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ABSTRACT

Two families of Ising models, one on the Kagomé lattice and the other on the square lattice, are studied. The interaction is chosen to be a particular mixture of ferro- and antiferromagnetic bonds. It is shown that the free energy is analytic at every positive temperature. For the Kagomé lattice models, the correlations in the totally symmetric equilibrium state are analytic and exponentially clustering at every temperature, including T=0.

АННОТАЦИЯ

Рассмотрены два семейства модели Изинга, одно определено на решетке Кагоме, а другое на квадратичной решетке. Взаимодействие выбрано как определенная смесь ферромагнитных и антиферромагнитных связей. Показано, что свободная энергия аналитична при всех температурах. Для модели Кагоме корреляции в полностью симметричном равновесном состоянии аналитичны и приводят к экспоненциальным кластерам при всех температурах, даже при T=0.

KIVONAT

Ebben a munkában az Ising modellek két családját vizsgáljuk: az egyik a Kagomé-, a másik a négyzetrácson értelmezett. A kölcsönhatásban ferro- és antiferromágneses kötések vegyesen szerepelnek. Megmutatjuk, hogy a vizsgált modellek szabadenergiája minden pozitiv hőmérsékleten analitikus. A Kagomé rácson vett modellek esetében a teljesen szimmetrikus egyensulyi állapothoz tartozó korrelációk analitikusak és exponenciálisan klaszter-képzők minden hőmérsékleten, T=0-n is.

1. Introduction

The mathematical investigation of non-ferromagnetic systems is much less extensive than that of ferromagnets. Many of the appearing new problems can be traced back to an incomplete knowledge of the ground state properties of the system. This is the case with potentials containing competing interactions: terms which cannot be minimized simultaneously. Antiferromagnetic bonds immersed into a ferromagnetic "see" represent a typical example. When they occur periodically and in suitable arrangements, a new quality appears: the infinity of periodic ground states. Such models were candidates to describe some properties of spin glasses and they were named "frustration models".

The aim of the present study is to discuss the analyticity properties of the free energy and correlations of some frustration models and to show examples when the competition inhibits the phase transitions. The method we apply to this end is the localization of the zeroes of the partition function on the complex $\tanh\beta$ plane. The Asano contraction ([1] and [2]) and relating theorems of Gruber et al [3] give bounds on the domain of analyticity but these results are not well fitted to the present problem.

In the most interesting cases of frustration, the low temperature phase cannot be considered as a small perturbation of some given spin configuration. This excludes the possibility to perform a low temperature expansion whereas high temperature (H.T.) expansion remains a promising tool. The usual H.T. expansion for Ising models results in an expression for the partition function in terms of a set of variables $\{z_b = \tanh \beta J_b\}$ where β is the inverse temperature, b is a finite set of lattice sites and J_b is the interaction among the spins belonging to the sites of b. In fact, z_b is the mean value of the product of these spins if J_b is the only interaction acting on them. The H.T. series is convergent if z_b is much less than 1 for all b and therefore if β is not very large. Now if B is the set of all bonds, i.e. of those b for which $J_b \neq 0$, one may consider a disjoint cover of B with bounded subsets, $B = \cup B^i$ (in contrast with the Asano contraction where the overlaps among B^i are essential). It turns out that, apart from unimportant factors, the partition function can be expressed in terms of spin correlations according to probability distributions each of which is determined by the bonds of a single B^i. Now the H.T. series thus obtained is convergent for the small values of these correlations which, in some frustration models, may be bounded by a value well below unity for whatever large β and hence the H.T. expansion may converge at any positive temperature.

The mathematical basis of our statements is an estimate on the values of a certain kind of polynomials, which we present in Section 2. Applications follow in Sections 3 and 4. Here we study two families of frustration models, one on the Kagomé and the other on the square lattice. With the method outlined above we show that the free energy is an analytic function of the temperature for any $1/\beta > 0$. The results are more complete for the Kagomé lattice models: we prove that the correlations in the totally symmetric equilibrium state (that obtained with zero boundary condition) are analytic and exponentially clustering at every temperature, including $1/\beta = 0$.

2. Bounds on polynomials

We are going to consider polynomials of K complex variables, $z_1,\ldots,z_K$. Let $Q=\{1,2,\ldots,K\}$ and P(Q) denote the set of all subsets of Q. We make use of two properties of P(Q).

(i) P(Q) is partially ordered with respect to the inclusion. If $G\subset P(Q)$ then inf G denotes the set of the minimal elements of $G-\{\emptyset\}$.

(ii) P(Q) is a group w.r.t. the symmetric difference of its elements: if g_1 and g_2 are parts of Q then

$$g_1 g_2=(g_1\cup g_2)-(g_1\cap g_2)$$

is their symmetric difference.

Now let G be a subgroup of P(Q) such that it is uniquely generated by inf G in the following sense: for any $g\in G$ there is a unique set

$$\{g_1,\ldots,g_k\}\subset \inf G$$

such that

$$g_i\cap g_j=\emptyset \text{ if } i\neq j$$

and

$$g=g_1\cup\ldots\cup g_k\ .$$

For any $i\in Q$ let $N_n(i)$ be the number of those elements of inf G which contain i and exactly n-1 other points of Q. For any $n>0$ we choose a number $N_n\geq N_n(i)$. Now the following statement is true.

Lemma 1.

Consider the polynomial

$$R(z) = \sum_{g \in G} \prod_{i \in g} z_i = \sum_{g \in G} z^g \qquad (1)$$

and suppose that

$$\sum_{n > 0} N_n x^n / (1-\varepsilon)^{n-1} \leq \varepsilon \qquad (2)$$

is satisfied by some $x > 0$ and $\varepsilon < 1$. Then

$$(1-\varepsilon)^K \leq |R(z)| \leq (1+\varepsilon)^K$$

if $|z_i| \leq x$ for all $i \in Q$.

Proof

For any $\alpha \subset Q$ let

$$G_\alpha = \{ g \in G: g \subset \alpha \}$$

$$R_\alpha = \sum_{g \in G_\alpha} z^g$$

$$r_\alpha^i = \sum_{i \in g \in G_{\alpha \cup \{i\}}} z^g / R_\alpha \qquad (3)$$

and

$$[i] = \{1,\ldots,i\} \ . \tag{4}$$

We have the following product representation of R.

$$\begin{aligned} R &= R_{[K-1]}\,(1+r^{K}_{[K-1]}) \\ &= R_{[K-2]}\,(1+r^{K-1}_{[K-2]})(1+r^{K}_{[K-1]}) \\ &= \ldots = \prod_{i=1}^{K} (1+r^{i}_{[i-1]}) \end{aligned} \tag{5}$$

