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DUALITY OF THE BLOCK TRANSFORMATION AND DECIMATION FOR QUANTUM SPIN SYSTEMS

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BUDAPEST

DUALITY OF THE BLOCK TRANSFORMATION
AND DECIMATION FOR QUANTUM SPIN SYSTEMS

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ABSTRACT

The zero temperature renormalization group transformations for quantum spin systems are analyzed. The block transformation and the decimation-type transformation used in the study of the one-dimensional Ising model in transverse field are extended to the quantum version of the Potts model and Ashkin-Teller model. It is shown that for these self-dual models the two kinds of renormalization group transformations are dual to each other and therefore give the same result for the critical behaviour. The duality persists even if the higher lying states are taken into account in a perturbational way.

АННОТАЦИЯ

Исследуются разработанные для изучения квантованных спиновых систем преобразования ренормализационной группы при нулевой температуре. Обобщаются преобразования блоков и метод десятичного деления, которые применялись при изучении одномерной модели Изинга в поперечном поле, для квантованного варианта модели Поттса, а также модели Ашкина-Теллера. Показано, что в случае этих самодуальных моделей указанные преобразования обоих типов являются дуальными по отношению друг к другу, и поэтому приведут к одинаковому критическому поведению. Дуальность имеет место и тогда, если высоколежащие состояния учтены методом теории возмущения.

KIVONAT

A kvantált spin-rendszerekre kidolgozott zérushőmérsékleti renormálási csoport transzformációkat vizsgáljuk. A merőleges térbe helyezett egy-dimenziós Ising modell tanulmányozásában használatos blokk-transzformációt és decimálást általánositjuk a Potts modell és Ashkin-Teller modell kvantált változatára. Megmutatjuk, hogy ezen önduális modellek esetén a kétféle renormálási csoport transzformáció egymás duálisa és ezért a kritikus viselkedésre azonos eredményt adnak. A dualitás akkor is érvényes, ha a magasan fekvő állapotokat perturbációs uton figyelembe vesszük.

I. INTRODUCTION

The renormalization group (RG) transformations have proved to be very powerful in the description of critical phenomena[1]. The momentum shell integration method, when combined with the large order perturbational calculation[2] can produce good values for the critical exponents of three-dimensional systems[3]. In the real space RG transformations there is some arbitrariness in the choice of the mapping and the results depend strongly on the mapping. The usual transformations can be classified into two broad categories. In the block transformation[4] a cluster of spins is mapped onto a single block spin according to an ad hoc rule. The coupling between the new block spins is obtained from the couplings of the individual spins between the neighboring blocks, weighted, however, with the weight with which the individual spins appear in the block spin state. The decimation transformation[5,6] is an alternative approach. There one eliminates a fraction of the spins by considering the effective couplings these spins mediate between the remaining spins. These transformations were originally invented for classical systems and the RG transformation is performed in a way that the partition function or free energy, from which the critical behavior is derived, should remain invariant during the RG mapping.

The extensions of the RG transformations to quantum systems[7-16] have been extensively used recently both to des-

cribe critical phenomena and to understand the properties of quantum field theories. Quantum effects are usually irrelevant for the critical behavior of systems near their phase transition point, nevertheless quantum models are often used to calculate critical properties, since d dimensional classical statistical mechanical problems can be mapped onto d-1 dimensional quantum mechanical problems[17]. The ground state energy and first excited state energy of the quantum problem are related to the partition function and coherence length of the classical problem. The critical exponents can also be calculated from the behavior of the quantum equivalent.

Since in the quantum problems one is interested in the ground state energy and low lying excited state energies, the number of degrees of freedom should be thinned in the quantum RG transformation in a way that these states should be well approximated. This is achieved by keeping the low lying states in each step of an iterative procedure and neglecting some higher lying states. This can be done for the quantum systems in several different ways. The method introduced by Jafarey et al.[7] is based on splitting the system into blocks. The eigenvalue problem of the finite block is solved and as many lowest levels are retained as it is necessary to map these states to the quantum states of a single block spin. The coupling between the blocks is obtained again from the couplings between the individual spins taking into account the wave function of the block state.

An alternative approach to the quantum RG treatment of the Ising model and lattice gauge theories has been proposed by Fradkin and Raby[16]. They decimate the number of lattice sites by fixing the quantum states on a fraction of the sites, keeping that state of the intermediate spins which gives the lowest energy with the fixed configuration of the selected spins and then mapping this state to a new state where only the selected sites have spins.

The two RG transformations seem quite different. The first one is similar to the classical block transformation, the second one is more like a decimation transformation. We will show in this paper that the two transformations are in fact very closely related. The decimation type RG transformation on quantum spin systems leads to the same result as a block transformation on the dual model.

The setup of the paper is as follows. A general description of block transformation and decimation for quantum spin systems is given in Sec. II. The quantum version of the Potts model (of which the Ising model in transverse field is a particular case) is studied in Sec. III. using both RG transformations. The duality relation between the two transformations is discussed in Sec. IV. These results are given for a scale factor $b = 2$. In Sec. V. the considerations are extended for arbitrary scale factor. A similar relationship between the two RG transformations for the Ashkin-Teller model is shown to exist in Sec. VI. The effect of the higher order perturbational correc-

tions is considered in Sec. VII. It is shown that the duality

persists even if new cuplings are introduced by these corrections.

Finally Sec. VIII. contains a discussion of the results.

II. GENERAL FORMALISM

In this paper we will be concerned with one-dimensional quantum systems on a lattice. Assuming a nearest neighbor interaction $T_{i,i+1}$ and a single site term U_i, the total Hamiltonian of the system has the form

$$H = \sum_{i=1}^{N} T_{i,i+1} + \sum_{i=1}^{N} U_i \quad , \qquad (2.1)$$

where periodic boundary condition has been imposed, though this is not important in the further calculations.

If the system can be in q states at each site, the total number of states is q^N. We are interested in the ground state and low lying excited states, either because the phase transition occurs at T=0, or because these quantities of the quantum mechanical problem are the analogues of relevant quantities of a statistical mechanical problem in higher dimensions. In the quantum RG transformation the number of degrees of freedom, the number of states is decreased by mapping the chain with N sites to a chain with N/b sites in a way that the $q^{N/b}$ states of the new system should possibly coincide with the $q^{N/b}$ lowest states of the original chain.

In the block transformation[7-15] this mapping is achieved by grouping the sites into cells (each having b sites) and mapping the lowest lying states of the cells onto equivalent new states. The sites will be indexed by a cell index ℓ ($\ell = 1,2,\ldots,N/b$) and a further index α ($\alpha = 1,2,\ldots,b$). The Hamiltonian is split

into intracell and intercell parts:

$$H = H_{intra} + H_{inter} \quad , \tag{2.2}$$

where the intracell part is

$$H_{intra} = \sum_{\ell=1}^{N/b} H_{cell}(\ell) \tag{2.3}$$

with

$$H_{cell}(\ell) = \sum_{\alpha=1}^{b-1} T_{\ell,\alpha;\,\ell,\,\alpha+1} + \sum_{\alpha=1}^{b} U_{\ell,\alpha} \quad , \tag{2.4}$$

while the intercell part is

$$H_{inter} = \sum_{\ell=1}^{N/b} H_{inter}(\ell,\ell+1) \tag{2.5}$$

with

$$H_{inter}(\ell,\ell+1) = T_{\ell,b;\,\ell+1,1} \quad . \tag{2.6}$$

Solving the eigenvalue problem of a single cell first, one finds q^b states. They have the form

$$\psi_\ell^{(\beta)} = \sum_{i_1,i_2,\ldots} a_{\ell,i_1,i_2,\ldots,i_b}^{(\beta)} \, |s_{i_1}>_{\ell,1} |s_{i_2}>_{\ell,2} \cdots |s_{i_b}>_{\ell,b}$$

$$\beta = 1,2,\ldots,q^b \quad , \tag{2.7}$$

where $|s_i>_{\ell,\alpha}$ is the ith state at site ℓ,α. Keeping the q lowest lying states, they can be identified as the q states of a renor-

malized entity μ .

$$\psi_{\ell}^{(\beta)} \rightarrow |\mu_{\beta}>_{\ell} \quad , \qquad \beta = 1,2,\ldots,q \quad . \tag{2.8}$$

The new Hamiltonian acting in the space of μ states should have the same form as the Hamiltonian in Eq. (2.1) acting in the space of s states, only the couplings can have renormalized values.

The new single site term is obtained from the energy spectrum of the low lying cell states. The coupling between the neighboring cells is calculated by requiring that the matrix-elements of the new Hamiltonian between the μ states should be the same as the matrix elements of the original Hamiltonian between the corresponding cell states in the s state representation.

