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THE FERMI GAS MODEL OF ONE-DIMENSIONAL CONDUCTORS

Hungarian Academy of Sciences

CENTRAL RESEARCH INSTITUTE FOR PHYSICS

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J. Sólyom
Central Research Institute for Physics
H-1525 Budapest, P.O.B.49. Hungary

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ABSTRACT

The Fermi gas model of one-dimensional conductors is reviewed. The exact solution known for particular values of the coupling constants in a single chain problem /Tomonaga model, Luther-Emery model/ are discussed. Renormalization group arguments are used to extend these solutions to arbitrary values of the couplings. The instabilities and possible ground states are studied by investigating the behaviour of the response functions. The relationship between this model and others is discussed and is used to obtain further information about the behaviour of the system. The model is generalized to a set of coupled chains to describe quasi-one-dimensional systems. The crossover from one-dimensional to three-dimensional behaviour and the type of ordering are discussed.

АННОТАЦИЯ

Рассматривается Ферми-газ модель одномерных проводников. Обсуждаются точные решения модели, существующие при определенных значениях константы связи в случае одной нити /модель Томонаги, модель Лютера-Эмери/. Метод группы ренормировок применяется для обобщения этих решений для произвольных значений константы связи. Исследуются свойства функции отклика для изучения возможных основных состояний и неустойчивостей. Обсуждается связь данной модели и других моделей и полученные соотношения используются для получения дальнейших информаций о свойствах системи. Модель далее обобщается на рассмотрение сети связанных нитей чтобы изучить квази-одномерных систем. Обсуждается переход от одномерных свойств к трехмерным и тип упорядочения.

KIVONAT

Áttekintést adunk az egydimenziós vezetők Fermi-gáz modelljéről. Az egylánc-problémában a csatolási állandók meghatározott értékénél /Tomonaga-modell, Luther-Emery modell/ egzakt megoldások léteznek. A renormálási csoport segitségével ezeket a megoldásokat általánosithatjuk a csatolások tetszőleges értékére. A válaszfüggvények vizsgálatával tanulmányozzuk a rendszer instabilitásait és a lehetséges alapállapotokat. Tárgyaljuk a Fermi gáz modell és más modellek kapcsolatát, és felhasználjuk ezt, hogy további információt kapjunk a rendszerről. Általánositjuk a modellt a csatolt láncok rendszerére, hogy kváziegydimenziós rendszereket is leirhassunk. Tárgyaljuk az egydimenziós viselkedésből a háromdimenziós viselkedésbe történő átmenetet és a rendeződés tipusait.

CONTENTS

§ 1. INTRODUCTION

Chemists have long been aware that many organic crystals are highly anisotropic due to the stacking of the molecules into loosely coupled chains. The anisotropy can be characterized by the ratio of the conductivities measured parallel and perpendicular to the chain direction. This ratio can be as large as 10^3. Elementary considerations can explain - at least partly - this large value. The building blocks of these crystals are usually large flat molecules lying rather closely to each other in one direction, thus forming chains which are nearly perpendicular to the plane of the molecules. The overlap of the wave functions in the chain direction allows for an easy motion of the electrons along the chains. The next chain is some distance away consequently there is less probability of interchain hopping,thus the motion of electrons is almost one-dimensional /1-d/.

Physicists realized only relatively recently that these systems have many interesting, unusual properties. They are due to the quasi-one-dimensional /quasi-1-d/ nature of these materials. It is well known that 1-d systems differ in many respect in their behaviour from 2-d and 3-d systems. By studying the properties of one-dimensional conductors it became possible for the first time to observe the peculiar dimensionality effects. Two different classes of quasi-1-d materials were investigated very intensively experimentally in the last few years, namely the mixed valence complexes and the charge transfer compounds. The best studied example of mixed valence complexes is KCP $[K_2 Pt(CN)_4 Br_{0.3} \cdot 3H_2O]$. The anisotropically oriented d orbitals of Pt represent an easy path for the motion of electrons in the direction of the Pt chains thus yielding a very large value for the ratio of the conductivities in the parallel and perpendicular directions.

A much richer class is that of the organic charge transfer compounds. In these materials two different kinds of molecules, namely donor and acceptor molecules are stacked separately into donor and acceptor chains. Once the charge transfer has taken place the motion of the electrons is confined almost exclusively to the chains. The richness of this class of materials is due to the large number of molecules which can be donors in these compounds. TTF-TCNQ is the most famous member of this group but there are many other interesting compounds showing widely differing behaviour. Useful reviews of the experimental background can be found in the proceedings of recent conferences: "One-Dimensional Conductors" /1975/, "Low-Dimensional Cooperative Phenomena" /1975/, "Chemistry and Physics of One-Dimensional Metals" /1977/ and "Organic Conductors and Semiconductors" /1977/.

The most interesting phenomena which are observed in many of these materials are the high conductivity at high temperatures, the transition to an insulating state at lower temperatures, the formation of charge density waves with wave vector $k = 2k_F$ / k_F is the Fermi momentum/ and the appearance of Peierls distortion /Peierls 1955/ in the ionic positions. This latter effect and the accompanying phonon softening above the transition temperature are typical 1-d effects in a coupled electron-phonon system. The other phenomena can possibly be understood by neglecting the effect of phonons and studying the electronic processes only. Accordingly basically there are two different theoretical approaches in the mathematical description of these systems. In one approach the starting Hamiltonian is the Fröhlich Hamiltonian /Fröhlich 1954/. The electron-phonon coupling leads to interesting dynamical effects such as the Kohn anomaly in the phonon dispersion relation and the phonon softening. This problem has a vast literature. Since we will not consider the effect of phonons we refer to a few papers where further references can be found: Rice and Strässler 1973a, 1973b, Horovitz et al. 1974, Horovitz et al. 1975, Bjeliš et al. 1974, Suzumura and Kurihara 1975, Brazovsky and Dzyaloshinsky 1976, Barišić 1978, Lukin 1978.

