows

$$\begin{array}{ccccc} \text{QST} & \xrightarrow{\hbar \to 0} & \text{RST} & \xrightarrow{v \ll c} & \text{NRST} \\ \text{QM} & \longrightarrow & \text{RQM} & \longrightarrow & \text{NRQM} \end{array}$$

i.e. q space-time include r space-time ($\mathbb{M}^4$) and non r space-time, and q mechanics over q space-time include rq mechanics and non rq mechanics as limiting cases, respectively.

<u>Remark</u>: As we saw in the theory of the one free scalar particle on q space-time, one can write for a $\phi \in H$ $\phi = \sum C(m^2, \mu, \ell) \times |m^2, \mu, \ell\rangle = \sum C(m^2)|m^2\rangle \, C(\ell, \mu)|\mu, \ell\rangle = \sum C(m^2)|m^2\rangle \varphi(\vartheta, \chi)$. Then an element ϕ of M can be written as $\phi = \sum C(m^2)|m^2\rangle = \phi(\bar{q}, \bar{t})$, i.e. $\phi(\bar{q}, \bar{t})$ is a superposition of different bare mass states of the scalar test particle. Furthermore the elements of M represent the smallest cells (bubbles) of the flat space-time $\mathbb{M}^4$, which cells possess, in principle, physically testable reality. Therefore the elements of M may be interpreted as the <u>quanta</u> of flat space-time $\mathbb{M}^4$. The following relations are valid for these space-time quanta: $\Delta r \Delta t \approx \hbar'$ where Δt is the lifetime of such a quatum, Δr is the size of such a quantum in space; $\Delta E \Delta t \approx \hbar$ where ΔE is the energy of such a quatum, $\Delta E \Delta p \approx \hbar' \hbar'^2 = \frac{\hbar^2}{\hbar'}$, where Δp is the momentum of such a quatum. If we choose the values $\Delta r \approx 10^{-15}$m and $\Delta t \approx 10^{-24}$sec as characteristic values for such space-time cells (quanta) then $\Delta E \approx 10^{-10} \text{kg}\frac{\text{m}^2}{\text{sec}} \approx 1$ Gev and $\Delta p \approx 10^{-19}$ kg $\frac{\text{m}}{\text{s}}$ (or $\Delta p \approx \frac{\hbar^2}{\hbar'} \cdot \frac{1}{\Delta E} \approx 0{,}2$ GeV in natural units).

8. Concluding remarks

(30) The idea that the space-time coordinates would have to be operators in a semantically consistent q theory of fields was emphasized by von Weizsäcker (1973, 1974). The "canonical" quantization of $\mathbb{M}^4$ proposed in this paper is a kind of realization of this idea. It is a well-known idea of Heisenberg that q uncertainties should emerge in the space-time metric in small space-time regions but the space-time points remain unchanged, i.e. the null cones should become "smeared out" in the q area. Then the causality becomes indefinite. In the twistor theory of Penrose (1972, 1975), space-time points arise as secondary concepts corresponding to linear sets in twistor space, and they, rather than the null cones should become "smeared out" on the passage to a q relativity theory. It is straightforward to see that both the space-time points and null cones become "smeared out" in our q space-time model. The causal relation of two q events is only decidable operationally by measuring the transition probability between them that is exactly formulated by the relation $\langle \phi_1 | \phi_2 \rangle$ describing the causality connection among q events in q space-time. We could say nothing about causality inside an $\hbar$ size region of $\mathbb{M}^4$.

As another comparison we note that our approach shows up a similarity, in its spirit, to that of Finkelstein (1974), however the differences between them are clear.

Comparing our model to the elementary length theories or to the time quantum theory of Finkelstein we see that our model does not assume the existence of an elementary length ℓ or of a time quantum τ rather it assumes the existence of an elementary cell $\hbar \sim \ell \cdot \tau$ in $\mathbb{M}^4$.

The inner topology of q space-time (the topology of a space-like hyperplane in q space-time) is much more discrete relative to the usual topology of $\mathbb{M}^4$ and it is very similar to the topology of a lattice on $\mathbb{M}^4$ nowadays used extensively in the practical calculations of finite results in qlft. For, considering the points of a lattice on a spacelike hyperplane of $\mathbb{M}^4$ as the isolated points in a spacelike hyperplane Γ of q space-time (cf. sec.2. paragraph(9)).

