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ABSTRACT

It is shown that the Bogomolny equations for the simplest static, axially symmetric gauge fields are equivalent to the Ernst equation. The BPS one monopole is obtained via Harrison's Bäcklund transformation.

АННОТАЦИЯ

Показывается, что в наиболее простом статическом аксиально симметрическом случае уравнения Богомольного эквивалентны уравнению Эрнста. Монополь БПШ генерируется при помощи преобразования Бэклунда, данного Гаррисоном.

KIVONAT

Megmutatjuk, hogy a Bogomolny egyenletek legegyszerübb sztatikus, tengelyszimmetrikus esetben az Ernst egyenlettel ekvivalensek. A BPS egy monopólust Harrison Backlund transzformációja segitségével generáljuk.

We are looking for axially symmetric SU /2/ monopoles in the BPS[0] limit in a classical non-Abelian gauge theory with a Higgs field in the adjoint representation, such that we can treat it as the A_t component of the potential A_μ[1]. Until now the only finite energy solution of this theory is the spherically symmetric BPS one monopole /1MP/. It is not known whether the theory has any other classical solution with finite energy, such as multimonopoles. In the BPS limit the scalar field becomes massless and can mediate a long range force which can cancel the magnetic forces. It has been shown[2,3] that the force between monopoles decreases faster than any inverse power of the separation. This encourages one to hope that static, noninteracting, finite energy monopoles may exist. On the other hand Montonen and Olive conjectured that there may exist a dual theory which looks like the original one with monopoles as basic constituents[4]. Furthermore from an entirely different point of view Adler[5] also investigated the axially symmetric case and conjectured the existence of solutions with an extended zero set of the Higgs field within the 1MP sector.

So far no systematic method has been given to generate solutions for the axially symmetric case. Our aim here is to show that there are such methods: Bäcklund transformations and the method of inverse scattering To do this we first reduce our axially symmetric equations to the Ernst equation[6], for which these transformations are worked out. Then as a first application of these techniques we generate the 1MP from a suitably chosen solution.

Manton constructed an ansatz for the Euclidean static, axially, symmetric, selfdual gauge fields in order to find multimonopole solutions of the Bogomolny equations.[1]

His ansatz in polar coordinates

$$A_t^a = (0, \phi_1, \phi_2) \qquad A_\varphi^a = -(0, \eta_1, \eta_2)$$

$$A_z^a = -(W_1, 0, 0) \qquad A_\varrho^a = -(W_2, 0, 0)$$

where $x_1 = \varrho\cos\varphi$, $x_2 = \varrho\sin\varphi$ and η_i, ϕ_i, W_i are functions of ϱ, z only. The Bogomolny eqs. take the following form

$$\partial_\varrho\phi_1 - W_2\phi_2 = -\varrho^{-1}(\partial_z\eta_1 - W_1\eta_2) \tag{2 a}$$
$$\partial_\varrho\phi_2 + W_2\phi_1 = -\varrho^{-1}(\partial_z\eta_2 + W_1\eta_1) \tag{2 b}$$
$$\partial_\varrho W_1 - \partial_z W_2 = \varrho^{-1}(\phi_1\eta_2 - \phi_2\eta_1) \tag{2 c}$$
$$\partial_z\phi_1 - W_1\phi_2 = \varrho^{-1}(\partial_\varrho\eta_1 - W_2\eta_2) \tag{2 d}$$
$$\partial_z\phi_2 + W_1\phi_1 = \varrho^{-1}(\partial_\varrho\eta_2 + W_2\eta_1) \tag{2 e}$$

Eqs. /2a-e/ are invariant under Abelian gauge transformations

$$W_i' = W_i + \partial_i\Lambda, \quad \begin{pmatrix}\phi_i' \\ \eta_i'\end{pmatrix} = \begin{pmatrix}\phi_i \\ \eta_i\end{pmatrix}\cos\Lambda + \varepsilon_{ij}\begin{pmatrix}\phi_j \\ \eta_j\end{pmatrix}\sin\Lambda, \quad \varepsilon_{12} = 1 = -\varepsilon_{21}$$