In the other approach only the electron system is considered, since - as it was mentioned - the particular features of the 1-d electron gas can already explain many properties of the one-dimensional conductors. Even when we restrict ourselves to consider the electron system only we still have two different approaches. For the description of systems where the conductivity along the chains is almost metallic, a Fermi gas model can be used. The electron-electron interactions are supposed to be weak and are taken into account in a consistent but perturbational way. In the other approach, which is more suitable for non-conducting systems, a Hubbard Hamiltonian /Hubbard 1963/ with strong intra-atomic correlation is used. These two models can be considered as limiting cases of a general model of interacting electrons written in different representations /momentum or site representation/. In this paper we will limit ourselves to reviewing the results obtained in the past few years for the Fermi gas model. The results for the Hubbard model will be mentioned only shortly to compare the two models.

The organization of the paper is as follows. Firstly, the model for the strictly 1-d case will be defined, then it will be shown that this model leads to a logarithmic problem where a perturbational treatment is not sufficient. A consequent summation of the subsequent logarithmic corrections can be achieved by using the renormalization group method. The real merit of the application of the renormalization group is to provide a means of scaling the original problem to other problems which might be simpler to solve. In fact the Fermi gas model can be solved exactly for particular values of the coupling constants. The exact solution of the Tomonaga model and the Luther-Emery solution of the backward scattering problem are presented. Following this, renormalization and scaling arguments will be used to extend these results for arbitrary values of the couplings. Based on this, the possible ground state configurations will be studied and the phase diagram will be presented. Further information can be obtained about the behaviour of the system described by this model if its relationship to other models is studied. The 2-d Coulomb plasma, spin models /e.g. 1-d X-Y-Z model and the 2-d X-Y model/ and field theoretical models are among those which are closely related to the Fermi gas model. The results for these models and their relations are also discussed. The choice of the cutoff is very important in proving the equivalence of the various models, thus the problem of cutoff will be considered. Finally the model will be generalized to a set of coupled chains to provide a more realistic model for quasi-1-d materials. The effect of interchain scattering and hopping in the stabilization of the ordered phase, the type of ordering and the crossover from 1-d to 3-d behaviour are described. We conclude the review by showing the possible application of this model to understand the properties of one-dimensional conductors.

§ 2. THE MODEL

The Fermi gas model is a model of weakly interacting electrons. Other excitations, such as e.g. phonons, and their interactions with the electrons are neglected. Though we may think that the effect of the electron-phonon interaction on the electronic properties can be accounted for by taking an effective electron-electron interaction, this interaction is a retarded one, whereas we will be concerned with a non-retarded interaction.

The unperturbed part of the Hamiltonian is

$$H_0 = \sum_{k,\alpha} \varepsilon_k \, c^+_{k\alpha} c_{k\alpha} \,, \qquad /2.1/$$

where $c^+_{k\alpha}$ $(c_{k\alpha})$ is a creation /annihilation/ operator of an electron with momentum k and spin α. The kinetic energy of the electrons measured from the Fermi energy is given by ε_k. In the first part of this paper a strictly 1-d model will be considered. The electrons can propagate along the chain only. The dispersion relation in a nearly free electron or tight binding approximation is illustrated schematically in fig. 1.

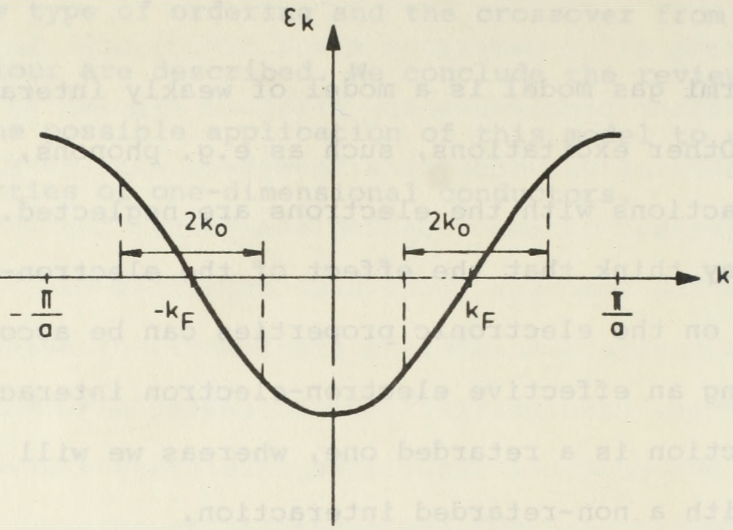

Fig. 1. Dispersion relation of a 1-d electron gas.

It is in general true that only those electrons lying near the Fermi surface are important in the physical processes. The Fermi surface of a strictly 1-d metal consists of two points: $+k_F$ and $-k_F$. In the neighbourhood of these points the dispersion curve can be approximated by straight lines and we get

$$\varepsilon_k = v_F \left(|k| - k_F \right). \qquad /2.2/$$

This approximation is a reasonable one in a finite range around the Fermi points. A momentum k_o is introduced to define the regions $(-k_F-k_0, -k_F+k_0)$ and (k_F-k_0, k_F+k_0) . Within these regions the linearized dispersion of eqn. /2.2/ will be used, whereas the states which are further away from the Fermi points will be neglected. Therefore k_0 serves as a cutoff for the allowed states. This cutoff is called bandwidth cutoff. In this model the bandwidth E_0 is determined by k_0 , $E_0 = 2 v_F k_0$. The dispersion relation is given in fig. 2.

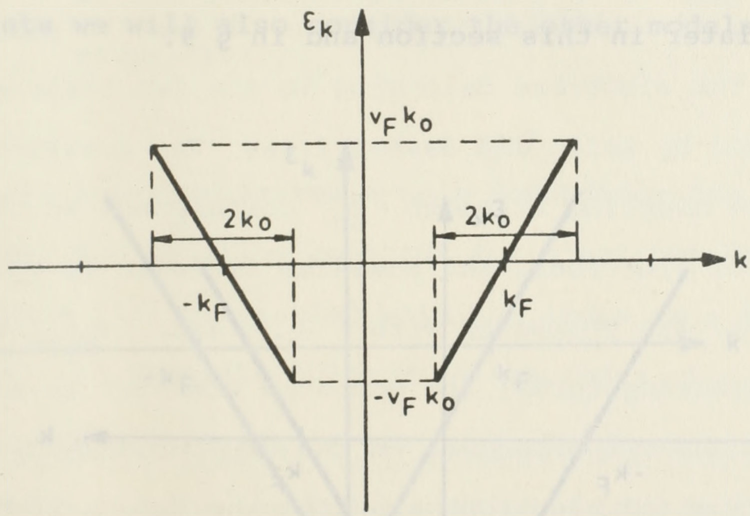

Fig. 2. Dispersion relation of the 1-d Fermi gas model with bandwidth cutoff.

