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WAVE FUNCTIONS OF A FREE ELECTRON IN AN EXTERNAL FIELD AND THEIR APPLICATION IN INTENSE FIELD INTERACTIONS, I NONRELATIVISTIC TREATMENT

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ABSTRACT

The behaviour of a free electron in a homogeneous but time varying external field is analysed and exact results are presented. Based on the exact wave function obtained a new perturbation method for treating intense field problems is proposed.

АННОТАЦИЯ

Анализировано поведение свободного электрона во внешнем однородном, но зависящем от времени поле, и получены экзактные результаты. На основе полученной экзактной волновой функции предложен новый метод возмущения для обсуждения проблем интенсивных полей.

KIVONAT

Szabad elektron viselkedését analizáltuk homogén időfüggő külső térben, egzakt megoldások segitségével. Az igy nyert egzakt hullámfüggvényre alapozva, intenziv térbeli problémák tárgyalására uj perturbációs módszert javasoltunk.

A detailed analysis is given here of the behaviour of a free electron in a homogeneous external field. We consider separately the case of the constant and the periodically time-dependent fields. For the description of the electron the Schrödinger equation is used /nonrelativistic treatment/ together with the dipole /long wavelength/ approximation of the field. The general solution of the problem is given and it is shown how it can be matched to different initial conditions. By choosing special initial conditions the stationary solution in a constant field /Landau-Lifshitz, 1963/ and the plane wave solution in a periodic field /Keldysh, 1965/ are reobtained. By using this last set of solutions we develop a perturbation method for treating intense field problems. The relationship between this method and other approximation methods /Henneberger, 1968; Faisal, 1973/ is also established and we give the expression for multiphoton transition matrix elements as well. The transition matrix element has the form predicted previously /Bergou, 1975/ for the case of a periodic Hamiltonian. The Schrödinger equation for an electron in a homogeneous constant electric field with amplitude $\underline{E}_o$ is

$$\left[\frac{\hat{\underline{p}}^2}{2m} - e\underline{E}_o\hat{\underline{x}}\right]\Psi = i\hbar\frac{\partial\Psi}{\partial t} \qquad /1/$$

Operators are denoted by ^. m is the mass of the electron, e its charge, $\hbar$ Planck's constant divided by 2π. Vector quantities are denoted by underlining.

As the Hamiltonian does not depend on time we can look for the stationary solution in the form $\Psi(\underline{x},t) = e^{-\frac{i}{\hbar}Et}\,u(\underline{x})$. Instead of solving the corresponding equation for $u(\underline{x})$ we perform the following gauge-transformation

$$\phi' = \phi - \frac{1}{c}\frac{\partial\chi}{\partial t}$$

$$\underline{A}' = \text{grad}\ \chi$$

$$\chi = -c\underline{E}_o\underline{x}t$$

Here ϕ and $\underline{A}$ are the scalar and vector potentials, respectively, and χ is chosen so that in the new gauge $\phi'=0$. If furthermore, in the new gauge $\Psi'=e^{\frac{i}{\hbar}\frac{e}{c}\chi}\Psi$ /Schiff, 1955/, then Ψ' satisfies the following wave equation:

$$\frac{1}{2m}\left[\underline{\hat{p}} - \frac{e}{c}\underline{A}'\right]^2 \Psi' = i\hbar\frac{\partial\Psi'}{\partial t} \qquad /1a/$$

As the transformation between Ψ and Ψ' is unitary, /1/ and /1a/ give completely equivalent descriptions of the same problem. Nevertheless, /1a/ is more convenient for practical calculation because in momentum representation the Hamiltonian becomes diagonal and the equation is readily integrable. Its general solution is

$$\Psi'(\underline{p},t) = f(\underline{p})\, e^{-\frac{i}{\hbar}\frac{1}{2m}\int_0^t (p + e\underline{E}_o\tau)^2 d\tau} \qquad /2/$$

Here $f(\underline{p})$ is an arbitrary function of the momentum, to be determined from the initial, boundary, or any subsidiary conditions. It is interesting to note at this point that in contrast to /1/, Eq. /1a/ has no stationary solution because its Hamiltonian explicitly depends on time. The stationary solution to /1/ corresponds to a special choice of $f(\underline{p})$ in /2/, namely / in one dimension/:

$$f_E(p) = \frac{1}{\sqrt{2\Pi\hbar eE_o}}\ e^{-\frac{i}{\hbar}\left(\frac{p^3}{6meE_o} - \frac{Ep}{eE_o}\right)} \qquad /3/$$

Here E is the energy of the stationary state /the separation constant of Eq. /1// and with this choice of the normalization constant the states are properly orthonormalized on energy scale /Landau-Lifshitz, 1963/. Thus, we can conclude that even the solutions of Eq. /1/ are generally non-stationary and only the very special choice $f(p)=f_E(p)$ leads to stationary states. These form, however, a complete orthonormal set and therefore any other state can be given as a linear combanation of them with coefficients which depend only on time.

