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GEOMETRIC CONVERGENCE OF SOME TWO-POINT PADÉ APPROXIMATIONS

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ABSTRACT

In this paper the geometric convergence of some two-point Padé approximations on certain infinite sets of the complex plane is considered.

АННОТАЦИЯ

В данной работе исследуется вопрос геометрической сходимости специальных приближений Паде на неограниченных областях комплексной плоскости.

KIVONAT

A cikkben speciális két-pont Padé közelitések geometriai konvergenciáját vizsgáljuk a komplex sik bizonyos végtelen halmazain.

1. INTRODUCTION

The main aims of this paper are to investigate the convergence of some two-point Padé approximations on certain infinite sets of the complex plane. The convergence of Padé approximations has received much interest, both for its application in numerical computations and for approximation theory problems.

In particular, we consider the function

$$F(x) = \mu\int_0^1 (1-u)^{\mu-1}e^{-ux}du \quad , \qquad \mu > 0 \quad . \tag{1}$$

For $\mu = \frac{1}{2}$ this function is the subject of numerical calculations connected with the plasma dispersion function [1]-[3]

$$Z(s) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{e^{-t^2}}{t-s}dt = i\sqrt{\pi}e^{-s^2} - 2e^{-s^2}\int_0^s e^{u^2}du \quad . \tag{2}$$

The case $\mu = 0$ (the exponential function) was considered by Saff et al. in their excellent papers [4]-[6].

After the development of some preliminary considerations in Section 2, we consider, in Section 3, the convergence of two-point Padé approximations to the function $F(x)$ on the real positive axis. We shall prove that these rational approximants of $R_k(x)$ type

$$R_k(x) = \frac{p_0+p_1x+\ldots+p_{k-1}x^{k-1}}{1+q_1x+\ldots+q_kx^k} \quad , \tag{3}$$

have a geometric convergence rate as of at least $\frac{1}{2^k}$: Theorem 2. In Theorem 3 we establish that the best generalized two-point Padé approximations have a geometric convergence rate like $\frac{1}{3^k}$. In Section 4 we consider some infinite parabolic-type domains of the complex plane in which the geometric convergence of two-point Padé approximations also holds. Our results, Theorem 4, is an application of the results of Saff and Varga [11]. In Theorem 5 we present infinite sectors of the complex plane in which the special generalized two-point Padé approximations converge in geometric order.

II. DEFINITIONS AND PRELIMINARY RESULTS

When a function $f(x)$ satisfies the conditions

$$f(x) \sim \sum_{k=0}^{\infty} c_k x^k \quad , \quad x \to 0 \quad ,$$
$$f(x) \sim \sum_{k=0}^{\infty} d_k x^{-k-1} \quad , \quad x \to \infty \quad , \tag{4}$$

we can determine rational fractions $R_k(x)$, (3), for which the following relations hold

$$f(x) - R_k(x) = O(x^k) \quad , \quad x \to 0 \quad ,$$
$$f(x) - R_k(x) = O(x^{-k-1}) \quad , \quad x \to \infty \quad . \tag{5}$$

Definition 1. The rationals $R_k(x)$ satisfying both previous conditions are called two-point Padé approximations to the function $f(x)$. There exists a more general conception of this definition.

Definition 2. The rationals $R_k^{(m)}(x)$

$$R_k^{(m)}(x) = \frac{p_0^{(m)} + p_1^{(m)} x + \ldots + p_{k-1}^{(m)} x^{k-1}}{1 + q_1^{(m)} x + \ldots + q_k^{(m)} x^k} \tag{6}$$

satisfying the conditions

$$f(x) - R_k^{(m)}(x) = O(x^{k+m}) \quad , \quad x \to 0 \quad ,$$
$$f(x) - R_k^{(m)}(x) = O(x^{-k+m-1}) \quad , \quad x \to \infty \quad , \tag{7}$$

where m is a positive integer $m = 0,1,2,\ldots,k$, we call generalized two-point Padé approximations to the function $f(x)$. The reason for this generalization is obvious: we take $k+m$ terms from the series near $t = 0$, and $k-m$ terms from the series near $t = \infty$ to calculate the coefficients of the rational $R_k^{(m)}(x)$. Let us mention that the case $m=0$ corresponds to Definition 1 and that $m=k$ is the classic (one-point) Padé approximation. For our function $F(x)$ we can solve exactly the problem of generalized two-point Padé approximation in closed form.

