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RELATIVISTIC TREATMENT

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ABSTRACT

The behaviour of a relativistic free electron in an external plane wave field is analysed and a review of the existing solutions of the corresponding Dirac equation is presented. Completeness and orthogonality of the Volkov states are also proved. Based on the exact wave function obtained a relativistic generalization of the perturbation method proposed in a previous paper is elaborated as a means of treating intense field problems in a covariant manner.

АННОТАЦИЯ

Анализировано поведение релятивистского свободного электрона во внешнем плоском волновом поле, и представлен обзор различных точных решений соответствующего уравнения Дирака. Обсуждены комплектность и ортогональность состояний Волкова. На основе точной волновой функции разработано обобщение метода возмущения, предложенного в изданной ранее работе для обсуждения проблем интенсивных полей при помощи ковариантного метода.

KIVONAT

Relativisztikus szabad elektron külső sikhullámtérbeli viselkedését analizáltuk a megfelelő Dirac egyenlet különböző egzakt megoldásainak áttekintésével. Diszkutáljuk a Volkov állapotok teljességét és ortogonalitását. Az egzakt hullámfüggvényre alapozva egy előző cikkben javasolt perturbációs módszer általánositását dolgoztuk ki intenziv térbeli problémák kovariáns tárgyalására.

INTRODUCTION

In a previous paper we have presented a review of the solutions of the nonrelativistic free electron external field interaction problem /Bergou, 1979/. We have shown the equivalence of some at least superficially different solutions and proposed a perturbation method to treat scattering problems in the presence of an intense external field. In this method we used the complete set of the exact wave functions of the free electron in the field as a basis and treated the scattering potential as a perturbation. In the present paper we give a similar account of some existing solutions of the corresponding Dirac equation, prove their equivalence, orthogonality and completeness and, using this complete set of relativistic wave functions, we give a simple generalization of the above mentioned perturbation method and determine the validity of the dipole approximation as well as the validity of the nonrelativistic Born-approximation in the present problem.

The exact solution of the Dirac equation of a relativistic free electron in an electromagnetic plane wave field has long been known /Volkov, 1935/. This famous result has, since that time, been reproduced by several authors using different methods. It was shown, for example, that this problem can also be solved by purely algebraic methods /Beers and Nickle, 1972/. In another paper the so-called projection technique led to the same result /Becker and Mitter, 1974/. The Dirac equation, however, can also be solved without the direct use of the special assumptions and specific methods applied in these papers. By choosing an appropriate coordinate system the system of the coupled differential equations for the spinor components can be reduced to an ordinary first order differential equation for each component separately if one uses Majorana representation instead of the standard representation for the Dirac matrices. In this context it is interesting to mention another method /Alperin, 1944/. It is of course well known that in the "derivation" of the Dirac equation the originally irrational Hamiltonian /given by a square root expression/ is rationalized by the usual Dirac matrices. The basic idea of Alperin's paper was to exploit the symmetry of the problem by a suitable choice of coordinates, ensuring that both the rational and irrational parts show the required symmetry. Based on the wave function such obtained he determined the scattering cross section of an arbitrarily intense classical e.m. field by an electron using the method of the transition currents. The paper did not, however, draw much attention at that time, nor since.

The next section is devoted to the orthogonality and completeness problem of the Volkov states - this being the central problem in perturbation theoretical applications. In Section 3, the solution in Majorana representation and rederivation of the Alperin solution are given and their unitary equivalence with the Volkov solution is proved. In Section 4, it is shown how the multiphoton radiative corrections to the scattering of a free electron on a background potential due to the interaction with an intense mode of the e.m. field /laser/ can be obtained by using the Volkov states. In the last section we deal with the connection of the present approach with the method introduced by one of us in a previous paper. The limits of validity of the nonrelativistic dipole approximation as well as other consequences of the relativistic generalization, are also discussed.

2. THE VOLKOV STATES

In an external electromagnetic field characterized by the A/x/ four-vector potential the relativistic wave equation of a spinor electron has the form /cf.Bjorken and Drell, 1964; we shall use the metric and notation as well as representation of the Dirac matrices of this reference/:

$$(i\not{\partial}-\varepsilon\not{A}-\varkappa)\Psi = 0 \qquad /2.1/$$

where

$$\varepsilon = \frac{e}{\hbar c} \ , \quad \varkappa = \frac{mc}{\hbar} \qquad /2.1a/$$

/here c is the velocity of light, $\hbar$ is Planck's constant divided by 2π/. We choose A/x/ as representing a transverse plane wave, i.e.:

$$A(x) = A(\xi) \ , \ \xi = k\cdot x \ , \quad k\cdot A = k^2 = 0 \qquad /2.2/$$

In the case of a general elliptically polarized wave

$$\left.\begin{aligned} &A(\xi) = e_1 A_1(\xi) + e_2 A_2(\xi) \\ &k\cdot e_i = 0, \ e_i\cdot e_j = -\delta_{ij} \quad /i = 1, 2/ \end{aligned}\right\} \qquad /2.2a/$$