The proof can be performed by showing that $|r^i_\alpha| \leq \varepsilon$ for any $i \in Q$ and $\alpha \subset Q$, provided that $|z_j| \leq x$ for any $j \in Q$. We do this by induction according to $|\alpha|$, the number of points in α . For $\alpha = \emptyset$ we have

$$r^i_\emptyset = \begin{cases} z_i & \text{if } \{i\} \in G \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, in the first case

$$|r^i_\emptyset| \leq x = N_1 x \leq \sum_n N_n x^n / (1-\varepsilon)^{n-1} \leq \varepsilon \quad .$$

Suppose now that $|r^j_\alpha| \leq \varepsilon$ is proved for any j and α with $|\alpha| < i$. It is sufficient to show that $|r^{i+1}_{[i]}| \leq \varepsilon$;

for other sets we get the result by permutation. Now

$$r^{i+1}_{[i]} = \sum_{i+1 \in g \in G_{[i+1]}} z^g \, R_{[i]-g} \, / \, R_{[i]} \tag{6}$$

where we used that g has a unique decomposition into the disjoint union of the elements of inf G. On the other hand, if

$$g = \{ i+1, j_2, \ldots, j_n \} \subset [i+1]$$

and

$$g'k = \{ j_2, \ldots, j_k \}$$

then

$$R_{[i]} = R_{[i]-g} \prod_{k=2}^{|g|} (1 + r^{j_k}_{[i]-g'k}) \tag{7}$$

where $|g|$ = card g = n. Putting (7) into (6) one obtains

$$r^{i+1}_{[i]} = \sum_{i+1 \in g \in \inf G_{[i+1]}} z^g \, / \, \prod_{k=2}^{|g|} (1 + r^{j_k}_{[i]-g'k}) \tag{8}$$

For each r^j_α in the denominator α has at most i-1 points and therefore is bounded by ε. Then Eqs(8) and (2) clearly imply that $r^{i+1}_{[i]}$ is bounded by ε.-

Remark

From the group properties of G we used only that it is closed w.r.t. subtraction: if $g \in G$ and $g_1 \in G$, $g_1 \subset g$ then also $g - g_1 \in G$.

In the following, we discuss a possibility to obtain bounds on the polynomial R of Eq.(1), even if G is not uniquely generated by inf G.

Let $\{Q^i\}_{i=1,\ldots,N}$ be a disjoint cover of Q:

$$Q = \bigcup_{i=1}^{N} Q^i \quad \text{and} \quad Q^i \cap Q^j = \emptyset \quad \text{if } i \neq j.$$

Let G° be a subgroup of G, defined by

$$G^\circ = \{ g \in G: \; g \cap Q^i \in G \text{ for any } i \} \tag{9}$$

Consider the quotient group, G/G°. We show that under certain conditions it may substitute G in Lemma 1. Let

$$G^i = \{ g \in G: \; g \subset Q^i \}$$

then, plainly, G^i is a subgroup of G° and also it is the projection of G° into Q^i. In general, if A is a coset of G according to G° then

$$\text{Proj}_i A = \{ g \cap Q^i \; : \; g \in A \} \tag{10}$$

is a coset of $P(Q^i)$ - the power set of Q^i - according to G^i. Any $A \in G/G^\circ$ is uniquely represented by the set of those projections (10) which differ from the corresponding G^i : if

$$a^m = \mathrm{Proj}_m A \begin{cases} \neq G^m & \text{for } m=i_1,\ldots,i_k \\ = G^m & \text{otherwise} \end{cases} \tag{11a}$$

then this set is

$$s_A = \{ a^{i_1},\ldots,a^{i_k} \} \tag{11b}$$

Now let

$$\hat{Q} = \bigcup_{i=1}^{N} \hat{Q}^i$$

where

$$\hat{Q}^i = (P(Q^i)/G^i) - G^i$$

and let

$$S = \{ s \subset \hat{Q} \; : \; s=s_A \text{ for some } A \in G/G^\circ \} \tag{12}$$

Plainly, if $s \in S$ then $\mathrm{card}(s \cap \hat{Q}^i) \leq 1$ for any i. The elements of S form a group: if $s,s' \in S$ then $s=s_A$, $s'=s_B$ for some $A,B \in G/G^\circ$; let now

$$\mathrm{Proj}_i A = a^i \quad \text{and} \quad \mathrm{Proj}_i B = b^i$$

then

$$ss' = \{a^i b^i\}_{i=1}^{N} - \{G^i\}_{i=1}^{N} \in S$$

defines the group operation. Here

$$a^i b^i = \{ gg' \subset Q^i : g \in a^i ,\ g' \in b^i \}$$

is a coset of $P(Q^i)$. The group S is isomorphic with $G/G°$. S is ordered w.r.t. the inclusion and inf S is the set of the minimal elements of $S-\{\emptyset\}$; inf $G/G°$ is that part of $G/G°$ which is isomorphic with inf S.

The cover of Q can always be chosen so that inf S uniquely generates S, in the sense we used it earlier (indeed, for instance, covers with at most three subsets all have this property). This can be told as inf $G/G°$ uniquely generates $G/G°$. To $G°$, one can assign a polynomial analogous to (1):

$$R°(z) = \sum_{g \in G°} z^g = \prod_{i=1}^{N} \sum_{g \in G^i} z^g \qquad (13)$$

Now one obtains the following result.

Lemma 2.

Let $Q=\{1,...,K\}$, G be the subgroup of P(Q) and a cover $\{Q^i\}_{i=1,...,N}$ be given so that, with $G°$ defined by (9), inf $G/G°$ uniquely generates $G/G°$. Let, moreover

$$N_n(i) = \text{card}\{ A \in \inf G/G° : \text{Proj}_k A \neq G^k \text{ for } k=i \text{ and for exactly } n-1 \text{ other values of } k \} \qquad (14)$$

and N_n be chosen so that

$$N_n \geq N_n(i) \quad \text{for } i=1,\dots,N \tag{15}$$

Suppose that (2) holds with these N_n and with some $x > 0$ and $\varepsilon < 1$. Let R and R° be the polynomials defined in (1) and (13), respectively. Then

$$(1-\varepsilon)^{N-1} \leq |R(z)/R^{\circ}(z)| \leq (1+\varepsilon)^{N-1} \tag{16}$$

provided that

$$\left| \sum_{g \in a^i} z^g \Big/ \sum_{g \in G^i} z^g \right| \leq x$$

for any $1 \leq i \leq N$ and $a^i \in \hat{Q}^i$.

