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ABSTRACT

$\frac{3}{2}$ A finite action solution of SU/2/ gauge theory with topological charge is given.

АННОТАЦИЯ

Найдено самодуальное решение конечного действия калибровочной SU/ 2/ теории с топологическим зарядом, равным 3/2.

KIVONAT

Egy egzakt, SU/2/, önduális, véges hatásu megoldást konstruálunk, $\frac{3}{2}$ topológikus töltéssel.

In this paper we give a new finite action solution of the selfduality equations /SDE/ with Pontryagin number 3/2. This runs counter to the common wisdom based on the pioneering work of Belavin, Polyakov, Schwartz and Tyupkin /BPST/[1] and supported by the outstanding work of Atiyah, Ward, Drinfeld, Hitchin and Manin[2]. Nevertheless, Crewter[3] pointed out that solutions of the SDE with fractional topological charge might exist.

Here we argue that our solution does not actually contradict to any of the existing theorems. BPST pointed out that the requirement of finite action implies that asymptotically in R^4 the gauge field A_μ should tend to a pure gauge $\partial_\mu g \cdot g^{-1}$, and with the assumption that g represents a continuous map of S^3 to SU(2) they concluded using homotopy theory that the topological charge must be an integer. However, finite action does not imply this mapping be continuous, and without continuity the concept of homotopy breaks down. If, however, g is not a continuous $S^3 \rightarrow S^3$ mapping, this automatically rules out the possibility of moving from R^4 to S^4 in the sense of the fibre bundle approach, thus the theorems of Atiyah et al. do not apply here[4]. On the other hand, Uhlenbeck[5] has recently shown that from finite action solutions of the Yang-Mills equations in R^4 point-like singularities are removable, so it is possible to extend this solution to S^4. Of course, this theorem is not applicable when the singularities of A_μ are not pointlike.

In fact, our solution has a singularity on a two dimensional sphere / S_0^2 / and it may be thought of as an extended object, to be contrasted with the point-like structure of instantons. We may interpret it as a closed string-like fluctuation of the vacuum; appearing at a certain instant / in Euclidean time / with zero radius, evolving to a maximal one and then shrinking back to zero radius again and finally disappearing. Alternatively, since in Euclidean space there is no preferred time variable we may describe our solution as a "balloon" / S_o^2 / with a given radius appearing at a given instant and then disappearing again.

We now proceed to describe this solution in some detail. It is perhaps somewhat surprising that our solution is in the well known Corrigan, Fairlie, 'tHooft, Wilczek /CFtHW/ [6] ansatz

$$A_\mu = \sigma_{\mu\nu} \partial^\nu \ln \rho \tag{1}$$

where the SDE $F_{\mu\nu} = {}^*F_{\mu\nu}$ [7] reduce to

$$\frac{1}{\rho} \Box \rho = 0 \tag{2}$$

In this gauge we need two coordinate patches to describe A_μ. In addition, even these two patches cover only $R^4 \setminus S_o^2$, and we have to define A_μ in the whole R^4 by an appropriate continuation as it will be explained later.

In the two patches the $A_\mu^{(i)}$-s are given by different superpotentials ρ_i:

$$A_\mu^{(i)} = \sigma_{\mu\nu} \partial^\nu \ln \rho_i \qquad (i=1,2) \tag{3}$$

Now for our solution both ρ_i-s depend only on z and r and they have the form:

$$\varrho_i = \varrho_0 h_i \tag{4}$$

are given by

$$\varrho_0 = (S^5 - S_-^5)\, r^{-1} S^{-5}$$

$$h_i = S^5 \left[S^5 + S_-^5 + H_i \right]^{-1}$$

$$H_1 = \sqrt{(S+S_-)^2 - 4\alpha^2} \left\{ \frac{1}{4} \left[4\alpha^2 - (S-S_-)^2 \right]^2 + S^2 S_-^2 - 12\alpha^2 (z-\beta)^2 \right\} \tag{5}$$

$$H_2 = \sqrt{4\alpha^2 - (S-S_-)^2} \left\{ \frac{1}{4} \left[4\alpha^2 - (S+S_-)^2 \right]^2 + S^2 S_-^2 - 12\alpha^2 (z-\beta)^2 \right\}$$

where

$$S = \sqrt{(r+\alpha)^2 + (z-\beta)^2} \quad , \quad S_- = \sqrt{(r-\alpha)^2 + (z-\beta)^2} \tag{6}$$

and $\alpha > 0$, β is an arbitrary real number.

