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Á. SEBESTYÉN

EINSTEIN'S THEORY RECOVERED

Hungarian Academy of Sciences

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Á. Sebestyén

Central Research Institute for Physics
H-1525 Budapest 114, P.O.B. 49, Hungary

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ABSTRACT

It is shown that a consequent treatment of local Lorentz invariance and of the group of translations as a gauge symmetry group necessarily leads to theories in which torsion has no place. It is also shown that the requirement of symmetry under Lorentz gauge transformations leads to the emergence of the conventional $\sqrt{-g}R$ additive term, responsible for the effects of gravitation, in the Lagrangian. It is thus proved that Einstein's general relativity is a unique consequence of the requirements of invariance under translations and Lorentz transformations.

АННОТАЦИЯ

Показано, что последовательная трактовка групп локальной инвариантности Лоренца и трансляции, как калибровочную симметрию, обязательно преведет таким теориям, в которых нет кручения. Показано также, что если потребуем симметрию относительно калибровочным преобразованиям типа Лоренца, то в функции Лагранжа появляется обычный аддитивный член $\sqrt{-g}R$, ответственный за гравитационные влияния. Таким образом доказывается, что обшая теория относительности Эйнштейна является однозначным последствием требования симметрии относительно трансляциям и преобразованиям Лоренца.

KIVONAT

Megmutatjuk, hogy a lokális Lorentz invariancia és transzlációs csoport mérték szimmetriaként történő következetes kezelése szükségképpen olyan elméletekhez vezet, amelyekben torzió nincs. Azt is megmutatjuk, hogy a Lorentz tipusu mértéktranszformációkkal szembeni szimmetria a Lagrange függvényében a gravitációs hatásokért felelős szokásos $\sqrt{-g}R$ additiv tag felbukkanásához vezet. Ily módon bebizonyitjuk, hogy Einstein általános relativitáselmélete a transzlációkkal és a Lorentz transzformációkkal szemben megkövetelt szimmetria egyértelmü következménye.

It is shown that a consequent treatment of local Lorentz invariance and of the group of translations as a gauge symmetry group necessarily leads to theories in which torsion has no place. It is also shown that the requirement of symmetry under Lorentz gauge transformations leads to the emergence of the conventional $\sqrt{-g}R$ additive term, responsible for the effects of gravitation, in the Lagrangian. It is thus proved that Einstein's general relativity is a unique consequence of the requirements of invariance under translations and Lorentz transformations.

INTRODUCTION

In the past fifteen years, that followed the appearance of Utiyama's fundamental paper [1], a great amount of work has been done in the foundation of various theories of gravitation making use of the gauge theoretical treatment of Poincare symmetry. This approach also appears in papers considering U_4 theories with spin and torsion [2], and the work in this field is still continued [3, 4, 5, 6].

In these considerations the Poincare group is treated as a gauge group and certain effort is made to cast the effect of infinitesimal Poincare transformations in a form that resembles, as closely as possible, the way internal symmetry groups act on fields. This can only be achieved by introducing a priori an orthonormal tetrad and metric structure on the space-time manifold. The Lagrangian is then supposed to depend on the projections of the fields and their derivatives.

This approach can be objected from a number of points of view.

First, translations are diffeomorphisms of space-time, whereas Lorentz transformations, at least in the form the experiments establish Lorentz symmetry, are isomorphisms of the tangent spaces of this manifold. These two kinds of mappings get entangled to produce the semidirect product called the Poincare group, only if the manifold becomes flat, thus to require Poincare gauge invariance for curved space-times seems to be artificial.

Second, in a correct gauge theoretical foundation of general relativity it is expected that the existence of a metric structure on space-time be the consequence of the basic symmetry requirements, thus the a priori supposition of an orthonormal tetrad and the relevant metric does not fit in this scheme of foundation of the theory.

Third, due to the previous objection the Lagrangian must depend on the fields and their derivatives and after showing the necessary appearance of a tetrad of gauge vectors it must be proved that if the equations of motion are projected out by means of this tetrad then the resulting equations are derivable from a Lagrangian that depends on the projections of the fields and their derivatives.

In addition to these objections we stress that the equations determining the structure of space-time are expected to be unambiguous consequences of the basic suppositions whereas in the papers mentioned above /see e.g. [2]/ some further requirements serve to establish the form of the part of the Lagrangian responsible for gravitation.

In order to determine the most general possible Lagrangian of a theory satisfying the requirements implied by these objections we consider first translations as a gauge symmetry group. It is to be noted that papers dealing with the same problem, although arrive at precious results, do not seem to exploit completely the equations resulting from the requirement of symmetry under translations /see e.g. [7]/. In the first section it is shown that four gauge fields and their dual one-formes must be introduced if translations are symmetries of the Lagrangian. It is proved that there exists a Lagrangian depending only on the projections of fields and their derivatives. It is also shown that local Lorentz invariance generate in a natural way a metric structure on space-time, and the Lagrangian can always be cast in a form that contains the usual covariant derivatives with respect to this metric thus no torsion can appear in the theory. It is also indicated that the projections of the equations of motion are the equations of motion of the projections, thus our foundation of the theory meets the requirement implied by the third objection.