(31) One of the main implications of our model is the confinement phenomenon for point particles ("quarks"). This phenomenon finally may lead to the explanation of the "quark puzzle". This explanation would be a combined kinematical and dynamical one, for, kinematical because it may follow from the q nature of space-time, and dynamical, too, because the q geometry of space-time provides the confining force. (Cf. this explanation with that of Saavedra and Utreras(1981).)

(32) Let us consider briefly the case of a particle closed inside a sphere with radius R, i.e. $0 \leq r \leq R$. One may demonstrate the physical meaning of the uncertainty relation (3.2) by thought experiments leading to such a problem, similarly to usual q mechanics where the Heisenberg uncertainty principle is demonstrated by thought experiments which are equivalent with the problem of a particle closed inside a box. In that case $H = L^2(-R, +R)$ and, as it is well-known, now the time operator $\hat{t} = -i\hbar' \frac{d}{dq}$ has a purely discrete spectrum $t_n = \frac{\pi\hbar'}{R} n$, $n = 0, \pm 1, \pm 2, \ldots$ with eigenstates $|t_n\rangle = \frac{1}{\sqrt{2R}} \exp(\frac{i\cdot\pi}{R} nq)$. Now we can write $\Delta r = 2R$ and $\Delta t = t_{n+1} - t_n = \frac{\pi\hbar'}{R}$, then $\Delta r \Delta t = 2\pi\hbar'$. One can consider $\Delta t = \tau$ as a time quatum and if $\tau \sim 10^{-24}$sec in a good agreement with the estimated magnitude of the Finkelstein's time quantum then $R \sim \sim \pi \cdot 6.10^{-40}$msec 10^{24}sec$^{-1} \sim 1{,}8.10^{-15}$m ~ 2 fermi. Then one might consider R as the radius of a hadronic "micro-universe" with the circular frequency $\omega = \frac{1}{\tau} \sim 10^{24}$sec^{-1}, which appears as a "bubble" embedded in an external, overall essentially flat, macroscopic world. (Cf. with P. Roman(1979).)

(33) In this paper we interpreted the lattice ℓ of the values of the local propositions of a local physical system described in Banai (1980) by a CROC-valued propositional system as the r causal logic of the corresponding q space-time; the Hilbert space H for wich $\mathcal{P}(H) = \ell$ was interpreted as the event space of this q space-time. Now in Banai (1980) it was mentioned that the CROC-valued logics are in a close connection with the ℓ-valued models of the Takeuti's (1981) q set theory. In this way, in the Takeuti's ℓ-valued models, the lattice ℓ of the truth values and the corresponding Hilbert space H get a physical interpretation, too. Each maximal Boolean algebra in $\mathcal{P}(H)$ determines a "spacelike" hyperplane in the corresponding q space-time, on which the physicists prepare the states of the local physical system. So every maximal Boolean algebra in $\mathcal{P}(H)$ may be interpreted as a Boolean reference frame in accordance with M. Davis (1977), first who established in q theory the relativity principle we used here to set up q space-time models.

(34) Along the C^*-module quantization program for rlft suggested in Banai (1981b) (and which quantization is partly done for non r local physical system in Banai (1981c)) we give in this paper a resolution for the problem appears in this program in connection with the space-time of a local physical system. Thus the C^*-algebra A over which the corresponding Hilbert module H_A must be constructed is determined, too: it is the von Neumann algebra generated by the lattice of projectors of the event space H. In the succeeding paper (in the second part) we will proceed along this program and the formalism of clft will be transferred over q space-time proposed in this part as a canonical example for general q space-time models. In this context this procedure will provide the <u>first</u> quantization for local field theories.

Acknowledgments

I should like to thank C.F. von Weizsäcker, L. Castell and T. Lehmeier for the hospitality on the 4th Symposium on Quantum Theory and the Structures of Time and Space at Tutzing, A.R. Marlow and C. Brans for the excellent hospitality in the Loyola University at New Orleans and fruitful discussions, M.J. Dupré for helpful discussions. I am deeply indebted to T. Matolcsi for valuable discussions especially in sec.3. , to Z. Perjés for helpful discussions and encouragement.

References

Atkinson, D. and Halpern, M.B.(1967) J.Math.Phys. 8, 373-387.

Banai, M.(1980) preprint, KFKI-1980-42, Int.J.Theor.Phys.(1981)

Banai, M.(1981a) in Current Issues in Quantum Logic, eds. E. Beltrametti, B. van Fraassen, Plenum, New York, London.