The well-known positive and negative frequency Volkov type solutions of the above Dirac equation represent modulated plane waves, where the modulation depends only on ξ. The plane wave itself can be parametrized by the 4-momentum lying on the free mass shell /initial conditions are not taken into account/:

$$\Psi = \Psi_p^{(\pm)}(x) \equiv \left(1 \pm \frac{\varepsilon\not{k}\not{A}(\xi)}{2k\cdot p}\right) u_p^{(\pm)} e^{\mp i[px + \int J_p^{(\pm)}(\xi)d\xi]} = $$
$$= E_p^{(\pm)}(x) u_p^{(\pm)} \qquad /2.3/$$

where

$$(\not{p}\mp\varkappa)u_p^{(\pm)} = 0 \ , \qquad p^2 = \varkappa^2 \ , \quad p^\mu = (|p_o|, \mathbf{p}) \qquad /2.3a/$$

and

$$J_p^{(\pm)}(\xi) = \frac{1}{2kp}[\pm 2\varepsilon p\cdot A(\xi) - \varepsilon^2 A^2(\xi)] \qquad /2.3b/$$

One can follow the method of obtaining the solution in this covariant form in a paper by Brown and Kibble /Brown and Kibble, 1964/.

These Volkov states were applied by several authors to treat the interaction of a free electron with an intense optical mode /accounted for by the external field approximation/. Interaction with another weak mode or some other weak potential can be taken into account by the usual perturbation theory. By this method absorption and emission from the intense mode can be directly computed up to any order contrary to the usual Feynman-Dyson approach.

For the $E_p^{(\pm)}(x)$ matrices introduced in /2.3/ one can easily verify the following relationships to hold /Ritus, 1972/:

$$\int d^4x\, \overline{E_p^{(\pm)}(x)} E_q^{(\pm)}(x) = (2\pi)^4 \delta^{(4)}(p-q) \ ,$$

$$\int \frac{d^4p}{(2\pi)^4} E_p^{(\pm)}(x) \overline{E_p^{(\pm)}(y)} = \delta^{(4)}(x-y)$$

where

$$\bar{E} = \gamma^0 E^+ \gamma^0$$

Orthogonality and completeness given in this form are not satisfactory for our purposes since the 4-momentum components are not on the free mass shell. Therefore in the following treatment we give different orthogonality and completeness relations. For further investigation of the Volkov states it is convenient to use the light-like components originally introduced by Neville and Rohrlich /Neville and Rohrlich, 1971a and b; see also Becker and Mitter, 1974/. This formalism is based on the fact that the vectors

$$n^\mu = \frac{c}{\omega\sqrt{2}} k^\mu = \frac{1}{\sqrt{2}}(1, \underline{n}),$$

$$\tilde{n}^\mu = \frac{1}{\sqrt{2}}(1, -\underline{n}), \quad e_i = (0, \underline{e}_i) \ i=1, 2 \qquad /2.4/$$

form a complete orthonormal set in Minkowski space, therefore any "a" four-vector can be given by its light-like components in the following way

$$a = a_u\, n + a_v\, \tilde{n} + a_1 e_1 + a_2 e_2 \qquad /2.4a/$$

where

$$a_u = \tilde{n}\cdot a\ ,\quad a_v = n\cdot a\ ,\quad a_i = -a\cdot e_i \quad i=1,\ 2 \qquad /2.4b/$$

Taking into accoung /2.4/ - /2.4b/ the solution /2.3/ can be brought to the form

$$\Psi_p^{(\pm)}(uvx_i) = [1\mp\frac{\varepsilon}{2p_v}\gamma_v\gamma_i a_i(u)]u_p^{(\pm)}\cdot$$
$$\cdot\, e^{-i[up_u+vp_v-x_ip_i+\int f_p^{(\pm)}(u)du]} \qquad /2.5/$$

where

$$u = x_v\ ,\quad v = x_u\ ,\quad A_i(\xi) = a_i(u) \quad i=1,\ 2 \qquad /2.5a/$$

and

$$f_p^{(\pm)}(u) = \frac{\varepsilon}{2p_v}[\mp 2p_i a_i(u) + \varepsilon a_i(u)a_i(u)] \qquad /2.5b/$$

The states /2.5/ /with normalization factor $\frac{1}{(2\pi)^{3/2}}(\frac{\varkappa}{p_v})^{1/2}$ form an orthogonal set in the sense

$$\left.\begin{aligned} &\int\overline{\Psi_{pr}^{(\pm)}(uvx_i)}\gamma_v\Psi_{p'r'}^{(\pm)}(uvx_i)dvd^2x_i = \\ &\qquad = \delta(p_v-p_v')\delta^{(2)}(p_i-p_i')\delta_{rr'} \\ &\int\overline{\Psi_{pr}^{(\pm)}(uvx_i)}\gamma_v\Psi_{p'r'}^{(\mp)}(uvx_i)dvd^2x_i = 0\ , \\ &\qquad \bar\Psi \equiv \Psi^+\gamma^o \end{aligned}\right\} \qquad /2.6/$$