<u>Proof</u>

Let S be the group (12) and for any $a \in \hat{Q}$, let ζ_a be a complex variable assigned to a. Consider the polynomial

$$T(\zeta) = \sum_{s \in S} \prod_{a \in s} \zeta_a = \sum_{s \in S} \zeta^s.$$

It is easy to show that

$$T(\zeta) = R(z) \,/\, R^{\circ}(z) \tag{17a}$$

if, for any $i=1,\ldots,N$ and $a^i \in \hat{Q}^i$, one makes the substitution

$$\zeta_{a^i} = \sum_{g \in a^i} z^g \Big/ \sum_{g \in G^i} z^g \tag{17b}$$

Hence, one has to prove only that the bounds (16) are valid for $T(\zeta)$ if $|\zeta_a| \leq x$ for any $a \in \hat{Q}$. We introduce the following notations: let $\alpha \subset \{1,\ldots,N\}$, then

$$S_\alpha = \{ s \in S : s \subset \bigcup_{i \in \alpha} \hat{Q}^i \}$$

$$\underline{s} = \{ i \in \{1,\ldots,N\} : s \cap \hat{Q}^i \neq \emptyset \}$$

$$T_\alpha = \sum_{s \in S_\alpha} \zeta^s$$

$$t^i_\alpha = \sum_{s \in S_{\alpha \cup \{i\}} ; i \in \underline{s}} \zeta^s \,/\, T_\alpha \tag{18}$$

The Equations (18) are analogous to Eqs (3), just as

$$T = \prod_{i=1}^{N-1} (1 + t^{i+1}_{[i]}) \tag{19}$$

and

$$t^{i+1}_{[i]} = \sum_{s \in \inf S_{[i+1]};\ i+1 \in \underline{s}} \zeta^s \Big/ \prod_{k=2}^{|s|} (1 + t^{j_k}_{[i]-s'k}) \tag{20}$$

are analogues of Eqs (5) and (8), respectively. In (20), j_k is the kth point of $\underline{s}$,

$$\underline{s} = \{i+1, j_2, \ldots, j_n\}$$

and

$$s'k = \{j_2, \ldots, j_k\};$$

the cardinals of s and $\underline{s}$ are the same: $|s| = |\underline{s}|$.
Noticing that

$$N_n(i) = \text{card}\,\{s \in \inf S : |s| = n \text{ and } s \cap \hat{Q}^i \neq \emptyset\}$$

one can conclude the proof by showing, in the same way as in Lemma 1, that

$$|t^i_\alpha(\zeta)| \leq \varepsilon \qquad (21)$$

for any $1 \leq i \leq N$, $\alpha \subset \{1, \ldots, N\}$.-

So far, we considered only the subgroup G of P(Q); now for any $D \in P(Q)/G$, one can define

$$R^D(z) = \sum_{g \in D} z^g$$

In view of applications, it is interesting to obtain bounds also on R^D/R. To this end, let us continue the earlier discussion. In fact, G° of Eq.(9) factorizes not only G but also the whole P(Q). Meanwhile, it factorizes the elements of P(Q)/G distinctly. For $D \in P(Q)/G$, let

$$D/G^{\circ} = \{ A \in P(Q)/G^{\circ} : A \subset D \} .$$

Now we extend the definition of s_A , as given in Eqs (10), (11), to any $A \in P(Q)/G^{\circ}$ and consider the set

$$S^D = \{ s \subset \hat{Q} : s = s_A \text{ for some } A \in D/G^{\circ} \} .$$

A polynomial

$$T^D(\zeta) = \sum_{s \in S^D} \zeta^s$$

can be assigned to S^D; it is easy to show that

$$T^D(\zeta) = R^D(z) / R^{\circ}(z) \tag{22}$$

if ζ is given by Eq.(17b). Equation (22) is a generalization of (17a): for D=G the two equations coincide. Dividing (22) by (17a) one obtains

$$R^D(z)/R(z) = T^D(\zeta)/T(\zeta) \tag{23}$$

For $D \in (P(Q)/G)-G$, let inf S^D denote the set of minimal elements of S^D. Any $s \in S^D$ can be written as

$$\begin{aligned} &s = s_1 \cup s_2 \\ &s_1 \cap s_2 = \emptyset \\ &s_1 \in \inf S^D \ , \ s_2 \in S \end{aligned} \tag{24}$$

though, in general, this decomposition is not unique. The cover of Q can always be chosen so that inf G/G° uniquely generates G and also, the decomposition (24) is unique for any $D \in (P(Q)/G)-G$ and $s \in D$. (This is true, for example, for the trivial cover $\{Q\}$ and the cover with two disjoint sets $\{Q^1, Q^2\}$.) Assume that the cover, we have chosen to Lemma 2, satisfies these conditions. Then we can write

$$T^D(\zeta)/T(\zeta) = \sum_{s \in \inf S^D} \zeta^s \, T_{[N]-\underline{s}} / T_{[N]}$$

where we applied the notations (4) and (18) (notice that $T_{[N]}=T$). The analogue of Eq.(7) gives then

$$T^D(\zeta)/T(\zeta) = \sum_{s \in \inf S^D} \zeta^s / \prod_{k=1}^{|s|} (1+t^{j_k}_{[N]-s'k}) \tag{25}$$

Let now

$$N^D_n = \text{card}\,\{s \in \inf S^D : \ |s| = n\} \tag{26}$$

From Eqs (21),(23),(25) and (26) we find

$$|R^D(z)/R(z)| \leq \sum_n N^D_n \, x^n / (1-\varepsilon)^n \tag{27}$$

provided that $|\zeta_{a^i}| \leq x$ for any ζ_{a^i} given by (17b).

In the following sections we apply these results to obtain bounds on Ising partition functions and correlations. Let $\mathbb{Z}$ be a lattice and $\sigma : \mathbb{Z} \to \pm 1$ be a spin configuration. The potential of a finite subsystem of spins is defined as

$$H_B(\sigma) = - \sum_{b \in B} J_b \prod_{x \in b} \sigma(x) = - \sum_{b \in B} J_b \sigma^b \qquad (28)$$

where B is a finite family of finite subsets of $\mathbb{Z}$.
Now H_B defines the probability distribution of the spins on

$$\Lambda = \bigcup_{b \in B} b$$

and the corresponding partition function can be written as

$$Z_B = \sum_{\sigma | \Lambda} \exp(-\beta H_B(\sigma)) = 2^{|\Lambda|} \left(\prod_{b \in B} \cosh \beta J_b \right) \tilde{R}$$

Here $\tilde{R}$ is defined by the H.T. expansion as

$$\tilde{R} = \sum_{g \in G} \prod_{b \in g} \tanh \beta J_b \qquad (29)$$

and G is the "High Temperature Group" [3]:

$$(b_1,\ldots,b_k) \in G$$

if and only if $b_i \in B$ and $b_1 b_2 \ldots b_k = \emptyset$ $(bc=(b \cup c)-(b \cap c))$.
Now $\tilde{R}$ can play the role of the polynomial (1) if one identifies the set of bounds B with the set Q and the complex variables z_i , $i \in Q$, with

$$z_b = \tanh \beta J_b \ , \ b \in B. \tag{30}$$

Furthermore, if $D \in P(Q)/G$, then there is a $d \subset \Lambda$ such that

$$\prod_{b \in g} b = d$$

for any $g \in D$: the cosets can be indexed with the subsets of the lattice. Now if

$$\tilde{R}^d = \sum_{g \in D} \prod_{b \in g} \tanh \beta J_b$$

then

$$\tilde{R}^d / \tilde{R} = \langle \sigma^d \rangle_B$$

where $\langle \cdot \rangle_B$ denotes the mean value according to the probability distribution defined by the potential (28). The bound (27) then refers to $\langle \sigma^d \rangle_B$. The variables, introduced in (17b), also correspond to correlations:

let $B^i \subset B$ and $G^i = G \cap P(B^i)$. The cosets of $P(B^i)$, according to G^i, can also be indexed with the subsets of

$$\Lambda^i = \bigcup_{b \in B^i} b \ .$$

Let $b \subset \Lambda^i$ correspond to $a \in P(B^i)/G^i$; then

$$\zeta_a = \langle \sigma^b \rangle_{B^i} \qquad (31)$$

where the mean value is taken according to the probability distribution

$$\sim \exp(-\beta H_{B^i}(\sigma)) = \exp(\beta \sum_{b \in B^i} J_b \sigma^b) \ .$$