The decimation transformation[16] starts from a different splitting of the Hamiltonian (2.1). Using the same convention as above for indexing the sites, the first site of each cell is selected to be kept while the other sites are to be eliminated. We separate the Hamiltonian into two parts

$$H = H_{\text{fixed spin}} + H_{\text{intermediate}} \tag{2.9}$$

where $H_{\text{fixed spin}}$ contains the single site terms on the selected sites,

$$H_{\text{fixed spin}} = \sum_{\ell=1}^{N/b} H_{\text{fixed spin}}(\ell) \quad , \tag{2.10}$$

with

$$H_{\text{fixed spin}}(\ell) = U_{\ell,1} \quad , \tag{2.11}$$

while $H_{intermediate}$ contains the single site terms on the inter-mediate spins and the coupling terms

$$H_{intermediate} = \sum_{\ell=1}^{N/b} H_{intermediate}(\ell,\ell+1) \; , \qquad (2.12)$$

with

$$H_{intermediate}(\ell,\ell+1) = \sum_{\alpha=1}^{b-1} T_{\ell,\alpha;\; \ell,\alpha+1} + T_{\ell,b;\; \ell+1,1} + \sum_{\alpha=2}^{b} U_{\ell,\alpha} \; . \qquad (2.13)$$

Fixing the states on the N/b selected sites gives $q^{N/b}$ possible configurations. For each such configuration the states on the intermediate sites are chosen in such a way that the energy be minimal. This is equivalent to finding the lowest lying eigenstate of $H_{\ell,\ell+1}$ with fixed states $s_{i_{\ell,1}}$ and $s_{i_{\ell+1,1}}$ on the end sites $\ell,1$ and $\ell+1,1$. Denoting this state by $\phi_{\ell,\ell+1}(s_{i_{\ell,1}}, s_{i_{\ell+1,1}})$,

$$\phi_{\ell,\ell+1}(s_{i_{\ell,1}}, s_{i_{\ell+1,1}}) = | s_{i_{\ell,1}}>_{\ell,1} \chi_{\ell,\ell+1}(s_{i_{\ell,1}}, s_{i_{\ell+1,1}}) \, | s_{i_{\ell+1,1}}>_{\ell+1,1} \, , \qquad (2.14)$$

where

$$\chi_{\ell,\ell+1}(s_{i_{\ell,1}}, s_{i_{\ell+1,1}}) = \sum_{s_{i_{\ell,\alpha}}} b(s_{i_{\ell,1}}, s_{i_{\ell+1,1}}, s_{i_{\ell,\alpha}}) \prod_{\alpha=2}^{b} | s_{i_{\ell,\alpha}}>_{\ell,\alpha} \qquad (2.15)$$

is a linear combination of the states of the intermediate sites. The states of the system which are kept in the RG transformation are of the form

$$| s_{i_{1,1}}>_{1,1} \chi_{1,2}(s_{i_{1,1}}, s_{i_{2,1}}) | s_{i_{2,1}}>_{2,1} \cdots \chi_{\ell,\ell+1}(s_{i_{\ell,1}}, s_{i_{\ell+1,1}}) | s_{i_{\ell+1,1}}>_{\ell+1,1} \cdots$$

$$(2.16)$$

This state will then be mapped onto the state

$$|\mu_{i_{1,1}}\rangle_1 |\mu_{i_{2,1}}\rangle_2 \cdots |\mu_{i_{\ell+1,1}}\rangle_{\ell+1} \cdots \quad . \tag{2.17}$$

The Hamiltonian acting in the space of the μ states should have the same form as the original Hamiltonian, the new couplings should be calculated from the requirement that the matrix-elements of the renormalized Hamiltonian between the states given in Eq. (2.17) should be the same as the matrix elements of the original Hamiltonian between the states given in Eq. (2.16).

From the formulation of the problem it is clear that the block transformation is conveniently done in such a representation where the s and μ states are eigenstates of the single site term, the decimation is conveniently done in a representation where the nearest neighbor coupling is diagonal.

In the RG transformations presented until now the higher lying states are completely neglected. The matrix elements are calculated between states which are products of low lying eigenstates of individual cells. Hirsch and Mazenko[14] have shown that a systematic improvement can be achieved by taking into account the higher lying states in a perturbational way. Using the same mapping of the s states to the μ states as before, the requirement is not simply that the matrix-elements of the Hamiltonian should be unchanged, but the shift due to virtual excitation of the higher lying states is taken into account. The states we are working with are eigenstates of a truncated Hamiltonian.

In the case of the block transformation they are eigenstates of the intracell part

$$H_o = H_{intra} = \sum_{\ell=1}^{N/b} H_{cell}(\ell) \ , \qquad (2.18)$$

with H_ℓ given in Eq. (2.4), while for the decimation transformation they are eigenstates of the Hamiltonian of the intermediate spins

$$H_o = H_{intermediate} = \sum_{\ell=1}^{N/b} H_{intermediate}(\ell,\ell+1) \ , \quad (2.19)$$

with $H_{\ell,\ell+1}$ given in Eq. (2.13). The rest of the Hamiltonian

$$V = H_{inter} = \sum_{\ell=1}^{N/b} H_{inter}(\ell,\ell+1) = \sum_{\ell=1}^{N/b} T_{\ell,b; \ell+1,1} \ , \qquad (2.20)$$

and

$$V = H_{fixed\ spin} = \sum_{\ell=1}^{N/b} H_{fixed\ spin}(\ell) = \sum_{\ell=1}^{N/b} U_{\ell,1} \ , \quad (2.21)$$

respectively, are treated as perturbations. If $|\psi_i>$ and $|\psi_j>$ are eigenstates of H_o with energies E_i and E_j, respectively, such that they are the products of the low lying cell states, while ψ_α is an eigenstate with energy E_α, such that at least one of the cells is in a higher lying state, the RG transformation should be done by comparing the matrix elements of the renormalized Hamiltonian to the matrix elements calculated in second order in V

$$\langle \psi_i | \left[H_o + V + \frac{1}{2} \sum_\alpha V | \psi_\alpha \rangle \langle \psi_\alpha | V \left(\frac{1}{E_i - E_\alpha} + \frac{1}{E_j - E_\alpha} \right) \right] | \psi_j \rangle \quad . \tag{2.22}$$

In one case the perturbation is a nearest neighbor coupling between the end sites of neighboring cells, while in the other case the single site term on selected sites serves as a perturbation.

The two transformation can in general lead to completely different approximation schemes. We will see in the next sections that for self-dual models the two RG transformations lead to equivalent results. Block transformation is the same as decimation in the dual model and therefore the critical exponents calculated in the two ways are equal.

III. RENORMALIZATION GROUP TRANSFORMATIONS FOR THE QUANTUM VERSION OF THE POTTS MODEL

The one-dimensional Ising model in transverse field has been used extensively as a test of various quantum RG transformations, since the exact solution of this model is known[18]. The Potts model[19] being a simple generalization of the Ising model, we will consider now the quantum RG transformations on this model.

The one-dimensional Hamiltonian version of the two-dimensional classical Potts model has been discussed in the preceding paper[20]. The Hamiltonian contains two terms

$$H = H_{Potts} + H_{field} \quad , \qquad (3.1)$$

where H_{Potts} is the usual Potts coupling between the neighboring spins,

$$H_{Potts} = - \lambda \sum_{i=1}^{N} \delta_{s_i, s_{i+1}} = - \frac{\lambda}{q} \sum_{i=1}^{N} \sum_{k=1}^{q-1} \Omega_i^k \, \Omega_{i+1}^{q-k} \quad , \qquad (3.2)$$

where $s_i = 1, 2, \ldots, q$; q is the number of components of the Potts spin, and Ω is a diagonal matrix

$$\Omega = \begin{pmatrix} 1 & & & & & \\ & \omega & & & & \\ & & \omega^2 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & & & & & \omega^{q-1} \end{pmatrix} \quad , \qquad \omega = \exp \left(\frac{2\pi i}{q} \right) \quad , \qquad (3.3)$$

while H_{field} is the "transverse field" which rotates the spins,

$$H_{field} = - h \sum_{i=1}^{N} \sum_{k=1}^{q-1} M_i^k \quad , \tag{3.4}$$

where

$$M = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{3.5}$$

A. The block transformation

The preceding paper[20] contains the results of the block transformation. Here I only quote the results. Starting with the Hamiltonian given in Eqs. (3.1) - (3.4), new couplings are generated, which correspond to the simultaneous flip of two neighboring spins. The strength of these new couplings is renormalized in such a way that a well defined relationship is maintained and in fact only a single new coupling is needed in terms of which all other couplings can be expressed. Using now the notations of Ref. 20, the new couplings are denoted by λ_1, λ_2 and λ_3 and they satisfy the relations

$$\lambda_2 = \times \lambda_1 \quad , \qquad \lambda_3 = \times^2 \lambda_1 \quad . \tag{3.6}$$

The recursion relations for the renormalized couplings are as follows[20]

$$\lambda_{1\,cell} = \frac{1}{[1+(q-1)a^2][q-2+2b^2\,]} \{ \lambda_1 \,(1+a)^2 b^2 + 2(q-2)\,\lambda_2(1+a)\,a\,b +$$

$$+ (q-2)^2\,\lambda_3\,a^2 \} \,, \tag{3.7}$$

$$\lambda_{2\,cell} = \frac{1}{\sqrt{1+(q-1)a^2}\,[q-2+2b^2]^{3/2}} \{ 2\,\lambda_1(1+a)b^2 + \lambda_2\,[\,(1+a)\,b(q-3+b^2) +$$

$$+ 2(q-2)\,a\,b\,] + (q-2)\,\lambda_3\,a(q-3+b^2) \} \,, \tag{3.8}$$

$$\lambda_{3\,cell} = \frac{1}{[\,q-2+2b^2]^2} \{ 4\,\lambda_1\,b^2 + 4\,\lambda_2\,b(q-3+b^2) + \lambda_3(q-3+b^2)^2 \} \,, \tag{3.9}$$

with

$$a = \frac{q}{(q-1)\lambda_1} \{ -q\,h + \frac{q-2}{2q}\,\lambda_3 + \sqrt{(q\,h - \frac{q-2}{2q}\,\lambda_3)^2 + \frac{q-1}{q^2}\,\lambda_1^2} \} \,, \tag{3.10}$$

$$b = \frac{q}{2\lambda_2} \{ \frac{q}{2}\,h + \frac{\lambda_1-(q-3)\lambda_3}{2q} + \sqrt{(\frac{q}{2}\,h + \frac{\lambda_1-(q-3)\lambda_3}{2q})^2 + \frac{2(q-2)}{q^2}\,\lambda_2^2} \} \,, \tag{3.11}$$

and

$$q\,h_{cell} = E_1' - E_1 \,, \tag{3.12}$$

where

$$E_1 = -\,(q-2)\,h - \frac{q-2}{2q}\,\lambda_3 - \sqrt{(qh - \frac{q-2}{2q}\,\lambda_3)^2 + \frac{q-1}{q^2}\,\lambda_1^2} \,, \tag{3.13}$$

$$E_1' = -\,\frac{q-4}{2}\,h - \frac{\lambda_1+(q-3)\lambda_3}{2q} - \sqrt{(\frac{q}{2}\,h + \frac{\lambda_1-(q-3)\lambda_3}{2q})^2 + \frac{2(q-2)}{q^2}\,\lambda_2^2} \,. \tag{3.14}$$

These recursion relations have been analyzed in Ref. 20. Here we want to compare them with the results of the decimation transformation.

