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AN ISU(3) HADRON MASS FORMULA

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ABSTRACT

A universal mass expression is derived for non-charm hadrons which unites GMO formulae for various multiplets. Mass splitting is achieved via the generators of the 14-parameter ISU(3) group. The spectrum depends on three parameters which vary with the spin. Comparison with particle data shows a nearly linear dependence.

АННОТАЦИЯ

Введено общее уравнение массы для неочарованных адронов, объединяющее GMO-формулы различных мультиплетов. Расщепление массы проводится генераторами группы ISU(3) с 14-ю параметрами. Спектр зависит от трех спинозависящих параметров. Сравнение с данными частиц дает приблизительно линейную зависимость.

KIVONAT

Általános tömegformulát vezetünk le nem-charmos hadronokra, amely különféle multiplettek GMO-formuláit egyesiti. A tömegfelhasitást a 14 paraméteres ISU(3) csoport generátorai végzik. A spektrum három, spintől függő paramétert tartalmaz. Összehasonlitás a részecskeadatokkal közel lineáris függést szolgáltat.

We derive a mass expression for non-charmed hadrons which connects Gell--Mann-Okubo (GMO) formulae for various multiplets. Our expression is quadratic both in meson and in baryon masses,

$$M^2 = m^2 + m_o^2 + m_1^2 Y + m_2^2 [I(I+1) - (\tfrac{1}{2}Y)^2] \qquad (1)$$

with m the bare mass whereas m_o^2, m_1^2 and m_2^2 are functions of the SU(3) Casimir labels λ and μ, and of the baryon number B. These functions will now be calculated explicitly. Thus the spectrum generation will be achieved by three real parameters a, b and m^2, each dependent on the spin eigenvalue j.

The internal SU(3) flavor group with generators A^i_k is enlarged by including the translation generators d^i and their conjugates $\bar{d}_i$. The resulting 14-parameter ISU(3) group has the Lie-algebra commutation rules

$$\begin{aligned}
&[A^i_j, A^k_\ell] = \delta^i_\ell A^k_j - \delta^k_j A^i_\ell \\
&[d^i, A^j_k] = \delta^i_k d^j - \tfrac{1}{3}\delta^j_k d^i \qquad\qquad [d^i, d^j] = 0 = [d^i, \bar{d}_j] \ . \qquad (2) \\
&[\bar{d}_i, A^j_k] = \tfrac{1}{3}\delta^j_k \bar{d}_i - \delta^j_i \bar{d}_k
\end{aligned}$$

In unitary representations, the A^i_k form a Hermitian and trace-less matrix. This larger hadronic symmetry group is suggested by the Penrose theory of twistors[1,2], together with the identification of internal-group infinitesimal operators with physical quantum numbers. We shall not make explicit use of twistor theory, however, in the present paper.

In the limit of perfect symmetry, the rest-mass is given by the Casimirian of the ISU(3) group[1]

$$m^2 = 2d^i \bar{d}_i \quad . \qquad (3)$$

We assume, as customary, that the physical masses are split by an operator H^3_3 belonging to a nonet:

$$\Delta M^2 \equiv M^2 - m^2 = \langle a | H^3_3 | a \rangle \quad . \qquad (4)$$

We build up this operator H_3^3 from the ISU(3) generators. Among the generators, it is d^i and $\bar{d}_i$ which have matrix elements connecting different SU(3) multiplets. We define the nonet operator

$$\Delta_k^i = d^i \bar{d}_k \quad . \tag{5}$$

Additional nonet operators may be obtained by contractions of Δ_k^i with A_k^i's from left or right. Such chains of operators can be kept linear in Δ_k^i, by using the dyadic structure (5) to factor out SU(3) scalars. We use Okubo's notation[3] for nonet chains of operators. For example, we write

$$(A.A.\Delta.A)_k^i = A_r^i A_s^r \Delta_t^s A_k^t \quad . \tag{6}$$

The number of algebraically independent nonet chains is limited by Okubo's theorem[3] which expresses the sum

$$(\Delta.A.A)_k^i + (A.\Delta.A)_k^i + (A.A.\Delta)_k^i \tag{7}$$

in terms of shorter nonet chains. We find that the algebraically independent Hermitian nonet operators are Δ_k^i and

$$\begin{gathered}(A.\Delta)_k^i + (\Delta.A)_k^i \ , \qquad i[(A.\Delta)_k^i - (\Delta.A)_k^i] \ , \qquad (A.\Delta.A)_k^i \ , \\ i[(\Delta.A.A)_k^i - (A.A.\Delta)_k^i] \ , \quad i[(A.\Delta.A.A)_k^i - (A.A.\Delta.A)_k^i] \quad .\end{gathered} \tag{8}$$