Here r = 1, 2 are the spin indexes and $\delta^{(2)}$ denotes the two-dimensional Dirac-delta function. The normalization of the $u_p^{(\pm)}$ bi-spinors is as usual

$$\overline{u_p^{(\pm)}}u_p^{(\pm)} = \pm 1 \qquad /2.6a/$$

To obtain the appropriate definition of completeness we deal in the first step with the completeness of free plane waves. The solutions of the Dirac equation of a free particle are:

$$\varphi^{(\pm)}_{pr}(x) = \frac{1}{(2\pi)^{3/2}}\left(\frac{\varkappa}{p_o}\right)^{1/2} u^{(\pm)}_{pr} e^{\mp i\, p\cdot x} \qquad (2.7)$$

The definition and the normalization condition of the $u^{(\pm)}_{pr}$ bi-spinors are again given by Eqs./2.3a/ and /2.6a/. The completeness relation of the set of positive and negative frequency solutions is

$$\sum_{r=1,2}\int d^3p\left[\varphi^{(+)}_{pr}(x)\overline{\varphi^{(+)}_{pr}(x')}+\varphi^{(-)}_{pr}(x)\overline{\varphi^{(-)}_{pr}(x')}\right]_{x_o=x'_o}\gamma_o=\delta^{(3)}(\underline{x}-\underline{x}') \qquad (2.8)$$

Here we made use of the fact that

$$\sum_r u^{(\pm)}_{pr}\overline{u^{(\pm)}_{pr}} = \frac{\not p\pm\varkappa}{2\varkappa} \qquad (2.8a)$$

Relation /2.8/ can be generalized in a covariant manner such that instead of the x_o=const 3-space we define completeness on a space-like hyperplane determined by an arbitrary timelike normal vector. For the symmetry of the external plane wave field the best choice is the u=const null-plane. Therefore in full analogy with /2.8/ to establish completeness of the Volkov states on the null plane we investigate the expression

$$V(x,x')=\sum_r\int d^3\tilde{p}\left[\Psi^{(+)}_{pr}(x)\overline{\Psi^{(+)}_{pr}(x')}+\Psi^{(-)}_{pr}(x)\overline{\Psi^{(-)}_{pr}(x')}\right]\gamma_v \qquad (2.9)$$

or in particular

$$V(x,x')\big|_{u=u'} =\int_{-\infty}^{\infty} dp_v\iint d^2p_i\left(1+\frac{\varepsilon\not k\not A}{2k\cdot p}\right)\frac{\not p+\varkappa}{2p_v}\left(1-\frac{\varepsilon\not k\not A}{2kp}\right)\gamma_v e^{-i\tilde{p}(\tilde{x}-\tilde{x}')} \qquad (2.9a)$$

Here and below

$$\int d^3\tilde{p} \equiv \int_0^{\infty} dp_v\iint_{-\infty}^{\infty} d^2p_i\ , \qquad \tilde{x}\tilde{p} \equiv vp_v - x_ip_i \qquad (2.9b)$$

In obtaining /2.9a/ from /2.9/ we used the relation /2.8a/ and changed p_v to $-p_v$ in the negative frequency term. Before giving the completeness relationship of the Volkov states, we investigate the meaning of the operators defined by /2.9/ and /2.9a/.

Using the /2.6/ orthogonality relations it is easy to prove the validity of the following projection properties

$$\left.\begin{aligned} &\int d^3\tilde{x}' V(x,x')\Psi_{pr}^{(\pm)}(x') = \Psi_{pr}^{(\pm)}(x) \\ &\int d^3\tilde{x}' \overline{\Psi_{pr}^{(\pm)}(x')}\gamma^0 V^+(x',x)\gamma^0 = \overline{\Psi_{pr}^{(\pm)}(x)} \end{aligned}\right\} \qquad /2.10/$$

Introducing the bra and ket vector notation, the algebraic meaning of the relations /2.10/ becomes even more apparent in the abstract state vector space:

$$\left.\begin{aligned} &V(u,u')|\Psi_{pr}^{(\pm)}(u')\rangle = |\Psi_{pr}^{(\pm)}(u)\rangle \\ &\langle\Psi_{pr}^{(\pm)}(u')|\gamma^0\bar{V}(u',u) = \langle\Psi_{pr}^{(\pm)}(u)|\gamma^0 \end{aligned}\right\} \qquad /2.11/$$

where

$$\bar{V}(u',u) = \gamma^0 V^+(u',u)\gamma^0 \qquad /2.11a/$$

From /2.11/ it is clear that $V(u,u')$ and $\bar{V}(u',u)$ represent propagators of the Volkov states and the Dirac-adjoint Volkov states, respectively. It is also clear that $V(u,u)$ and $\bar{V}(u,u)$ are the projectors of the corresponding states. Other abstract algebraic properties of the Volkov states will be discussed in more detail in a subsequent paper.