B. Decimation transformation

We will outline the decimation for the $b = 2$ case, when only the odd sites are retained after the decimation. According to the prescriptions given in Sec. II, the Hamiltonian is split as

$$H = H_o + V \ , \tag{3.15}$$

where

$$H_o = -\lambda \sum_{\ell=1}^{N/2} (\delta_{s_{\ell,1}s_{\ell,2}} + \delta_{s_{\ell,2}s_{\ell+1,1}}) - h \sum_{\ell=1}^{N/2} \sum_{k=1}^{q-1} M_{\ell,2}^{k} \ , \tag{3.16}$$

and

$$V = -h \sum_{\ell=1}^{N/2} \sum_{k=1}^{q-1} M_{\ell,1}^{k} \ . \tag{3.17}$$

Fixing the states at the $(\ell,1)$ sites, the eigenfunctions of H_o are easily obtained. If two neighboring fixed spins are in the same state i, the lowest energy configuration of the intermediate spin is

$$\chi_1(ii) = \frac{1}{\sqrt{1+(q-1)c^2}} \{c|1\rangle + c\{2\rangle + \ldots + c|i-1\rangle + |i\rangle + c|i+1\rangle + \ldots + c|q\rangle\},$$

$$\tag{3.18}$$

where

$$c = \frac{1}{(q-1)h} \{ -\lambda + \frac{q-2}{2} h + \sqrt{(\lambda - \frac{q-2}{2} h)^2 + (q-1) h^2} \} \ . \tag{3.19}$$

and the energy is

$$E_1(ii) = -\lambda - \frac{q-2}{2} h - \sqrt{\left(\lambda - \frac{q-2}{2} h\right)^2 + (q-1) h^2} \quad . \tag{3.20}$$

If the two end spins are in different states, the lowest energy configuration is

$$\chi_1(i \neq j) = \frac{1}{\sqrt{q-2+2d^2}} \{ |1> + |2> + \ldots + |i-1> + d|i> + |i+1> + \ldots$$

$$\tag{3.21}$$

$$+ |j-1> + d|j> + |j+1> + \ldots + |q> \} ,$$

with

$$d = \frac{1}{2h} \left\{ \frac{1}{2} \lambda - \frac{1}{2} (q-4)h + \sqrt{\left(\frac{1}{2} \lambda - \frac{1}{2} (q-4)h\right)^2 + 2(q-2)h^2} \right. \quad , \tag{3.22}$$

and the energy is

$$E_1(i \neq j) = -\frac{1}{2} \lambda - \frac{1}{2} (q-2)h - \sqrt{\left(\frac{1}{2} \lambda - \frac{1}{2}(q-4)h\right)^2 + 2(q-2)h^2} \quad . \tag{3.23}$$

Performing now the mapping as discussed in Sec. II, the new Potts coupling between the renormalized spins is obtained from the energy difference of the configurations when the neighboring fixed spins are in identical or different states:

$$\lambda_{cell} = E_1(i \neq j) - E_1(ii) \quad . \tag{3.24}$$

The renormalized field is given by the matrix element between states, where one spin is different. It turns out that the

matrix element will depend on the configuration of the neighbors.
We have to introduce three different fields defined by the matrix
elements:

$$-h_1 = <i \ j \ i \ |H| \ i \ i \ i> = < j \ j \ i \ |H| \ i \ i \ j> \ , \ i \neq j \qquad (3.25)$$

$$-h_2 = <j \ k \ i \ |H| \ i \ i \ j> \ , \ i \neq j, \ i \neq k, \ j \neq k \qquad (3.26)$$

$$-h_3 = <j \ \ell \ i \ |H| \ i \ k \ j> \ , \ i \neq k, \ i \neq \ell, \ j \neq k, \ j \neq \ell \qquad (3.27)$$

The difference in the renormalization comes clearly from the
fact, that in calculating these matrix elements, identical
neighbors appear twice in (3.25), once only in (3.26) and there
are no identical neighbors in (3.27). For a self-consistent
renormalization we have to introduce these couplings from the
very beginning. $\chi_1(ii)$ and $\chi_1(i \neq j)$ still have the same form as
in Eqs. (3.18) and (3.21), but now

$$c = \frac{1}{(q-1)h_1} \ \{ - \lambda + \frac{q-2}{2} \ h_3 + \sqrt{(\lambda - \frac{q-2}{2} \ h_3)^2 + (q-1) \ h_1^2} \ \} \qquad (3.28)$$

$$E_1(ii) = - \lambda - \frac{q-2}{2} \ h_3 - \sqrt{(\lambda - \frac{q-2}{2} \ h_3)^2 + (q-1) \ h_1^2} \ , \qquad (3.29)$$

$$d = \frac{1}{2h_2} \ \{ \frac{1}{2} \ \lambda + \frac{1}{2} \ [h_1 - (q-3)h_3] + \sqrt{[\frac{1}{2} \ \lambda + \frac{1}{2}(h_1 - (q-3)h_3)]^2 + 2(q-2)h_2^2} \ \} \ , \qquad (3.30)$$

$$E_1(i \neq j) = - \frac{1}{2} \ \lambda - \frac{1}{2} \ [h_1 + (q-3)h_3] - \sqrt{[\frac{1}{2} \ \lambda + \frac{1}{2}(h_1 - (q-3)h_3)]^2 + 2(q-2)h_2^2} \ , \qquad (3.31)$$

The renormalized value of λ is still given by the energy

difference as in Eq. (3.24), but with the new energy expressions. The renormalization of the three fields is obtained by calculating the matrix elements (3.25) - (3.27) between the renormalized states. We get

$$
h_{1 \text{ cell}} = \frac{1}{(1+(q-1)c^2)(q-2+2d^2)} \{ h_1(1+c)^2 d^2 + 2(q-2)h_2(1+c)cd +
$$

$$
+ (q-2)^2 h_3 c^2 \} , \tag{3.32}
$$

$$
h_{2 \text{ cell}} = \frac{1}{\sqrt{1+(q-1)c^2}(q-2+2d^2)^{3/2}} \{ 2h_1(1+c)d^2 + h_2[(1+c)d(q-3+d^2) +
$$

$$
+2(q-2)cd] + (q-2)h_3c(q-3+d^2) \} , \tag{3.33}
$$

$$
h_{3 \text{ cell}} = \frac{1}{(q-2+2d^2)^2} \{ 4h_1 d^2 + 4h_2 d(q-3+d^2) + h_3(q-3+d^2)^2 \} . \tag{3.34}
$$

The three fields are not really independent of each other. Since in the unrenormalized model $h_1 = h_2 = h_3$, in each step of the iteration

$$
h_{3 \text{ cell}} / h_{1 \text{ cell}} = (h_{2 \text{ cell}} / h_{1 \text{ cell}})^2 \tag{3.35}
$$

Comparing now these recursion relations with those obtained in the block transformation (see Eqs. (3.7) - (3.14)), it is seen immediately that the substitution

$$
\lambda \to qh, \quad qh_1 \to \lambda_1 , \quad qh_2 \to \lambda_2 , \quad qh_3 \to \lambda_3 \tag{3.36}
$$

generates the results of the block transformation from the deci-

mation transformation. We will show in the next section that these relations are the consequence of the self-duality of the Potts model.

IV.   DUALITY RELATIONS IN THE POTTS MODEL

The Hamiltonian of the quantum version of the Potts model has been given in terms of the matrices Ω_i and M_i. These matrices commute if they belong to different sites, while on the same site they satisfy the following algebra:

$$\Omega_i^k \, \Omega_i^\ell = \Omega_i^{k+\ell} \quad , \tag{4.1}$$

$$M_i^k \, M_i^\ell = M_i^{k+\ell} \quad , \tag{4.2}$$

$$M_i^k \, \Omega_i^\ell = \omega^{k\ell} \Omega_i^\ell \, M_i^k \quad . \tag{4.3}$$

Let us introduce the dual lattice and define the operators $\Omega_i'^{\,k}$ and $M_i'^{\,k}$ on the sites of the dual lattice, i.e. on the links of the original lattice:

$$\Omega_i'^{\,k} = \prod_{j<i} M_j^k \tag{4.4}$$

$$M_i'^{\,k} = \Omega_i^{q-k} \, \Omega_{i+1}^k \quad . \tag{4.5}$$

It is easy to see, that these operators satisfy the same algebra. For example

$$M_i'^{\,k} \, \Omega_i'^{\,\ell} = \omega^{k\ell} \Omega_i'^{\,\ell} \, M_i'^{\,k} \tag{4.6}$$

The Hamiltonian can be written in terms of these new operators as

$$H = H_{Potts} + H_{field} \tag{4.7}$$

where now

$$H_{Potts} = -\frac{\lambda}{q} \sum_{i=1}^{N} \sum_{k=1}^{q-1} M_i'^k \qquad (4.8)$$

and

$$H_{field} = -h \sum_{i=1}^{N} \sum_{k=1}^{q-1} \Omega_i'^k \Omega_{i+1}'^{q-k} \qquad (4.9)$$

We have used the relation $\omega^q = 1$ in deriving this form of the Hamiltonian.