Theorem 1. For the generalized two-point Padé approximations to the function $F(x)$ the following results hold:

(i) the denominator of the rationals

$$R_k^{(m)}(x) = \frac{P_k^{(m)}(x)}{Q_k^{(m)}(x)} \quad ,$$

$Q_k^{(m)}(x)$, in hypergeometric notation, is

$$Q_k^{(m)}(x) = {}_1\mathcal{F}_1(-k; 1-\mu-m-k; x) \quad ,$$

(ii) the numerator of the error term

$$E_k^{(m)}(x) = F(x) - R_k^{(m)}(x) = \frac{S_k^{(m)}(x)}{Q_k^{(m)}(x)} ,$$

$S_k^{(m)}(x)$, in integral form, is

$$S_k^{(m)}(x) = (-1)^m \frac{\Gamma(1+\mu)}{\Gamma(m+k+\mu)} x^{k+m} \int_0^1 e^{-xu} u^k (1-u)^{m+\mu-1} du ,$$

(iii) the functions $P_k^{(m)}(x)$, $Q_k^{(m)}(x)$ (and $S_k^{(m)}(x)$ too) satisfy the second order difference equations with respect to k

$$(k+m+\mu-1)(k+m+\mu)y_{k+1} = (k+m+\mu-1)(k+m+\mu+x)y_k - kxy_{k-1} , \quad k=1,2,\ldots ,$$

(iv) the error function has a more economic representation:

$$E_k^{(m)}(x) = (-1)^m \Gamma(1+\mu)\Gamma(m+\mu) \sum_{j=k}^{\infty} \frac{j!\,t^{j+m}}{\Gamma(j+m+\mu)} \frac{1}{\Gamma(j+m+\mu+1) Q_j^{(m)}(x) Q_{j+1}^{(m)}(x)} .$$

Proofs. First we mention that the function F(x) has the series representations

$$\left.\begin{aligned} F(x) &= \sum_{k=0}^{\infty} \frac{(-x)^k}{(1+\mu)_k} , \quad x \to 0 , \\ F(x) &\sim \mu \sum_{k=0}^{\infty} (1-\mu)_k x^{-k-1} , \quad x \to \infty \end{aligned}\right\} \mu > 0 , .$$

The coefficients of the rationals $R_k^{(m)}(x)$ are determined (corresponding to Definition 2) from the equations

$$\sum_{j=0}^{\ell} \frac{(-1)^j}{(1+\mu)_j} q_{\ell-j} = p_\ell , \quad \ell=0,1,\ldots,k-1,$$

$$\sum_{j=0}^{k} q_j \frac{(-1)^{i-j}}{(1+\mu)_{i-j}} = 0 , \quad i=k,k+1,\ldots,k+m-1,$$

$$\mu \sum_{j=0}^{i} (1-\mu)_{i-j} q_{k-j} = p_{k-i-1} , \quad i=0,1,\ldots,k-m-1 .$$

This is a system of 2k simultaneous equations in 2k unknowns: $p_0, p_1, \ldots, p_{k-1}$, $q_1, q_2, \ldots, q_k$. We determine explicitly the q_k numbers only. When we eliminate the numbers p_k we get the system:

$$\sum_{j=0}^{\ell} \frac{(-1)^j}{(1+\mu)_j} q_{\ell-j} = \mu \sum_{j=0}^{k-1-\ell} (1-\mu)_{k-1-\ell-j} q_{k-j} , \quad \ell=m,m+1,\ldots,k-1,$$

$$\sum_{j=0}^{k} q_j \frac{(-1)^{\ell-j}}{(1+\mu)_{\ell-j}} = 0 , \quad \ell=k,k+1,\ldots,k+m-1 ,$$

or in simpler form

$$\sum_{j=0}^{k} q_j \frac{(-1)^{\ell-j}}{(1+\mu)_{\ell-j}} = 0 , \quad \ell=m,m+1,\ldots,m+k-1 .$$

By elementary manipulations we can transform this system to

$$\sum_{j=0}^{k}(j+1)_{\ell}\Gamma(j-k-m-\mu+1)q_j = 0 \quad , \qquad \ell=0,1,\ldots k-1 \quad .$$