The use of a linear dispersion relation has many calculational advantages. If the states far from the

Fermi points do not contribute essentially to the physical properties of the system, a linearized dispersion relation without bandwidth cutoff such as that given in fig. 3 could be used. We will see, however, later that the introduction of a cutoff is always necessary to avoid unphysical singularities, though it may be different from the bandwidth cutoff. It is possible in some cases to use the whole linearized dispersion curve and to assume a cutoff for the momentum transferred in a scattering process. The difference between the two cutoff procedures will be discussed later in this section and in § 9.

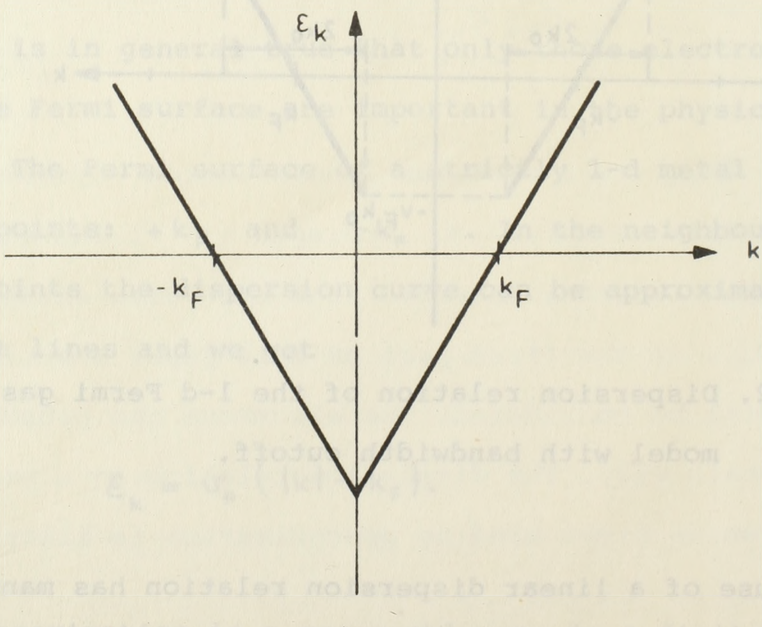

Fig. 3. Linearized dispersion relation without bandwidth cutoff.

A mathematically more rigorous treatment of the model can be made if the two branches of the dispersion curve do not terminate at $k=0$, both branches go from $-\infty$ to $+\infty$ as shown in fig. 4. This is the dispersion relation of the Luttinger model /Luttinger 1963/. The newly introduced non-physical states do not modify the physical results - at least not if the interaction is not very strong - but they do make the mathematical treatment easier. In most of the considerations of the present paper we will limit ourselves to the model with bandwidth cutoff, but at some points we will also consider the other models.

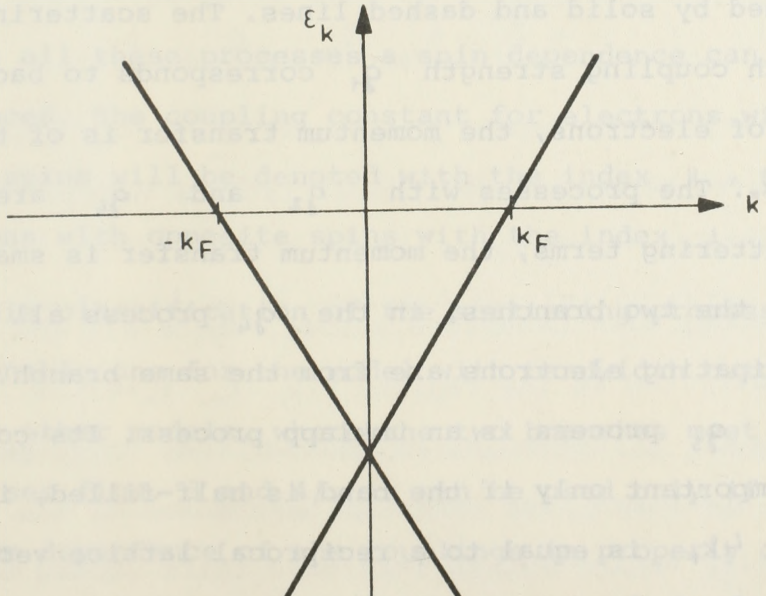

Fig. 4. Dispersion relation of the Luttinger model.

In any case there are two well defined branches of the dispersion relation. The operators for the electrons belonging to the branch containing the Fermi point $+k_F$ $(-k_F)$ are denoted by $a^+_{k\alpha}$ and $a_{k\alpha}$ ($b^+_{k\alpha}$ and $b_{k\alpha}$). In terms of these operators the free Hamiltonian is

$$H_o = \sum_{k,\alpha} v_F (k-k_F)\, a^+_{k\alpha} a_{k\alpha} + \sum_{k,\alpha} v_F (-k-k_F)\, b^+_{k\alpha} b_{k\alpha} . \qquad /2.3/$$

Turning now to the interaction processes, they can be conveniently classified into the four different types shown in fig. 5. The electrons belonging to the two branches are distinguished by solid and dashed lines. The scattering process with coupling strength g_1 corresponds to backward scattering of electrons, the momentum transfer is of the order of $2k_F$. The processes with g_2 and g_4 are forward scattering terms, the momentum transfer is small. g_2 couples the two branches, in the g_4 process all the four participating electrons are from the same branch. Finally the g_3 process is an umklapp process. Its contribution is important only if the band is half-filled, in which case $4k_F$ is equal to a reciprocal lattice vector and all the four electrons can be near the Fermi surface.

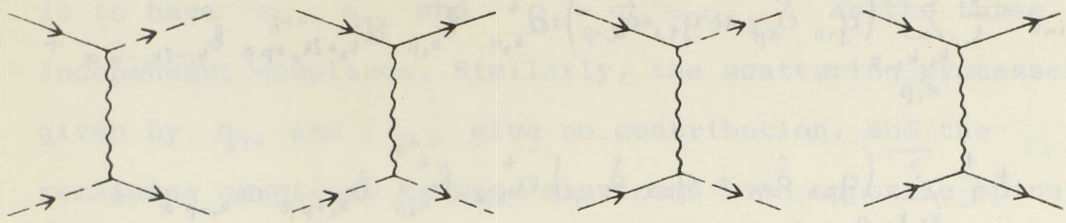

Fig. 5. Possible scattering processes. Solid and dashed lines correspond to electrons belonging to the branches containing $+k_F$ and $-k_F$ respectively.