Completeness of the Volkov states on a light-like hyperplane taking into account Eqs. /2.10/ and /2.11/ can now be expressed by the formula

$$\begin{aligned} [V(x,x')+\gamma^0 V^+(x',x)\gamma^0]_{u=u'} &= \delta^{(3)}(\tilde{x}-\tilde{x}') - \\ &- i\varkappa\frac{1}{2}\,\varepsilon(v-v')\,\delta^{(2)}(x_i-x_i')\gamma_v \end{aligned} \qquad /2.12/$$

where the definition of $\varepsilon(v)$ is given by the integral

$$\varepsilon(v) = \frac{2}{\pi}\int_0^\infty \frac{\sin vp_v}{p_v}\, dp_v = \begin{cases} 1 & v>0 \\ 0 & v=0 \\ -1 & v<0 \end{cases}$$

/see also Neville and Rohrlich, 1971a, App II./.
On the basis of /2.12/ we accept the assumption that any bispinor function can be represented as a /generalized/ linear combination in terms of Volkov states.

3. CONNECTION WITH OTHER SOLUTIONS

It is obvious from the preceding section that the problem of a free spinor electron interacting with an external plane wave field can be solved exactly in a covariant manner by using the light-like formalism. In this section we show two important examples where the Hamiltonian form of the corresponding Dirac equation, with appropriate coordinate systems, can also be solved exactly. Let us chose the y-axis of our coordinate system as coinciding with the direction of the wave vector of the light field given by the A/x/ vector potential, and polarization parallel to the x-axis. The Dirac equation of the problem in this coordinate system is

$$[\alpha_x(-i\frac{\partial}{\partial x} - \varepsilon A_x) + \alpha_y(-i\frac{\partial}{\partial y}) + \alpha_z(-i\frac{\partial}{\partial z}) + \beta\varkappa]\Psi = i\frac{\partial}{\partial x_o}\Psi \qquad /3.1/$$

where

$$A_x = A(\frac{\omega}{c}\xi), \quad \xi = x_o - y \qquad /3.1a/$$

A_x is an otherwise arbitrary function of ξ.

We look for the solution of /3.1/ again in the form of a plane wave modulated by the external field:

$$\Psi = \exp\{-i(x_o p_o - xp_x - yp_y - zp_z)\}\Phi(\xi) \qquad /3.2/$$

Here $\Phi(\xi)$ is a bispinor function for which, after substituting /3.2/ into /3.1/, we get the following ordinary differential equation

$$\{\alpha_x[p_x-\varepsilon A_x(\xi)] + \alpha_y p_y + \alpha_z p_z + \beta\varkappa - p_o\}\Phi =$$

$$= (1-\alpha_y) i \frac{d\Phi}{d\xi} \qquad /3.3/$$

In the representation of the Dirac matrices used throughout the present paper this is a system of coupled equations since α_y on the r.h.s. couples the derivatives of the different bispinor components of Φ. It is easy, however, to get rid of this difficulty in Majorana representation /Majorana, 1937/, where the /3.3/ system of equations is decoupled:

$$[\alpha_x(-p_x+\varepsilon A_x)+\beta p_y-\alpha_z p_z+\alpha_y\varkappa-p_o]\Phi' = (1-\beta) i \frac{d\Phi'}{d\xi} \qquad /3.4/$$

If we introduce φ' and χ' the upper and lower components of Φ' resp., /3.4/ gives a simple algebraic equation between φ' and χ':

$$\varphi' = \frac{1}{p_o-p_y}[(-p_x+\varepsilon A_x)\sigma_x-p_z\sigma_z+\varkappa\sigma_y]\chi' \qquad /3.4a/$$

where σ-s are 2x2 Pauli matrices.
If /3.4a/ is substituted into the lower component equation of /3.4/ we get

$$2i\frac{d\chi'}{d\xi} = \{\frac{1}{p_\eta}[(p_x-\varepsilon A_x)^2+p_z^2+\varkappa^2]-p_\xi\}\chi' \qquad /3.4b/$$

where

$$p_\eta = p_o-p_y \quad , \quad p_\xi = p_o+p_y \qquad /3.4c/$$

Without loss of generality we can assume that the parameter p satisfies the usual free mass-shell relationship

$$p_o^2 = p_x^2+p_y^2+p_z^2+\varkappa^2 \quad , \quad p_\eta p_\xi = p_x^2+p_z^2+\varkappa^2 \qquad /3.4d/$$

Then from /3.4b/

$$\chi' = \exp[i\int I_p(\xi)d\xi]\chi'_o \quad ,$$

$$I_p(\xi) = \frac{1}{2p_\eta}[2\varepsilon p_x A_x(\xi) - \varepsilon^2 A_x^2(\xi)] = -J_p^{(+)}(\xi) \qquad /3.4e/$$

Here χ'_o is a constant spinor. The final form of the solution of /3.4/ using /3.4a/ then becomes

$$\Psi' = \begin{bmatrix} \frac{1}{p_\eta}(-p_x\sigma_x - p_z\sigma_z + \varkappa\sigma_y)\chi'_o + \frac{1}{p_\eta}\varepsilon A_x\sigma_x\chi'_o \\ \chi'_o \end{bmatrix} e^{-i[p\cdot x + \int J_p^{(+)}(\xi)d\xi]} \qquad /3.5/$$

The exponent of /3.5/ agrees with that of the Volkov states, while the proof of the equivalence of the bispinor amplitudes will be given in what follows. In the special coordinate system introduced in the above calculation the Volkov-bispinor reads

$$u_p^{(v)} = (1+\frac{\varepsilon k\!\!\!/ A\!\!\!/}{2k\cdot p})u_p = \left[1-(1+\alpha_y)\alpha_x\frac{\varepsilon A_x}{2p_\eta}\right]u_p \qquad /3.6/$$