Banai, M.(1981b) preprint, KFKI-1981-01, and in Quantum Theory and the Structures of Time and Space, Vol.4., eds. L. Castell, M. Drieschner, C.F. von Weizsäcker, Carl Hanser Verlag, München.

Banai, M.(1981c) submitted to J.Math.Phys.

Borchers, H.J. and Hegerfeldt, G.C.(1972) Commun.math.Phys. 28, 259-266.

Cegła, W. and Jadczyk, A.Z.(1976) Rep.Math.Phys. 9, 377.

Cegła, W. and Jadczyk, A.Z.(1977) Commun.math.Phys. 57, 213-217.

Cegła, W. and Jadczyk, A.Z. (1979) Lett.Math.Phys. 3, 109-112.

Das, A. (1966) J.Math.Phys. 7, 52-60.

Davis, M. (1977) Int.J.Theor.Phys. 16, 867-874.

Eilers, M. and Horst, E. (1975) Int.J.Theor.Phys. 13, 419-424.

Finkelstein, D. (1974) Phys.Rev. D9, 2219-2231.

Finkelstien, D. et al. (1974) Phys.Rev. D9, 2231-2236.

Gleason, A.M. (1957) J.Math. and Mechanics, 6, 885-893.

Gudder, S.P. (1968) SIAM J.Appl.Math. 16, 1011-1019.

Gudder, S.P. (1970) in Probabilistic Methods in Applied Mathematics, Vol.2. 53-129, ed.A.T. Bharucha-Reid, Acad. Press, New York-London.

Jakobsen, H.P. et al. (1978) Proc.Nath.Acad.Sci. USA, 75, 1609-1611.

Mackey, A. (1963) The Mathematical Foundations of Quantum Mechanics, Benjamin, New York.

Marlow, A.R. (1981a) in Current Issues in Quantum Logic, eds. E. Beltrametti, B. van Fraassen, Plenum, New York, London.

Marlow, A.R. (1981b) in Quantum Theory and the Structures of Time and Space, Vol.4, eds. L. Castell, M.Drieschner, C.F. von Weizcäcker, Carl Hanser Verlag, München.

Penrose, R. and MacCallum, M.A.H. (1972) Phys.Rep. C6, No.4, 241--316.

Penrose, R. (1975) in Quantum Gravity - An Oxford Symposium, eds. C.F. Isham, R. Penrose, D.W. Sciama, Claredon, Oxford.

Piron, C. (1976) Foundations of Quantum Physics, W.A. Benjamin, Inc., Reading, Mass.

Roman, P. (1979) in Quantum Theory and the Structures of Time and Space, Vol.3, eds. L. Castell, M. Drieschner, C.F. von Weizsäcker, Carl Hanser Verlag, München.

Saavedra, I. and Utreras, C. (1981) Phys.Lett. 98B, 74-76.

Schweber, S.S. (1962) An Introduction to Relativistic Quantum Field Theory, Harper & Row, Publishers, New York.

Segal, I.E.(1965) Physical and parametric Space in Quantum Field Theory, Séminaire de Mecanique Celeste et Mecanique Analytique, manuscript.

Segal, I.E. (1980) in Symmetries in Science, eds. B. Gruber, R.S. Millman, Plenum, New York, Lodon.

Takeuti, G. (1981) in Current Issues in Quantum Logic, eds. E. Beltrametti, B. van Fraassen, Plenum, New York, London.

Tryon, E.P. (1976) Phys.Rev.Lett. 36, 455.

Von Neumann, J. (1932) Annals of Math. 33, 567.

Von Neumann, J. (1955) Mathematical Foundations of Quantum Mechanics, Princeton University Press, Princeton, p. 235-236.

Von Weizcäcker, C.F. (1973) Classical and Quantum Descriptions, in The Physicist's Conseption of Nature, ed. F. Mehra, D. Reidel Publ. Co., Dordrecht - Holland.

Von Weizsäcker, C.F. (1974) Geometrie und Physik, in Physical Reality and Mathematical Description, eds. Ch.P. Enz, F. Mehra, D. Reidel Publ. Co., Dordrecht - Holland.

Von Weizsäcker, C.F. et al. (1975) in Quantum Theory and the Structures of Time and Space,eds. L. Castell, M. Drieschner, C.F. von Weizsäcker, Carl Hanser Verlag, München.