In the following we shall consider the interaction of a free electron with a periodic external field. The corresponding Schrödinger equation with scalar potentional reads

$$\left[\frac{\hat{p}^2}{2m} - e\underline{E}_o\hat{\underline{x}}\cos\omega t\right]\Psi = i\hbar\frac{\partial\Psi}{\partial t}\ ;\ \underline{A} = 0 \qquad /4/$$

with vector potential

$$\frac{1}{2m}\left[\hat{\underline{p}} + \frac{e\underline{E}_o}{\omega}\sin\omega t\right]^2\Psi' = i\hbar\frac{\partial\Psi'}{\partial t}\ ;$$

$$\underline{A}' = -\frac{c}{\omega}\underline{E}_o\sin\omega t\ ;\ \Phi' = 0 \qquad /4a/$$

The above described gauge transformation is effected in this case by $\chi=-\frac{c}{\omega}\underline{E}_o x\sin\omega t$. Considering /4a/ in momentum representation, it is readily integrable again and its general solution is

$$\Psi'(\underline{p},t) = g(\underline{p})e^{-\frac{i}{\hbar}\frac{1}{2m}\int_o^t(\underline{p} + \frac{e\underline{E}_o}{\omega}\sin\omega\tau)^2 d\tau} \qquad /5/$$

In the limit $\omega\to 0$ this solution coincides with /2/, g(p) is again an arbitrary function of the momentum. The solution of /4/ is given by $\Psi(p,t)=\Psi'(p-\frac{e\underline{E}_o}{\omega}\sin\omega t)$ and the corresponding solution in coordinate representation

$$\Psi'(\underline{x},t) = \frac{1}{\sqrt{2\Pi\hbar}}\int d^3p' g(p') e^{\frac{i}{\hbar}\left[p'\underline{x} - \frac{1}{2m}\int_0^t (p' + \frac{eE_o}{\omega}\sin\omega\tau)^2 d\tau\right]} \qquad /5a/$$

and for $\Psi(\underline{x},t)$ we have $\Psi(\underline{x},t)=e^{\frac{i}{\hbar}\frac{eE_o}{\omega}\underline{x}\sin\omega t}\;\Psi'(\underline{x},t)$.

The usual plane wave solution given by Keldysh /1965/ can be obtained by substituting $g(p')=\delta(p-p')$. The meaning of this solution becomes clearer if we consider the time evolution operator $\hat{U}(t)$ of Eq. /4a/. From the definition of the evolution operator we have

$$\hat{U}(t) = e^{-\frac{i}{\hbar}\frac{1}{2m}\int_0^t (\hat{p}+\frac{e}{\omega}\underline{E}_o\sin\omega\tau)^2 d\tau}$$

$$\hat{U}(0) = \hat{1} \qquad /6/$$

When applying it to a momentum state $|p\rangle$ /with the bra and ket vector notation/, one obtains the time development of the state. As $\hat{U}(t)$ is diagonal in p representation, its only effect on $|p\rangle$ is to multiply it by a complex c number which is a function of p and t. Furthermore the modulus of this number is unity, which is a consequence of the unitary nature of $\hat{U}(t)$:

$$\hat{U}(t)|p\rangle = U(p,t)|p\rangle$$

$$|U(p,t)|^2 = 1 \qquad /7/$$

Here $U(p,t)$ is the matrix element of the time evolution operator and from /6/ it can be seen that it has the form predicted by the Floquet theorem for the solution of differential equations with periodic coefficients /Shirley, 1965/. From /7/ we see that a given momentum state remains always the same, only its phase will change in time. The same result can be expressed using a somewhat different language. The equation of motion for the density matrix which corresponds to /1a/ or /4a/ reads in momentum representation:

$$i\hbar \frac{\partial \hat{\rho}(p_1,p_2,t)}{\partial t} = H(p_1)\rho(p_1,p_2,t) - H(p_2)\rho(p_1,p_2,t) = \left[\frac{1}{2m}(p_1^2-p_2^2) - \frac{e}{mc}\,\underline{A}(t)(p_1-p_2)\right]\rho(p_1,p_2,t) \qquad /8/$$

The solution of this equation is again easily obtained by direct integration:

$$\rho(p_1,p_2,t) = \rho(p_1,p_2,0)\otimes \otimes \exp\left\{-\frac{i}{\hbar}\int_0^t\left[\frac{1}{2m}(p_1^2-p_2^2) - \frac{e}{mc}\,\underline{A}(\tau)(p_1-p_2)\right]d\tau\right\} \qquad /9/$$

By setting $p_1=p_2=p$ we see that the initial momentum distribution function /given by the diagonal elements of ρ at $t=0$ /remains unchanged:

$$\rho(p,p,t)=\rho(p,p,0) \qquad /10/$$

From /10/ we may conclude, in agreement with /7/, that the momentum distribution of a free electron in an external field remains unchanged in dipole approximation. Deviation from this result is expected only if the dipole approximation is dropped.