This system we solve by the orthogonal polynomial method:

$$\int_0^\infty e^{-u}u^{\ell}\sum_{j=0}^{k}q_j u^j \frac{\Gamma(j-k-m-\mu+1)}{j!}du = 0 \quad , \qquad \ell=0,1,2,\ldots,k-1 \quad ,$$

and

$$\sum_{j=0}^{k}q_j u^j \frac{\Gamma(j-k-m-\mu+1)}{j!} = \frac{\Gamma(-k-m-\mu+1)}{k!}e^u\frac{d^k}{du^k}(e^{-u}u^k) \quad ,$$

therefore

$$q_j = \frac{(-k)_j}{j!(1-\mu-m-k)_j} \quad , \qquad j=0,1,\ldots,k \quad .$$

Now (i) is proved. Next, to get $S_k^{(m)}(x)$ we must compute the series

$$S_k^{(m)}(x) = \sum_{j=k+m}^{\infty}x^j\sum_{\ell=0}^{k}q_{\ell}\frac{(-1)^{j-\ell}}{(1+\mu)_{j-\ell}} =$$

$$= (-1)^m\Gamma(1+\mu)\sum_{j=0}^{\infty}(-1)^j x^{j+m+k}\frac{\Gamma(m+\mu)}{\Gamma(k+m+\mu)\Gamma(j+m+\mu+1)}\sum_{\ell=0}^{k}\frac{(-k)_{\ell}}{\ell} \cdot$$

$$\cdot \frac{(m+\mu)_{\ell}}{(j+m+\mu+1)_{\ell}} = (-1)^m k!x^{k+m}\frac{\Gamma(1+\mu)\Gamma(m+\mu)}{\Gamma(k+m+\mu)\Gamma(k+m+\mu+1)} \cdot$$

$$\cdot \sum_{j=0}^{\infty}\frac{(k+1)_j}{j!(k+m+\mu+1)_j}(-x)^j \quad .$$

This is a hypergeometric function ($_1\mathcal{F}_1$ type). It is not difficult to see [7] that is satisfies the difference equation (iii). The function $Q_k^{(m)}(x)$ also satisfies (iii) and therefore $P_k^{(m)}(x)$ is the solution of the same equation. Thus (iii) is proved. When we apply the usual integral representation [8] to this function we get (ii). Finally we prove (iv). Let us consider the difference

$$\frac{S_{k+1}^{(m)}(x)}{Q_{k+1}^{(m)}(x)} - \frac{S_k^{(m)}(x)}{Q_k^{(m)}(x)} = \frac{H_k(x)}{Q_k^{(m)}(x)Q_{k+1}^{(m)}(x)} \quad .$$

Applying (iii) we arrive at the difference equation for H_k:

$$H_k(x) = \frac{kx}{(k+m+\mu)(k+m+\mu-1)}H_{k-1}(x)$$

from which

$$H_k(x) = (-1)^m \frac{\Gamma(1+\mu)\Gamma(m+\mu)}{\Gamma(k+m+\mu)\Gamma(k+m+\mu+1)} x^{k+m} \quad .$$

Summing the previous difference relation we get formula (iv).

III. NEW RESULTS ON GEOMETRIC CONVERGENCE

With the aid of the results of the previous sections, we now establish the convergence of generalized two-point Padé approximations to the function $F(x)$. First we deal with the parameter m having a bounded, fixed value.

Theorem 2. For the maximum value of the error function

$$E_k^{(m)} = \max_{0 \le x < \infty} |E_k^{(m)}(x)|$$

for $m+\mu > 1$ the following estimation holds

$$E_k^{(m)} < \frac{1}{2} E_{k-1}^{(m)} \quad , \qquad k=m,m+1,\ldots, \tag{8}$$

Proof. From (ii) it is not difficult to check that $E_k^{(m)}(0) = E_k^{(m)}(\infty) = 0$, and thus undoubtedly there exists a positive value x_k where

$$E_k^{(m)} = |E_k^{(m)}(x_k)| \ ,$$

and it naturally holds that $\frac{d}{dx} E_k^{(m)}(x_k) = 0$. When we differentiate the integral form of the error function we get

$$0 = -\frac{\mu}{x_k} E_k^{(m)}(x_k) - \frac{k}{k+m+\mu-1} \frac{Q_{k-1}^{(m)}(x_k)}{Q_k^{(m)}(x_k)} E_k^{(m)}(x_k) - E_k^{(m)}(x_k) +$$

$$+ \frac{k}{k+m+\mu-1} \frac{Q_{k-1}^{(m)}(x_k)}{Q_k^{(m)}(x_k)} E_{k-1}^{(m)}(x_k) \ .$$