Von Weizsäcker, C.F. et al. (1977) in Quantum Theory and the Structures of Time and Space, Vol.2, eds. L. Castell, M. Drieschner, C.F. von Weizsäcker, Carl Hanser Verlag, München.

Von Weizsäcker, C.F. et al. (1979) in Quantum Theory and the Structures of Time and Space, Vol.3, eds. L. Castell, M. Drieschner, C.F. von Weizsäcker, Carl Hanser Verlag, München.

Von Weizsäcker, C.F. et al. (1981) in Quantum Theory and the Structures of Time and Space, Vol.4, eds. L. Castell, M. Drieschner, C.F. von Weizsäcker, Carl Hanser Verlag, München.

Footnotes

[1] It is not easy to give a complete references on the related papers because of their enormous numbers and their numbers are growing year by year. Nevertheless we list here other related works, too, which are known explicitly by the author; Das(1966), Atkinson and Halpern(1967) and Gudder(1968), these papers concern on the elementary length theories. Along the algebraic approach of q theory it is worthwhile to mention here some papers of Segal - among others - : Segal(1965, 1980) and Jakobsen et al.(1978). Futhermore, related papers can be found in a high concentration, e.g., in the proceedings of the conferences on "the q theory and the structures of Time and Space" held bianually by the Max Planck Institute at Starnberg: von Weizsäcker et al. (1975, 1977, 1979, 1981), especially under the names Finkelstein, von Weizsäcker, Castell, Segal, Roman, Barut, Biedenharn, Penrose, Wheeler.

[2] This argument is due to W. Cegła.

[3] The use of the c light velocity c in $\hat{x}_0 = c\hat{t}$ is not too consequent because of the complete spreadoutness of the light cone structure of $\mathbb{M}^4$ in the following model, rather it should write $\hat{c}$ instead of c, which operator $\hat{c}$ would produce c in the c limit. However, in this first draft of the theory, we insist on the use of c.

[4] Several trials give this relation which will satisfy all expected physical criteria. The relations, e.g., $\Delta x_0 \Delta x_i \geq \frac{1}{2}\hbar$, i=1,2,3, in Banai(1981b) could not imply a definite c limit (correspondence principle!) for the corresponding q space-time.

[5] The attention is paid to the followings. Let $(H, \phi) = \{L^2_\varrho(0,+\infty) \otimes L^2_\nu(0,\pi) \otimes L^2(0, 2\pi)$, $\langle f_1, f_2 \rangle = \int_0^\infty dr\, r^2 \int_0^\pi d\vartheta \sin\vartheta \int_0^{2\pi} d\chi\, \bar{f}_1(r,\vartheta,\chi) f_2(r,\vartheta,\chi)\}$, then the operators $-i\hbar' \frac{\partial}{\partial r}$ and r in H satisfy formally (3.3) but $-i\frac{\partial}{\partial r}$ does not has a unique spectral decomposition, as it is well-known. The operator $-i\hbar' \frac{1}{r}\frac{\partial}{\partial r}(r\phi) = -i\hbar'(\frac{\partial}{\partial r} + \frac{1}{r})\phi$ satisfies (3.3) with $r\phi$, and it is symmetric but it has not a unique self-adjoint extention because it is unitarily equivalent with $-i\frac{d}{dh}$ in $L^2(0,\infty)$. The author got considerable aid from prof. T. Matolcsi to reîsolve this problem in the trick.

[6]As supplementary conditions to (3.3) we should require the following formal CR's for the angles ϑ and χ and for t: $[\hat{t}, \hat{\vartheta}] = 0$, $[\hat{t}, \hat{\chi}] = 0$, and these relations is satisfied by $\hat{t}$ in (3.5d), indeed.

[7]If we define the rotations, similarly to the translations in (4.1), as follows:for infinitesimal $\omega_{\mu\nu}$'s, let $\hat{x}'_{\mu} = U(\omega)\, \hat{x}_{\mu} U^{-1}(\omega) = \hat{x}_{\mu} - \omega_{\mu}{}^{\nu}\, \hat{x}_{\nu}$, then this would lead a generator $\hat{M}'_{\mu\nu}$ different from (4.14).

63.160

Kiadja a Központi Fizikai Kutató Intézet
Felelős kiadó: Szegő Károly
Szakmai lektor: Lukács Béla
Nyelvi lektor: Forgács Péter
Példányszám: 355 Törzsszám: 81-358
Készült a KFKI sokszorosító üzemében
Felelős vezető: Nagy Károly
Budapest, 1981. junius hó