In the second section Lorentz transformations are considered. By introducing a unique divergence term, in accord with Noether's original ideas, it is possible to fix the structure of the term in the Lagrangian responsible for gravitation. It is also shown that Einstein's equations are recovered, showing that general relativity is the only possible form of a space-time theory.

Finally the results of the paper are summarized.

1. TRANSLATIONS AND LOCAL LORENTZ INVARIANCE OF THE DERIVATIVES

We assume that all the fields that are considered belong to various classes of tensor fields. L the Lagrangian is supposed to depend on the coordinates of a space-time point, on the fields and their first derivatives at this point, as only theories in which the equations of motion are at most second order differential equations will be considered.

We require that the point transformations of the form

$$x'^i = x^i + \xi^i \qquad /1/$$

with the ξ^i-s being the components of an arbitrary infinitesimal vector field, be a symmetry, that is the variation δS of the action

$$S = \int L d^4x$$

vanish for these transformations

The Principle of Complete Freedom /P.C.F./ of gauge theories will be adopted:

L can be any kind of function of the physical fields and their derivatives, and these fields can be of any number and of any tensorial class.

Due to P.C.F. we may restrict our consideration to a Lagrangian depending on a single contravariant vector field U^i and its derivatives $U^i{}_{,k}$.

If L did not depend on any other field, then we would get in the well-known manner

$$\delta S \equiv \int \Big[\frac{\partial L}{\partial x^r} \xi^r + \\ + \Big(L\delta^s_r + \frac{\partial L}{\partial U^r} U^s + \frac{\partial L}{\partial U^r{}_{,t}} U^s{}_{,t} - \frac{\partial L}{\partial U^t{}_{,s}} U^t{}_{,r} \Big) \xi^r{}_{,s} + \qquad /2/ \\ + \frac{\partial L}{\partial U^t{}_{,r}} U^s \xi^t{}_{,rs} \Big] d^4x = 0 ,$$

here the relations

$$\delta U^i = \xi^i{}_{,r} U^r$$

$$\delta U^i{}_{,k} = \xi^i{}_{,rk} U^r + \xi^i{}_{,r} U^r{}_{,k} - \xi^r{}_{,k} U^i{}_{,r}$$

$$\xi^i{}_{,k\ell} = \xi^i{}_{,\ell k}$$

were used.

The volume of integration in /2/ and the values of ξ^i and $\xi^i{}_{,k}$ are arbitrary at a fixed point, thus we get three equations

$$\frac{\partial L}{\partial x^i} = 0$$

$$L\delta^i_k + \frac{\partial L}{\partial U^k}U^i + \frac{\partial L}{\partial U^k_{,r}}U^i_{,r} - \frac{\partial L}{\partial U^r_{,i}}U^r_{,k} = 0 \,, \qquad /3/$$

$$\frac{\partial L}{\partial U^\ell_{,i}}U^k + \frac{\partial L}{\partial U^\ell_{,k}}U^i = 0 \quad .$$

It can easily be seen than /3/ leads to $L \equiv 0$. To overcome this contradiction to P.C.F. it is supposed that L depends on certain gauge fields in addition to the physical fields. Any one of these gauge fields can be written in the form of a multilinear combination of four independent covariant vectors h^α_i /$\alpha = 0,1,2,3$/ of a base of one-forms and their duals h^i_α:

$$h^\alpha_r h^r_\beta = \delta^\alpha_\beta, \quad h^\rho_i h^k_\rho = \delta^k_i \quad .$$

The third of /3/ as it stands leads to $\frac{\partial L}{\partial U^i_{,k}} \equiv 0$ contradicting P.C.F., thus we have to suppose that L depends on the derivatives of the gauge fields too. We have to stress that in contrast to the case of internal gauge symmetries, here it does not make sense to search for a separate "free" Lagrangian independent of the physical fields. Thus L is of the form

$$L = L(U^i,\ U^i_{,k},\ h^\alpha_i, h^\alpha_{i,k},\ \varphi^A,\ \varphi^A_{,k}) \qquad /4/$$

where φ^A-s are the scalar coefficiens of the gauge fields if the later are written in the form of the multilinear products mentioned above, and L does not depend on x^i because of the first of /3/.

If δS is calculated for L of /4/ then instead of the second and third of /3/ we have

$$L\delta^i_k + \frac{\partial L}{\partial U^k}U^i + \frac{\partial L}{\partial U^k{}_{,r}}U^i{}_{,r} - \frac{\partial L}{\partial U^r{}_{,i}}U^r{}_{,k} -$$

$$- \frac{\partial L}{\partial h^\rho_i}h^\rho_k - \frac{\partial L}{\partial h^\rho_{i,r}}h^\rho_{k,r} - \frac{\partial L}{\partial h^\rho_{r,i}}h^\rho_{r,k} - \qquad /5/$$

$$- \frac{\partial L}{\partial \varphi^R{}_{,i}}\varphi^R{}_{,k} = 0 \ ,$$

$$\frac{\partial L}{\partial U^\ell{}_{,i}}U^k + \frac{\partial L}{\partial U^\ell{}_{,k}}U^i - \left(\frac{\partial L}{\partial h^\rho_{i,k}} + \frac{\partial L}{\partial h^\rho_{k,i}}\right)h^\rho_\ell = 0 \ .$$

We have to remark that as the gauge fields that appear in L are certain combinations of the φ^A-s and the vectors h^α_i and h^i_α, these equations have to be supplemented by further differential equations. However as /5/ itself is a complete Jacobian system of independent first order partial differential equations, we first consider this set alone.