In the following, we write T^d and ζ_b instead of T^D and ζ_a, if $d, b \subset \mathbb{Z}$ are the subsets corresponding to the cosets D and a, respectively; also we omit the tilde: R and R^d will refer to the polynomials of z_b of Eq.(30).

3. Frustration models on the Kagomé lattice

The Kagomé lattice is a plane lattice built up from regular triangles and hexagons so that every edge is shared by two different types of polygons. Therefore, if $J_b \neq 0$ only for nearest neighbour pairs (nnp), their set B can be covered with the set $\{B^i\}$ of pairwise disjoint triangles:

$$B^i = (b^1, b^2, b^3)$$

where b^k are nnp forming a triangle. The elements of the H.T.Group can be visualised as graphs of even order, and the members of inf G are the simple (non-crossing) polygons. Now inf G does not generate G uniquely because crossing graphs have more than one decomposition. It is easy to see, however, that $\{B^i\}$ may play the role of the cover $\{Q^i\}$ of Lemma 2: if G° is the subgroup associated with the cover $\{B^i = Q^i\}$ via Eq.(9) then G/G° is uniquely generated by inf G/G°. (This is true because every lattice site belongs to only two triangles and for any $A \in G/G°$, s_A has at most one element common with $P(B^i)/G^i$.) Let us consider now G^i and the cosets of $P(B^i)$ according to G^i. The corresponding quotient group is

$$P(B^i)/G^i = \{G^i, a^1, a^2, a^3\}$$

where

$$G^i = \{\emptyset, \{b^1, b^2, b^3\}\}$$

and

$$a^k = \{b^k, \{b^\ell, b^m\}\}$$

with $(k,\ell,m)=(1,2,3)$ and its cyclic permutations. The co-set a^k can be indexed with the nnp b^k; the variable, assigned to a^k through Eq.(17b), is

$$\zeta_{b^k} = (z_{b^k} + z_{b^\ell} z_{b^m})/(1+z_{b^k} z_{b^\ell} z_{b^m}) \qquad (32)$$

where z_b are given by Eq.(30). According to (31), for real non-negative values of β, ζ_{b^k} is a pair correlation function belonging to the nnp b^k. It is useful to introduce the variable w_b with the equation

$$w_b = \tanh|J_b|\beta \qquad (33)$$

Then (32) becomes

$$\zeta_{b^k} = \mathrm{sgn}\, J_b\, (w_{b^k} + p(i) w_{b^\ell} w_{b^m})/(1+p(i) w_{b^k} w_{b^\ell} w_{b^m})$$

where

$$p(i) = \prod_{b \in B^i} \mathrm{sgn}\, J_b \qquad (34)$$

Let us notice that (34) simplifies to

$$\zeta_b = (\mathrm{sgn}\, J_b)\; w_i/(1-p(i)w_i+w_i^2) \tag{35}$$

if $w_b=w_i$ for all $b \in B^i$. This can be reached by choosing $|J_b|$ to be the same for all nnp in a given triangle.

Let now

$$B = \bigcup_{i=1}^{N} B^i$$

and consider the function

$$\psi(\beta) = \lim_{N\to\infty} \psi_N(\beta) \tag{36a}$$

with

$$\begin{aligned}\psi_N(\beta) &= (1/N)\;\log[R(\beta)/R^\circ(\beta)]\\ &= (1/N)\sum_{i=1}^{N-1} \log(1+t_{[i]}^{i+1}(\beta)\;)\end{aligned} \tag{36b}$$

Here R is defined by (29), R° corresponds to (13):

$$R^\circ(\beta) = \prod_{i=1}^{N} (1+p(i) \prod_{b\in B^i} w_b(\beta)\;) \tag{37}$$

and $t_{[i]}^{i+1}(\beta)$ is determined by R and R° through Eqs(17)-(20) and (30). Apart from a term analytic in β for $\beta \in [0,\infty)$, $\psi(\beta)$ is the specific free energy of the system; we know the existence of the limit (36a) for real β if the potential is periodic. Notice that $\psi(\beta)$ depends on β only through ζ_b . We have the following result.

<u>Theorem 1</u>

Consider a periodic nnp potential on the Kagomé lattice which satisfies the condition that for any triangle B^i

$$|J_b| = |J_{b'}| \quad \text{for } b,b' \in B^i . \tag{38a}$$

Then

(i) $\Psi(\beta)$ is an analytic function inside the domain

$$\mathcal{D} = \{ \beta \in \mathbb{C} : |\zeta_b(\beta)| < 0.34 \text{ for all nnp } b \}$$

(ii) the limit

$$\lim_{N\to\infty} \langle \sigma^d \rangle_B = \langle \sigma^d \rangle \tag{39}$$

exists and is an analytic function of β inside $\mathcal{D}$, for any finite subset d of the lattice; moreover,

$$|\langle \sigma^{d_1} \sigma^{d_2} \rangle - \langle \sigma^{d_1} \rangle \langle \sigma^{d_2} \rangle| \leq 10^6 \, e^{-0.09\, \varrho(d_1,d_2)} \tag{40}$$

holds in $\mathcal{D}$ ($\varrho(d_1,d_2)$ is the distance between d_1 and d_2).

Remarks

1. The theorem refers to a family of potentials. Apart from the freedom in choosing $|J_b|$ to be different in different triangles one can choose the signs p(i) and

$$q(k) = \operatorname{sgn} \prod_{b \in k\text{th hexagon}} J_b$$

independently. This is a general property of two dimensional lattices.

2. Let us consider the case when

$$p(i) = -1 \text{ for all } B^i . \tag{38b}$$

From (35) and (38a,b) it follows that $|\xi_b| \leq 1/3$ for any real β. As a consequence, $\Psi(\beta)$ and the correlations in the totally symmetric equilibrium state are analytic at any real β and they can be analytically continued to $1/\beta = 0$, even preserving the exponential clustering. It is easy to check that to any potential satisfying (38a,b) there exist infinitely many periodic ground states. The simplest example is the antiferromagnet, $J_b = -1$ for all nnp; this corresponds to $q(k) \equiv 1$.