In Majorana representation this becomes

$$u_p^{(v)\prime} = \left[1+(1+\beta)\alpha_x\frac{\varepsilon A_x}{2p_\eta}\right]u'_p \qquad /3.7/$$

where now u'_p satisfies the transformed free energy eigenvalue--equation

$$p_o u'_p = [\underline{\alpha}'\underline{p}+\beta'\varkappa]u'_p = [-\alpha_x p_x + \beta p_y - \alpha_z p_z + \alpha_y\varkappa]u'_p \qquad /3.7a/$$

Equation /3.7/ written out explicitly reads now

$$\left[1+\begin{pmatrix} 0 & \sigma_x \\ 0 & 0 \end{pmatrix}\frac{\varepsilon A_x}{p_\eta}\right]\begin{pmatrix} \varphi'_o \\ \chi'_o \end{pmatrix} = \begin{pmatrix} \varphi'_o + \frac{\varepsilon A_x}{p_\eta}\sigma_x\chi'_o \\ \chi'_o \end{pmatrix} \qquad /3.8/$$

On the other hand from /3.7a/

$$\varphi'_o = \frac{1}{p_\eta}(-p_x\sigma_x - p_z\sigma_z + \varkappa\sigma_y)\chi'_o \qquad /3.8a/$$

Substituting /3.8a/ into /3.8/ and comparing the result with /3.5/ one can see that the solution given here is equivalent with the Volkov solution.

Another interesting solution of the present problem was given by Alperin /Alperin, 1944/. In the following we shall repeat with some modification the original derivation of Alperin's solution. We start from the relativistic energy-momentum formula

$$\frac{E}{\hbar c} = \sqrt{\left(\frac{\underline{p}}{\hbar} - \varepsilon\underline{A}\right)^2 + \varkappa^2} \qquad /3.9/$$

or in operator form

$$\hat{p}_o = \sqrt{(\hat{\underline{p}}-\varepsilon\underline{A})^2+\varkappa^2}\ , \quad \hat{p}_o = i\frac{\partial}{\partial x_o}\ , \quad \hat{\underline{p}} = -i\frac{\partial}{\partial \underline{r}} \qquad /3.9a/$$

In the special coordinate system used throughout in the preceding calculation it is more convenient to take the square root in a different way, namely

$$\hat{p}_x-\varepsilon A_x = i\sqrt{-\hat{p}_\xi\hat{p}_\eta+\hat{p}_z^2+\varkappa^2} \qquad /3.10/$$

where

$$\hat{p}_\xi = i\left(\frac{\partial}{\partial x_o} - \frac{\partial}{\partial y}\right) = 2i\frac{\partial}{\partial \xi}\ , \quad \hat{p}_\eta = i\left(\frac{\partial}{\partial x_o} + \frac{\partial}{\partial y}\right) = 2i\frac{\partial}{\partial \eta}\ ,$$

$$\xi = x_o-y\ , \qquad \eta = x_o+y \qquad /3.10a/$$

The matrices $\alpha_\xi = \frac{i}{2}(\alpha_y+i\alpha_x)$ and $\alpha_\eta = \frac{i}{2}(\alpha_y-i\alpha_x)$ satisfy the commutation relation

$$\alpha_\xi\alpha_\eta + \alpha_\eta\alpha_\xi = -1 \qquad /3.11/$$

By using these matrices the Dirac equation corresponding to /3.10/ has the following rational form:

$$(\hat{p}_x-\varepsilon A_x)\Psi = i(\alpha_\xi\hat{p}_\xi+\alpha_\eta\hat{p}_\eta+\alpha_z\hat{p}_z+\beta\varkappa)\Psi \qquad /3.12/$$

The solution can be looked for again with the usual ansatz

$$\Psi = e^{i(xp_x+zp_z-\frac{1}{2}\xi p_\xi-\frac{1}{2}\eta p_\eta)}\Phi(\xi) ,$$

$$p_\xi = p_o+p_y, \qquad p_\eta = p_o-p_y \qquad /3.13/$$

After substituting /3.13/ into /3.12/ we obtain a coupled system of equations for the components of Φ:

$$-i[p_x-\varepsilon A_x(\xi)]\Phi_1 = -p_\eta\Phi_4+p_z\Phi_3+\varkappa\Phi_1 \qquad /3.14a/$$

$$-i[p_x-\varepsilon A_x(\xi)]\Phi_2 = p_\xi\Phi_3+2i\frac{d\Phi_3}{d\xi}-p_z\Phi_4+\varkappa\Phi_2 \qquad /3.14b/$$

$$-i[p_x-\varepsilon A_x(\xi)]\Phi_3 = -p_\eta\Phi_2+p_z\Phi_1-\varkappa\Phi_3 \qquad /3.14c/$$

$$-i[p_x-\varepsilon A_x(\xi)]\Phi_4 = p_\xi\Phi_1+2i\frac{d\Phi_1}{d\xi}-p_z\Phi_2-\varkappa\Phi_4 \qquad /3.14d/$$