In order to satisfy CPT symmetry, we must select positive C-parity operators. The C-conjugation is described by the involution

$$C: \qquad A_k^i \longleftrightarrow -A_i^k \ , \qquad d^i \longleftrightarrow \bar{d}_i \tag{9}$$

and by the operator product rule $C(XY) = (CX)(CY)$.

The baryon number operator B defined by[4]

$$m^2 B \equiv 2\Delta_k^i A_i^k \tag{10}$$

has negative C-parity. Any odd functional f of the baryon number, $f(B)=-f(-B)$ may be used to revert the charge parity of some operator. The resulting operator would not, however, contribute to the masses of B = 0 states. If we exclude them, we find that there is a unique combination of operators (8) with positive C-parity:

$$\Gamma_k^i = 2\bar{f}_k f^i - 3(\bar{d}_k f^i + \bar{f}_k d^i) + A_k^i + \delta_k^i B \ , \tag{11}$$

where we employ the notation $f^i \equiv d^k A_k^i$ and $\Delta \equiv m^2/2$.

We select as our mass splitting nonet operator the linear combination

$$H^i_k = a\Delta^i_k + b\Gamma^i_k \quad . \tag{12}$$

We may alternatively write H^i_k as

$$H^i_k = \sum_{A,B} g_A{}^B \bar{\zeta}^A_k \zeta^i_B + b(A^i_k + \delta^i_k B\Delta) \quad , \tag{13}$$

where

$$\zeta^i_1 = d^i \ , \qquad \zeta^i_2 = f^i \ , \qquad \zeta^i_3 = f^k A^i_k \tag{14}$$

and[7]

$$\left[g_A{}^B\right] = \begin{bmatrix} a & -3b & 0 \\ -3b & 2b & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad . \tag{15}$$

The matrix elements of operator H^3_3 may be evaluated by use of the Wigner--Eckart theorem $\langle b|\zeta^3_B|a\rangle = \langle a;(1,0)^3|b\rangle\langle b\|\zeta_B\|a\rangle$ where $(1,0)^3$ denotes the s-quark state with SU(3) quantum numbers $\{(\lambda,\mu),YII_z\} = \{(1,0), -\frac{2}{3}oo\}$ and $\langle b\|\zeta_B\|a\rangle$ is a reduced matrix element (RME) independent of the quantum numbers $\Pi = \{YII_z\}$. Substituting Eq.(13) in (4) we have

$$\begin{aligned} \Delta M^2 &= \sum_{b,A,B} g_A{}^B \langle a|\bar{\zeta}^A_3|b\rangle\langle b|\zeta^3_B|a\rangle = \\ &= \sum g_A{}^B \overline{\langle b|\zeta^3_A|a\rangle}\langle b|\zeta^3_B|a\rangle = \\ &= \sum g_A{}^B \langle a;(1,0)^3|b\rangle^2 \overline{\langle b\|\zeta_A\|a\rangle}\langle b\|\zeta_B\|a\rangle \ . \end{aligned} \tag{16}$$

The values of the quark coupling Clebsch-Gordan coefficients $\langle a;(1,0)^3|b\rangle$ have been worked out by Asherova and Smirnov[5]. On inserting these values and coefficients (15) in Eq.(16) we obtain our mass formula (1) with the functions m_o^2, m_1^2 and m_2^2 given by

$$\begin{aligned} m_o^2 &= \frac{(\lambda+2\mu)(\lambda+2\mu+3)}{9\mu(\lambda+\mu+1)}N^2 + \frac{(3+2\lambda+\mu)(6+2\lambda+\mu)}{9(\lambda+2)(\lambda+\mu+3)}P^2 + \frac{(\lambda-\mu)(3-\lambda+\mu)}{9\lambda(\mu+2)}S^2 + bB \\ m_1^2 &= \frac{\lambda+2\mu+3/2}{3\mu(\lambda+\mu+1)}N^2 - \frac{2\lambda+\mu+9/2}{3(\lambda+2)(\lambda+\mu+3)}P^2 - \frac{\lambda-\mu-3/2}{3\lambda(\mu+2)}S^2 + b \\ m_2^2 &= \frac{-N^2}{\mu(\lambda+\mu+1)} - \frac{P^2}{(\lambda+2)(\lambda+\mu+3)} + \frac{S^2}{\lambda(\mu+2)} \quad . \end{aligned} \tag{17}$$