We study first the second of /5/ which consists of 40 equations for the derivatives of L with respect to the 80 variables $U^i{}_{,k}$ and $h^\alpha_{i,k}$. From the theory of characteristic curves it is known, that there exist 40 independent linear combinations of these variables which themselves fulfil these equations and L is an arbitrary function of these linear combinations. Moreover the rank of the matrix of the derivatives of these 40 linear combinations with respect to the $U^i{}_{,k}$-s must be 16 otherwise it would be possible to write down at least one equation of the form

$$A^r_s \frac{\partial L}{\partial U^r{}_{,s}} = 0 \ ,$$

which would contradict P.C.F. As a result there must exist 16 variables, which we call "bar" derivatives and denote by

$U^i{}_{|k}$ defined by

$$U^i{}_{|k} = U^i{}_{,k} + P^{i\ rs}_{k\rho} h^{\rho}_{r,s} \ , \qquad /6/$$

which must satisfy the second of /5/. The remaining 24 combinations are easily proves to be

$$F^{\alpha}{}_{ik} = h^{\alpha}_{i,k} - h^{\alpha}_{k,i} \qquad /7/$$

and L is an arbitrary function of the variables /6/ and /7/.

To find the quantities $P^{i\ rs}_{k\rho}$ in /6/ we substitute the $U^i{}_{|k}$-s into the second of /5/ to get

$$P^{i\ rs}_{k\rho} h^{\rho}_{r,s} = \frac{1}{2} h^{i}_{\rho} (h^{\rho}_{r,k} + h^{\rho}_{k,r}) U^r + \frac{1}{2} Q^{i\ rs}_{k\rho} h^{\rho}_{r,s} \ , \qquad /8/$$

where the $Q^{i\ rs}_{k\rho}$ are arbitrary apart from the restriction

$$Q^{i\ rs}_{k\rho} = -Q^{i\ sr}_{k\rho} \qquad /9/$$

and we have

$$U^i{}_{|k} = U^i{}_{,k} + \frac{1}{2} h^{i}_{\rho} (h^{\rho}_{r,k} + h^{\rho}_{k,r}) U^r + \frac{1}{4} Q^{i\ rs}_{k\rho} F^{\rho}{}_{rs} \ . \qquad /10/$$

In order to find further restrictions for the quantities $Q^{i\ rs}_{k\rho}$ we turn to the consideration of local Lorentz invariance.

The local Lorentz transform h'^{α}_{i} of the vectors h^{α}_{i} is defined by

$$h'^{\alpha}_{i} = \Lambda^{\alpha}{}_{\rho} h^{\rho}_{i} \ , \qquad /11/$$

where the functions $\Lambda^{\alpha}{}_{\beta}$ satisfy the conditions

$$\Lambda^{\alpha}{}_{\rho} \Lambda^{\beta}{}_{\sigma} \gamma^{\rho\sigma} = \gamma^{\alpha\beta} \ , \qquad /12/$$

with $\gamma^{\alpha\beta}$ the usual Minkowski matrix

$$\gamma^{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1) \quad .$$

Using $\gamma^{\alpha\beta}$ and its inverse to raise and lower greek indeces it is easy to find the transform h'^{i}_{α} of h^{i}_{α}:

$$h'^{i}_{\alpha} = \Lambda_{\alpha}{}^{\rho} h^{i}_{\rho} \quad . \qquad /13/$$

We require now that the bar derivatives $U^{i}{}_{|k}$ be invariant under the local Lorentz transformations /11/. Note that the second of /5/ is invariant, which is necessary for setting up this requirement.

We write down $U^{i}{}_{|k}$ for h'^{α}_{i} and h'^{i}_{α}:

$$U^{i}{}_{|k} = U^{i}{}_{,k} + \frac{1}{2} h'^{i}_{\rho} (h'^{\rho}{}_{r,k} + h'^{\rho}{}_{k,r}) U^{r} + \frac{1}{4} Q'^{i}{}_{k\rho}{}^{rs} F'^{\rho}{}_{rs} \quad ,$$

where

$$F'^{\rho}{}_{rs} = h'^{\rho}{}_{r,s} - h'^{\rho}{}_{s,r}$$

and /10/ was taken into account. This expression must be equal to the corresponding expression of $U^{i}{}_{|k}$ with the unprimed h^{α}_{i}-s and h^{i}_{α}-s. From this equality one gets the necessary conditions

$$Q'^{i}{}_{k\alpha}{}^{rs} = \Lambda_{\alpha}{}^{\rho} Q^{i}{}_{k\rho}{}^{rs} \quad ,$$
$$/14/$$
$$\Lambda_{\rho}{}^{\sigma} h^{i}_{\sigma} U^{r} (\Lambda^{\rho}{}_{\tau,k} h^{\tau}_{r} + \Lambda^{\rho}{}_{\tau,r} h^{\tau}_{k}) + Q^{i}{}_{k\rho}{}^{rs} \Lambda_{\sigma}{}^{\rho} \Lambda^{\sigma}{}_{\tau,s} h^{\tau}_{r} = 0 \quad .$$