In a more compact form this is

$$E_k^{(m)}(x_k) = \frac{k x_k Q_{k-1}^{(m)}(x_k)}{k x_k Q_{k-1}^{(m)}(x_k) + (k+m+\mu-1)(x_k+\mu) Q_k^{(m)}(x_k)} E_{k-1}^{(m)}(x_k)$$

Next we show that for $0 \le x \le \infty$ and $m+\mu > 1$

$$Q_k^{(m)}(x) \ge Q_{k-1}^{(m)}(x) \ , \quad k=1,2,\ldots \ .$$

A short comparison of the coefficients of the same powers in the polynomials shows

$$\frac{(-k)_j}{j!\,(1-\mu-m-k)_j} \ge \frac{(-k+1)_j}{j!\,(2-\mu-m-k)_j} \ , \qquad j=0,1,\ldots,k-1 \ ,$$

because of trivial inequality

$$\frac{k}{k+m+\mu-1} \geq \frac{k-j}{k+m+\mu-1-j} \quad .$$

Now applying the inequality and the fact that $E_{k-1}^{(m)}(x_k) \leq E_{k-1}^{(m)}$ we get

$$E_k^{(m)} \leq \frac{kx_k Q_{k-1}^{(m)}(x_k)}{kx_k Q_{k-1}^{(m)}(x_k)+(k+m+\mu-1)(x_k+\mu)Q_k^{(m)}(x_k)} E_{k-1}^{(m)} \leq$$

$$\leq \frac{kx_k}{kx_k+(k+m+\mu-1)(x_k+\mu)} E_{k-1}^{(m)} \leq \frac{k}{2k+m+\mu-1} E_{k-1}^{(m)} < \frac{1}{2} E_{k-1}^{(m)} \quad .$$

Theorem 2 is now proved.

Next we shall show that there exists an optimal choice of parameter m in connection with the convergence rate of the generalized two-point Padé approximations to the function F(x). We treat the case when the parameter $m \to \infty$, if $k \to \infty$, in a suitable manner.

Theorem 3. Let us suppose $m \to \infty$,

$$\lim_{k\to\infty} \frac{m}{k} = \beta \quad , \quad 0 \leq \beta \leq 1 \quad , \tag{9}$$

and $\mu > 0$, then the generalized two-point Padé approximations to the function F(x) have geometric convergence rate

$$\lim_{k\to\infty} \{E_k^{(m)}\}^{\frac{1}{k}} = \varphi(\beta) = \beta^\beta (1-\beta)^{1-\beta} 2^{\beta-1} < 1 \quad . \tag{10}$$

Before proving this we would comment on our result. From the form of the function $\varphi(\beta)$ one can see that for $\beta = 0$ (two-point Padé approximation), the geometric convergence rate is $\frac{1}{2}$; for $\beta = 1$ (classic Padé approximation), $\varphi(1)=1$: geometric convergence does not exist; for $\beta = \frac{1}{3}$ (this is the minimal position of the function $\varphi(\beta)$), the geometric convergence rate is $\frac{1}{3}$. In view of this, we can state that the optimal choice of parameter m is $m = [\frac{k}{3}]$ with regard to the convergence rate of the generalized two-point Padé approximations to the function F(x).

Proof. We shall apply formula (iv) and Lemma 1 to investigate the function $L_\ell(x)$ in the error function

$$E_k^{(m)}(x) = (-1)^m \Gamma(1+\mu) \sum_{\ell=k}^{\infty} L_\ell(x) \quad ;$$

$$L_\ell(x) = \frac{\Gamma(m+\mu)\,\ell!\,x^{\ell+m}}{\Gamma(\ell+m+\mu)\Gamma(\ell+m+\mu+1)Q_\ell^{(m)}(x)Q_{\ell+1}^{(m)}(x)} \quad .$$

Lemma 1. Let us suppose that $\mu > 0$, $\ell \to \infty$, $m \to \infty$ and

$$\lim_{\ell\to\infty} \frac{m}{\ell} = \beta \quad , \quad 0 \leq \beta \leq 1 \quad ,$$

then

$$\lim_{\ell\to\infty} \{\max_{0\le x\le\infty} |L_\ell(x)|\}^{\frac{1}{\ell}} = \varphi(\beta) \quad . \qquad (11)$$