The quantities N^2, P^2 and S^2 containing the RME's are defined

$$N^2 = \sum_{A,B} g_A{}^B N_B^{2A} \quad , \quad P^2 = \sum_{A,B} g_A{}^B P_B^{2A} \ , \quad S^2 = \sum_{A,B} g_A{}^B S_B^{2A} \ ,$$

$$
\begin{aligned}
N_B^{2A} &= \overline{\langle\lambda,\mu-1\| \zeta_A\| \lambda\mu\rangle}\langle\lambda,\mu-1\| \zeta_B\| \lambda,\mu\rangle\frac{\mu(\lambda+\mu+1)}{(\mu+1)(\lambda+\mu+2)} \\
P_B^{2A} &= \overline{\langle\lambda+1,\mu\| \zeta_A\| \lambda\mu\rangle}\langle\lambda+1,\mu\| \zeta_B\| \lambda\mu\rangle\frac{(\lambda+2)(\lambda+\mu+3)}{(\lambda+1)(\lambda+\mu+2)} \qquad (18) \\
S_B^{2A} &= \overline{\langle\lambda-1,\mu+1\| \zeta_A\| \lambda\mu\rangle}\langle\lambda-1,\mu+1\| \zeta_B\| \lambda\mu\rangle\frac{\lambda(\mu+2)}{(\lambda+1)(\mu+1)} \quad .
\end{aligned}
$$

The RME's of operators ζ_1^i have been calculated, using the ISU(3) algebra, by Perjés and Sparling[6]. These are

$$
\begin{aligned}
N_1^{21} &= \frac{(\frac{1}{2}B+j+1+y)(\frac{1}{2}B-j+y)}{(\mu+1)(\lambda+\mu+2)} \\
P_1^{21} &= \frac{(\frac{1}{2}B+j-1-x)(\frac{1}{2}B-j-2-x)}{(\lambda+1)(\lambda+\mu+2)} \qquad (19) \\
S_1^{21} &= \frac{(\frac{1}{2}B+j+z)(\frac{1}{2}B-j-1+z)}{(\lambda+1)(\mu+1)}
\end{aligned}
$$

where

$$x = \frac{2\lambda+\mu}{3} , \qquad y = \frac{\lambda+2\mu}{3} , \qquad z = \frac{\lambda-\mu}{3} \quad . \qquad (20)$$

These RME's satisfy

$$N_1^{21} + P_1^{21} + S_1^{21} = 1 \qquad (21)$$

and they are non-negative within unitary representations of ISU(3).

The RME's of operators ζ_2^i and ζ_3^i can be easily produced by using definitions (14) between states $|a\rangle$ (chosen with the lowest weight) and $|b\rangle = |\lambda+1,\mu,\Pi\rangle$:

$$
\begin{aligned}
\langle b \| \zeta_2 \| a\rangle &= - \frac{2\lambda+\mu}{3}\langle b \| \zeta_1 \| a\rangle \\
\langle b \| \zeta_3 \| a\rangle &= (\frac{2\lambda+\mu}{3})^2\langle b \| \zeta_1 \| a\rangle \quad .
\end{aligned}
\qquad (22)
$$

The remaining RME's are obtained by choosing $|b\rangle = |\lambda-1,\mu+1\rangle$ and $|b\rangle = |\lambda,\mu-1\rangle$ with the results

$$
\begin{aligned}
N_2^2 &= \frac{\lambda+2\mu+6}{3}N_1^1 & N_3^3 &= \frac{(\lambda+2\mu+6)^2}{9}N_1^1 \\
P_2^2 &= - \frac{2\lambda+\mu}{3}P_1^1 & P_3^3 &= \frac{(2\lambda+\mu)^2}{9}P_1^1 \qquad (23) \\
S_2^2 &= \frac{\lambda-\mu+3}{3}S_1^1 & S_3^3 &= \frac{(\lambda-\mu+3)^2}{9}S_1^1 \quad .
\end{aligned}
$$