The plot of the domain

$$|w/(1+w+w^2)| < 0.34,$$

relevant in the case (38b), is shown on Fig.1.

Proof

(i) A brief inspection may convince us that for N_n of Eq.(15) the following values can be chosen:

$$N_n = 0 \quad \text{for n odd and } n = 2,4,8$$

$$N_6 = 3, \quad N_{10} = 15, \quad N_{12} = 6$$

$$N_{2n} = 2^n \quad \text{for } n \geq 7$$

Putting these values and x = 0.34, ε = 0.1 into (2) one finds that the inequality is satisfied. The statement is then a consequence of the uniform boundedness of $\{\psi_N(\beta)\}$ in $\mathfrak{D}$ and Vitali's convergence theorem (or, the convergence of ψ_N in $\mathfrak{D}$ can be proved directly, using (21) and (36b)).

(ii) The existence and analyticity of the correlations $\langle \sigma^d \rangle$ follow from the uniform boundedness of $\langle \sigma^d \rangle_B$ in the domain $\mathfrak{D}$ and from the H.T. existence of the limit (39). The former is a consequence of (27) and the estimate

$$N_n^d \leq 2^n \tag{41}$$

which is valid for any $d \in \mathbb{Z}$. (A direct proof of the convergence in $\mathfrak{D}$, using Eq.(25), is also possible.)

For $|d|$ odd, the correlations vanish identically. If $|d_1|$ and $|d_2|$ are odd numbers then

$$|\langle \sigma^{d_1 \cup d_2} \rangle| \leq 4.1\, e^{-0.27\, \rho(d_1,d_2)}$$

follows immediately from (27) and (41) and the fact that

$$N_n^{d_1 \cup d_2} = 0$$

for $n < \rho(d_1,d_2)$.

For $|d_1|$ and $|d_2|$ even, the weaker bound (40) can be obtained. The proof is lengthy and we leave it to an Appendix.-

4. Frustration models on the square lattice

Again, we confine ourselves to nnp potentials. Consider the squares of $\mathbb{Z}^2$ forming an infinite chessboard and let B^i be the set of the four nnp bordering the i th black square:

$$B^i = \{b^1, \ldots, b^4\} .$$

Now $\bigcup B^i$ covers the whole set of nnp. Due to the crossing graphs, inf G does not generate G uniquely while G/G^o (where G^o is defined by (9)) is uniquely generated by inf G/G^o. Using the variables introduced in (33) and (34) we find

$$\zeta_{b^1} = \operatorname{sgn} J_{b^1} \left(w_{b^1} + p(i) \prod_{j=2}^{4} w_{b^j}\right) / \left(1+p(i) \prod_{b \in B^i} w_b\right)$$

and

$$\zeta_c = (\operatorname{sgn} J_{b^1} J_{b^2})(w_{b^1} w_{b^2} + p(i) w_{b^3} w_{b^4}) / \left(1+p(i) \prod_{b \in B^i} w_b\right)$$

where c is a diagonal pair of sites in the i th black square. If $w_b = w_i$ for all $b \in B^i$ and $p(i) = -1$ then ζ_c vanishes and

$$\zeta_b = (\operatorname{sgn} J_b) w_i / (1+w_i^2) = (\operatorname{sgn} J_b)\, \xi_i . \tag{42}$$

Let $\Psi(\beta)$ be defined for the present group G and cover B, by Eqs(36) and (37). We obtain the following theorem for $\Psi(\beta)$.

Theorem 2

Consider a periodic nnp potential on the square lattice which satisfies the conditions (38a,b).

Then $\Psi(\beta)$ is an analytic function inside the domain

$$\mathcal{D}_{sq} = \{ \beta \in \mathbb{C} : |\zeta_b| < 1/2 \text{ for all } b \} .$$

Remarks

1. The theorem refers to a family of potentials. $|J_b|$ may vary from square to square and the signs

$$q(k) = \text{sgn} \prod_{b \in k \text{ th white square}} J_b$$

can be chosen independently. For periodic potentials satisfying (38) there are infinitely many periodic ground states. The simplest examples can be obtained by fixing $|J_b| = 1$ and choosing either $q(k) \equiv -1$ or $q(k) \equiv 1$. These give the so called "odd" and "chessboard" models, respectively, whose free energies were calculated exactly and found to be analytic for any positive temperature ([4], [5]).

2. The analyticity of $\Psi(\beta)$ follows for any real finite β . The domain $|w/(1+w^2)| < 1/2$ is shown on Fig.2. We cannot prove the analyticity and clustering of the correlations for all $\beta \in [0,\infty)$, the reason of which becomes obvious from the proof of the theorem. However, these properties could be shown, with the method applied for the Kagomé lattice, in a relatively large H.T. domain.

Proof

Asymptotically, $N_n = 2^n$ is an upper bound for $N_n(i)$ and one would find difficult to improve it. On the other hand, as (42) shows, $|\zeta_b| = 1/2$ if $1/\beta = 0$. One should choose x= 1/2 in Eq.(2) in order to obtain analyticity for all $\beta \in [0, \infty)$. However, with these N_n and x the inequality (2) cannot be satisfied for any $\varepsilon < 1$.

We need to use some special properties of the lattice and the potential. Let us rewrite the formula (18) for t^i_α, $i \notin \alpha$, with a change of notations as indicated at the end of Section 2.

$$t^i_\alpha = \sum_{b \in B^i} \zeta_b \sum_{s \in S^b_\alpha} \zeta^s / T_\alpha = \sum_{b \in B^i} \zeta_b \, T^b_\alpha / T_\alpha$$

$$S^b_\alpha = \left\{ s \subset \bigcup_{j \in \alpha} B^j : \; |s \cap B^j| \leq 1 \text{ and } \prod_{b' \in s} b' = b \right\} \tag{43}$$

Now t^i_α depends on β through the set of variables $\xi = \{ \xi_k = (\operatorname{sgn} J_b) \zeta_b \}_{k=1,2,\dots}$, each belonging to a square B^k.

Below we show that, with a suitable choice for the numbering of the set $\{B^i\}$, one can obtain the bound

$$| t^{i+1}_{[i]} (\xi) | \leq 1/2$$

for all i, if $|\xi_k| \leq 1/2$ for all k.

(i) If $0 \leq \xi_k \leq 1/2$ for all k then it is possible to define a potential and some $\beta \geq 0$ so that they determine just these ξ_k through Eqs (33) and (42). As a consequence, T^b_α / T_α is a correlation and

$$t^i_\alpha(\xi) = \sum_{b \in B^i} \zeta_b \langle \sigma^b \rangle_{B_\alpha}$$

where

$$B_\alpha = \bigcup_{j \in \alpha} B^j \quad .$$

Now if $n(i,\alpha)$ is the number of those vertices which are shared between B^i and the squares of B_α then

$$|t^i_\alpha(\xi)| \leq \begin{cases} 1 & \text{if } n(i,\alpha) \leq 3 \\ 1/2 & \text{if } n(i,\alpha) \leq 2 \end{cases} \tag{44}$$

because $S^b_\alpha \neq \emptyset$ for at most two or one $b \in B^i$, respectively.