From /3.14a/ and /3.14c/ Φ_2 and Φ_4 can be expressed by Φ_1 and Φ_3:

$$\Phi_2 = \frac{1}{p_\eta}\{i[(p_x-\varepsilon A_x)+i\varkappa]\Phi_3+p_z\Phi_1\} \qquad /3.15a/$$

$$\Phi_4 = \frac{1}{p_\eta}\{i[(p_x-\varepsilon A_x)-i\varkappa]\Phi_1+p_z\Phi_3\} \qquad /3.15b/$$

and substituting these expressions into /3.14b/ and /3.14d/ we obtain two similar uncoupled equations for Φ_1 and Φ_3:

$$2i\frac{d\Phi_{1,3}}{d\xi} = \{\frac{1}{p_\eta}[(p_x-\varepsilon A_x)^2+p_z^2+\varkappa^2]-p_\xi\}\Phi_{1,3} \qquad /3.16/$$

The solution of /3.16/ taking into account /3.4d/ will be

$$\Phi_{1,3} = \Phi_{1,3}(0)e^{-i\int J_p^{(+)}(\xi)d\xi} , \quad \Phi_{1,3}(0) = \text{const} \qquad /3.16a/$$

Through /3.15a/ - /3.15b/, all four components of Φ are known, thus another solution of the Dirac equation is found. The function in the exponent of this wave function coincides with exponents of the Volkov states and the state found in Majorana representation. All we have to show is the equivalence of the bispinor part with the previous solutions. From /3.4a/

$$-i\varphi_1 = \frac{1}{p_\eta}\{[i(p_x-\varepsilon A_x)-\varkappa]\chi_2+p_z i\chi_1\} \qquad /3.17a/$$

$$\varphi_2 = \frac{1}{p}\{[i(p_x-\varepsilon A_x)+\varkappa]i\chi_1+p_z\chi_2\} \qquad /3.17b/$$

/primes are omitted for the sake of simpler notation/. Comparing the above relations with /3.15a/ and /3.15b/ we can immediately see that the same relation holds between $\begin{pmatrix}-i\varphi_1\\ \varphi_2\end{pmatrix}$ and $\begin{pmatrix}i\chi_1\\ \chi_2\end{pmatrix}$ as between $\begin{pmatrix}\Phi_2\\ \Phi_4\end{pmatrix}$ and $\begin{pmatrix}\Phi_1\\ \Phi_3\end{pmatrix}$, therefore if we make the identifications $i\chi_1 \to \Phi_1$ and $\chi_2 \to \Phi_3$ the corresponding $-i\varphi_1 \to \Phi_2$ and $\varphi_2 \to \Phi_4$ id identification must also hold. From this consideration the connection between $\Phi = \binom{\phi}{\chi}$ and $\binom{\varphi}{\chi}$ can be written in the following compact form

$$\binom{\phi}{\chi} = \begin{pmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\\ \Phi_4\end{pmatrix} = \begin{pmatrix}0 & 0 & i & 0\\ -i & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\\ \chi_1\\ \chi_2\end{pmatrix} \equiv T\binom{\varphi}{\chi} \qquad /3.18/$$

As the T matrix defined here is unitary the Alperin type solutions are equivalent with the solutions obtained in the Majorana representation and due to the transitivity property of the unitary transformations the three formally different wave functions considered so far are interrelated by unitary transformations and they are therefore equivalent from the physical point of view.

4. AN APPLICATION OF THE VOLKOV STATES

In a previous paper of one of the present authors /J.B./, the wave functions of a nonrelativistic free electron moving in a homogeneous external field /dipole approximation/ were used as the basis set of a perturbation method to calculate the cross section of the inverse as well as induced multiphoton bremsstrahlung process /Bergou, 1979/. In this section we work out an obvious generalization of the method for the relativistic case and beyond dipole approximation by using the Volkov states as the basis set. Similar problems were touched on earlier /Denisov and Fedorov, 1967; and Brehme, 1971/ where the analytical and the numerical behaviour of the relativistic cross section formulae of the scattering by a Coulomb background were investigated by different methods and in a recent paper /Ehlotzky, 1978/ results beyond dipole approximation but using nonrelativistic description were published.

Consider the problem of the scattering of a relativistic free electron by a V/r/ scalar background potential in the presence of an intense electromagnetic mode /laser light/. The intense mode can be accounted for by the external field approximation and the corresponding Dirac equation reads /using light-like formalism/

$$\{\gamma_v i\partial_v + \gamma_u i\partial_u - \gamma_i[i\partial_i - \varepsilon a_i(u)] - \varepsilon \not{V}(v-u,\ x_i) - \varkappa\}\Psi = 0 \qquad /4.1/$$

$$\partial_u = \frac{\partial}{\partial u}\ ,\quad \partial_v = \frac{\partial}{\partial v}\ ,\quad \partial_i = \frac{\partial}{\partial x^i}$$