The two coordinate patches P_1, P_2 are chosen in such a way that $A_\mu^{(i)}$ be free of any singularities in patch P_i; their projections on the /z,r/ half plane are depicted on Fig. 1. The points A,B,C,D on the z-axis are excluded from the corresponding patches as the h_i functions have poles there; the line segments /$z=\beta$, $r<\alpha$ / and /$z=\beta$, $r>\alpha$ / are excluded from P_1 and P_2 respectively as $\partial_\mu h_1$ and $\partial_\mu h_2$ respectively are discontinuous there. Note, that S_0^2 / $z=\beta$,$r=\alpha$ / belongs to none of the two patches.

In both domains of the overlapping region, the two $A_\mu^{(i)}$-s are connected by a continuous gauge transformation

$$A_\mu^{(1)} = \Omega A_\mu^{(2)} \Omega^{-1} + i\, \partial_\mu \Omega\, \Omega^{-1} \tag{7}$$

where $\Omega = \exp\left\{ i\alpha(z,r) \frac{\vec{\sigma}\cdot\vec{x}}{2r} \right\}$ with

$$\alpha(z,r) = \frac{\pi}{2}\,\mathrm{sign}(z-\beta) + 2\,\mathrm{Arctan}\,\frac{R_1}{1-T_1} - 2\,\mathrm{Arctan}\,\frac{R_2}{1-T_2}$$

with R_i, T_i are given by

$$R_1 = \frac{\mathrm{sign}(z-\beta)}{2S^5}\,H_2 \qquad T_1 = \frac{1}{2S^5}\,H_1$$

$$R_2 = \frac{\mathrm{sign}(\beta-z)}{2S^5}\,H_1 \qquad T_2 = \frac{1}{2S^5}\,H_2 \tag{8}$$

We are forced to leave out S_o^2 from the overlapping region since the transition function Ω is not continuous there. However, from both patches the $A_\mu^{(i)}$ -s can be continued back making use of /3,4/ to this sphere where $A_\mu^{(1)} = A_\mu^{(2)}$. Here we argue that the SDE are fulfilled even on S_o^2. If we extend the ρ_i -s to the whole R^4 their derivatives become /singular/ distributions; however, the main point here is to realize that on S_o^2 they give no contribution / in the sense that the appearing δ -s are multiplied by coefficients vanishing on S_o^2/. It is in this sense that our solution satisfies the SDE on the whole R^4. This situation is not unfamiliar because in the case of the well known instanton solutions the SDE are satisfied in this gauge in a similar / distributional / sense, because of point-like singularities in the connection (A_μ). Our case is different since $A_\mu^{(i)}$-s are free of singularities in $P_i \cup S_o^2$, but the transition function is not regular on S_o^2. It can be interpreted as a singularity of the bundle itself.

We now proceed to calculate the topological charge

$$q = \frac{1}{8\pi^2} \int d^4x \, F^a_{\mu\nu} \, {}^*F^{a\mu\nu} \tag{9}$$

which is given by

$$q = -\frac{1}{16\pi^2} \int d^4x \, \Box\Box \ln \rho \tag{10}$$

The correct prescription for evaluating /10/:

$$q = -\frac{1}{16\pi^2} \lim_{\varepsilon \to 0} \int dz \, d\Omega \int_{\varepsilon}^{\infty} r^2 dr \, \Box\Box \ln \rho \tag{11}$$

Since the action density $\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$ is regular at r=0. In fact, /10/ should be interpreted as the sum of the integrals of $\Box\Box \ln \rho_i$ in $P_i \cup S^2_0$ substracting the contribution coming from the overlapping region.

Now, observing that $\Box\Box \ln h_i$ is identically zero as a consequence of $(\partial_z^2 + \partial_r^2) \ln h_i = 0$ in the domain of the integral, the topological charge is

$$q = -\frac{1}{16\pi^2} \int dS^\mu \partial_\mu \Box \ln \frac{S^5_- S^5_-}{r S^5} , \tag{12}$$

using Gauss theorem. /12/ is readily evaluated, and its value is found to be 3/2, contributions to /12/ coming from S^3 at $R \to \infty$ and the hypercylinder surrounding the z-axis.