(ii) Let

$$\tilde{t}^i_\alpha(\xi) = \sum_{\substack{s \in S_{\alpha \cup \{i\}} \\ i \in \underline{s}}} \prod_{b \in s} e_b \prod_{j \in \underline{s}} \xi_j \Big/ \sum_{s \in S_\alpha} \prod_{b \in s} e_b \prod_{j \in \underline{s}} \xi_j$$

where e_b is defined for each nnp so that

$$e_b = \begin{cases} -1 & \text{if } b \text{ is the lower nnp of some } B^i \\ 1 & \text{otherwise} \end{cases}$$

Clearly, $\tilde{t}_\alpha^i$ is a function of the form of (18) or (43): it corresponds to a particular choice for the signs of the interactions. One can show by elementary methods that

$$\prod_{b \in s} e_b = -1$$

for all $s \in \inf S$. Now let $0 \leq \xi_k \leq 1/2$ for all k, then

$$\tilde{t}_\alpha^i(\xi) \leq 0 \tag{45}$$

for all α and $i \notin \alpha$. Indeed,

$$\tilde{t}_\alpha^i(\xi) = - \sum_{\substack{s \in \inf S_{\alpha \cup \{i\}} \\ i \in \underline{s}}} \prod_{j \in \underline{s}} \xi_j \Big/ \prod_{k=2}^{|s|} (1 + \tilde{t}_{\alpha - s'k}^{j_k}(\xi)) \tag{46}$$

Now

$$\underline{s} = \{i, j_2, \ldots, j_n\}$$

is a set of indices of squares which form a ring by joining via vertices. The numbering can be chosen so that neighbouring indices belong to joining squares. Therefore,

$$n(j_k, \alpha - s'k) = n(j_k, \alpha - \{j_2, \ldots, j_k\}) \leq 3$$

which implies

$$|\tilde{t}_{\alpha - s'k}^{j_k}(\xi)| \leq 1 \tag{47}$$

by Eq.(44). Equations (46) and (47) then prove (45).

(iii) Let $|\xi|$ denote the set $\{|\xi_k|\}_{k=1,2,\ldots}$. Suppose that $|\xi_k| \leq 1/2$ for all k. Then

$$|t_{\alpha}^{i}(\xi)| \leq -\tilde{t}_{\alpha}^{i}(|\xi|) \tag{48}$$

This can be shown by induction. For, $t_{\alpha}^{i} = 0$ if $|\alpha| < 3$; if $\alpha = \{1,2,3\}$ and $B^1,\ldots,B^4$ surround a white square then

$$|t^4_{\{1,2,3\}}| = |\xi_1|\cdot|\xi_2|\cdot|\xi_3|\cdot|\xi_4| = -\tilde{t}^4_{\{1,2,3\}}(|\xi|).$$

In the n th step,

$$|t_{\alpha}^{i}(\xi)| \leq \sum_{\substack{s \in \inf S_{\alpha\vee\{i\}} \\ i\in \underline{s}}} \prod_{j\in\underline{s}} |\xi_j| / \prod_{k=2}^{|s|} |1-|t^{j_k}_{\alpha-s'k}(\xi)||$$

$$\leq \sum_{\substack{s \in \inf S_{\alpha\vee\{i\}} \\ i\in \underline{s}}} \prod_{j\in\underline{s}} |\xi_j| / \prod_{k=2}^{|s|} |1+\tilde{t}^{j_k}_{\alpha-s'k}(|\xi|)|$$

$$= -\tilde{t}_{\alpha}^{i}(|\xi|)$$

Here we applied the induction together with (47).

(iv) Consider now the set $\{B^1,B^2,\ldots\}$ which covers all nnp of the lattice. Let the numbering be chosen so that B^i joins B^{i+1} through a vertex and the whole set forms an infinite spiral of squares. Then

$$n(i+1, [i]) \leq 2$$

and (44) implies that

$$-\tilde{t}^{\,i+1}_{[i]}(|\xi|) \leq 1/2 .$$

This, together with (48) proves that

$$| t^{i+1}_{[i]}(\xi)| \leq 1/2 \qquad (49)$$

for all i. The remaining is an application of Vitali's theorem or a direct proof of the convergence of ψ_N in $\mathfrak{D}_{sq}$, using (36) and (49).

Appendix. Exponential clustering

Let $d_1, d_2 \subset \Lambda$ be disjoint sets with even number of points. Now we have to estimate

$$\langle \sigma^{d_1} \sigma^{d_2} \rangle_B - \langle \sigma^{d_1} \rangle_B \langle \sigma^{d_2} \rangle_B$$
$$= (T^{d_1 \cup d_2} T - T^{d_1} T^{d_2}) / T^2 \qquad (A.1)$$

This can be done by dividing (A.1) into terms and estimating them distinctly.

$$T^{d_1 \cup d_2} = \sum_{s \in \inf S^{d_1 \cup d_2}} \zeta^s T_{[N]-\underline{s}} = T_1 + T_2$$

where

$$T_1 = \sum_{\substack{s_1 \in \inf S^{d_1} \\ \underline{s}_1 \cap \underline{s}_2 = \emptyset}} \sum_{s_2 \in \inf S^{d_2}} \zeta^{s_1} \zeta^{s_2} T_{[N]-\underline{s}_1-\underline{s}_2}$$

and

$$T_2 = \sum_{s \in \inf S^{d_1 \cup d_2}}{}' \zeta^s T_{[N]-\underline{s}}$$

the prime indicating that no part of s is an element of $\inf S^{d_1}$. If s occurs in the summation for T_2 then $|s| \geq \varrho$, the distance of d_1 and d_2, and

$$|T_2/T| \leq \sum_{n \geq \rho} (2x/(1-\varepsilon))^n = (1-y)^{-1} y^{\rho} \tag{A.2}$$

Here we used the bound (41) and

$$|\zeta^{s}| \leq x^{|s|}$$
$$|t^{i}_{\alpha}| \leq \varepsilon$$
$$y = 2x/(1-\varepsilon) \tag{A.3}$$

On the other hand,

$$T^{d_1} T^{d_2} = U_1 + U_2$$

where

$$U_1 = \sum_{s_1 \in \inf S^{d_1}} \sum_{\substack{s_2 \in \inf S^{d_2} \\ \underline{s}_1 \cap \underline{s}_2 = \emptyset}} \zeta^{s_1} \zeta^{s_2} T_{[N]-\underline{s}_1} T_{[N]-\underline{s}_2}$$

and

$$U_2 = \sum_{s_1 \in \inf S^{d_1}} \sum_{\substack{s_2 \in \inf S^{d_2} \\ \underline{s}_1 \cap \underline{s}_2 \neq \emptyset}} \zeta^{s_1} \zeta^{s_2} T_{[N]-\underline{s}_1} T_{[N]-\underline{s}_2}$$