These equations must be fulfilled for any set of functions $\Lambda^{\alpha}{}_{\beta}$ satisfying /12/. The meaning of the first of /14/ is obvious, the evaluation of the second is done by taking the equations

$$\Lambda^{\alpha}{}_{\rho,k} \Lambda^{\beta\rho} + \Lambda^{\alpha}{}_{\rho} \Lambda^{\beta\rho}{}_{,k} = 0 \qquad /15/$$

into account by means of Lagrange's multiplicators. Note that /15/ is a consequence of /12/. We get:

$$h^i_\gamma \delta^m_k h_{\beta r} U^r + h^i_\gamma U^m h_{\beta k} + Q^{i\ rm}_{k\gamma} h_{\beta r} = \qquad /16/$$

$$= \lambda^{im}_{k\ \rho\sigma} (\Lambda^\rho{}_\beta \Lambda^\sigma_\gamma + \Lambda^\rho{}_\gamma \Lambda^\sigma{}_\beta) \quad ,$$

where the $\lambda^{im}_{k\ \rho\sigma}$-s are the multiplicators. Taking the antisymmetric part of /16/ in β and γ one sees that

$$Q^{i\ rm}_{k\gamma} h_{\beta r} - Q^{i\ rm}_{k\beta} h_{\gamma r} = h^i_\beta \delta^m_k h_{\gamma r} U^r - h^i_\gamma \delta^m_k h_{\beta r} U^r +$$

$$+ h^i_\beta U^m h_{\gamma k} - h^i_\gamma U^m h_{\beta k} \quad .$$

Contracting with $h^{\beta n}$ this equation yields

$$Q^{i\ nm}_{k\gamma} - Q^{i\ rm}_{k\rho} h_{\gamma r} h^{\rho n} = g^{in} \delta^m_k h_{\gamma r} U^r - h^i_\gamma \delta^m_k U^n +$$

$$+ g^{in} \delta^m h_{\gamma k} - h^i_\gamma \delta^n_k U^m \quad ,$$

which after contracting with $h^{\gamma p}$ gives

$$Q^{i\ nm}_{k\rho} h^{\rho p} - Q^{i\ pm}_{k\rho} h^{\rho n} = g^{in} \delta^m_k U^p - g^{ip} \delta^m_k U^n + \qquad /17/$$

$$+ g^{in} \delta^p_k U^m - g^{ip} \delta^n_k U^m \ ,$$

where in the last two equations the abbreviation

$$g^{ik} \equiv h^i_\rho h^{\rho k}$$

was used. /17/ and

$$Q^{i\ nm}_{k\rho} h^{\rho p} = -Q^{i\ mn}_{k\rho} h^{\rho p}$$

which is the consequence of /9/, suffice to determine

$\frac{1}{4}\Omega^{i}_{k\rho}{}^{rs}F^{\rho}{}_{rs}$ of /10/. As a result we get

$$U^{i}{}_{|k} = U^{i}{}_{,k}+\frac{1}{2}\{h^{i}_{\rho}(h^{\rho}_{t,k}+h^{\rho}_{k,t})+g^{ir}[h_{\rho k}(h_{r,t}-h_{t,r}) + h_{\rho t}(h^{\rho}_{r,k}-h^{\rho}_{k,r})]\}U^{t} \quad . \qquad /19/$$

This equation is one of our most important conclusions. If we take the non-singular g^{ik} of /18/ and its inverse

$$g_{ik} = h^{\rho}_{i}h_{\rho k} \qquad /20/$$

and form the usual Christoffel symbols

$$\Gamma^{i}_{k\ell} = \frac{1}{2}g^{ir}(g_{rk,\ell} + g_{r\ell,k} - g_{k\ell,r})$$

and take /20/ into account, we can easily see that /19/ may by written in the form

$$U^{i}{}_{|k} \equiv U^{i}{}_{;k} = U^{i}{}_{,k} + \Gamma^{i}_{kt}U^{t} \quad , \qquad /21/$$

thus the bar derivative is nothing else but the well-known covariant derivative $U^{i}{}_{;k}$ of U^{i}.

Thus we have established the result that in consequence of the second of /5/ and of the requirement that the derivative of U^{i} be invariant under local Lorentz transformations, the Lagrangian must depend on the variables /7/, the usual covariant derivative /21/ and on the other variables appearing in /4/, that is

$$L = L(U^{i}, U^{i}{}_{;k}, h^{\alpha}_{i}, F^{\alpha}{}_{ik}, \varphi^{A}, \varphi^{A}{}_{,k}) \quad . \qquad /22/$$