Since the new operators satisfy the same algebra as the original ones, comparison of the two forms of the Hamiltonian leads to the duality relationship[21]: for any value of $\frac{\lambda}{q}$ and h the model should behave in the same way as the model in which $\frac{\lambda}{q}$ and h are interchanged. The relations in Eq. (3.35) are the generalizations of this duality relationship for the renormalized Hamiltonian.

V. CALCULATION FOR LARGER CELLS

The quantum RG transformations usually do not give good values for the critical exponents when the scale factor b=2 . One way to improve the results is to take larger scale factors[13]. The analysis becomes very cumbersome and the lowest energy states of the cell problem can be found numerically only. It is, however, possible to see, without solving the problem, that the duality of the block transformation and decimation persists even for b>2 .

Let us look at the case b=3 . It is convenient to use in the block transformation the states defined by

$$| \ell' > = \frac{1}{\sqrt{q}} \sum_{k=1}^{q} \omega^{(\ell-1)(k-1)} | k > . \tag{5.1}$$

They are eigenstates of H_{field} ,

$$H_{field} | 1' > = - (q-1) h | 1' > ,$$

$$\tag{5.2}$$

$$H_{field} | \ell' > = h | \ell' > \quad \text{for} \quad \ell \neq 1 ,$$

while H_{Potts} will now flip the neighboring spins simultaneously. It is convenient for the further comparison to shift the energies by - h so that

$$H_{field} | \ell' > = - q h \delta_{\ell,1} | \ell' > . \tag{5.3}$$

The intercell part of the Hamiltonian will mix the following q^2 states:

$$|1'\ 1'\ 1'>\ ,\quad |1'\ 2'\ q'>\ ,\quad |1'\ 3'(q-1)'>\ ,\ ...\quad |1'\ q'\ 2'>\ ,$$

$$|2'\ q'\ 1'>\ ,\quad |3'(q-1)'\ 1'>\ ,\ ...\quad |q'\ 2'\ 1'>\ ,$$

$$|2'\ 1'\ q'>\ ,\quad |3'\ 1'(q-1)'>\ ,\ ...\quad |q'\ 1'\ 2'>\ ,\qquad (5.4)$$

$$|2'\ 2'(q-1)'>\ ,\quad |2'\ 3'(q-2)'>\ ,\ ...\quad |2'(q-1)'\ 2'>\ ,$$

$$|3'\ 2'(q-2)'>\ ,\ ...\qquad ...\quad |q'\ q'\ 3'>\ .$$

One has to find the lowest energy eigenstate of the intercell Hamiltonian in the subspace spanned by these states.

Alternatively in the decimation transformation one has to find the lowest energy configuration of two intermediate spins. Fixing the two end spins, the intermediate spins can be in q^2 configurations

$$|1\ 1>\ ,\quad |1\ 2>\ ,\ ...\ |1\ q>\ ,$$

$$|2\ 1>\ ,\quad |3\ 1>\ ,\ ...\ |q\ 1>\ ,$$

$$\qquad\qquad\qquad\qquad\qquad\qquad (5.5)$$

$$|2\ 2>\ ,\quad |3\ 3>\ ,\ ...\ |q\ q>\ ,$$

$$|2\ 3>\ ,\ ...\qquad ...\ |q\ (q-1)>\ .$$

We have to find the lowest energy configuration of the two intermediate spins when the end spins are fixed e.g. in the $|1>...|1>$ state. One can easily convince oneself that the eigenvalue matrices in the two transformations are related by the duality relations given in Eq. (3.36).

In the same way as for the b=2 case, the next lowest lying states of the cell problem in the block transformation are in an ortogonal subspace which can be generated starting from the state $|1'\ 1'\ 2'>$. Analogously one can look for the lowest energy state of the two intermediate spins in the decimation transformation when the two end spins are fixed in the states $|1> \ldots |2>$. Again the two eigenvalue matrices are related by the duality relations.

So in general one can show by writing down the eigenvalue matrices and the wave functions of the lowest energy configurations, that the two RG transformations are dual to each other. This is a consequence of the fact that $H_{intermediate}$ in Eqs. (2.12)-(2.13) and H_{intra} in Eqs. (2.3) - (2.4), of which the lowest lying states are considered and $H_{fixed\ spin}$ in Eqs. (2.10)-(2.11) and H_{inter} in Eqs. (2.5) - (2.6) which are treated as perturbations, are dual to each other. It is important to emphasize that this duality persist even after renormalization, when new couplings are generated.

VI. THE QUANTUM Z(4) MODEL OR ASHKIN-TELLER MODEL

Let us consider now the RG transformations for the quantum version of the Ashkin-Teller model. In the classical Ashkin-Teller model[22] there are four possible states at each lattice site. The energy of the system depends on the configuration of the nearest neighbors. It is $-\frac{1}{2}\lambda_1$ if the neighbors are in the same state, i.e. for the configurations $|11\rangle$, $|22\rangle$, $|33\rangle$ and $|44\rangle$. The energy is $+\frac{1}{2}\lambda_1$ for the configurations $|13\rangle$ and $|24\rangle$, while for the configurations $|12\rangle$, $|14\rangle$, $|23\rangle$ and $|34\rangle$ the energy is $\frac{1}{2}\lambda_2$. In the case when $\lambda_1 = \lambda_2$, we recover the four-state Potts model, while $\lambda_2 = 0$ is the usual clock model.

In the quantum version of the Ashkin-Teller model spin-flip terms are introduced. The transverse field which flips the spins can be defined by the relations

$$H_h \ |1\rangle = + \ h_2 \ |1\rangle - h_1 |2\rangle - h_2 |3\rangle - h_1 |4\rangle \ ,$$

$$H_h \ |2\rangle = - \ h_1 \ |1\rangle + h_2 |2\rangle - h_1 |3\rangle - h_2 |4\rangle \ ,$$

$$H_h \ |3\rangle = - \ h_2 \ |1\rangle - h_1 |2\rangle + h_2 |3\rangle - h_1 |4\rangle \ , \qquad (6.1)$$

$$H_h \ |4\rangle = - \ h_1 \ |1\rangle - h_2 |2\rangle - h_1 |3\rangle + h_2 |4\rangle \ ,$$

where H_h is the field term in the Hamiltonian.

Equivalently we could use a linear combination of the states

$$|1'\rangle = \frac{1}{2} \ (\ |1\rangle + |2\rangle + |3\rangle + |4\rangle \ ) \ ,$$

$$|2'\rangle = \frac{1}{2} \ (\ |1\rangle + i|2\rangle - |3\rangle - i|4\rangle \ ) \ ,$$

$$|3'\rangle = \frac{1}{2} \ (\ |1\rangle - |2\rangle + |3\rangle - |4\rangle \ ) \ , \qquad (6.2)$$

$$|4'\rangle = \frac{1}{2} \ (\ |1\rangle - i|2\rangle - |3\rangle + i|4\rangle \ ) \ ,$$

which are eigenstates of H_h ,

$$H_h|1'> = -2h_1|1'> , \quad H_h|2'> = 2h_2|2'> ,$$

$$H_h|3'> = 2h_1|3'> , \quad H_h|4'> = 2h_2|4'> . \tag{6.3}$$

The Ashkin-Teller coupling part of the Hamiltonian, H_λ , in this representation will flip the neighboring spins, e.g.

$$H_\lambda|1'1'> = \frac{\lambda_2}{4}|1'1'> - \frac{\lambda_1}{4}|2'4'> - \frac{\lambda_2}{4}|3'3'> - \frac{\lambda_1}{4}|4'2'> ,$$

$$H_\lambda|2'4'> = -\frac{\lambda_1}{4}|1'1'> + \frac{\lambda_2}{4}|2'4'> - \frac{\lambda_1}{4}|3'3'> - \frac{\lambda_2}{4}|4'2'> ,$$

$$\tag{6.4}$$

$$H_\lambda|3'3'> = -\frac{\lambda_2}{4}|1'1'> - \frac{\lambda_1}{4}|2'4'> + \frac{\lambda_2}{4}|3'3'> - \frac{\lambda_1}{4}|4'2'> ,$$

$$H_\lambda|4'2'> = -\frac{\lambda_1}{4}|1'1'> - \frac{\lambda_2}{4}|2'4'> - \frac{\lambda_1}{4}|3'3'> + \frac{\lambda_2}{4}|4'2'> .$$

Similar relations hold for the states $|1'2'>$, $|2'1'>$, $|3'4'>$ and $|4'3'>$, for $|1'3'>$, $|2'2'>$, $|3'1'>$ and $|4'4'>$ as well as for $|1'4'>$, $|2'3'>$, $|3'2'>$ and $|4'1'>$.