Now our first observation is that eqs /2a-e/ are in fact equivalent to the Ernst equation[6]. To show this let us consider the line element

$$ds^2 = A\,d\varrho^2 + 2B\,d\varrho\,dz + C\,dz^2 = g_{ij}\,dy^i dy^j \tag{3}$$

where $A = \eta^2/\varrho^2$, $B = -\phi\cdot\eta/\varrho$, $C = \phi^2$ using the notation of ref.[1] The "zweibeins" for this metric are

$$e_{1a} = -\frac{1}{\varrho}(\eta_1, \eta_2) \quad , \qquad e_{2a} = (\phi_1, \phi_2)$$

Introducing the one forms $\omega^a = e^a{}_i\,dy^i$,and the connection one forms $\omega^1{}_2 = -\omega^2{}_1 = -W_i\,dy^i$, eqs./2a-c/ are

$$d\omega^a + \omega^a{}_b \wedge \omega^b = 0 \tag{4}$$
$$d\omega^1{}_2 + \omega^1 \wedge \omega^2 = 0 \tag{5}$$

which means the Gaussian curvature of the metric /3/ is just R=-1, /i.e. it describes a pseudosphere/. The remaining two equations /2d-e/ can be written as

$$d\bar{\omega}^a + \omega^a{}_b \wedge \bar{\omega}^b = 0 \qquad \text{where} \qquad \bar{\omega}^a = {}^*e^a{}_i\, dy^i ,$$

$$^*e_{1a} = \rho(\phi_1, \phi_2) \quad , \quad ^*e_{2a} = (\eta_1, \eta_2)$$

The geometrical meaning of these equations is not quite clear.

On can now proceed by solving eqs /2a-c/, choosing an appropriate parametrization of the pseudosphere, and the remaining two equations will determine the dependence of the parameter functions on the coordinates (ρ, z) . In what follows, we adopt a well-known form of the Poincaré metric on the pseudosphere

$$ds^2 = \frac{df^2 + d\psi^2}{f^2} \qquad f = f(\rho, z) \ , \quad \psi = \psi(\rho, z)$$

corresponding to

$$\omega^1 = f^{-1}(f_{,\rho}\, d\rho + f_{,z}\, dz)$$

$$\omega^2 = f^{-1}(\psi_{,\rho}\, d\rho + \psi_{,z}\, dz)$$

Now expressing ϕ_i, η_i, W_i in terms of f, ψ , we get

$$\phi_1 = f^{-1}\psi_{,z} \ ; \quad \phi_2 = -f^{-1} f_{,z} \ ; \quad \eta_1 = -\rho f^{-1} \psi_{,\rho} \ ;$$

$$\eta_2 = \rho f^{-1} f_{,\rho} \ ; \quad W_1 = -f^{-1}\psi_{,z} \ ; \quad W_2 = -f^{-1}\psi_{,\rho} \qquad (6)$$

Substituting /6/ into /2d-e/ we obtain the celebrated form of the Ernst equation of General Relativity

$$\mathrm{Re}\,\epsilon \,\Delta \epsilon - (\nabla \epsilon)^2 = 0 \qquad (7)$$

where $\epsilon = f + i\psi$, $\Delta = \partial_\rho^2 + \partial_z^2 + \rho^{-1}\partial_\rho$

One remark is in order in connection with /6/. The two conditions $W_1 = -\phi_1$; $\rho W_2 = \eta_1$ implied in /6/ cannot be imposed simultaneously as a gauge fixing, however, for a solution of /2a-c/ this is possible.

This is so because adopting a certain parametrization of the pseudosphere is more than fixing a gauge; in fact - as was mentioned above - it simultaneously ensures the solution of some of the selfduality equations, Now, if for some ϕ_i, η_i and W_i configuration these equations are satisfied, this fact guarantees that the configuration can be gauge transformed into a gauge where both relations are valid at once[7].