In all these processes a spin dependence can be introduced. The coupling constant for electrons with parallel spins will be denoted with the index $\parallel$, that for electrons with opposite spins with the index $\perp$.

This classification of the scattering processes is a reasonable one for the model with bandwidth cutoff. For the other models, where the two branches meet at one point /see figs. 3 and 4/, it can be used only if the momentum dependence of the couplings is properly chosen, as will be discussed. The interaction part of the Hamiltonian can be written in terms of the $a_{k\alpha}$ and $b_{k\alpha}$

operators as follows:

$$H_{int} = \frac{1}{L} \sum_{\substack{k_1, k_2, p \\ \alpha, \beta}} \left(g_{1\parallel} \delta_{\alpha\beta} + g_{1\perp} \delta_{\alpha,-\beta} \right) a^+_{k_1\alpha} b^+_{k_2\beta} a_{k_2+2k_F+p\,\beta} b_{k_1-2k_F-p\,\alpha} +$$

$$+ \frac{1}{L} \sum_{\substack{k_1, k_2, p \\ \alpha, \beta}} \left(g_{2\parallel} \delta_{\alpha\beta} + g_{2\perp} \delta_{\alpha,-\beta} \right) a^+_{k_1\alpha} b^+_{k_2\beta} b_{k_2+p\,\beta} a_{k_1-p\,\alpha} +$$

$$+ \frac{1}{2L} \sum_{\substack{k_1, k_2, p \\ \alpha, \beta}} \left(g_{3\parallel} \delta_{\alpha\beta} + g_{3\perp} \delta_{\alpha,-\beta} \right) \Big(a^+_{k_1\alpha} a^+_{k_2\beta} b_{k_2-2k_F+p\,\beta} b_{k_1+2k_F-p-G\,\alpha} +$$

$$+ b^+_{k_1\alpha} b^+_{k_2\beta} a_{k_2+2k_F+p\,\beta} a_{k_1-2k_F-p+G\,\alpha} \Big) +$$

$$+ \frac{1}{2L} \sum_{\substack{k_1, k_2, p \\ \alpha, \beta}} \left(g_{4\parallel} \delta_{\alpha\beta} + g_{4\perp} \delta_{\alpha,-\beta} \right) \Big(a^+_{k_1\alpha} a^+_{k_2\beta} a_{k_2+p\,\beta} a_{k_1-p\,\alpha} +$$

$$+ b^+_{k_1\alpha} b^+_{k_2\beta} b_{k_2+p\,\beta} b_{k_1-p\,\alpha} \Big). \qquad (2.4)$$

The coupling constants depend in general on k_1, k_2 and p. G in the third term is a reciprocal lattice vector. For a half-filled band $G = 4k_F$.

In the model with bandwidth cutoff, where all the momenta are restricted to the regions $(-k_F - k_0, -k_F + k_0)$ or $(k_F - k_0, k_F + k_0)$, depending on whether they correspond to b or a operators, the momentum dependence of the couplings is usually neglected. In this case there should be no distinction between $g_{1\parallel}$ and $g_{2\parallel}$, they correspond to the same process. Instead of having the four couplings $g_{1\parallel}$, $g_{1\perp}$, $g_{2\parallel}$ and $g_{2\perp}$, only three independent couplings should be used. A common choice in the literature is to have $g_{1\parallel}$, $g_{1\perp}$ and $g_2 = g_{2\parallel} = g_{2\perp}$ as the three independent couplings. Similarly, the scattering processes given by $g_{3\parallel}$ and $g_{4\parallel}$ give no contribution, and the remaining couplings between electrons with opposite spins are simply denoted by g_3 and g_4.

This situation changes somewhat if one uses a momentum transfer cutoff, i.e. a cutoff on p. As we will see, in this case $g_{4\parallel}$ gives a trivial but non-vanishing contribution and therefore it should be kept. $g_{3\parallel}$, however, gives no contribution and can be neglected in this case as well. The terms with $g_{1\parallel}$ and $g_{2\parallel}$ do not correspond to exactly the same processes if the cutoff is only on the transfer p, while k_1 and k_2 can be anywhere in the respective branch. They become equivalent only if in addition to the transfer cutoff a bandwidth cutoff is also used as it should be if backward scattering terms are also present.

Finally it should be mentioned that the other convention for the coupling constants often used in the literature can be translated to the language of 'g' -ology by use of the identification

$$U_{\parallel} \to g_{1\parallel}, \quad U_{\perp} \to g_{1\perp}, \quad V \to g_2, \quad W_{\perp} \to g_3$$

/2.5/

and

$$W_{\parallel} = 2V - U_{\parallel} \to 2g_2 - g_{1\parallel}$$

/2.6/

since this combination plays particular role.

§ 3. PERTURBATIONAL TREATMENT OF THE MODEL

To obtain some sort of idea as to what may happen in our model, let us start by treating the model in a perturbational way. The behaviour of the Fermi gas can be studied by calculating the vertex function which describes the scattering of electrons and can reflect the instabilities of the system. The bare vertex functions are the same as the bare interaction processes shown diagrammatically in fig. 5. The first corrections to the scattering of two electrons from different branches are given in fig. 6. They are partly effective g_1 type /backward scattering/ and partly effective g_2 type /forward scattering/ processes. There are three types of diagrams. The four diagrams in fig. 6.a have the same structure: they contain in the intermediate state two electrons, one from each branch. These are the so-called Cooper pair diagrams. The other diagrams contain an electron-hole pair in the intermediate state. In the zero sound type diagrams /fig.6.b/ the electron and hole are on different branches; in the diagrams of fig. 6.c they are on the same branch. The structure of the diagrams is shown in fig. 7, where the interactions are represented by dots instead of wavy lines.