We now want to discuss the topological behaviour of our solution. Since the gauge fixed by eq. /1/ is not suitable for discussing the asymptotics of the gauge fields at $R \to \infty$ — $A^{(i)}_\mu$-s are vanishing faster than $\partial_\mu g \cdot g^{-1}$ —

we make a gauge transformation in the following way: first, in P_2 we carry out a gauge transformation S_2 on $A_\mu^{(2)}$ which makes $A_\mu^{(2)'}$ regular on the z-axis. We now deform P_i to P_i' in such a way that P_1' does not contain the z-axis. We then transform $A_\mu^{(1)}$ by S_1, where the S_i-s are given by

$$S_i = \exp\left\{-i\theta_i(z,r)\,\frac{\vec{\sigma}\cdot\vec{x}}{2r}\right\} \tag{13}$$

with $\theta_i = \pi + 2\,\mathrm{Arctan}\,\dfrac{R_i}{1-T_i} + 5\,\mathrm{Arctan}\,\dfrac{\beta-z}{\alpha+r}$.

As a result the new transition function $\Omega^{(N)}$ is given as

$$\Omega^{(N)} = S_1\,\Omega\,S_2^{-1} = \exp\left\{i\,\frac{\pi}{2}\,\mathrm{sign}(z-\beta)\,\frac{\vec{\sigma}\cdot\vec{x}}{2r}\right\}.$$

The asymptotics of $A_\mu^{(1)'}$ in P_i' is

$$A_\mu^{(1)'} = i\,\partial_\mu g_1\cdot g_1^{-1},$$

with $g_1 = \exp\left\{i(3\varphi+\pi)\,\dfrac{\vec{\sigma}\cdot\vec{x}}{2r}\right\}$

where $\varphi = \arctan\dfrac{z}{r}$.

$A_\mu^{(2)'} = i\,\partial_\mu g_{2(+)}g_{2(+)}^{-1}$ with $g_{2(+)} = \exp\left\{i\left(3\varphi+\dfrac{\pi}{2}\right)\dfrac{\vec{\sigma}\cdot\vec{x}}{2r}\right\}$

for $z-\beta>0$, while for $z-\beta<0$

$A_\mu^{(2)'} = i\,\partial_\mu g_{2(-)}g_{2(-)}^{-1}$ with $g_{2(-)} = \exp\left\{i\left(3\varphi+\dfrac{3\pi}{2}\right)\dfrac{\vec{\sigma}\cdot\vec{x}}{2r}\right\}$

Note, that in P_2 the asymptotic domain consists of two disconnected parts, therefore, it is not surprising that $A_\mu^{(2)'}$ behaves differently in these regions. We remark, that in this gauge on S_0^2 there is the same bundle type singularity as in the previous one.

One can now see the reason in this gauge for the fractional Pontryagin number: although A_μ falls off as a pure gauge at infinity, it cannot be represented by a global pure gauge[8].

The calculation of the topological charge /9/ requires some care. Usually, /9/ is given by the surface integral of the topological current, $J_\mu = \mathrm{Tr}\, \varepsilon_{\mu\nu\rho\sigma}(A^\nu \partial^\rho A^\sigma + i\frac{2}{3} A^\nu A^\rho A^\sigma)$, on S^3 at infinity. Since there are several patches in our case, applying Gauss theorem there are additional contributions coming from the boundaries. However, shrinking the overlapping region to the hypersurface $\alpha^2 - r^2 + (z-\beta)^2 = 0$ there are no additional contributions. This means, that on the asymptotic S^3 P'_{2+}, P'_1, P'_{2-} are defined as $\frac{\pi}{2} \geq \varphi \geq \frac{\pi}{4}$, $\frac{\pi}{4} \geq \varphi \geq -\frac{\pi}{4}$, $-\frac{\pi}{4} \geq \varphi \geq -\frac{\pi}{2}$ respectively.

The existence of this solution may be relevant for the following problems: understanding the structure of the QCD vacuum[9]; it may provide a solution of the U(1) problem as advocated by Crewther[3]. It needs further clarification what is the relevance of these closed string-like fluctuations to the confinement problem.

We would like to mention that solutions of the SDE with topological charge other than half-integer exist, work is in progress in this direction and we shall present these results later.

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6. E. Corrigan, D.B. Fairlie, Phys. Lett. 67B, 69 /1977/; G. 'tHooft, unpublished; F. Wilczek, Quark Confinement and Field Theory, ed. D. Stump & D. Weingarten / John Wiley, N.Y. 1977/.
7. Our notations and conventions:

$$A_\mu = A_\mu^a \frac{\sigma^a}{2}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\,[A_\mu, A_\nu]$$

$$\sigma_{\mu\nu} = \begin{cases} \sigma_{ij} = \frac{1}{4i}\,[\sigma_i, \sigma_j] \\ \sigma_{i0} = \frac{1}{2}\sigma_i \end{cases}$$