Now if (s_1, s_2) occurs in the summation for U_2 then $|s_1| + |s_2| \geq \rho$. We can use the simple estimate

$$\text{card}\,\{(s_1, s_2) \in \inf S^{d_1} \times \inf S^{d_2} : |s_1| + |s_2| = n\} \leq n\, 2^n \tag{A.4}$$

to obtain the bound

$$| U_2/ T^2| \leq \sum_{n \geq \varrho} n\, y^n \leq (1-y)^{-2}\, \varrho\, y^{\varrho} \qquad \text{(A.5)}$$

Let us consider TT_1-U_1 .

$$TT_1-U_1 = \sum_{\substack{s_1 \in \inf S^{d_1} \\ \underline{s}_1 \cap \underline{s}_2 = \emptyset}} \sum_{s_2 \in \inf S^{d_2}} \zeta^{s_1} \zeta^{s_2} (T_{[N]} T_{[N]-\underline{s}_1-\underline{s}_2}$$

$$- T_{[N]-\underline{s}_1} T_{[N]-\underline{s}_2}) = \sum_{(s_1,s_2)}{}' \zeta^{s_1} \zeta^{s_2} \Delta(s_1,s_2) \qquad \text{(A.6)}$$

where $T_{[N]} = T$. Writing up the difference in the parentheses explicitly, one can see that many terms cancel out. Omitting a lengthy intermediate speculation, we present the surviving terms:

$$\Delta(s_1,s_2) = \sum_{v \in V_o} \Big(\prod_{s \in v} \zeta^s \Big) T_{[N]-\underline{s}_1-\underline{s}_2-\underline{v}} T_{[N]-\underline{s}_1-\underline{s}_2}$$

$$- \sum_{(v_1,v_2) \in V_{12}} \prod_{i=1}^{2} \Big(\prod_{s \in v_i} \zeta^s \Big) T_{[N]-\underline{s}_1-\underline{s}_2-\underline{v}_1} T_{[N]-\underline{s}_1-\underline{s}_2-\underline{v}_2}$$

$$+ \sum_{(v_1,v_2,v_3) \in V_{123}} \prod_{i=1}^{3} \Big(\prod_{s \in v_i} \zeta^s \Big) T_{[N]-\underline{s}_1-\underline{s}_2-\underline{v}_1-\underline{v}_2} T_{[N]-\underline{s}_1-\underline{s}_2-\underline{v}_3}$$

(A.7)

With the definition

$$[S] = \{ \{s'_1,\dots,s'_k\} \subset \inf S : \underline{s}'_i \cap \underline{s}'_j = \emptyset \text{ for } i \neq j \} ,$$

the sets V_o, V_{12}, V_{123} are given as follows:

$$V_o = \{ \{s'_1,\dots,s'_k\} \in [S] : \underline{s}'_i \cap (\underline{s}_1 \cup \underline{s}_2) \neq \emptyset \text{ for } i=1,\dots,k$$
$$\text{and } \underline{s}'_j \cap \underline{s}_1 \neq \emptyset \text{ , } \underline{s}'_j \cap \underline{s}_2 \neq \emptyset \text{ for some } 1 \leq j \leq k \}$$

$$V_1 = \{ \{s'_1,\dots,s'_k\} \in [S_{[N]-\underline{s}_2}] : \underline{s}'_i \cap \underline{s}_1 \neq \emptyset \text{ for } i=1,\dots,k \}$$

$$V_2 = \{ \{s'_1,\dots,s'_k\} \in [S_{[N]-\underline{s}_1}] : \underline{s}'_i \cap \underline{s}_2 \neq \emptyset \text{ for } i=1,\dots,k \}$$

$$V_{12} = \{ (v_1,v_2) \in V_1 \times V_2 : \underline{v}_1 \cap \underline{v}_2 \neq \emptyset \}$$

where

$$\underline{v} = \bigcup_{s \in v} \underline{s} \quad ;$$

$$V_{123} = \{ (v_1,v_2,v_3) \in V_1 \times V_2 \times [S_{[N]-\underline{s}_1-\underline{s}_2}] : (v_1,v_2) \in (V_1 \times V_2) - V_{12}$$
$$\text{and } \underline{s} \cap (\underline{v}_1 \cup \underline{v}_2) \neq \emptyset \text{ for any } s \in v_3 \text{ and}$$
$$\text{and } \underline{s} \cap \underline{v}_1 \neq \emptyset \text{ , } \underline{s} \cap \underline{v}_2 \neq \emptyset \text{ for some } s \in v_3 \}$$

Dividing (A.7) by T^2 and applying (A.3) we find

$$|\Delta(s_1,s_2)/T^2| \leq (1-\varepsilon)^{-2|s_1|-2|s_2|} \sum_{v\in V_o\cup V_{12}\cup V_{123}} (y/2)^{|v|} \tag{A.8}$$

where $|v| = \sum_{s\in v} |s|$. Let M_n^o , M_n^{12} and M_n^{123} be the numbers of elements with length n in V_o , V_{12} and V_{123} , respectively. It is easy to show that

$$M_n^o\ ,\ M_n^{12} \leq 2^{|s_1|+|s_2|+n}$$

$$M_n^{123} \leq \sum_m 2^{|s_1|+|s_2|+m}\, 2^{\min\{m/2,(n-m)/6\}+(n-m)}$$

$$\leq 7\cdot 2^{|s_1|+|s_2|+9n/8}$$

and therefore

$$M_n = M_n^o + M_n^{12} + M_n^{123} \leq 9\cdot 2^{|s_1|+|s_2|+9n/8} \tag{A.9}$$

Moreover, if $L(s_1,s_2)$ denotes the distance of the supports of s_1 and s_2 , then

$$M_n = 0 \quad \text{for } n < 2L(s_1,s_2) \tag{A.10}$$

because every $v \in V_o\cup V_{12}\cup V_{123}$ connects s_1 and s_2 with at least two chains of triangles. From (A.8)-(A.10) one finds

$$|\Delta(s_1,s_2)/T^2| \leq 9(1-2^{1/8}y)^{-1}(2/(1-\varepsilon)^2)^{|s_1|+|s_2|}(2^{1/8}y)^{2L(s_1,s_2)} \tag{A.11}$$