Proof. First we determine the asymptotic approximation of the denominator polynomial. We apply its integral representation:

$$\Gamma(\ell+m+\mu)Q_\ell^{(m)}(x) = \int_0^\infty u^{m+\mu-1}(x+u)^\ell e^{-u}du \quad .$$

Taking $m = \beta\ell$, $x = \alpha\ell$ we use Laplace's method to obtain the main term of the integral for $\ell \to \infty$. The main contribution comes from the neighbourhood of the point $u_0 = \ell s$ where s is the root of equation

$$-1 + \frac{1}{s+\alpha} + \frac{\beta}{s} = 0 \quad .$$

In the usual manner of doing the calculations the main term is

$$\int_0^\infty u^{\beta\ell+\mu-1}(\alpha\ell+u)^\ell e^{-u}du \sim a\cdot\ell^b\cdot\exp\{\ell(-s+\ln(s+\alpha)+\beta\ln s)+(1+\beta)\ell\ln\ell\} ,$$

$$\ell \to \infty ,$$

where a and b are constans independent of ℓ. With the aid of this result we get an asymptotic representation of $L_\ell(x)$ for $\ell \to \infty$:

$$L_\ell(x) \sim a^*\ell^{b^*}\cdot\exp\{\ell(-1-\beta+(1+\beta)\ln\alpha+\beta\ln\beta+2s-2\ln(s+\alpha)-2\beta\ln\beta)\} \quad .$$

Because $L_\ell(0) = L_\ell(\infty) = 0$ the function $L_\ell(x)$ has its maximum value where $\frac{d}{dx}L_\ell(x) = 0$ or $\frac{d}{d\alpha}L_\ell(x) = 0$. By differentiating the main term of $L_\ell(x)$ we obtain the equation

$$\frac{1+\beta}{\alpha} - \frac{2}{s+\alpha} = 0$$

and therefore we can solve the equations for α and s explicitly:

$$\alpha = \frac{(1+\beta)^2}{2(1-\beta)} , \qquad s = \frac{1}{2}(1+\beta) \quad .$$

Eliminating α and s in the asymptotic expression of $L_\ell(x)$ we get the required result.

Returning to the proof of Theorem 3 the following estimations are obvious

$$\Gamma(1+\mu)L_k(x) < |E_k^{(m)}(x)| \le \Gamma(1+\mu)(L_k(x)+L_{k+1}(x)+\ldots) \quad .$$

Here, when we raise this inequality to the k^{-1}-th power, then on letting $k \to \infty$ we obtain the statement of Theorem 3.

IV. CONVERGENCE ON THE COMPLEX PLANE

In the previous section we considered convergence on the real positive axis. Here we shall be concerned with the convergence on unbounded domains of the complex plane that are symmetric with respect to the real positive axis. Such an extension of the convergence to larger domains of the complex plane "overconvergence problem" very much depends on the knowledge of the location of the poles of the two-point Padé approximations to $F(z)$. It is clear from formula (ii) that the poles of these approximants are the zeros of polynomial $Q_k^{(m)}(z)$. Our next results come from investigations of the location of the zeros for the polynomial $Q_k^{(m)}(z)$.

First of all we show that the convergence of the generalized two-point Padé approximation holds for any bounded domain of the complex plane.

From representation (i), by the Theorem of Tannery [9], it follows that

$$Q_k^{(m)}(z) \to \exp\{z/(1+\beta)\}\ ,\quad k\to\infty\ ,\quad |z|<K=\text{const.}\ , \tag{12}$$

and

$$\lim_{k\to\infty} \frac{m}{k} = \beta \ \text{(where } \beta = 0\text{, when m is bounded)}.$$

As a consequence of this result all zeros of the polynomial $Q_k^{(m)}(z)$ tend to infinity when $k \to \infty$.

Another consequence is that the rationals $R_k^{(m)}(z)$ converge to $F(z)$ faster than geometrically

$$\lim_{k\to\infty} \{\max_{|z|<K=\text{const.}} |F(z)-R_k^{(m)}(z)|\}^{\frac{1}{k}} = 0\quad . \tag{13}$$

This follows easily from the integral representation (ii) of the error.

Next we shall consider the convergence problem in parabolic type unbounded domain of the complex plane. We state another result on the location of the poles of the rational $R_k^{(m)}(z)$.