$$x^0 = z, \qquad r = \sqrt{x^2+y^2+t^2}, \qquad R = \sqrt{z^2+r^2}$$

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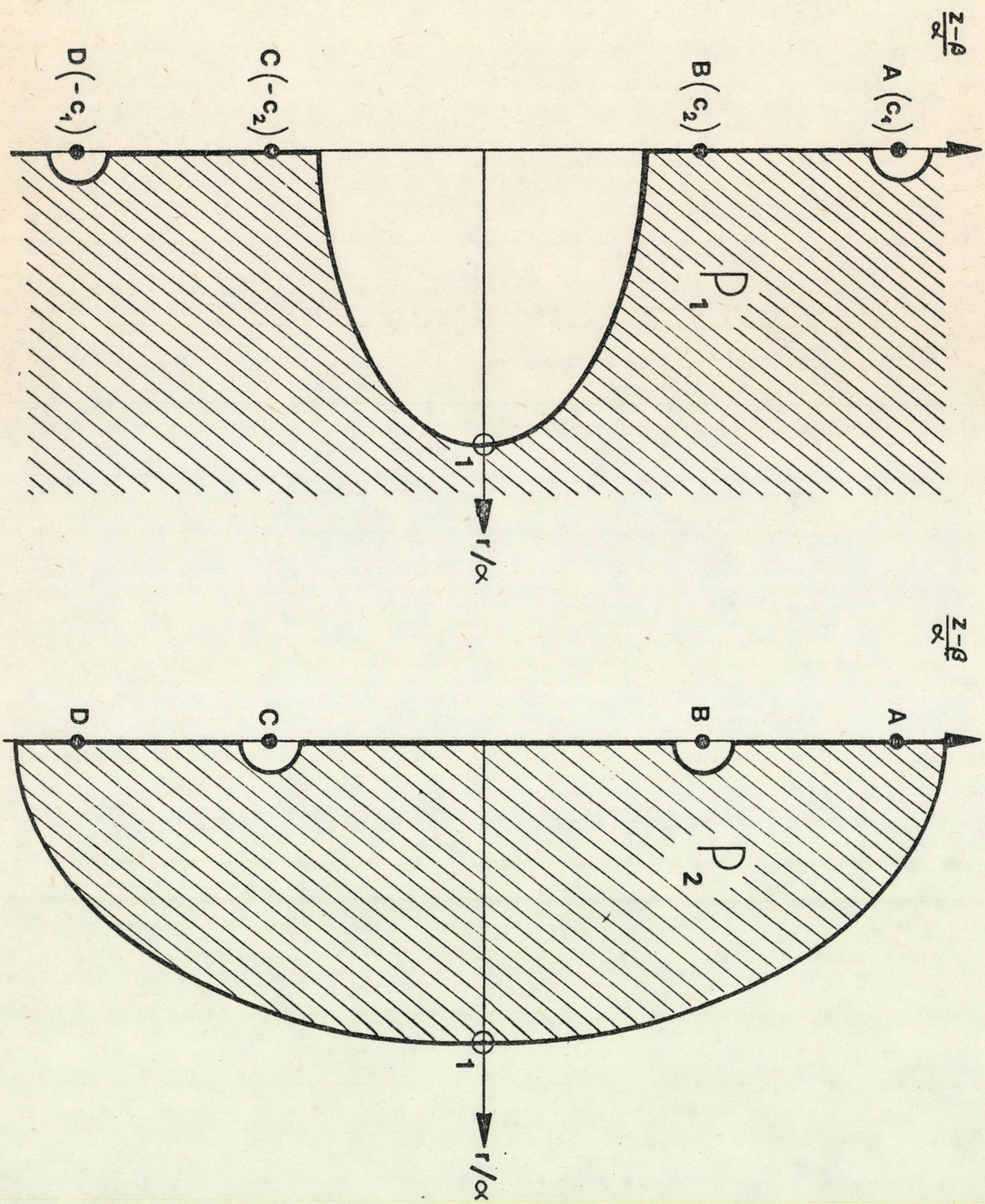
FIGURE CAPTION

Figure 1. The position of the poles are given by

$$c_1 = \sqrt{1+\frac{2}{\sqrt{5}}}\ ,\quad c_2 = c_1^{-1}\ .$$

figure 1.



$\frac{z-\beta}{\alpha}$
A (c_1)
B (c_2)
C ($-c_2$)
D ($-c_1$)
P_1
1
r/α
$\frac{z-\beta}{\alpha}$
A
B
C
D
P_2
1
r/α

63.036

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