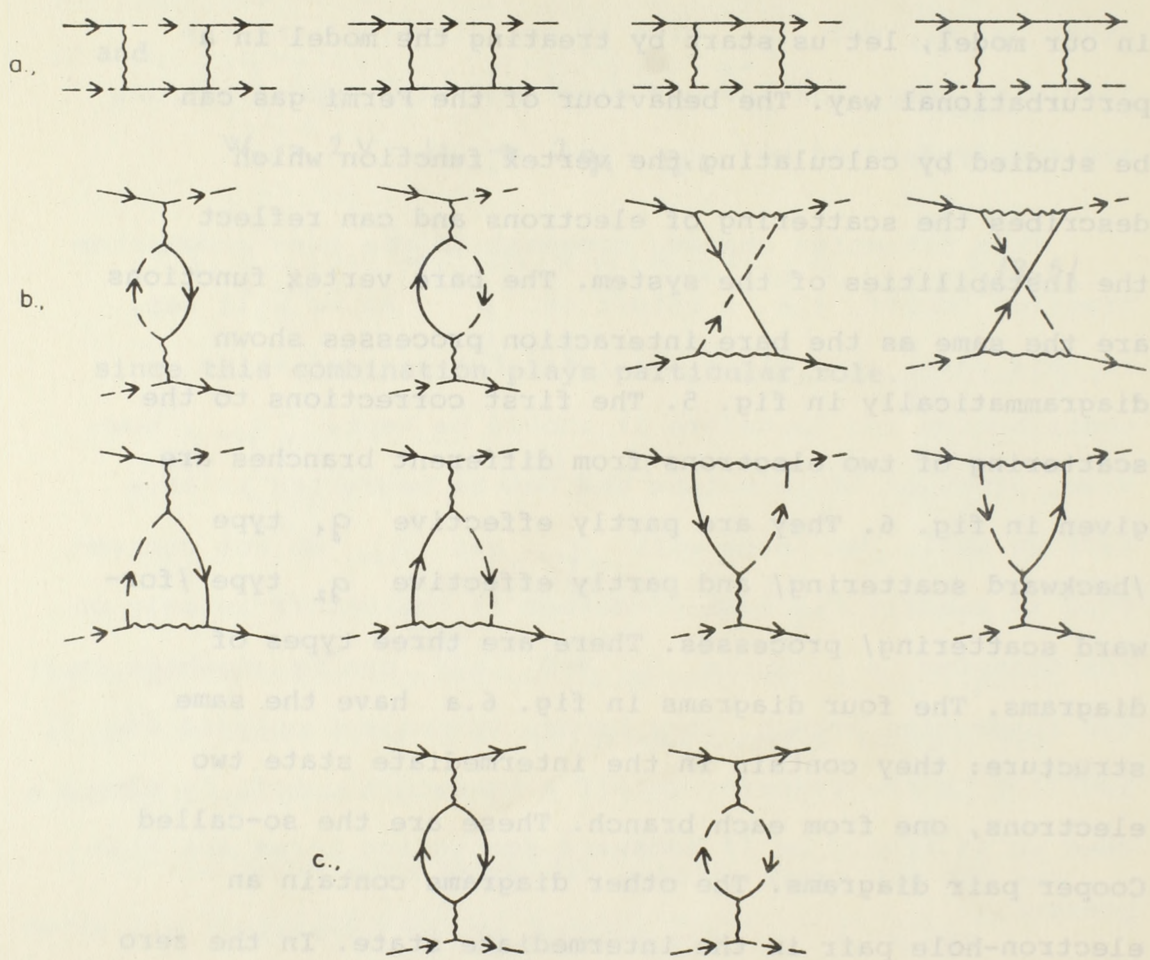

Fig. 6. First corrections to the vertex in the a/ Cooper pair, b/ zero sound and c/ third channel.

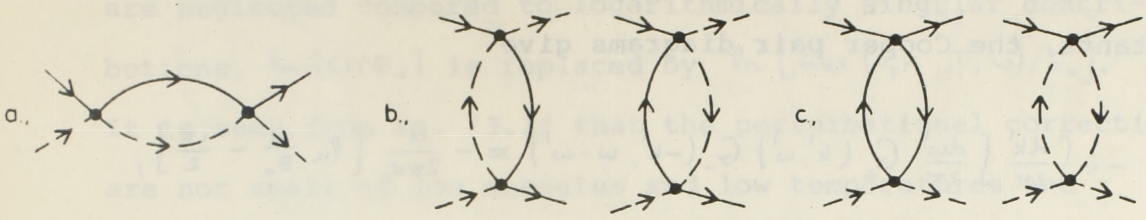

Fig. 7. The structure of the vertex diagram in the a/ Cooper pair, b/ zero sound and c/ third channel.

In the calculation of the analytic contribution we will restrict ourselves to a particular choice of the variables of the vertex, since here we only wish to illustrate the problems we have to face. The momenta are fixed at the Fermi momentum / $+k_F$ and $-k_F$, respectively/ and the energy variables are chosen in such a way that the usual combinations $\omega_1+\omega_2$, $\omega_1-\omega_3$ and $\omega_1-\omega_4$ are all equal to ω, a single energy variable. This special choice of the momentum and energy variables is shown in fig. 8.

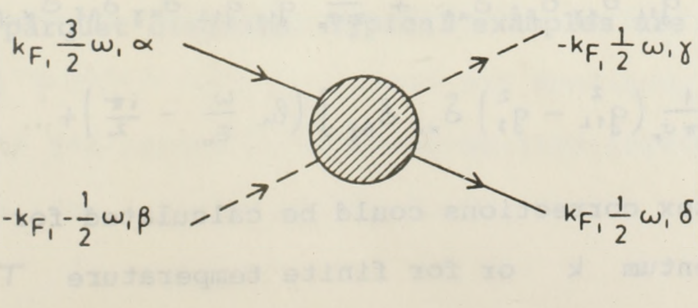

Fig. 8. General vertex diagram showing the special choice of the external variables.

Apart from the factors coming from the coupling constants, the Cooper pair diagrams give

$$-i\int\frac{dk'}{2\pi}\int\frac{d\omega'}{2\pi}\,G_+(k',\omega')\,G_-(-k',\omega-\omega') = -\frac{1}{2\pi v_F}\left[\ln\frac{\omega}{E_0}-\frac{i\pi}{2}\right], \qquad /3.1/$$

where G_+ and G_- are the Green's functions of electrons for the two branches of the spectrum. The diagram is calculated with a bandwidth cutoff E_0 .