Consider now the problem of the interaction of an electron with an external field in the presence of a background potential. The Schrödinger equation is

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[\frac{1}{2m}(\hat{p} + \frac{eE_o}{\omega}\sin\omega t)^2 + V(\underline{r})\right]\Psi \qquad /11/$$

Depending on the nature of the problem, different approximation methods for the solution of Eq. /11/ are worked out. If the field is of low intensity the usual perturbation theory applies /Gontier and Trahin, 1971/. If, however, the external field is of the same /or higher/ order of magnitude than the static field given by $V(\underline{r})$, then other methods would be necessary. The other limiting case is when the external field is so strong that the background potential $V(\underline{r})$ can be treated as a perturbation. It seems to be quite natural, at least in scattering problems which frequently occur in highly ionized plasma, to build up a perturbation series in powers of $V(\underline{r})$, where the complete set of the plane wave solutions of /4a/ is used as a basis. Let us denote $\Psi'(\underline{x},t)$ of /5a/ in the case of $g(p')=\delta(p-p')$ by Ψ_{p}, and look for the solution of Eq. /11/ in the following form

$$\Psi_i(\underline{x},t) = \Psi_{p_i} + \Psi_1; \quad \Psi_1 = \int a_p(t)\Psi_p d^3p \qquad /12/$$

with the initial condition $\Psi_i(t=o)=\Psi_{p_i}$. By substituting into /11/ and neglecting terms higher than first order in $V(\underline{r})$, one obtains the following differential equation for $a_p(t)$

$$i\hbar \frac{da_p(t)}{dt} = V(p-p_i)\otimes$$

$$\otimes\exp\{\frac{i}{\hbar}\frac{1}{2m}\int_o^t\Big[(p+\frac{eE_o}{\omega}\sin\omega\tau)^2-(p_i+\frac{eE_o}{\omega}\sin\omega\tau)^2\Big]d\tau \qquad /13/$$

The solution of this equation is simple, and from /12/ we have an explicit expression for $\Psi_1(\underline{x},t)$ in the first approximation. From this latter expression the transition matrix element for the scattering process has the form

$$T_{fi} = (\Psi_{p_f}, \Psi_1) = -\frac{i}{\hbar} V(p_f-p_i)\otimes$$

$$\otimes\int_o^t \exp\{\frac{i}{\hbar}\frac{1}{2m}\int_o^{t'}\Big[(p_f+\frac{eE_o}{\omega}\sin\omega\tau)^2-(p_i+\frac{eE_o}{\omega}\sin\omega\tau)^2\Big]d\tau\}dt'$$

/14/

where Ψ_{p_f} is the final-state plane wave function, satisfying the free particle equation /4a/. The above expression for the transition amplitude is in agreement with the more general one in the case of a Hamiltonian which is periodic in time /Reiss, 1970; Bergou, 1975/.

The term $-\frac{i}{\hbar} V(\underline{p}_f-\underline{p}_i)$ is the usual scattering amplitude in Born approximation. Using the definition of the $J_n(z)$ Bessel functions the periodically time dependent part of the exponential term in the integrand can be expanded into power series of the absorbed and emitted photons:

$$e^{\frac{i}{\hbar} z\cos\omega t} = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{\frac{i}{\hbar} n\hbar\omega t} \quad ; \quad z = \frac{e\underline{E}_o}{m\hbar\omega^2}\underline{Q}$$

where

$$\underline{Q} = \underline{p}_i - \underline{p}_f \quad \text{and} \quad \frac{p_f^2}{2m} = \frac{p_i^2}{2m} - n\hbar\omega$$

p_i and p_f are the initial and final momenta, respectively, and so $\underline{Q}$ accounts for the momentum change in the scattering. If this expansion is introduced into /14/ the time integration can easily be carried out and one obtains the following final result for the scattering cross section

$$\frac{d\sigma}{d\Omega} = \sum_n \frac{d\sigma^{(n)}}{d\Omega} \quad ;$$

$$\frac{d\sigma^{(n)}}{d\Omega} = \frac{p_f}{p_i} J_n^2(z) \frac{d\sigma_{Born}^{(el)}}{d\Omega} \qquad /15/$$

Here $\frac{d\sigma_{Born}^{(el)}}{d\Omega}$ is the differential cross section of the elastic scattering on a $V(\underline{r})$ background potential in Born approximation and the Bessel function of n-order accounts for the modification of it due to n-photon processes. The scattering process is elastic with respect to the background potential but inelastic with respect to the external field.

The result /15/ was obtained earlier by several authors, however by a somewhat different method, by the so-called space translation transformation /Faisal, 1973; Gontier and Rahman, 1974; Bergou, 1976/. The main advantage of the method outlined in the present paper is the simplicity of obtaining higher order approximations in $V(\underline{r})$ /the background potential/ in contrast to the space translation method, where the perturbation potential is much more complicated and therefore the extension of the Born approximation is difficult. The generalization of this method to relativistic electrons as well as its extension beyond the dipole approximation is in progress.

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