In the block transformation it is convenient to use this representation. In the same way as in the Potts model, new couplings are generated by the renormalization. Accordingly, we will generalize the Ashkin-Teller coupling part of the Hamiltonian, H_λ , to have seven couplings:

$$H_\lambda|1'1'> = \frac{\lambda_o}{4}|1'1'> - \frac{\lambda_1}{4}|2'4'> - \frac{\lambda_2}{4}|3'3'> - \frac{\lambda_1}{4}|4'2'> ,$$

$$H_\lambda|2'4'> = -\frac{\lambda_1}{4}|1'1'> + \frac{\lambda_o}{4}|2'4'> - \frac{\lambda_3}{4}|3'3'> - \frac{\lambda_4}{4}|4'2'> ,$$

$$\tag{6.5}$$

$$H_\lambda|3'3'> = -\frac{\lambda_2}{4}|1'1'> - \frac{\lambda_3}{4}|2'4'> + \frac{\lambda_o}{4}|3'3'> - \frac{\lambda_3}{4}|4'2'> ,$$

$$H_\lambda|4'2'> = -\frac{\lambda_1}{4}|1'1'> - \frac{\lambda_4}{4}|2'4'> - \frac{\lambda_3}{4}|3'3'> + \frac{\lambda_o}{4}|4'2'> ,$$

and

$$H_\lambda |1'2'\rangle = \frac{\lambda_o}{4} |1'2'\rangle - \frac{\lambda_1}{4} |2'1'\rangle - \frac{\lambda_6}{4} |3'4'\rangle - \frac{\lambda_5}{4} |4'3'\rangle \ ,$$

$$H_\lambda |2'1'\rangle = - \frac{\lambda_1}{4} |1'2'\rangle + \frac{\lambda_o}{4} |2'1'\rangle - \frac{\lambda_5}{4} |3'4'\rangle - \frac{\lambda_6}{4} |4'3'\rangle \ ,$$

$$H_\lambda |3'4'\rangle = - \frac{\lambda_6}{4} |1'2'\rangle - \frac{\lambda_5}{4} |2'1'\rangle + \frac{\lambda_o}{4} |3'4'\rangle - \frac{\lambda_3}{4} |4'3'\rangle \ ,$$

$$H_\lambda |4'3'\rangle = - \frac{\lambda_5}{4} |1'2'\rangle - \frac{\lambda_6}{4} |2'1'\rangle - \frac{\lambda_3}{4} |3'4'\rangle + \frac{\lambda_o}{4} |4'3'\rangle \ ,$$

(6.6)

The relations for the set of states $|1'4'\rangle$, $|4'1'\rangle$, $|3'2'\rangle$, $|2'3'\rangle$ are the same as in (6.6), while

$$H_\lambda |1'3'\rangle = \frac{\lambda_o}{4} |1'3'\rangle - \frac{\lambda_5}{4} |2'2'\rangle - \frac{\lambda_2}{4} |3'1'\rangle - \frac{\lambda_5}{4} |4'4'\rangle \ ,$$

$$H_\lambda |2'2'\rangle = - \frac{\lambda_5}{4} |1'3'\rangle + \frac{\lambda_o}{4} |2'2'\rangle - \frac{\lambda_5}{4} |3'1'\rangle - \frac{\lambda_4}{4} |4'4'\rangle \ ,$$

(6.7)

$$H_\lambda |3'1'\rangle = - \frac{\lambda_2}{4} |1'3'\rangle - \frac{\lambda_5}{4} |2'2'\rangle + \frac{\lambda_o}{4} |3'1'\rangle - \frac{\lambda_5}{4} |4'4'\rangle \ ,$$

$$H_\lambda |4'4'\rangle = - \frac{\lambda_5}{4} |1'3'\rangle - \frac{\lambda_4}{4} |2'2'\rangle - \frac{\lambda_5}{4} |3'1'\rangle + \frac{\lambda_o}{4} |4'4'\rangle \ .$$

λ_o can be set equal to zero, it does not play any role in the renormalization of the other couplings.

Solving first the cell problem with two sites, the four lowest lying levels of the cell are two non-degenerate and one doubly degenerate levels. A non-degenerate level is at E_1 , which is the solution of

$$\begin{vmatrix} -E_1 - 4h_1 & -\frac{\lambda_1}{4} & -\frac{\lambda_2}{4} & -\frac{\lambda_1}{4} \\ -\frac{\lambda_1}{4} & -E_1 + 4h_2 & -\frac{\lambda_3}{4} & -\frac{\lambda_4}{4} \\ -\frac{\lambda_2}{4} & -\frac{\lambda_3}{4} & -E_1 + 4h_1 & -\frac{\lambda_3}{4} \\ -\frac{\lambda_1}{4} & -\frac{\lambda_4}{4} & -\frac{\lambda_3}{4} & -E_1 + 4h_2 \end{vmatrix} = 0 \ , \quad (6.8)$$

a doubly degenerate level is at

$$E_2 = 2h_2 - \frac{\lambda_1 + \lambda_3}{8} - \sqrt{(2h_1 + \frac{\lambda_1 - \lambda_3}{8})^2 + (\frac{\lambda_5 + \lambda_6}{4})^2} \quad , \tag{6.9}$$

and another non-degenerate level is at

$$E_3 = 2h_2 - \frac{\lambda_2 + \lambda_4}{8} - \sqrt{(2h_2 + \frac{\lambda_2 - \lambda_4}{8})^2 + (\frac{\lambda_5}{2})^2} \quad . \tag{6.10}$$

The corresponding wave functions are:

$$\psi_1 = \frac{1}{\sqrt{1 + 2a_1^2 + a_2^2}} \{ |1'1'> + a_1|2'4'> + a_2|3'3'> + a_1|4'2'> \} \tag{6.11}$$

with

$$a_1 = \frac{(-E_1 - 4h_1)\lambda_3 + \frac{1}{4}\lambda_1\lambda_2}{(-E_1 + 4h_2 - \frac{\lambda_4}{4})\lambda_2 + \frac{1}{2}\lambda_1\lambda_3} \quad ,$$

$$\tag{6.12}$$

$$a_2 = \frac{(-E_1 - 4h_1)\lambda_3 + \frac{1}{4}\lambda_1\lambda_2}{(-E_1 + 4h_1)\lambda_1 + \frac{1}{4}\lambda_2\lambda_3} \quad ,$$

$$\psi_2^{(1)} = \frac{1}{\sqrt{2 + 2b^2}} \{ b|1'2'> + b|2'1'> + |3'4'> + |4'3'> \} \quad ,$$

$$\tag{6.13}$$

$$\psi_2^{(2)} = \frac{1}{\sqrt{2 + 2b^2}} \{ b|1'4'> + b|4'1'> + |3'2'> + |2'3'> \} \quad ,$$

with

$$b = \frac{4}{\lambda_5 + \lambda_6} \left\{ 2h_1 + \frac{\lambda_1 - \lambda_3}{8} + \sqrt{(2h_1 + \frac{\lambda_1 - \lambda_3}{8})^2 + (\frac{\lambda_5 + \lambda_6}{4})^2} \right\} , \quad (6.14)$$

and

$$\psi_3 = \frac{1}{\sqrt{2 + 2c^2}} \left\{ c|1'3'> + c|3'1'> + |2'2'> + |4'4'> \right\} , \quad (6.15)$$

with

$$c = \frac{2}{\lambda_5} \left\{ 2h_2 + \frac{\lambda_2 - \lambda_4}{8} + \sqrt{(2h_2 + \frac{\lambda_2 - \lambda_4}{8})^2 + (\frac{\lambda_5}{2})^2} \right\} . \quad (6.16)$$

The mapping of these four states of the cell to the states of the renormalized spin is chosen as:

$$\psi_1 \to |1'>^{cell} , \quad \psi_2^{(1)} \to |2'>^{cell} , \quad \psi_2^{(2)} \to |4'>^{cell} , \quad \psi_3 \to |3'>^{cell} . \quad (6.17)$$

The renormalized values of the fields are obtained from the energy spectrum of the renormalized states:

$$h_{1\ cell} = \frac{1}{4} (E_3 - E_1) ,$$

$$\quad (6.18)$$

$$h_{2\ cell} = \frac{1}{4} (E_2 - E_1) - \frac{1}{4} (E_3 - E_2) .$$

The renormalized λ couplings can be calculated from the matrix elements between states differing by two spin flips.