The zero sound channel gives

$$-i\int\frac{dk'}{2\pi}\int\frac{d\omega'}{2\pi}\,G_+(k',\omega')\,G_-(k'-2k_F,\omega'-\omega) = \frac{1}{2\pi v_F}\left[\ln\frac{\omega}{E_0}-\frac{i\pi}{2}\right]. \qquad /3.2/$$

Since the third type of diagrams /fig. 6.c/ does not give logarithmic correction, it will be neglected for energies $\omega \ll E_0$. Taking now all the numerical factors into account, the analytic expression of the vertex is

$$\Gamma = g_{1\parallel}\,\delta_{\alpha\gamma}\,\delta_{\beta\delta}\,\delta_{\alpha\beta} + g_{1\perp}\,\delta_{\alpha\gamma}\,\delta_{\beta\delta}\,\delta_{\alpha,-\beta} - g_2\,\delta_{\alpha\delta}\,\delta_{\beta\gamma} +$$
$$+\Big[\frac{1}{\pi v_F}\,g_{1\perp}^2\,\delta_{\alpha\gamma}\,\delta_{\beta\delta}\,\delta_{\alpha\beta} + \frac{1}{\pi v_F}\,g_{1\parallel}\,g_{1\perp}\,\delta_{\alpha\gamma}\,\delta_{\beta\delta}\,\delta_{\alpha,-\beta} -$$
$$-\frac{1}{2\pi v_F}\left(g_{1\perp}^2-g_3^2\right)\delta_{\alpha\delta}\,\delta_{\beta\gamma}\Big]\left(\ln\frac{\omega}{E_0}-\frac{i\pi}{2}\right)+\ldots \qquad /3.3/$$

The vertex corrections could be calculated for finite incoming momentum k or for finite temperature T .

In the logarithmic approximation, where non-singular terms are neglected compared to logarithmically singular contributions, $\ln(\omega/E_0)$ is replaced by $\ln[\max(v_F k, T, \omega)/E_0]$. It is seen from eq. /3.3/ that the perturbational corrections are not small at low energies and low temperatures and higher order corrections should also be considered.

The first attempt to take into account the higher order corrections in a consistent way was that of Bychkov et al. /1966/. They pointed out that the 1-d Fermi gas is a typical logarithmic problem, i.e. the vertex is logarithmically singular in every order of the perturbational calculation. In the n-th order the correction is proportional to $\ln^{n-1}(\omega/E_0)$. In a consistent calculation all these corrections have to be summed. This can be done by realizing that the leading logarithmic corrections come from a particular set of diagrams, the so-called parquet diagrams. These diagrams are built up of the two basic logarithmic bubbles, the Cooper pair and zero sound bubbles. Starting from these elementary bubbles, the higher order diagrams can be constructed by inserting these bubbles instead of the bare vertex. A repetition of this procedure generates the so-called parquet diagrams. Typical examples are shown in fig. 9.

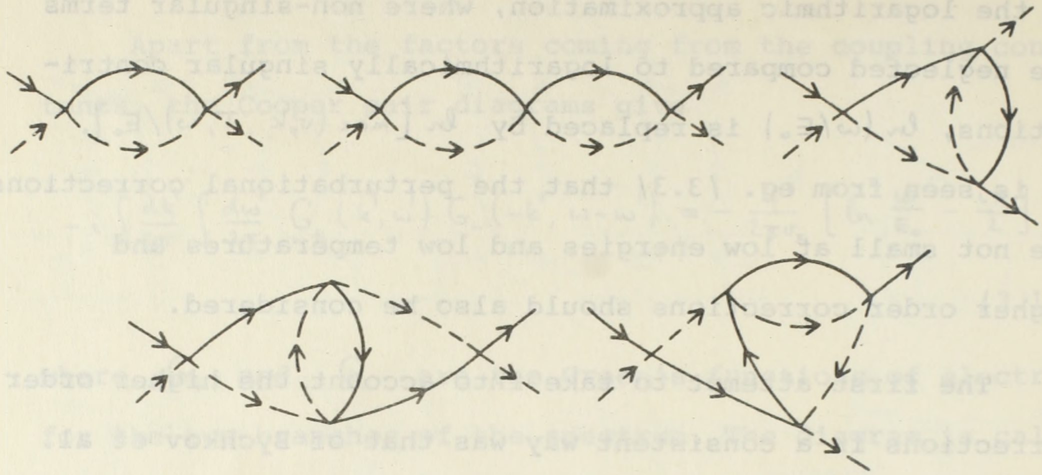

Fig. 9. Typical parquet diagrams.

The contribution of the parquet diagrams can be summed by writing a closed set of integral equations for the vertex /Diatlov et al. 1957/. This is a usual approximation in logarithmic problems and has been used e.g. by Abrikosov /1965/ for the Kondo problem, by Roulet et al. /1969/ and Nozières et al. /1969/ for the X-ray absorption problem and by Ginzburg /1974/ for the critical phenomena in $4-\varepsilon$ dimensions. A detailed description of the method can be found in the papers by Roulet et al. /1969/ and Nozières et al. /1969/. Bychkov et al. /1966/ used the parquet summation to calculate the vertex in the particular case when all the interesting couplings are equal: $g_{1\parallel} = g_{1\perp} = g_2 = g$. The umklapp processes have been neglected since they are important for a half-filled band only. They obtained

$$\Gamma = \frac{g}{1 - \frac{g}{\pi v_F} \ln \frac{\omega}{E_0}} \delta_{\alpha\gamma} \delta_{\beta\delta} - \left(\frac{1}{2} g + \frac{1}{2} \frac{g}{1 - \frac{g}{\pi v_F} \ln \frac{\omega}{E_0}} \right) \delta_{\alpha\delta} \delta_{\beta\gamma} . \qquad /3.4/$$

This result immediately shows the inadequacy of this approximation. The vertex is singular at a finite frequency or finite temperature given by

$$T_c = E_0 \exp\left(- \frac{\pi v_F}{|g|} \right) , \qquad /3.5/$$

if $g < 0$. This singularity is an indication of an instability, a phase transition in the system at this temperature. This is in contradiction with the well known theorem which states that a 1-d system with short range forces cannot have a phase transition at any finite temperature. The failure of the parquet approximation follows from neglecting the lower order logarithmic corrections which, at low temperatures, are not negligible. A method which is capable of taking into account these subsequent corrections is presented in the following paragraphs.