We proceed now to evaluate the first of /5/. If the form /22/ is substituted in this equation the following obvious rules for calculating partial derivatives must be used:

$$\frac{\partial L}{\partial U^i} \to \frac{\partial L}{\partial U^i} + \frac{\partial L}{\partial U^r{}_{;s}} \frac{\partial U^r{}_{;s}}{\partial U^i} \quad ,$$

$$\frac{\partial L}{\partial U^i{}_{,k}} \to \frac{\partial L}{\partial U^i{}_{;k}} \quad ,$$

$$\frac{\partial L}{\partial h^\alpha_i} \to \frac{\partial L}{\partial h^\alpha_i} + \frac{\partial L}{\partial U^r{}_{;s}} \frac{\partial U^r{}_{;s}}{\partial h^\alpha_i} \quad ,$$

$$\frac{\partial L}{\partial h^\alpha_{i,k}} \to \frac{\partial L}{\partial U^r{}_{;s}} \frac{\partial U^r{}_{;s}}{\partial h^\alpha_{i,k}} + \frac{1}{2} \frac{\partial L}{\partial F^\rho{}_{rs}} \frac{\partial F^\rho{}_{rs}}{\partial h^\alpha_{i,k}} \quad ,$$

the $\frac{1}{2}$ appears in the last term of the last equations because of the antisymmetry of $F^\alpha{}_{ik}$. To get rid of the term $L\delta^i_k$ of the first of /5/ we substitute

$$L = \det(h^\alpha_i)\mathcal{L} \quad . \qquad /23/$$

A lengthy but straightforward calcualtion leads to

$$\frac{\partial \mathcal{L}}{\partial U^k} U^i + \frac{\partial \mathcal{L}}{\partial U^k{}_{;r}} U^i{}_{;r} - \frac{\partial \mathcal{L}}{\partial U^r{}_{;i}} U^r{}_{;k} -$$

$$- \frac{\partial \mathcal{L}}{\partial h^\rho_i} h^\rho_k - \frac{\partial \mathcal{L}}{\partial F^\rho{}_{ir}} F^\rho{}_{kr} - \frac{\partial \mathcal{L}}{\partial \varphi^R{}_{,i}} \varphi^R{}_{,k} = 0 \quad . \qquad /24/$$

It is to be observed that the set /24/ is again a complete Jacobian system of 16 independent, linear, homogeneous, partial differential equations in the variables U^i, $U^i{}_{;k}$, h^α_i, $F^\alpha{}_{ik}$, $\varphi^A{}_{,k}$. Let N be the number of these variables. Then it is easy to find the N - 16 combinations which themselves fulfil /24/:

$$U^{\alpha} \equiv U^{r} h^{\alpha}_{r} \quad ,$$

$$U^{\alpha}{}_{;\beta} \equiv U^{r}{}_{;s} h^{\alpha}_{r} h^{s}_{\beta} \quad ,$$

$$F^{\alpha}{}_{\beta\gamma} \equiv F^{\alpha}{}_{rs} h^{r}_{\beta} h^{s}_{\gamma} \quad , \qquad /25/$$

$$\varphi^{A}{}_{,\alpha} \equiv \varphi^{A}{}_{,r} h^{r}_{\alpha}$$

and L is an arbitrary function of these new variables:

$$L = L(U^{\alpha},\ U^{\alpha}{}_{;\beta},\ F^{\alpha}{}_{\beta\gamma},\ \varphi^{A}{}_{,\alpha}) \quad . \qquad /26/$$

Finally we consider the supplementary differential equations mentioned in the remark that followed /5/. These equations, if L of /26/ is substituted into them, can give only further restrictions for U^{α}, $U^{\alpha}{}_{;\beta}$, $F^{\alpha}{}_{\beta\gamma}$, $\varphi^{A}{}_{,\alpha}$, in the sense, that it can happen that L depends only on certain combinations of these variables. However, these restrictions do not concern our further analysis, thus we continue to work with the form /26/.

We have arrived at the result that the complete Lagrangian is

$$\mathrm{L} = \det(h^{\alpha}_{i})\, L(U^{\alpha},\ U^{\alpha}{}_{;\beta},\ F^{\alpha}{}_{\beta\gamma},\ \varphi^{A}{}_{,\alpha}) \quad . \qquad /27/$$

If /18/ and /20/ are taken into account it can be seen that

$$\det(h^{\alpha}_{i}) = \sqrt{-\det(g_{ik})} \quad ,$$

the well-known factor of Lagrangians in general relativity. The variables in /27/ are projections with the tetrad h^{i}_{α}, h^{α}_{i}. This tetrad is orthonormal, and the existence of a metric structure is not to be supposed a priori but arises in a natural way in consequence of the requirements of translational gauge symmetry and of local Lorentz invariance of the bar derivative.

It is seen that the Lagrangian can always be written in a form in which torsion does not appear, showing that theories with torsion and with a Lagrangian that can not be written in the form /27/ necessarily violate translational gauge symmetry or local Lorentz invariance. We remark that the quantities $F^{\alpha}{}_{\beta\gamma}$ are the projections of the so-called coefficients of anholonomity.