We get:

$$\lambda_{1\ cell} = \frac{1}{(1+2a_1^2+a_2^2)(2+2b^2)} \{ \lambda_1(1+a_1)^2b^2 + 2\lambda_5(1+a_1)b(a_1+a_2) + \lambda_3(a_1+a_2)^2 \} ,$$

$$\lambda_{2\ cell} = \frac{1}{(1+2a_1^2+a_2^2)(2+2c^2)} \{ \lambda_2(1+a_2)^2c^2 + 4\lambda_6(1+a_2)c\,a_1 + 4\lambda_4\,a_1^2 \} ,$$

$$\lambda_{3\ cell} = \frac{1}{(2+2b^2)(2+2c^2)} \{ \lambda_1(b+c)^2 + 2\lambda_5(b+c)(1+bc) + \lambda_3(1+bc)^2 \} ,$$

$$\lambda_{4\ cell} = \frac{1}{(2+2b^2)^2} \{ 4\lambda_2\,b^2 + 4\lambda_6 b(1+b^2) + \lambda_4(1+b^2)^2 \} , \qquad (6.19)$$

$$\lambda_{5\ cell} = \frac{1}{\sqrt{1+2a_1^2+a_2^2}\,(2+2b^2)\,\sqrt{2+2c^2}} \{ \lambda_1(1+a_1)b(b+c) + \lambda_5[(1+a_1)b(1+bc) +$$

$$+ (b+c)(a_1+a_2)] + \lambda_3(a_1+a_2)(1+bc) \} .$$

$$\lambda_{6\ cell} = \frac{1}{\sqrt{(1+2a_1^2+a_2^2}\,(2+2b^2)\,\sqrt{2+2c^2}} \cdot \{ 2\lambda_2(1+a_2)cb + \lambda_6[(1+a_2)c(1+b^2)+4a_1b\,] $$

$$+ 2\lambda_4\,a_1(1+b^2) \} ,$$

In the physical model, before renormalization there are only two couplings, $\lambda_1 = \lambda_3 = \lambda_5$ and $\lambda_2 = \lambda_4 = \lambda_6$, and as it is easy to see, the generated new couplings are not independent of each other. There are in fact two new couplings, since

$$\lambda_{3\ cell}/\lambda_{1\ cell} = (\lambda_{5\ cell}/\lambda_{1\ cell})^2 \quad \text{and} \qquad (6.20)$$

$$\lambda_{4\ cell}/\lambda_{2\ cell} = (\lambda_{6\ cell}/\lambda_{2\ cell})^2 .$$

Let us look now at the decimation transformation. Here it is more convenient to work in that representation, in which the Ashkin-Teller coupling is diagonal. The renormalization will lead to new spin-flip terms, more precisely the spin flip amplitude will depend on the two neighboring spins. We will introduce six spin-flip terms defined by the matrix elements:

$$-h_1 = \langle 121|H_h|111\rangle = \langle 221|H_h|112\rangle \ ,$$

$$-h_2 = \langle 131|H_h|111\rangle = \langle 331|H_h|113\rangle$$

$$-h_3 = \langle 131|H_h|121\rangle = \langle 241|H_h|132\rangle$$

$$-h_4 = \langle 141|H_h|121\rangle = \langle 341|H_h|123\rangle$$

$$-h_5 = \langle 241|H_h|112\rangle = \langle 231|H_h|122\rangle$$

$$-h_6 = \langle 231|H_h|112\rangle = \langle 241|H_h|122\rangle$$

(6.21)

Fixing now every other spin on the chain, the eigenvalue problem for the fixed configuration is easily solved. If the two endspins are in the same state, say in $|1\rangle$, the lowest energy configuration of the intermediate spin is

$$\chi_1(11) = \frac{1}{\sqrt{1+2d_1^2+d_2^2}} \{ \ |1\rangle + d_1|2\rangle + d_2|3\rangle + d_1|4\rangle \ \} \ , \qquad (6.22)$$

with

$$d_1 = \frac{(-E_1(11)-\lambda_1)h_3+h_1h_2}{(-E_1(11)+\lambda_2-h_4)h_2+2h_1h_3} \quad , \quad d_2 = \frac{(-E(11)-\lambda_1)h_3+h_1h_2}{(-E_1(11)+\lambda_1)h_1+h_2h_3} \quad .$$

(6.23)

and $E_1(11)$ is the lowest energy solution of

$$
\begin{vmatrix}
-E_1-\lambda_1 & -h_1 & -h_2 & -h_1 \\
-h_1 & -E_1+\lambda_2 & -h_3 & -h_4 \\
-h_2 & -h_3 & -E_1+\lambda_1 & -h_3 \\
-h_1 & -h_4 & -h_3 & -E_1+\lambda_2
\end{vmatrix} = 0 \ . \qquad (6.24)
$$

If the two end-spins are in the $|12\rangle$, $|14\rangle$, $|23\rangle$ or $|34\rangle$ configurations, the lowest energy configuration of the intermediate spin is different. For the $|12\rangle$ state, e.g.

$$
\chi_1(12) = \frac{1}{\sqrt{2+2e^2}} \{ e|1\rangle + e|2\rangle + |3\rangle + |4\rangle \} \ , \qquad (6.25)
$$

with

$$
e = \frac{1}{h_5+h_6} \{ \tfrac{1}{2} \lambda_1 + \tfrac{1}{2} (h_1-h_3) + \sqrt{(\tfrac{1}{2} \lambda_1 + \tfrac{1}{2} (h_1-h_3))^2 + (h_5+h_6)^2} \} \ ,
$$

$$
(6.26)
$$

and the energy is

$$
E_1(12) = \tfrac{1}{2} (\lambda_2 -h_1 -h_3) - \sqrt{\tfrac{1}{4}(\lambda_1+h_1-h_3)^2 + (h_5+h_6)^2} \ . \qquad (6.27)
$$

Similarly, when the two endspins are in the $|13\rangle$ configuration, we get

$$
\chi_1(13) = \frac{1}{\sqrt{2+2f^2}} \{ f|1\rangle + |2\rangle + f|3\rangle + |4\rangle \} \ , \qquad (6.28)
$$

with

$$
f = \frac{1}{2h_5} \{ \tfrac{1}{2} (\lambda_2 +h_2 -h_4) + \sqrt{\tfrac{1}{4} (\lambda_2 +h_2 -h_4)^2 + 4h_5^2} \} \ , \qquad (6.29)
$$

and the energy is

$$E_1(13) = \frac{1}{2}(\lambda_2 - h_2 - h_4) - \sqrt{\frac{1}{4}(\lambda_2 + h_2 - h_4)^2 + 4h_5^2} \quad . \tag{6.30}$$

The renormalized Ashkin-Teller couplings are simply given by the energy differences of the various configurations:

$$\lambda_{1\ cell} = E_1(13) - E_1(11) \quad ,$$

$$\lambda_{2\ cell} = (E_1(12) - E_1(11)) - (E_1(13) - E_1(12)) \quad . \tag{6.31}$$

The spin-flip amplitudes can be obtained by calculating the matrix elements of Eq. (6.21) between the renormalized states. We get:

$$h_{1\ cell} = \frac{1}{(1+2d_1^2+d_2^2)(2+2e^2)} \{ h_1(1+d_1)^2 e^2 + 2h_5(1+d_1) e(d_1+d_2) + h_3(d_1+d_2)^2 \} ,$$

$$h_{2\ cell} = \frac{1}{(1+2d_1^2+d_2^2)(2+2f^2)} \{ h_2(1+d_2)^2 f^2 + 4h_6(1+d_2) f d_1 + 4h_4 d_1^2 \} ,$$

$$h_{3\ cell} = \frac{1}{(2+2e^2)(2+2f^2)} \{ h_1(e+f)^2 + 2h_5(e+f)(1+ef) + h_3(1+ef)^2 \} ,$$

$$h_{4\ cell} = \frac{1}{(2+2e^2)^2} \{ 4h_2 e^2 + 4h_6 e(1+e^2) + h_4(1+e^2)^2 \} , \tag{6.32}$$

$$h_{5\ cell} = \frac{1}{\sqrt{1+2d_1^2 + d_2^2}(2+2e^2)\sqrt{2+2f^2}} \{ h_1(1+d_1)e(e+f) + h_5[(1+d_1)e(1+ef) +$$

$$+ (d_1+d_2)(e+f)] + h_3(d_1+d_2)(1+ef) \} ,$$

$$h_{6\ cell} = \frac{1}{\sqrt{1+2d_1^2 + d_2^2}(2+2e^2)\sqrt{2+2f^2}} \{ 2h_2(1+d_2)ef + h_6[(1+d_2)f(1+e^2) + 4d_1 e] +$$

$$+ 2h_4 d_1(1+e^2) \} \quad .$$

Comparing now these recursion relations with those obtained in the block transformation, Eqs. (6.18) - (6.19), we see that after the substitution

$$\lambda_i \rightarrow 4h_i \quad , \quad i = 1,2,$$

$$4h_j \rightarrow \lambda_j \quad , \quad j = 1,2,\ldots,6,$$

(6.33)

the two transformations lead to identical recursion relations. In the block transformation new λ couplings are introduced, while in the decimation new spin-flip terms entered, but in a dual manner. So the decimation transformation in these general four state models is equivalent to the block transformation in the dual model.

VII. PERTURBATIONAL CORRECTIONS TO THE RECURSION RELATIONS

A consistent way to improve the results of the block trans-
formation has been suggested by Hirsch and Mazenko[14]. The inter-
cell Hamiltonian is treated as a perturbation and the higher
order corrections are calculated in a consistent perturbational
way. In second order e.g. the renormalized Hamiltonian is obtained
to match the second order matrix elements given in Eq. (2.22).

A consistent treatment of the Potts model or Ashkin-Teller
model is prohabitively difficult due to the large number of new
couplings generated in higher orders. We will show here for the
Ising model that the new couplings are again dual to each other.

We write the Hamiltonian of the Ising chain in transverse
field in the form

$$H = - h \sum_{i=1}^{N} \sigma_i^x - \frac{\lambda}{2} \sum_{i=1}^{N} \sigma_i^z \sigma_{i+1}^z \quad , \qquad (7.1)$$

where σ^x and σ^z are the Pauli operators. Hirsch and Mazenko[14]
have shown that in second order a new coupling of the form

$$- K \sigma_{i-1}^z \sigma_i^x \sigma_{i+1}^z \qquad (7.2)$$

is generated. Note that in Ref. 14. both the Hamiltonian and the
new coupling are written with a different choice of the coordinate
system.