Inserting the values (15) and (23) in Eqs.(18) we compute the contribution of the RME's in the mass matrix elements:

$$N^2 = \{a + 2b[(y+2)^2 - 3(y+2)]\}N_1^{21}$$

$$P^2 = \{a + 2b[x^2+3x]\}P_1^{21} \tag{24}$$

$$S^2 = \{a + 2b[(z+1)^2 - 3(z+1)]\}S_1^{21} \quad .$$

All the coefficients in our mass formula (1) can now be explicitly calculated. For a given spin j and with suitably chosen values of the parameters m, a and b, we insert the RME's (19) in expressions (24) which we use, in turn, for evaluating the coefficients m_o, m_1 and m_2 given in Eqs.(17).

Our procedure may first seem to offer a wide range of applications. Yet we meet severe limitations in its use. The strongest constraint follows from the unitarity of the ISU(3) representations. The allowed range of quantum numbers in unitary irreps has been obtained in Ref. 6. We now recall that the inequality $\lambda+\mu \geq 2j$ holds for ISU(3) unitary irreps. The unitary singlet $(\lambda,\mu) = (0,0)$ can only occur with vanishing spin. The octet (1,1) has only the spin values 0, 1/2 and 1 whereas the decuplet (3,0) can have spin values not exceeding 3/2. Regge recurrences and the vector singlet do not lie in unitary ISU(3) points. Several propositions have been made to incorporate these hadrons in the present scheme[6,8] but it is not yet clear which of them if any is acceptable.

Another problem is posed by the SU(3) mixing. This means effectively that singlet masses are uncertain to the degree the mixing angle is. In conclusion, our mass formula can reliably be tested with the octets of pseudo--scalar and vector mesons, the spin-1/2 baryons and with the j=3/2 baryon decuplet.

For octets we obtain

$$m_o^2 = \tfrac{2}{3}aN_1^{21} + \tfrac{2}{5}(a+8b)P_1^{21} + bB$$

$$m_1^2 = \tfrac{1}{2}aN_1^{21} - \tfrac{1}{6}(a+8b)P_1^{21} + \tfrac{1}{6}(a-4b)S_1^{21} + b \tag{25}$$

$$m_2^2 = -\tfrac{1}{3}aN_1^{21} - \tfrac{1}{15}(a+8b)P_1^{21} + \tfrac{1}{3}(a-4b)S_1^{21} \quad ,$$

where

$$N_1^{21} = \tfrac{1}{8}(B/2+j+2)(B/2-j+1)$$

$$P_1^{21} = \tfrac{1}{8}(B/2+j-2)(B/2-j-3) \tag{26}$$

$$S_1^{21} = \tfrac{1}{4}(B/2+j)(-B/2+j+1) \quad .$$

For triangular SU(3) representations the mass formula (1) greatly simplifies. We obtain for the baryon decuplet with j=3/2, $(\lambda,\mu) = (3,0)$ and B = 1:

$$M^2 = m^2 + \frac{1}{3}(a+5b) + (\frac{1}{12}a - \frac{1}{3}b)Y \quad . \tag{27}$$

We now want to use the experimental values of the masses M to determine the values of the parameters m, a and b. Without a further assumption, however, this is only possible with the baryon octet. A least-square fit to the octet masses yields the ratio (*Table 1*)

$$b/a = -0.066 \pm 0.013 \quad . \tag{28}$$

Clearly, the contribution of operator Γ^3_3 to the mass is small.

In order to test the sensitivity of the parameter values to the mass data, we repeat the fit with different assumptions. For example, we may generally ignore the isosinglet particle in an octet because of possible mixing problems. Calculation of the baryon octet parameters with Λ ignored, results in a drastic reduction of standard errors. This is because the GMO fit contains now only the experimental mass errors. The values of m^2 and a do not change appreciably with respect to the corresponding least-square standard errors. However, the parameter b does change. Conversely, we may say that the masses vary little with the ratio b/a.

The masses in the meson octets and in the decuplet are also insensitive to the ratio b/a. We shall exploit this phenomenon and adopt the value (28) throughout. (The remaining multiplets contain insufficient data for evaluating b/a from the experimental masses. Cf. Eq.(27), for example.) In this way we are able to calculate m^2 and a for the pseudo-scalar and vector meson octets and for the j=3/2 decimet (*Table 1*).