Another trivial bound comes from (19) and (21):

$$|\Delta(s_1,s_2)/T^2| \leq 2(1-\varepsilon)^{-|s_1|-|s_2|} \qquad (A.12)$$

Now we are able to estimate $(TT_1-U_1)/T^2$. Let η be some positive number; we get

$$(TT_1-U_1)/T^2 = \sum_{\substack{(s_1,s_2):\\ |s_1|+|s_2|<\eta\rho}}{}' \zeta^{s_1}\zeta^{s_2}\Delta(s_1,s_2)/T^2$$

$$+ \sum_{\substack{(s_1,s_2):\\ |s_1|+|s_2|\geq\eta\rho}}{}' \zeta^{s_1}\zeta^{s_2}\Delta(s_1,s_2)/T^2$$

$$= A_< + A_>$$

Using (A.3),(A.4) and (A.11) together with

$$L(s_1,s_2) \geq \rho - |s_1| - |s_2|$$

we find that

$$|A_<| \leq a(x,\varepsilon)\, b(x,\varepsilon;\eta)^{\rho}$$

$$a(x,\varepsilon) = 10.7(1-\varepsilon-2^{9/8}x)(1-\varepsilon)^{-1}(1-2^{1/4}x)^{-1}x$$

$$b(x,\varepsilon;\eta) = 2^{(9-\eta)/4}\, x^{2-\eta}\, (1-\varepsilon)^{-2} \qquad (A.13)$$

To obtain a bound for $A_>$, we apply (A.3),(A.4) and (A.12). These give

$$|A_{>}| \leq 2\,(1-y)^{-2}\,\eta\varrho\;y^{\eta\varrho} \qquad \text{(A.14)}$$

Now we put

$$x = 0.34$$

$$\varepsilon = 0.1$$

which were found to satisfy (2), and choose η so that

$$b(x,\varepsilon\,;\eta) = (2x/(1-\varepsilon))^{\eta} \quad .$$

Sustituting these values into Eqs (A.2),(A.5),(A.13) and (A.14) we obtain

$$|\langle\sigma^{d_1}\,\sigma^{d_2}\rangle_B - \langle\sigma^{d_1}\rangle_B\,\langle\sigma^{d_2}\rangle_B|$$

$$\leq\ |T_2/T| + |U_2/T^2| + |A_{<}| + |A_{>}|$$

$$\leq\ 60\,\varrho\,e^{-0.091\varrho} < 10^6\cdot e^{-0.09\varrho}$$

which is true for all B and therefore gives Eq.(40).

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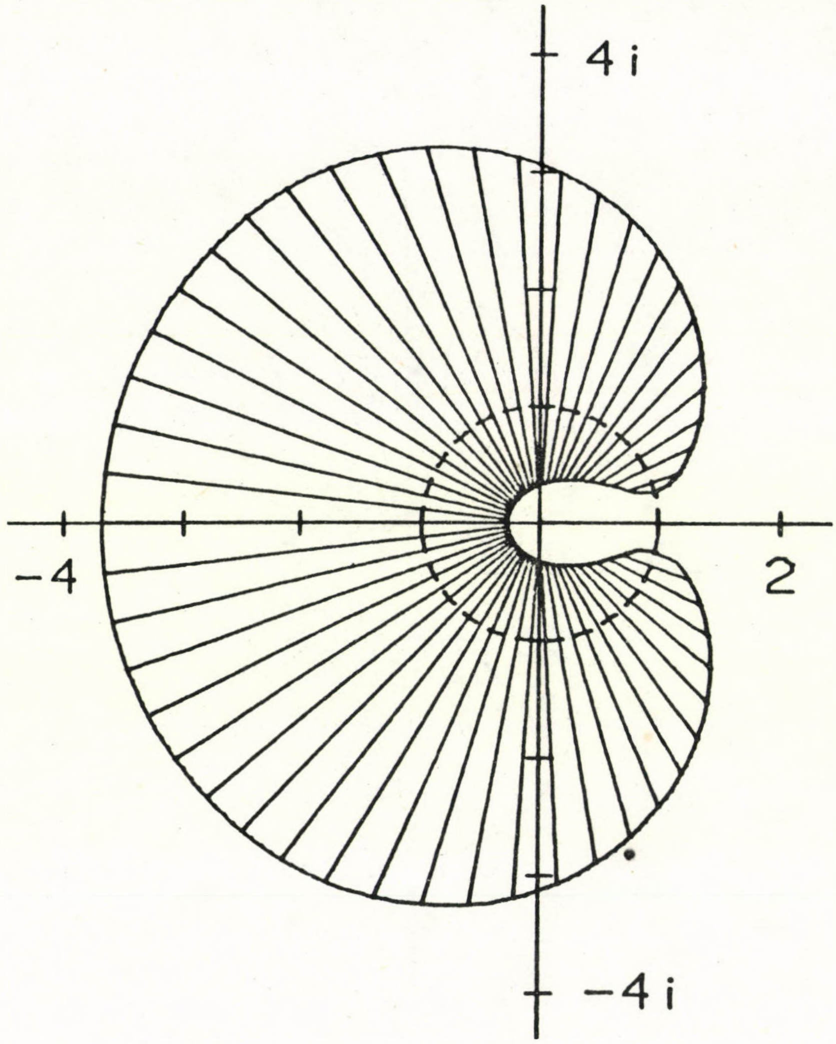

Fig. 1

The domain of analyticity (outside the shaded region) on the complex tanh $|J_b|\beta$ plane, for the Kagomé lattice models

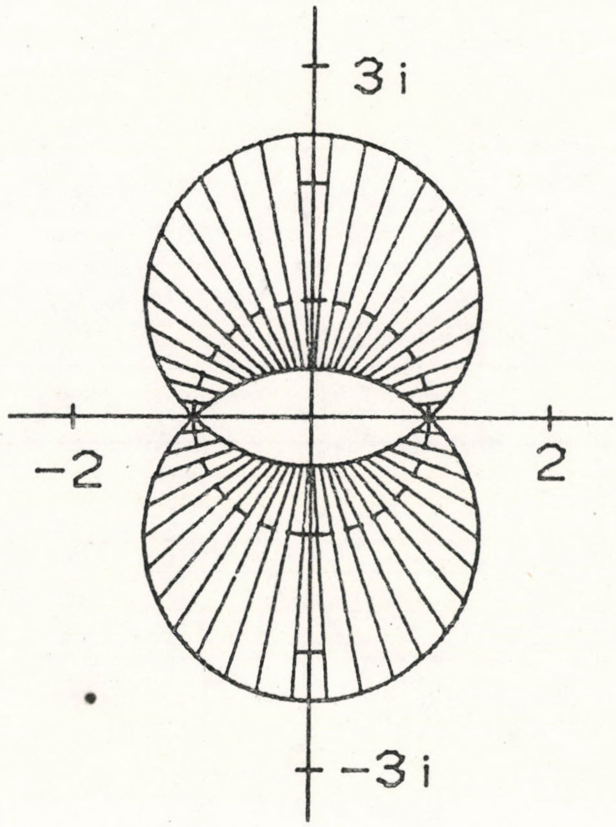

Fig. 2

The domain of analyticity (outside the shaded region) on the complex tanh $|J_b|\beta$ plane, for the square lattice models

63.101

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