The recursion relations obtained by Hirsch and Mazenko[14]
can be written in the form

$$h_{cell} = \frac{1}{2}(E_1 - E_o) + (\frac{\lambda}{2})^2 \frac{(1-a^2)^2}{8(1+a^2)^2} (\frac{1}{E_1} - \frac{1}{E_o}) \quad , \tag{7.3}$$

$$\lambda_{cell} = \lambda \frac{(1+a)^2}{2(1+a^2)} - 2K \frac{1-a^2}{1+a^2} \quad , \tag{7.4}$$

$$K_{cell} = (\frac{\lambda}{2})^2 \frac{(1-a^2)^2}{8(1+a^2)^2} (\frac{1}{E_1} + \frac{1}{E_o}) \quad , \tag{7.5}$$

with

$$a = \frac{2}{\lambda} [\sqrt{(\frac{\lambda}{2})^2 + 4h^2} - 2h] \quad , \tag{7.6}$$

$$E_o = - \sqrt{(\frac{\lambda}{2})^2 + 4h^2} \quad , \tag{7.7}$$

$$E_1 = - \frac{\lambda}{2} \quad . \tag{7.8}$$

In the decimation transformation the Hamiltonian is split as

$$H = H_o + V \quad , \tag{7.9}$$

with

$$H_o = - h \sum_{\ell=1}^{N/b} \sum_{\alpha=2}^{b} \sigma_{\ell,\alpha}^x - \frac{\lambda}{2} \sum_{\ell=1}^{N/b} \sum_{\alpha=1}^{b-1} \sigma_{\ell,\alpha}^z \sigma_{\ell,\alpha+1}^z - \frac{\lambda}{2} \sum_{\ell=1}^{N/b} \sigma_{\ell,b}^z \sigma_{\ell+1,1}^z \quad , \tag{7.10}$$

and

$$V = - h \sum_{\ell=1}^{N/b} \sigma_{\ell,1}^x \quad . \tag{7.11}$$

The eigenstates of H_o are easily obtained by fixing the spins on the selected sites and solving the cell problem with the fixed end spins. For a scale factor $b = 2$, the wave functions with both

end spins up are

$$\psi_1(\uparrow\uparrow) = \frac{1}{\sqrt{1+c^2}} \; |\uparrow> (|\uparrow> + c|\downarrow>) |\uparrow> = |\uparrow> \chi_1(\uparrow\uparrow) |\uparrow> \; ,$$

$$\psi_2(\uparrow\uparrow) = \frac{1}{\sqrt{1+c^2}} \; |\uparrow> (-c|\uparrow> + |\downarrow>) |\uparrow> = |\uparrow> \chi_2(\uparrow\uparrow) |\uparrow> \; ,$$
(7.12)

where

$$c = \frac{1}{h} \{ \sqrt{\lambda^2 + h^2} - \lambda \} \tag{7.13}$$

and the energies are

$$E_1(\uparrow\uparrow) = - \sqrt{\lambda^2 + h^2} \; , \qquad E_2(\uparrow\uparrow) = \sqrt{\lambda^2 + h^2} \; , \tag{7.14}$$

The wave functions when both end spins point down can be obtained by flipping all spins.

When the two end spins are in different states, the wave functions are:

$$\psi_1(\uparrow\downarrow) = \frac{1}{\sqrt{2}} \; |\uparrow> (|\uparrow>+|\downarrow>) |\downarrow> = |\uparrow> \chi_1(\uparrow\downarrow) |\downarrow> \; ,$$

$$\psi_2(\uparrow\downarrow) = \frac{1}{\sqrt{2}} \; |\uparrow> (|\uparrow>-|\downarrow>) |\downarrow> = |\uparrow> \chi_2(\uparrow\downarrow) |\downarrow> \; ,$$
(7.15)

with energies

$$E_1(\uparrow\downarrow) = - h \; , \qquad E_2(\uparrow\downarrow) = h \; . \tag{7.16}$$

The intermediate spins are eliminated by taking the lowest energy state for each configuration, i.e. keeping $\psi_1(\uparrow\uparrow)$ and $\psi_1(\uparrow\downarrow)$ and mapping these states onto the $|\uparrow\uparrow>$ and $|\uparrow\downarrow>$ con-

figurations of the renormalized spins. The first order renorma-
lized value of λ is obtained from the energy difference of the
$|\uparrow\uparrow>$ and $|\uparrow\downarrow>$ configurations:

$$\lambda_{cell} = E_1(\uparrow\downarrow) - E_1(\uparrow\uparrow) \quad . \tag{7.17}$$

The renormalization of the transverse field can be calcula-
ted from the matrix element between two configurations differing
by one spin flip on a selected site. This gives

$$h_{cell} = h \frac{(1+c)^2}{2(1+c^2)} \quad . \tag{7.18}$$

It is straightforward to calculate the matrix elements
appearing in the second-order correction in Eq. (2.17). The second-
-order energy shifts of the configurations, when the fixed spin
orientations are $...\uparrow\uparrow\uparrow...$ or $...\uparrow\downarrow\uparrow...$, lead to a second-order
correction to the coupling between the renormalized spins:

$$\Delta\lambda_{cell} = h^2 \frac{(1-c^2)^2}{4(1+c^2)^2} \left(\frac{1}{E_1(\uparrow\downarrow)} - \frac{1}{E_1(\uparrow\uparrow)} \right) \tag{7.19}$$

In second order there is also a possibility for the simul-
taneous flip of two neighboring spins, a process which is not
present in the original Hamiltonian. The matrix element of the
second-order term of Eq. (2.17) between the states

$$...|i>\chi_1(i\uparrow)|\uparrow>\chi_1(\uparrow\uparrow)|\uparrow>\chi_1(\uparrow j)|j> ...$$

and

$$...|i>\chi_1(i\downarrow)|\downarrow>\chi_1(\downarrow\downarrow)|\downarrow>\chi_1(\downarrow j)|j> ...$$

is

$$-h^2 \; \frac{(1-c^2)^2}{8(1+c^2)^2} \; \left(\; \frac{1}{E_1(\uparrow\uparrow)} \; + \; \frac{1}{E_1(\uparrow\downarrow)} \; \right) \quad . \tag{7.20}$$

whereas the matrix element between the states

$$\ldots |i\rangle \chi_1(i\uparrow) |\uparrow\rangle \chi_1(\uparrow\downarrow) |\downarrow\rangle \chi_1(\downarrow j) |j\rangle \ldots$$

and

$$\ldots |i\rangle \chi_1(i\downarrow) |\downarrow\rangle \chi_1(\downarrow\uparrow) |\uparrow\rangle \chi_1(\uparrow j) |j\rangle \ldots$$

is

$$+h^2 \; \frac{(1-c^2)^2}{8(1+c^2)^2} \; \left(\; \frac{1}{E_1(\uparrow\uparrow)} \; + \; \frac{1}{E_1(\uparrow\downarrow)} \; \right) \quad . \tag{7.21}$$

Thus the sign of the two-spin flip process depends on whether the two spins are parallel or antiparallel. The corresponding term in the Hamiltonian can be written as

$$- K \; \sigma_i^z \; \sigma_{i+1}^z \; \sigma_i^x \; \sigma_{i+1}^x = K \; \sigma_i^y \; \sigma_{i+1}^y \quad , \tag{7.22}$$

where to second order in h

$$K = h^2 \; \frac{(1-c^2)^2}{8(1+c^2)^2} \; \left(\frac{1}{E_1(\uparrow\uparrow)} + \frac{1}{E_1(\uparrow\downarrow)} \right) \tag{7.23}$$

Since in a consistent RG calculation this generated new coupling has to be introduced from the outset, its effect on the other couplings should also be considered. This new coupling

contributes to the matrix elements between the states

$\dots |i\rangle\chi_1(i\uparrow)|\uparrow\rangle\chi_1(\uparrow j)|j\rangle \dots$ and $\dots |i\rangle\chi_1(i\downarrow)|\downarrow\rangle\chi_1(\downarrow j)|j\rangle \dots$ and leads

to an extra renormalization of h'

$$\Delta h_{cell} = -K\,\frac{1-c^2}{1+c^2} \quad . \tag{7.24}$$

Collecting the various contributions, finally the recursion relations are:

$$\lambda_{cell} = E_1(\uparrow\downarrow) - E_1(\uparrow\uparrow) + h^2\,\frac{(1-c^2)^2}{4(1+c^2)^2}\left(\frac{1}{E_1(\uparrow\downarrow)} - \frac{1}{E_1(\uparrow\uparrow)}\right) \tag{7.25}$$

$$h_{cell} = h\,\frac{(1+c)^2}{2(1+c^2)} - K\,\frac{1-c^2}{1+c^2} \quad , \tag{7.26}$$

$$K_{cell} = h^2\,\frac{(1-c^2)^2}{8(1+c^2)^2}\left(\frac{1}{E_1(\uparrow\downarrow)} + \frac{1}{E_1(\uparrow\uparrow)}\right) \quad . \tag{7.27}$$

A comparison of these relations with those given in Eqs. (7.3) - - (7.8) shows again that the two transformations lead to identical results if the $h\leftrightarrow\lambda/2$ interchange is made.