Before considering the equations of motion of the fields U^i, we note that the projection $U^{\alpha}{}_{;\beta}$ of /25/ is a certain type of directional derivative of the projection U^{α}. Indeed we have

$$U^{\alpha}{}_{;\beta} = U^{r}{}_{;s}h^{\alpha}_{r}h^{s}_{\beta} = U^{\alpha}{}_{,s}h^{s}_{\beta}-h^{\alpha}_{r;s}h^{s}_{\beta}U^{r}=U^{\alpha}{}_{,s}h^{s}_{\beta} -$$

$$- h^{\alpha}_{r;s}h^{s}_{\beta}h^{r}_{\sigma}U^{\sigma} = U^{\alpha}{}_{,\beta}+\Gamma^{\alpha}_{\sigma\beta}U^{\sigma} \quad ,$$

where

$$U^{\alpha}{}_{,\beta} \equiv U^{\alpha}{}_{,r}h^{r}_{\beta}$$

and

$$\Gamma^{\alpha}_{\sigma\beta} \equiv h^{r}_{\sigma;s}h^{\alpha}_{r}h^{s}_{\beta}$$

This formula for $U^{\alpha}{}_{;\beta}$, which can be easily generalized for tensors of any class, strongly resembles that of the usual covariant derivative. It contains the projection U^{α} and its directional derivative $U^{\alpha}{}_{,\beta}$. The quantities $\Gamma^{\alpha}_{\beta\gamma}$ are the counterparts of Christoffel's symbols but it must be emphasized that

$$\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}$$

if and only if the tetrad h^{α}_{i} is holonomic:

$$h^{\alpha}_{i,k} = h^{\alpha}_{k,i}$$

in which case the $F^{\alpha}{}_{\beta\gamma}$-s of /25/ vanish.

The equations of motion of the field U^i are formed by taking the Lagrange derivative

$$\frac{\partial}{\partial U^i} - \frac{\partial}{\partial x^r}\frac{\partial}{\partial U^i{}_{,r}}$$

of /27/ and equating the resulting expression to zero. If the derivatives with respect to U^i and $U^i{}_{,k}$ are expressed by the derivatives with respect to the variables of /25/ and the preceeding consideration of $U^\alpha{}_{;\beta}$ is taken into account, then it is easy to prove that the equation of motion of U^i take the following form

$$\frac{\partial L}{\partial U^\alpha} - \left(\frac{\partial L}{\partial U^\alpha{}_{;\rho}}\right)_{;\rho} = 0 \quad ,$$

which is the desired result implied by the third objection of the introduction, namely if the principle of least action is assumed to hold for the fields then the principle is formally valid for projections too, although in the procedure of the variation the anholonomic coordinates associated to the tetrad must be used. It is to be stressed that the equation of motion of projections is of great importance as these later are generally interpreted as the measurable quantities of general relativity.

In the next section we consider local Lorentz invariance of the Lagrangian /27/.

2. LOCAL LORENTZ INVARIANCE OF THE LAGRANGIAN

We require that the Lagrangian /27/ be invariant under transformations /11/. For an infinitesimal $\Lambda^\alpha{}_\beta$ we have

$$\Lambda^\alpha{}_\beta = \delta^\alpha_\beta + \varepsilon^\alpha{}_\beta \quad ,$$

where

$$\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha} \; , \quad \varepsilon^{\alpha\beta}{}_{,i} \neq 0 \quad .$$

/remember that greek indices are raised and lowered by means of the Minkowsky $\gamma^{\alpha\beta}$/.

For the variables in /27/ we have

$$\delta U^{\alpha} = \varepsilon^{\alpha}{}_{\rho} U^{\rho} \quad ,$$

$$\delta U^{\alpha}{}_{;\beta} = \varepsilon^{\alpha}{}_{\rho} U^{\rho}{}_{;\beta} + \varepsilon_{\beta}{}^{\rho} U^{\alpha}{}_{;\rho} \quad ,$$

$$\delta\varphi^{A}{}_{,\alpha} = \varepsilon_{\alpha}{}^{\rho} \varphi^{A}{}_{,\rho} \, , \qquad /28/$$

$$\delta F^{\alpha}{}_{\beta\gamma} = \varepsilon^{\alpha}{}_{\beta,\gamma} - \varepsilon^{\alpha}{}_{\gamma,\beta} + \varepsilon^{\alpha}{}_{\rho} F^{\rho}{}_{\beta\gamma} +$$

$$+ \varepsilon_{\beta}{}^{\rho} F^{\alpha}{}_{\rho\gamma} + \varepsilon_{\gamma}{}^{\rho} F^{\alpha}{}_{\beta\rho} \quad ,$$

where

$$\varepsilon^{\alpha}{}_{\beta,\gamma} = \varepsilon^{\alpha}{}_{\beta,r} h^{r}{}_{\gamma} \quad .$$

The second of /28/ is a consequence of the second of /25/ in which $U^{i}{}_{;k}$ is formed by means of the g_{ik} of /20/ which is invariant under transformations /11/. /28/ is made use of in the evaluation of the variation of the action:

$$\delta S = \int \det(h^{\alpha}_{i}) \Big(\frac{\partial L}{\partial U^{\rho}} \delta U^{\rho} + \frac{\partial L}{\partial U^{\rho}{}_{;\sigma}} \delta U^{\rho}{}_{;\sigma} +$$

$$+ \frac{1}{2} \frac{\partial L}{\partial F^{\rho}{}_{\sigma\tau}} \delta F^{\rho}{}_{\sigma\tau} + \frac{\partial L}{\partial \varphi^{R}{}_{,\rho}} \delta\varphi^{R}{}_{,\rho} \Big) d^{4}x \quad .$$