We look for the solution in the form

$$\Psi = \Psi^{(+)}_{qr} + \Psi_k \qquad /4.2/$$

where q,r are determined by the parameters of the initial state and the correction term Ψ_k is a superposition of the /2.5/ Volkov states taking into account the /2.12/ completeness relation:

$$\Psi_k(uvx_i) = \sum_r \int_0^\infty dp_v \iint_\infty^\infty d^2p_i\,[c^{(+)}_{p_v p_i r}(u)\Psi^{(+)}_{pr}(uvx_i) + c^{(-)}_{p_v p_i r}(u)\Psi^{(-)}_{pr}(uvx_i)] \qquad /4.3/$$

Here $c^{/+/}$ and $c^{/-/}$ are scalar amplitudes to be determined. For the sake of simplicity we choose the initial conditions

$$c^{(\pm)}_{pr}(u \le u_i) = 0 \qquad \text{for all p and r} \qquad /4.3a/$$

Upon substitution of /4.3/ into /4.1/ we obtain the equation /in more compact notation/

$$\oint \gamma_v i \left[\frac{dc^{(+)}}{du}\Psi^{(+)} + \frac{dc^{(-)}}{du}\Psi^{(-)}\right] = \varepsilon \not{V} \Psi^{(+)}_{qr} + \varepsilon \not{V} \Psi_k \qquad /4.4/$$

In first approximation we neglect the second term on the r.h.s. which contains the product of the correction term and the perturbing potential thus giving higher order corrections only. Then we take the scalar product of the remaining terms with $\overline{\Psi^{(+)}_{q'r'}}$ from the left and obtain the following ordinary differential equation for $c^{/+/}_{pr}$ $/u,u_i/$

$$i\frac{dc^{(+)}_{q'r'}}{du} = \int dvd^2x_i \overline{\Psi^{(+)}_{q'r'}}(uvx_i)\varepsilon \not{V} \Psi^{(+)}_{qr}(uvx_i) \qquad /4.5/$$

Here we directly made use of the /2.6/ orthogonality relations and that for arbitrary spin orientation $\bar{u}_p\gamma_v u_p = \frac{p_v}{\varkappa}\bar{u}_p u_p$ and $\overline{u_p}\gamma_v v_p = 0$. Equation /4.5/ can be integrated in a simple way leading to

$$c^{(+)}_{q'r'}(u,u_i) = -i\varepsilon\int_{u_i}^{u} du' \int dvd^2x_i \overline{\Psi^{(+)}_{q'r'}} \not{V} \Psi^{(+)}_{qr} \qquad /4.5a/$$

The transition matrix element of the $qr \to q'r'$ process is connected to the $c^{(+)}_{q'r'}(u,u')$ amplitude in the following way

$$T_{fi} = c^{(+)}_{q'r'}(u \to \infty,\ u_i \to -\infty) \qquad /4.6/$$

or

$$T_{fi} = -i\varepsilon\int d^4x \overline{\Psi^{(+)}_{q'r'}} \not{V} \Psi^{(+)}_{qr} \qquad /4.6a/$$

We perform the calculation for a circularly polarized wave

$$A_1(\xi) = a\cos\xi\ , \quad A_2(\xi) = a\sin\xi \qquad /4.7/$$

Then from /4.6/

$$T_{fi} = -i\varepsilon\int d^4x\, \overline{u}_{q'r'}\left(1-\frac{\varepsilon \not{k}\not{A}}{2q'\cdot k}\right)\gamma^o\left(1+\frac{\varepsilon\not{k}\not{A}}{2q\cdot k}\right)u_{qr}\cdot$$
$$\cdot\, V(\underline{x})e^{i[(\tilde{q}'-\tilde{q})\cdot x + z\sin(kx-\chi)]}\quad , \qquad /4.8/$$

$$z=\sqrt{\alpha_1^2+\alpha_2^2}\ ,\quad \alpha_{1,2} = \varepsilon a\left(\frac{q'\cdot e_{1,2}}{2q'\cdot k} - \frac{qe_{1,2}}{2q\cdot k}\right)\ ,$$
$$\sin\chi = \frac{\alpha_2}{z}\ ,\qquad \tilde{q} = q+\frac{\varepsilon^2 a^2}{2k\cdot q}k \qquad /4.8a/$$

To evaluate the integration in /4.8/ we use the well-known Fourier expansion

$$e^{iz\sin\varphi} = \sum_{n=-\infty}^{\infty} J_n(z)e^{in\varphi}$$

J_n denotes a Bessel function of integer order. Thus the transition matrix element represents an infinite sum of photon absorbed and photon emitted terms

$$T_{fi} = \sum_{n=-\infty}^{\infty} T_{fi}^{(n)}\ ,\quad T_{fi}^{(n)} = -2\pi i\ t_{fi}^{(n)}\delta(\tilde{q}_o'-\tilde{q}_o+nk) \qquad /4.9/$$

$$t_{fi}^{(n)} = \varepsilon V(\underline{Q}_n)(\bar{u}_{q'r}M_n u_{qr})\ ,\quad \underline{Q}_n = \tilde{\underline{q}}'-\tilde{\underline{q}}+n\underline{k}\ ,$$
$$V(\underline{Q}_n) = \int d^3x\, V(\underline{x})e^{-i\underline{Q}_n\underline{x}} \qquad /4.9a/$$