It is furthermore apparent that the new coupling generated in the block transformation, Eq. (7.2) and the one generated in the decimation transformation are dual to each other. For the special case of the Ising model the duality relations in Eqs. (4.4) - (4.5) can be written in the usual form:

$$\sigma_i'^z = \prod_{j<i}\sigma_j^x \quad , \tag{7.28}$$

$$\sigma_i'^x = \sigma_i^z\,\sigma_{i+1}^z \quad , \tag{7.29}$$

and therefore

$$- K \; \sigma_{i-1}^{\prime z} \; \sigma_i^{\prime x} \; \sigma_{i+1}^{\prime z} \; = \; - \; K \; \sigma_i^z \; \sigma_{i+1}^z \; \sigma_i^x \; \sigma_{i+1}^x \quad . \qquad (7.30)$$

This proves that the higher order corrections do not destroy the duality of the two transformations.

VIII. DISCUSSION

In this paper we compared two types of quantum RG transformations. In the block transformation[7,8] the low lying levels of independent cells are mapped onto new spin states, the coupling between the cells, a nearest neighbor coupling between the two adjacent end spins, is treated in a perturbational way. In the decimation transformation[16] the spin states of selected sites are mapped onto the states of new spins, by taking the lowest energy configuration for the intermediate spins. The single site term of the Hamiltonian acting on the selected sites is used now as a perturbation.

We have shown that the two RG transformations lead to equivalent results when applied to the 1-dimensional quantum versions of the Ising model, Potts model or general Z(4) model. We have seen that the results of the decimation transformation are identical to those obtained by the block transformation in the dual model. This is true even if several couplings are introduced, as in the Potts model or Z(4) model calculation, and also in higher orders of perturbation theory, where further new couplings are generated.

It was apparent in the first applications of the quantum RG transformation to the quantum Ising model that the transformations do not conserve the self-duality of the model. By treating the Ising coupling and the transverse field on equal

footing, Fernandez-Pacheco[23] was able to find an RG transformation which conserves self-duality and therefore gives the critical coupling exactly.

Except for the critical exponent ν , the other exponents are not given exactly. When applied to the Potts model[20], the critical coupling is again obtained exactly, the critical exponents, however, are not exact and become worse as the number of components increases. There is no indication of the crossover from second-order to first-order transition around $q = 4$. The other RG transformations, the usual block transformation and its dual, the decimation transformation have the merit, that the second-order to first-order crossover is reproduced[20]. Due to the generation of new couplings, these transformations can give a more realistic description of the behaviour of the Potts model.

We have not looked in this paper at the solutions of the recursion relations for the Z(4) model. Our aim was just to establish the duality of the two transformations. We will return to the solution of the equations in a subsequent publication.
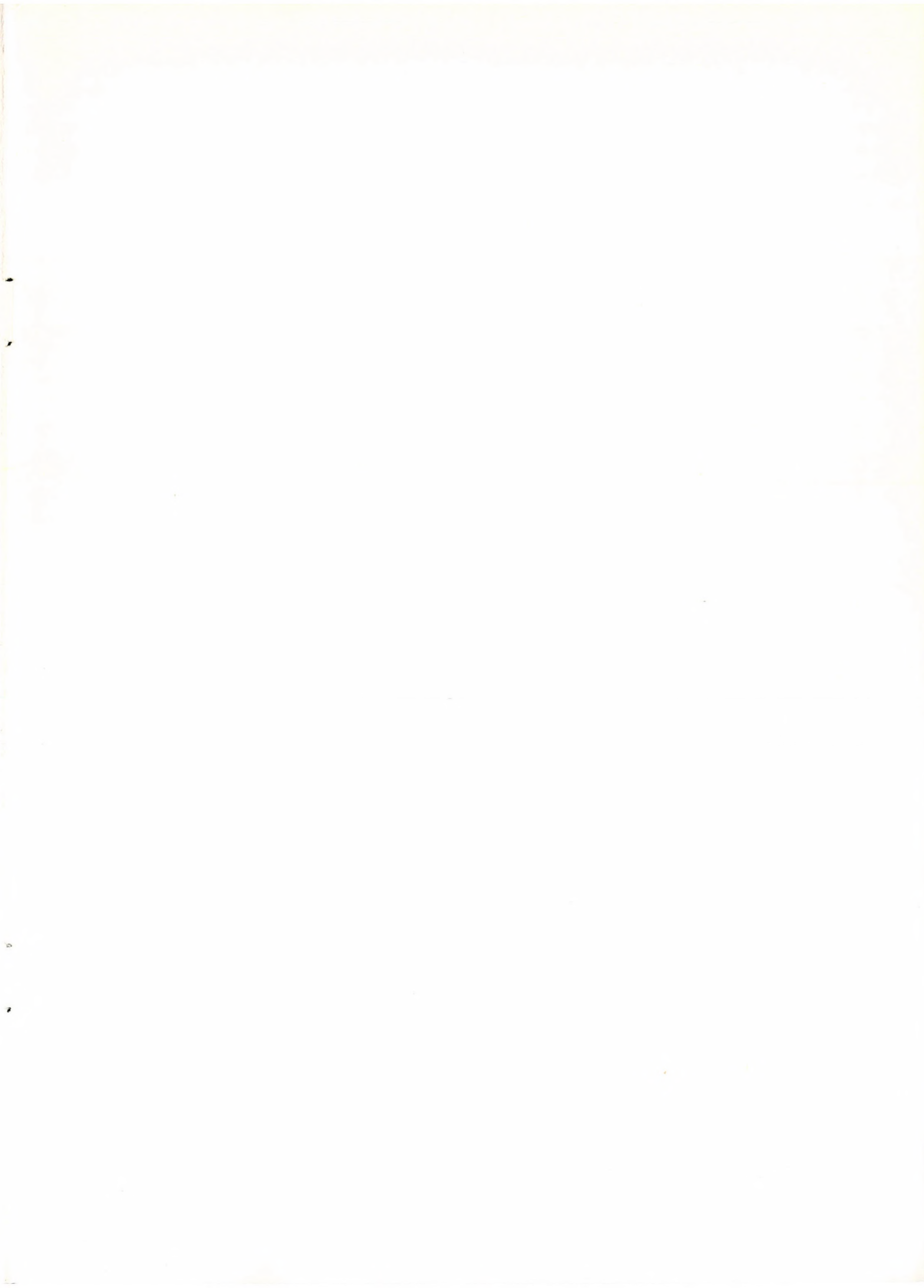
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REFERENCES

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[1] See e.g. K.G. Wilson and J. Kogut, Phys. Reports 12 C, 75 (1974); G. Toulouse and P. Pfeuty, "Introduction au groupe de renormalisation et à ses applications", Presses Universitaires de Grenoble, 1975; S.-K. Ma, "Modern Theory of Critical Phenomena", W.A. Benjamin, Inc. Reading, 1976.

[2] E. Brèzin, J.C. Le Guillou and J. Zinn-Justin, Phys. Rev. D15, 1544 (1977).

[3] J.C. Le Guillou and J. Zinn-Justin, Phys. Rev. Lett. 39, 95 (1977).

[4] Th. Niemeijer and J.M.J. van Leeuwen, Phys. Rev. Lett. 31, 1411 (1973), Physica 71, 17 (1974).

[5] L.P. Kadanoff and A. Houghton, Phys. Rev. B11, 377 (1975).

[6] K.G. Wilson, Rev. Mod. Phys. 47, 773 (1975).

[7] S. Jafarey, R. Pearson, D.J. Scalapino and B. Stoeckly (unpublished).

[8] S.D. Drell, M. Weinstein and S. Yankielowicz, Phys. Rev. D16, 1769 (1977); S.D. Drell, B. Svetitsky and M. Weinstein, ibid. 17, 523 (1978).

[9] R. Jullien, J.N. Fields and S. Doniach, Phys. Rev. Lett. 38, 1500 (1977); Phys. Rev. B16, 4889 (1977).

[10] Z. Friedman, Phys. Rev. Lett. 36, 1326 (1976).

[11] K. Subbarao, Phys. Rev. Lett. $\underline{37}$, 1712 (1976).

[12] G. Um, Phys. Rev. B$\underline{15}$, 2736 (1977); ibid. $\underline{17}$, 3670 (1978).

[13] R. Jullien, P. Pfeuty, J.N. Fields and S. Doniach, Phys. Rev.

B$\underline{18}$, 3568 (1978);

R. Jullien and P. Pfeuty, ibid. $\underline{19}$, 4646 (1979);

K. Penson, R. Jullien and P. Pfeuty, ibid. $\underline{19}$, 4653 (1979).

[14] J.E. Hirsch and G.F. Mazenko, Phys. Rev. B$\underline{19}$, 2656 (1979);

J.E. Hirsch, ibid. $\underline{20}$, 3907 (1979).

[15] J.N. Fields, Phys. Rev. B$\underline{19}$, 2637 (1979).

[16] E. Fradkin and S. Raby, Phys. Rev. D$\underline{20}$, 2566 (1979).

[17] M. Suzuki, Progr. Theor. Phys. $\underline{56}$, 1454 (1978).

[18] P. Pfeuty, Ann. of Phys. $\underline{57}$, 79 (1970).

[19] R.B. Potts, Proc. Cambridge Philos. Soc. $\underline{48}$, 106 (1952).

[20] J. Sólyom and P. Pfeuty, Phys. Rev. preceding paper.

[21] H.A. Kramers and G.H. Wannier, Phys. Rev. $\underline{60}$, 252 (1941);

G.H. Wannier, Rev. Mod. Phys. $\underline{17}$, 50 (1945).

[22] J. Ashkin and E. Teller, Phys. Rev. $\underline{64}$, 178 (1943).

[23] A. Fernandez-Pacheco, Phys. Rev. D$\underline{19}$, 3173 (1979).

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