It is straightforward to determine the functions f and ψ for the 1MP: we have to transform the monopole into the R gauge[7] and integrate /6/ once; the result is[8]

$$f_{1MP} = \frac{\varsigma}{F} \qquad \psi_{1MP} = \frac{P}{F}$$

$$F = \frac{r}{\sinh r} + r\cosh z \coth r - z\sinh z \tag{8}$$

$$P = z\cosh z - r\sinh z\coth r \qquad r = \sqrt{\varsigma^2 + z^2}$$

The direct equivalence of eqs /2a-e/ and /7/ is important because it makes possible the use of solution generating techniques that exist for the Ernst equation. Recently several authors proposed various group-theoretic or soliton-theoretic methods for generating new solutions of /7/[9]. These techniques offer the possibility of iterations, thus are capable of generating infinitely many solutions from an initial one. Next we want to indicate how the soliton-theoretic techniques /Bäcklund transformations found by Harrison /HB[10]/ and Neugebauer /NB/[11] and the "inverse scattering method" of Belinsky and Zakharov /BZ/[12]/ may be applied to the monopole problem. However, before entering into the details we must clarify the meaning of Bäcklund transformation here.

Corrigan et al.[13] have found nonlinear transformations-that they also called Backlund transformation - which connect solutions of the selfduality equations in the R gauge. These transformations were adopted to the static case in Ref.8. in an attempt to generate new solutions from 1MP / the result was a singular 3MP/. Having realized that the system /2a-e/ reduce to the Ernst equation /7/, one can identify the analogues of these transformations for the Ernst equation. In fact one finds that the transformations(β) and (γ) in Ref.8. are the so called discrete Neugebauer-Kramer mapping /I/[14] and Ehlers transformations[15] respectively /Lohe applied the special product $\beta\gamma\beta$/. Now, these two transformations are elements of the infinite dimensional Geroch group[16]. Applying a finite number of these transformations usually changes the properties of the solution in an uncontrollable way. If we want to preserve certain properties of the seed solution, then we have to find suitable subgroups of the Geroch group. In practice, it means summing up in a suitable way an infinite number of similar infinitesimal transformations. See Kinnersley et al[17]. On the other hand - as it was shown by Cosgrove[9] - the /HB/ and /NB/ transformations are not contained in the Geroch group, and a finite number of these BTs preserve certain asymptotic properties of the solution[18]. Therefore, the transformations we are considering are different from those of Ref.8.

To apply /HB/ or /NB/ transformations to a given solution requires to solve Riccati-type equations /see below/. These equations depend essentially on the form of the known solution, and given the fact that the 1MP is somewhat complicated it does not appear straightforward to apply these transformations directly.

Knowing that the Bts can be iterated algebraically it seems to be sufficient to generate the lMP solution from a suitable ground state as a first step. By carrying this out we achieve two things. On the one hand we see that the lMP /8/ can be interpreted as a "soliton" of the Ernst equation as well. On the other side, it may make it possible to find a finite energy nonlinear superposition of monopoles. We applied both the /HB/ and the /BZ/ transformations to generate the lMP, here we give in some details the application of /HB/ because we think it is more transparent and selfcontained.

To apply /HB/ transformation one defines from the known solution $\epsilon = f + i\psi$ the quatities

$$M_1^o = \frac{1}{2f}\,\epsilon_{,1}\,; \quad M_2^o = \frac{1}{2f}\,\epsilon^*_{,1}\,; \quad N_1^o = \frac{1}{2f}\,\epsilon^*_{,2}\,; \qquad (9)$$

$$N_2^o = \frac{1}{2f}\,\epsilon_{,2}\,, \quad \epsilon_{,1} = \frac{\partial\epsilon}{\partial\zeta_1}, \dots \quad \zeta_1 = \rho + iz\,; \quad \zeta_2 = \rho - iz$$