The 1-d Fermi gas has many similarities to the Kondo problem /for a review on the Kondo problem see Grüner and

Zawadowski 1974/. Both problems have an infrared divergence due to the continuum of low energy excitations. Formally this gives rise to the logarithmic corrections in the perturbational expansion. The parquet diagram summation in the Kondo problem gives a sharp transition at a temperature T_K, below which a bound polarization cloud is formed around the impurity. The expression for T_K is analogous to T_c in eqn. /3.5/. The sharp transition is non-physical and the corrections beyond the parquet approximation should also be considered. The renormalization group /Abrikosov and Migdal 1970, Fowler and Zawadowski 1971 and Wilson 1975/ proved to be very useful in the Kondo problem to get a full understanding of the physics of magnetic impurities in normal metals. A similar treatment is attempted in the next paragraph.

An alternative approach is the application of skeleton graph technique /Ohmi et al. 1976/. This method allows - in the same way as the renormalization group - to go beyond the parquet approximation. The same results, which will be presented in the next paragraph, can be obtained in this way as well.

§ 4. RENORMALIZATION GROUP TREATMENT

The multiplicative renormalization group has been known in field theory for a long time as a method to improve the results obtained in perturbation theory /see e.g. Bogoliubov and Shirkov 1959 and Bjorken and Drell 1965/. This method is particularly useful in logarithmic problems, where only a few diagrams have to be calculated and the summation of the higher order contributions is achieved by solving the renormalization group equations. The method has been applied to the logarithmic problems mentioned earlier. Abrikosov and Migdal /1970/ as well as Fowler and Zawadowski /1971/ studied the Kondo problem, Di Castro /1972/ and Brézin et al. /1973/ formulated the critical phenomena in $4-\varepsilon$ dimensions in this framework. Since the 1-d Fermi gas model is a logarithmic problem and the parquet approximation which is equivalent to summing the leading logarithmic corrections is not sufficient, it is hoped that a renormalization group treatment will allow a better approximation.

That aspect of the renormalization group that starting from a perturbational calculation a partial summation is obtained by solving the group equations is almost absent in the modern formulation of the renormalization group /see Wilson and Kogut 1974/, though it is present in most of the applications in a hidden form. Instead of that the scaling aspect is dominant. The basic idea is that one can find a set of equivalent problems which are described by Hamiltonians of similar form. The coupling constants and other parameters of the Hamiltonians may be different, but the systems should have the same physical behaviour. If there is a model among the equivalent ones which can be solved, the solution of the original problem can also be obtained.

The renormalization group transformations which relate the equivalent problems may be different to a very large extent depending on the problem at hand. One procedure which is very often used is to eliminate degrees of freedom near the cutoff - if there is a natural cutoff in the problem - and compensate their effect by choosing different coupling constants. This idea can be realized in many ways depending on how the equivalence of the problems is defined. It is very common to require that the free energy or partition function be invariant under the renormalization transformation. This is very suitable when thermodynamic properties are studied. Other conditions may be more convenient when scattering properties and instabilities are considered. First a simple treatment, then a more sophisticated approach will be presented.

4.1. Poor man's scaling

Anderson /1970/ suggested that the equivalent problems can be obtained by requiring that the scattering properties be the same in the systems, i.e. the scattering matrix T be invariant under the renormalization transformation. This poor man's approach to scaling can easily be applied to the 1-d Fermi gas model.

The scattering matrix T obeys the following equation

$$T(\omega) = H_{int} + H_{int} \frac{1}{\omega - H_0} T(\omega) , \qquad /4.1/$$

where the free and the interaction parts of the Hamiltonian are given in eqns. /2.3/ and /2.4/. As it has already been mentioned, there is a natural cutoff in the model with bandwidth cutoff and this will serve as a scaling parameter.

If the cutoff E_0 is changed to a smaller value $E_0 - dE_0$, states which were allowed as intermediate states in scattering processes are no longer available and the Hamiltonian should be modified to compensate for these lost states. A straightforward rearrangement of eqn. /4.1/ /Sólyom and Zawadowski 1974/ leads to the following form for the new Hamiltonian:

$$H'_{int}(\omega) = (1-\mathcal{P})\Big[H_{int} + H_{int}\,\mathcal{P}\frac{1}{\omega-H_0}H_{int} + $$

$$+ H_{int}\,\mathcal{P}\frac{1}{\omega-H_0}H_{int}\,\mathcal{P}\frac{1}{\omega-H_0}H_{int} + \dots\Big](1-\mathcal{P}),$$

/4.2/

where the projection operator $\mathcal{P}$ selects those states which contain at least one electron in the energy range $(E_0 - dE_0,\ E_0)$ or at least one hole in the range $(-E_0,\ -E_0+dE_0)$. Strictly speaking the scattering matrix calculated with this new Hamiltonian and new cutoff is not the same as the original one. The new T matrix has matrix elements between states only in which the electrons before and after the scattering belong to the restricted band. These matrix elements are the same in the original and the new systems.

The Hamiltonian /4.2/ of the scaled system can be very different from that of the original one. It is energy dependent and may contain - in addition to two-particle scattering - terms which correspond to scattering of three, four, etc. particles. This scaling procedure is useful only if it does not lead to a large number of new types of couplings which are essential and if the dependence of the scaled couplings on the other parameters of the scattering, like the energy and momentum of the electrons, is not important.

This is the case for the 1-d Fermi gas. The matrix element is taken between initial and final states which both contain two extra electrons from the neighbourhood of the Fermi surface, one from each branch, added to the filled Fermi sea. The initial state, $|i\rangle$ and final state, $|f\rangle$ are given as

$$|i\rangle = a^{+}_{k_1\alpha}\, b^{+}_{k_2\beta}\, |0\rangle \quad , \quad |f\rangle = b^{+}_{k_3\gamma}\, a^{+}_{k_4\delta}\, |0\rangle , \tag{4.3}$$

where $|0\rangle$ is the state vector of the filled Fermi sea. The momentum conservation requires that $k_1 + k_2 = k_3 + k_4$.