As expected, the mass parameters m^2 and a do vary with the spin. It would be of interest to model this variance. A further insight is obtained from the experimental values of *Table 1* if we plot m^2 against a (*Figure 1*). The nearly linear behaviour of the parameters found in this way gives a hint that the twistor particle model in which our mass formula originates may contain valid ingredients of hadron structure.

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j	1/2 octet (least--square fit)	1/2 octet (Λ omitted)	0 octet$^{x)}$ (η omitted)	1 octet$^{x)}$ (φ omitted)	3/2 decuplet$^{x)}$ (least--square)
a (GeV)^2	-2.81 ±0.09	-2.998 ±0.006	1.43 ±0.07	-0.703 ±0.05	-4.12 ±0.17
b (GeV)^2	0.186 ±0.03	0.229 ±0.002	-	-	-
m^2 (GeV)^2	1.96 ±0.08	1.9612 ±0.0028	-0.117 ±0.05	0.942 ±0.02	2.85 ±0.13

Table 1

Fit of ISU(3) parameters to experimental mass^2 values[9]. Errors in middle columns contain only experimental mass uncertainties since the number of mass data is insufficient for a least-square fit.

x) *With input value b/a = -0.066; Cf. text*

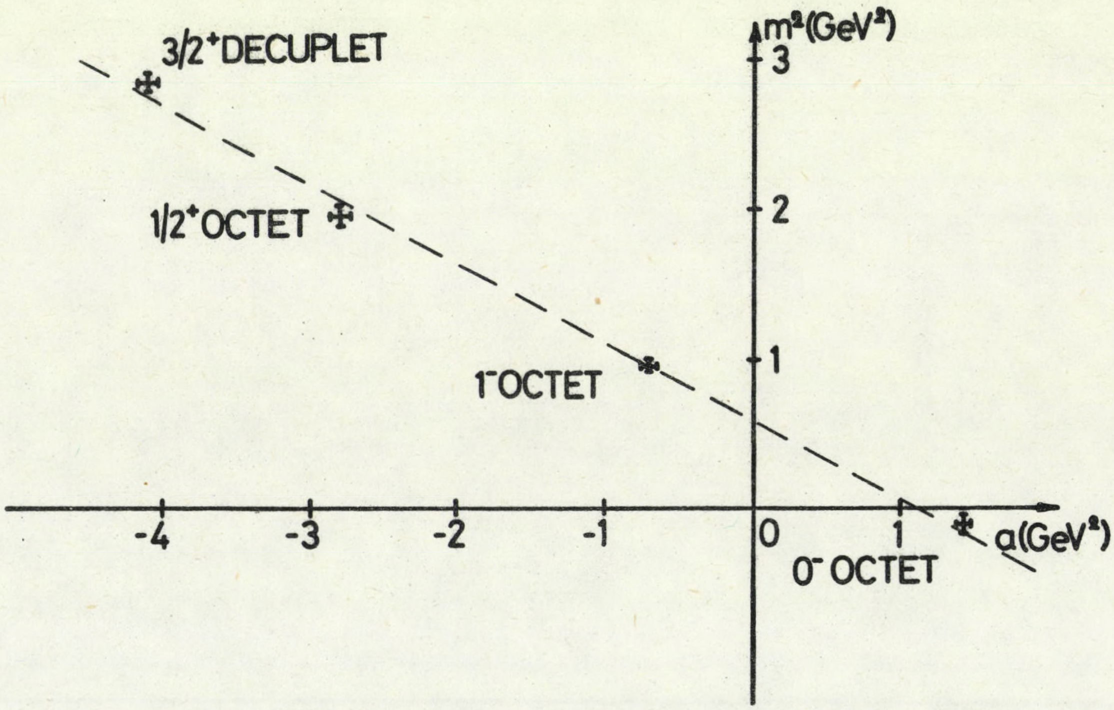

Figure 1

Experimental values of the mass parameters m^2 and a (least-square fit used for baryon octet)

Kiadja a Központi Fizikai Kutató Intézet
Felelős kiadó: Szegő Károly
Szakmai lektor: Forgács Péter
Nyelvi lektor: Sebestyén Ákos
Gépelte: Polgár Julianna
Példányszám: 370 Törzsszám: 80-757
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Felelős vezető: Nagy Károly
Budapest, 1980. december hó