and solves the total Riccati equation for the pseudopotential $q(\zeta_1, \zeta_2)$:

$$dq = \left[(M_2^o - M_1^o)q + \gamma(s)(M_2^o - M_1^o q^2)\right]d\zeta_1 + \\ + \left[(N_1^o - N_2^o)q + \gamma^{-1}(s)(N_1^o - N_2^o q^2)\right]d\zeta_2 \qquad (10)$$

where $\gamma(s) = \sqrt{\frac{1 - 2si\zeta_2}{1 + 2si\zeta_1}}$ s being a real parameter. The new /transformed/ M_i-s are given in terms of M^o_i, q and $\gamma(s)$ as follows

$$M_1 = -\frac{q(1+\gamma q)}{\gamma + q}M_1^o - \frac{q(\gamma^2 - 1)}{\gamma + q}\frac{1}{4\rho}$$

$$M_2 = -\frac{\gamma + q}{q(1+\gamma q)}M_2^o - \frac{\gamma^2 - 1}{1 + \gamma q}\frac{1}{4\rho} \qquad (11)$$

It is and advantage of the /HB/ transformation that it acts on the M_i-s as these are in direct connection - via /6/ - with the fields ϕ_i, η_i and W_i of our interest. In particular the gauge invariant length of the Higgs field / that determines the magnetic charge/ $\phi^2 = \phi_1^2 + \phi_2^2$ is given as

$$\phi^2 = \frac{f_{,z}^2 + \psi_{,z}^2}{f^2} = 4(M_1 - M_2^*)(M_1^* - M_2) \tag{12}$$

Now, the particular solution of /7/ from which the 1MP can be obtained by a /HB/ transformation is $\epsilon_o = \rho \sinh z + \frac{i}{2}\rho^2$. The naive expectation would be to choose a solution for which $\phi^2 = 1$, but it does not work[19]. In fact applying the /HB/ once to a family of these states we obtained singularities in the gauge invariant quantity ϕ^2 . It is important to realize that the Ehlers transformations[15] do not change ϕ^2 , however, the final result of the BT depends on which Ehlers equivalent state we start with. Note that for ϵ_o $\phi^2 = \coth^2 z$;however, it can be obtained by a Neugebauer-Kramer mapping /I/ from a complex solution $f' = \sinh^{-1} z,\ \psi' = i\coth z$ having $\phi^2 = 1$.
If we allow complex solutions we can say that we generated the 1MP by the product /HB/ I from this "natural" ground state. In the case of ϵ_o the solution of /10/ is given by

$$q = \frac{\rho\cosh(z + \frac{\mu}{2} + \beta) + i s^{-1} D^2(s) \cosh(\frac{\mu}{2} + \beta)}{\rho\cosh(z + \frac{\mu}{2} + \beta) - i s^{-1} D^2(s) \cosh(\frac{\mu}{2} + \beta)} \tag{13}$$

where β is a constant of integration and

$$\frac{\mu}{2} = -\frac{z}{2} + \frac{1}{4s}\left(1 - \sqrt{(1-2sz)^2 + 4s^2\rho^2}\right) = -\frac{z}{2} + \frac{1}{4s}\left(1 - R(s)\right),$$

while $D^2(s) = \frac{1}{2}(1 + R(s)) - sz$. Combining /13/ and /11/ choosing $\beta = -1/4s$ we obtain

$$M_1 - M_2^* = \frac{iq}{2}\left(\frac{2s}{R(s)} - \coth\frac{R(s)}{2s}\right); \quad i.e. \quad \phi^2 = \left(\frac{2s}{R(s)} - \coth\frac{R(s)}{2s}\right)^2,$$

which is really the length of the Higgs field in the case of 1MP. This is the main result of our paper. As we mentioned earlier the fact that we generated the 1MP from a suitable state may open the way to obtain a decisive answer to the question of the existence of multimonopole solutions by the repeated applications of these transformations.