After introducing /28/ we get an expression in which the coefficients of $\varepsilon^{\alpha\beta}$ and $\varepsilon^{\alpha\beta}{}_{,\gamma}$ must vanish. We get the following two equations

$$\frac{\partial L}{\partial U^{\alpha}} U_{\beta} + \frac{\partial L}{\partial U^{\rho}{}_{;\sigma}} (\delta^{\rho}_{\alpha} U_{\beta;\sigma} + \gamma_{\sigma\alpha} U^{\rho}{}_{;\beta}) + \frac{1}{2} \frac{\partial L}{\partial F^{\rho}{}_{\sigma\tau}} (\delta^{\rho}_{\alpha} F_{\beta\sigma\tau} + \qquad /29a/$$

$$+ \gamma_{\sigma\alpha} F^{\rho}{}_{\beta\tau} + \gamma_{\tau\alpha} F^{\rho}{}_{\sigma\beta}) + \frac{\partial L}{\partial \varphi^{R}{}_{,\rho}} \gamma_{\rho\alpha} \varphi^{R}{}_{,\beta} - (\alpha \leftrightarrow \beta) = 0 \quad ,$$

$$\frac{\partial L}{\partial F^{\alpha\beta\gamma}} - \frac{\partial L}{\partial F^{\beta\alpha\gamma}} = 0 \quad . \qquad /29b/$$

The second of /29/ gives, together with

$$\frac{\partial L}{\partial F^{\alpha\beta\gamma}} = - \frac{\partial L}{\partial F^{\alpha\gamma\beta}} \quad , \qquad /30/$$

which is a consequence of the antisymmetry of $F^{\alpha\beta\gamma}$,

$$\frac{\partial L}{\partial F^{\alpha\beta\gamma}} = 0 \quad .$$

However, according to the original ideas which led Noether to her famous theorems, in δS divergence terms may appear. Such a term is proportional to $\varepsilon^{\alpha\beta}$ and its derivatives and must not contain derivatives of the fields [8].

The only non-vanishing possible term of this kind compatible with equations /29/ is of the form

$$\mu[\det(h^{\alpha}_{i})\varepsilon^{\rho\sigma}{}_{,r}h^{r}_{\rho}h^{s}_{\sigma}]_{,s} = \qquad /31/$$

$$= \mu\det(h^{\alpha}_{i})(\varepsilon^{\rho\sigma}{}_{,r}h^{r}_{\rho}h^{s}_{\sigma})_{;s} \; ,$$

where μ is some constant. In δS /31/ will give a contribution only to the coefficient of $\varepsilon^{\alpha\beta}{}_{,\gamma}$. This contribution can be easily calculated using the well-known manipulations with the covariant derivatives of the tetrad h^{α}_{i}. As a result we get instead of the second of /29/

$$\frac{\partial L}{\partial F^{\alpha\beta\gamma}} - \frac{\partial L}{\partial F^{\beta\alpha\gamma}} = \qquad /32/$$

$$= \mu(-F_{\gamma\alpha\beta}+\gamma_{\gamma\beta}F^{\rho}{}_{\alpha\rho}-\gamma_{\gamma\alpha}F^{\rho}{}_{\beta\rho}) \quad .$$

/32/ and /30/ suffice to determine

$$\frac{\partial L}{\partial F^{\alpha\beta\gamma}} = \mu[\tfrac{1}{2}(F_{\alpha\beta\gamma}+F_{\gamma\beta\alpha}-F_{\beta\gamma\alpha}) + $$

$$+ \gamma_{\alpha\beta}F^{\rho}{}_{\gamma\rho}-\gamma_{\alpha\gamma}F^{\rho}{}_{\beta\rho}] \ .$$

The solution of this differential equation can be readily written down:

$$L = \frac{\mu}{8}F_{\rho\sigma\tau}F^{\rho\sigma\tau} + \frac{\mu}{4}F_{\sigma\rho\tau}F^{\rho\tau\sigma} - $$

$$- \frac{\mu}{2}F^{\rho\sigma}{}_{\rho}F^{\tau}{}_{\sigma\tau} + L_o \equiv L_g + L_o \quad , \qquad /33/$$

where

$$\frac{\partial L_o}{\partial F^{\alpha\beta\gamma}} = 0 \quad .$$

If we substitute /33/ into the first of /29/ we get

$$\frac{\partial L_o}{\partial U^{\alpha}}U_{\beta} + \frac{\partial L_o}{\partial U^{\rho}{}_{;\sigma}}(\delta^{\rho}_{\alpha}U_{\beta;\sigma}+\gamma_{\sigma\alpha}U^{\rho}{}_{;\beta}) + $$

$$+ \frac{\partial L_o}{\partial \varphi^{R}{}_{,\rho}}\gamma_{\rho\alpha}\varphi^{R}{}_{,\beta} - (\alpha\leftrightarrow\beta) = 0 \quad . \qquad /34/$$

This equation states that L_o as a function of the variables U^{α}, $U^{\alpha}{}_{;\beta}$ and $\varphi^{A}{}_{,\alpha}$ must be invariant under the Lorentz transformations /11/ regarded formally as global transformations /$\Lambda^{\alpha}{}_{\beta,i} = 0$/.