Lemma 2. The polynomials $Q_k^{(m)}(z)$ have no zeros in the parabolic domain

$$S = \{z=x+iy\in C;\ y^2<4(m+\mu)(x+m+\mu)\}\quad . \tag{14}$$

Proof. This statement immediately follows from a Theorem of Henrici [10], when we use the identifications $z_k = z$, $\beta_k = k+m+\mu-1$, $\varepsilon_k = k-1$, $q_k = \Gamma(k+m+\mu-1)Q_k^{(m)}(z)$, $k=1,2,\ldots$, $\alpha = m+\mu$.

Now we define the parabolic type unbouded domains

$$S_r = \{z=x+iy\in C;\ y^2<4r(m+\mu)(x+m+\mu)\}\quad . \tag{15}$$

The following theorem gives the estimation of the convergence rate to the $R_k^{(m)}(z)$ in S_r.

Theorem 4. Let us suppose, for the number r, that

$$0 < r < 3-2\sqrt{2} \quad , \tag{16}$$

holds; then the rationals $R_k^{(m)}(z)$ converge to $F(z)$ in the domain S_r with the geometric convergence rate

$$\lim_{k\to\infty} \{\max_{z\in S_r} |F(z)-R_k^{(m)}(z)|\}^{\frac{1}{k}} < \frac{1}{2}\left(\frac{1+r}{1-r}\right)^2 < 1 \quad . \tag{17}$$

Proof. We can apply a general Theorem of Saff and Varga [11]. For our special result we need the identifications $q = 2$, $r_k = Q_k^{(m)}(z)$. Their exceptional bounded subset K_d is missing here, i.e. we proved in previous considerations that on every bounded set stronger than geometric convergence holds.
Finally we consider the problem of convergence on unbounded sectors

$$W = \{z=x+iy\in C,\ |\arg z| < \Theta\} \quad . \tag{18}$$

When m has a finite value there exists no infinite sector of this type which is devoid of zeros of $Q_k^{(m)}(z)$, $k=1,2,\ldots$; consequently, there is no infinite sector in which the geometric convergence of the rationals $R_k^{(m)}(z)$ can hold. But when we consider the polynomial $Q_k^{[\beta k]}(z)$, $k=1,2,\ldots$ such a sector does exist nevertheless.
Lemma 3. For $k=1,2,\ldots$ the polynomial $Q_k^{[\beta k]}(z)$ has no zeros in the infinite sector

$$W_\beta=\{z=x+iy\in C,\ |\arg z|<\arccos\frac{1-\beta}{1+\beta}\} \quad , \quad 0 < \beta < 1 \quad . \tag{19}$$

Proof. We can apply a Theorem of Saff and Varga [12]. Instead of their ν (which is an integer value) must take $\beta k+\mu-1$. In this case the (rather long) proof is easy, therefore we omit it for the sake of brevity.
The following result gives the estimation of the convergence rate of $R_k^{[\beta k]}(z)$ in the infinite sector W.
Theorem 5. Let us suppose that for $\Theta_o = \arccos\frac{1-\beta}{1+\beta}$, $0<\beta<1$ the sector W_β contains no poles of $R_k^{[\beta k]}(z)$, then for every Θ satisfying the inequality

$$0 < \Theta < 4\arctan\left[\frac{1-\sqrt{\varphi(\beta)}}{1+\sqrt{\varphi(\beta)}}\cdot\tan\frac{\Theta_o}{4}\right] \quad , \tag{20}$$

the rationals $R_k^{[\beta k]}(z)$ converge to $F(z)$ in the infinite sector W with the geometric convergence rate

$$\lim_{k\to\infty}\{\max_{z\in W}|F(z)-R_k^{[\beta k]}(z)|\}^{\frac{1}{k}} < \varphi(\beta)\left\{\frac{\sin\frac{1}{4}(\Theta_o+\Theta)}{\sin\frac{1}{4}(\Theta_o-\Theta)}\right\}^2 < 1 \quad . \tag{21}$$

Proof. We can apply a general Theorem of Saff and Varga [11]. For our special result we need the identifications $q = \frac{1}{\varphi(\beta)}$, $r_k = R_k^{[\beta k]}(z)$, $\Theta_o = \arccos\frac{1-\beta}{1+\beta}$. The cited author's exceptional part $|z| \leq \mu$ of the sector is missing here because of its boundedness (on the bounded domain the stronger than geometric convergence holds).

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