Relation /4.9/ expresses, in an explicit way, energy conservation, the wavy line stands for the fact that electron energies in the presence of the external field are different from those of a bare electron /see e.g. the last of the /4.8a/ relations/. In /4.9a/

$$M_n = \left(\gamma_o+\frac{\varepsilon^2a^2}{4q_vq'_v}\gamma_v\right)c_n + \left(\frac{\varepsilon a}{2q_v}\gamma_o\gamma_v + \right.$$
$$\left. + \frac{\varepsilon a}{2q'_v}\gamma_v\gamma_o\right)\left[\not{e}^{(+)}c_{n-1} + \not{e}^{(-)}c_{n+1}\right] \qquad /4.10/$$

where

$$c_n = J_n(z)e^{-in\chi} \quad , \quad e^{(+)} = \frac{1}{2}(e_1 - ie_2) = e^{(-)*} \qquad /4.10a/$$

After averaging over initial and summing up for final spin variables one obtains finally for the scattering cross section

$$\frac{d\sigma^{(n)}}{d\Omega} = \frac{|\underline{q}'|}{|\underline{q}|} J_n^2(z) \frac{d\sigma_B^{(n)}}{d\Omega} \frac{1}{2}\left(1 + \frac{q_o q_o' + \underline{q}\,\underline{q}'}{\varkappa^2}\right)\left(\frac{q_o'}{\varkappa}\right) + \qquad /4.11/$$

$$+ \frac{|\underline{q}'|}{|\underline{q}|} \frac{d\sigma_B^{(n)}}{d\Omega}\left(\frac{q_o'}{\varkappa}\right)(\alpha_n \nu + \beta_n \nu^2 + \gamma_n \nu^3 + \delta_n \nu^4)$$

where

$$\frac{d\sigma_B^{(n)}}{d\Omega} = \left| \frac{\varkappa}{2\pi} \varepsilon V(\underline{Q}_n) \right|^2 \quad , \quad \nu = \frac{\varepsilon a}{\varkappa}\left[\nu^2 \approx 8 \cdot 10^{-11} \lambda^2 I\right] \qquad /4.11a/$$

The parameters α_n, β_n, γ_n and δ_n depend upon the four-momentum of the electron as well as on the frequency and polarization of the light field but they are almost independent of light intensity. Their analytical expression is rather complicated and does not give a better insight into the physical process involved therefore we omit them here. In /4.11/ the first term is just the generalization of the result obtained by the nonrelativistic dipole approximation, while the second term comes from the interaction of the spin momentum with the e.m. field and is exact in the sense that in all orders it is given by a fourth order polynomial of the intensity parameter ν, only the coefficients being slowly dependent on the order of the process and intensity.

5. DISCUSSION AND SUMMARY

As is well known, an intense mode of the electromagnetic field can be represented by a c-number plane wave field. The central problem of the semiclassical theory is, therefore, the solution of the wave equations of charged particles in such a surrounding. As an extension of previous work /Bergou, 1979/ on exact wave functions, in the present paper we have given a detailed study of the Volkov states from some special aspects. In Section 2 using the light-like formalism we have shown that the Volkov states paremetrized by the four-momentum on the free mass-shell form a complete orthonormal set on the $k \cdot x$ = const null-plane. The orthogonality and the completeness of this kind are consequences of the special symmetry of the external plane wave field, i.e. of the dependence of the vector potential on the quantity $k \cdot x$ only. As several authors have made direct use of these solutions in perturbation theoretical calculations of different kinds, it seemed to us to be important to prove the completeness of this system and to examine in what sense they can be applied as a basis set.

In the next section we gave two simple methods for the solution of the Dirac equation under consideration, each of the methods was based on the fact that with appropriate choice of the coordinate system the coupled system of equations for the bispinor components can be decoupled into ordinary differential equations for each component separately in a suitable representation for the Dirac matrices. This was first performed in the Majorana representation and another solution was found by a suitable rationalization of the relativistic energy-momentum formula. We note here that neither of these two methods of solution required the solution of a second order equation as was done in the original derivation by Volkov. We have shown that the bispinor amplitudes of the solutions found in this way are related to the Volkov amplitudes through unitary transformations /the agreement of phases is obvious/ and consequently they are equivalent with each other from the physical point of view.

In the last section the use of the Volkov states was demonstrated in the derivation of the nonlinear inverse and induced bremsstrahlung scattering cross section. The expression obtained can be considered as a relativistic generalization of the results obtained in the nonrelativistic dipole approximation. Scattering is elastic with respect to the background potential and inelastic with respect to the external field. This last property is expressed by the Bessel functions, while corrections to this result were found from two different origins. The first is what one would expect when nonrelativistic dipole approximation is dropped /relativistic non-dipole part/ and the second comes from the relativistic interaction of a spin momentum with an external field. It is interesting to note at this point that this second correction is given by similar finite /fourth order/ polynominal of the intensity parameter in all orders, the coefficients of the polynomial being only slowly dependent on the order and intensity. From here we may conclude that in a sufficiently intense external field, relativistic effects may become important.

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