We now turn to the discussion of the be significance of the term L_g defined in /33/. First, it can be seen by manipulating again with the covariant derivatives of the tetrad that

$$L_g = \frac{\mu}{2}(h^{\rho r}{}_{;s}h^{s}{}_{\rho;r}-h^{r}{}_{\rho;r}h^{\rho s}{}_{;s}) \quad . \qquad /33/$$

Second, in our way of establishing the possible form of the theory the tetrad h^α_i plays the essential role, while the metric tensor /18/ and /20/ is merely an abbreviation for a Lorentz invariant non-singular symmetric tensor formed by means of the h^α_i-s. Now the obvious question arises: what is the form of the term corresponding to L_g of /33/ in the Euler-Lagrange equations for h^α_i if the later are considered as dynamical variables and are subject to a variational principle? A straightforward but lengthy calculation in which the form /33/ is used shows that

$$\frac{\partial[\det(h^\alpha_i)L_g]}{\partial h^\alpha_i} - \left\{\frac{\partial[\det(h^\alpha_i)L_g]}{\partial h^\alpha_{i,r}}\right\}_{,r} = \frac{\mu}{2}h^r_\alpha R_r{}^i \quad , \qquad /36/$$

where $R_r{}^i$ is the Ricci tensor of g_{ik}. If the form /35/ is used we have

$$\begin{aligned} L_g &= \frac{\mu}{2}(h^{\rho r}{}_{;s}h^s_{\rho;r} - h^{\rho r}{}_{;r}h^s_{\rho;s}) = \\ &= \frac{\mu}{2}(h^{\rho r}{}_{;s}h^s_\rho - h^{\rho s}{}_{;s}h^r_\rho)_{;r} - \\ &- \frac{\mu}{2}(h^{\rho r}{}_{;sr} - h^{\rho r}{}_{;rs})h^s_\rho = \qquad /37/ \\ &= \frac{\mu}{2}h^{\rho t}R_t{}^r{}_{rs}h^s_\rho + \frac{\mu}{2}(h^{\rho r}{}_{;s}h^s_\rho - h^{\rho s}{}_{;s}h^r_\rho)_{;r} = \\ &= \frac{\mu}{2}R + \frac{\mu}{2}(h^{\rho r}{}_{;s}h^s_\rho - h^{\rho s}{}_{;s}h^r_\rho)_{;r} \quad , \end{aligned}$$

where R is the invariant Riemannian curvature

$$R = R^r{}_r = R_t{}^r{}_r{}^t \quad .$$

The second term in the last row of /37/ is a complete divergence if it is multiplied by

$$\det(h^\alpha_i) = \sqrt{-g}$$

and can be dropped from the Lagrangian, the first term gives the well-known gravitational Lagrangian of general relativity:

$$\frac{\mu}{2}\sqrt{-g}R \quad .$$

We have arrived at the result that if local Lorentz invariance is required for the Lagrangian L of /27/ in the previous section, then L is the sum of a term L_o which is an invariant function of U^α, $U^\alpha{}_{;\beta}$, $\varphi^A{}_{,\alpha}$, and of L_g which is equal to R. Also if h^α_i are regarded as dynamical variables and are subject to a variational principle Einstein's equations are recovered. Naturally the energy-momentum tensor is then derived from

$$\frac{\partial[\det(h^\alpha_i)L_o]}{\partial h^\alpha_i} - \left(\frac{\partial[\det(h^\alpha_i)L_o]}{\partial h^\alpha_{i,r}}\right)_{,r} = \frac{2}{\mu}(T^i_r - \frac{1}{2}\delta^i_r T^s{}_s)h^r_\alpha \quad .$$

CONCLUSIONS

In the paper the consequent treatments of translations as a gauge symmetry and of local Lorentz invariance have been studied. It has been shown that in consequence of the requirement of invariance under these transformations the Lagrangian is a sum of two terms L_o and L_g. L_o is an invariant fucntion of the projections of the fields and their derivatives; the projections being made by means of a tetrad, the covariant derivatives are formed using the invariant metric tensor derived from this tetrad. L_g is the well-known gravitational term of general relativity. The metric tensor appearing in the theory is a consequence of Lorentz invariance. In this form of the Lagrangian no torsion appears.

The equations of motion of the projected physical fields are the projections of the equations of motion of these fields. If the fields of the tetrad are dynamical variables subject to the variational principle, the equations of motion for the tetrad are Einstein's equation.

We have to stress that our results can be generalized to Lagrangians of physical fields of any class and of any number.

As a conclusion we state that the form Einstein established for general relativity is a unique consequence of our symmetry requirements. Thus any theory that cannot be cast in the form indicated in the paper must necessarily violate either translational or Lorentz invariance or both.

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Felelős kiadó: Szegő Károly
Szakmai lektor: Perjés Zoltán
Nyelvi lektor: Tóth Kálmán
Gépelte: Polgár Julianna
Példányszám: 320 Törzsszám: 80-652
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