In conclusion we have shown the equivalence of Manton's equations /2a-e/ to the Ernst equation /7/. This enabled us to generate the 1MP from a complex state with $\phi^2 = 1$ using Harrison's Bäcklund transformation and the Neugebauer-Kramer mapping. Details of applying the other techniques to the monopole problem will be published elsewhere.

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References and Footnotes

0. Bogomolny - Prasad - Sommerfield
1. N.S. Manton, Nucl. Phys. B135, 319 /1978/.
2. N.S. Manton, Nucl. Phys. B126, 525 /1977/.
3. W. Nahm, CERN Report No. TH 2642 /unpublished/.
4. C. Montonen and D. Olive, Phys. Letters 72B,117 /1977/.
5. S.L. Adler, Phys.Rev. D20, 1386 /1979/.
6. F.J. Ernst, Phys. Rev. 167, 1175 /1968/.
7. C.N.Yang, Phys. Rev.Lett. 38, 1377 /1977/. Note that combining /1/ and /6/ one obtains expressions for A_μ^a that are identical to the formulae of vectorpotentials in the R gauge with the special choice for the ϕ and ρ functions

$$\phi = \rho f(\rho, z) \; ; \quad \rho = \sqrt{2}\,\bar{y}\,\psi(\rho, z) \; ; \quad \sqrt{2}\,y = x_1 + i x_2$$

Indeed substituting this ansatz into the selfduality equations in the R gauge we obtain the Ernst equation /7/.
8. M.A. Lohe, Nucl. Phys. B142, 236 /1978/. We take the vacuum expectation value of the Higgs field v=1.
9. C.M. Cosgrove, Lecture given at the Second Marcel Grossmann Meeting on Recent developments of general relativity /Trieste, Italy, July 1979/; Montana State University Report /1980, unpublished/.
10. B.K. Harrison, Phys.Rev.Lett. 41, 1197 /1978/.
11. G. Neugebauer, J.Phys. A.12, L67 /1979/.
12. V. A. Belinski and V.E, Zakharov, Zh, Eksp. Teor. Fiz. 75, 1953/1978/; ibid. 77, 3 /1979/.

13. E. Corrigan, D.B. Fairlie, P. Goddard and R.G. Yates, Comm. Math. Phys. 58, 223 /1978/.

14. G. Neugebauer and D. Kramer, Ann. Phys. Lpz. 24, 62 /1969/. One defines a function ω by

$$\partial_\rho \psi = \frac{f^2}{\rho} \partial_z \omega , \quad \partial_z \psi = -\frac{f^2}{\rho} \partial_\rho \omega$$

then (I) acts as : $(\mathrm{I}) \{ f, \omega, \psi \} = \{ \rho f^{-1} ; \; i\psi , -i\omega \}$.

15. J. Ehlers in Les Theories Relativistes de la Gravitation /CNRS, Paris, 1959/. It acts on ϵ as

$$\epsilon' = \frac{\alpha_4 \epsilon - i\alpha_3}{\alpha_1 + i\alpha_2 \epsilon} \qquad \alpha_1\alpha_4 - \alpha_2\alpha_3 = 1$$

16. R. Geroch, J. Math. Phys. 12, 918 /1971/; ibid. 13, 394 /1972/.

17. W. Kinnersley, J. Math. Phys. 18, 1529 /1977/; W. Kinnersley and D.M. Chitre, ibid. 18, 1538 /1977/; 19, 1926 /1978/; 19, 2037 /1978/; C. Hoenselaers, W. Kinnersley and B.C. Xanthopoulos, ibid. 20, 2530 /1979/.

18. In particular, an even number of /HB/ transformations preserves asymptotic flatness /Ref. 9./, however, since our asymptotic conditions are different it is far from being obvious what the implications are in our case.

19. Even in general relativity one /HB/ changes the asymptotic flatness, sec. Ref. 9.

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