The matrix element of H_{int} between these states is

$$\langle i | H_{int} | f \rangle = \frac{1}{L}\Big[g_{1\parallel}\, \delta_{\alpha\gamma}\, \delta_{\beta\delta}\, \delta_{\alpha\beta} + g_{1\perp}\, \delta_{\alpha\gamma}\, \delta_{\beta\delta}\, \delta_{\alpha,-\beta} - g_2\, \delta_{\alpha\delta}\, \delta_{\beta\gamma} \Big]. \tag{4.4}$$

A straightforward calculation of the next term in eqn. /4.2/ gives

$$\langle i | H_{int}' | f \rangle = \frac{1}{L}\Big\{\Big[g_{1\parallel}\,\delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{\alpha\beta} + g_{1\perp}\,\delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{\alpha,-\beta} - g_2\,\delta_{\alpha\delta}\delta_{\beta\gamma}\Big] +$$

$$+ \frac{dE_0}{E_0}\Big[\frac{1}{\pi v_F} g_{1\perp}^2\,\delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{\alpha\beta} + \frac{1}{\pi v_F} g_{1\parallel} g_{1\perp}\,\delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{\alpha,-\beta} -$$

$$- \frac{1}{2\pi v_F}\left(g_{1\perp}^2 - g_3^2\right)\delta_{\alpha\delta}\delta_{\beta\gamma}\Big] + \ldots\Big\}. \qquad /4.5/$$

We have neglected ω and the energies of the scattered electrons with respect to E_0.

The structure of eqn. /4.5/ is the same as that of eqn. /4.4/. If the new coupling constants of the scaled Hamiltonian are chosen in the form $g_i + dg_i$, the following relations are obtained:

$$dg_{1\parallel} = \frac{dE_0}{E_0}\left[\frac{1}{\pi v_F} g_{1\perp}^2 + \ldots\right], \qquad /4.6/$$

$$dg_{1\perp} = \frac{dE_0}{E_0}\left[\frac{1}{\pi v_F} g_{1\parallel} g_{1\perp} + \ldots\right], \qquad /4.7/$$

$$dg_2 = \frac{dE_0}{E_0}\left[\frac{1}{2\pi v_F}\left(g_{1\perp}^2 - g_3^2\right) + \ldots\right] \qquad /4.8/$$

Since we have calculated everywhere the first correction only, this approximation, as it can immediately be seen, is equivalent to the parquet approximation. The solution of these equations for spin-independent couplings and for non-half-filled band, where g_3 can be neglected, gives

$$g_1' = \frac{g_1}{1 - \frac{g_1}{\pi v_F} \ln \frac{E_0'}{E_0}} , \qquad g_2' = g_2 - \frac{1}{2} g_1 + \frac{1}{2} \frac{g_1}{1 - \frac{g_1}{\pi v_F} \ln \frac{E_0'}{E_0}} . \tag{4.9}$$

Here g_1' and g_2' are the couplings if the cutoff E_0 is scaled to E_0'. Similar singular behaviour appears as in the expression of the vertex in the parquet approximation /eqn. /3.4//. In fact the use of the first order scaling equations is equivalent to summing the leading logarithmic corrections. In order to go beyond the parquet approximation, we have to calculate the next corrections in eqns. /4.6/ - /4.8/.

The couplings in the scaled Hamiltonian are non-physical quantities in the sense that they depend on what invariance property has been required. When calculated from the T matrix, the couplings are different whether the self-energy corrections of the electrons in the initial and final states are considered or not. Since we want to use the scaling procedure to calculate physical quantities, namely Green's functions, response functions etc., a new formulation will be presented in the next section which allows this in a convenient form.

4.2. Multiplicative renormalization generated by cutoff scaling

The simple physical picture of renormalization by successive elimination of degrees of freedom through cutoff scaling is absent in the usual formulation of multiplicative renormalization. It turns out, however, that in logarithmic problems cutoff scaling generates a multiplicative renormalization of Green's functions, vertices and other related quantities. Thus a new renormalization procedure can be worked out for these problems which combines the physical idea of successive elimination of degrees of freedom with the mathematical framework of multiplicative renormalization. This approach reproduces the known results in the X-ray absorption and Kondo problems /Sólyom 1974/ and in the critical phenomena /Forgács et al. 1978/. Here we will present the ideas applied to the 1-d Fermi gas /Menyhárd and Sólyom 1973/. The conventional field theoretical renormalization group has been applied to this system independently by Kimura /1973/, but in the lowest approximation only.

Before formulating the multiplicative renormalization transformation in a mathematical way, the dimensionless Green's function and vertices are introduced. In order to simplify the formulae we will restrict ourselves to the case when $g_3 = 0$, i.e. umklapp processes are not allowed and g_4 type processes are also neglected. The general case will be discussed in § 7.

The dimensionless Green's function d is defined by

$$G(k, \omega) = d\left(\frac{k}{k_0}, \frac{\omega}{E_0}\right) G^{(0)}(k, \omega). \tag{4.10}$$

The total vertex describing the scattering of two electrons from different branches is decomposed into three parts corresponding to the three different elementary scattering processes:

$$\begin{aligned}\Gamma_{\alpha\beta\gamma\delta}(k_i, \omega_i) = {} & g_{1\parallel} \tilde{\Gamma}_{1\parallel}\left(\frac{k_i}{k_0}, \frac{\omega_i}{E_0}\right) \delta_{\alpha\gamma} \delta_{\beta\delta} \delta_{\alpha\beta} + \\ & + g_{1\perp} \tilde{\Gamma}_{1\perp}\left(\frac{k_i}{k_0}, \frac{\omega_i}{E_0}\right) \delta_{\alpha\gamma} \delta_{\beta\delta} \delta_{\alpha,-\beta} - \\ & - g_2 \tilde{\Gamma}_2\left(\frac{k_i}{k_0}, \frac{\omega_i}{E_0}\right) \delta_{\alpha\delta} \delta_{\beta\gamma}.\end{aligned} \tag{4.11}$$

The dimensionless vertices $\tilde{\Gamma}_i$ $(i = 1\parallel, 1\perp, 2)$ are defined through this relation.

Multiplicative renormalization is usually defined by the transformation

$$G \rightarrow z G \qquad \text{or} \qquad d \rightarrow z d \tag{4.12}$$

$$\tilde{\Gamma}_i \rightarrow z_i^{-1} \tilde{\Gamma}_i \,, \qquad i = 1\parallel, 1\perp, 2, \tag{4.13}$$

$$g_i \rightarrow z^{-2} z_i g_i \,, \qquad i = 1\parallel, 1\perp, 2, \tag{4.14}$$

Working with these transformed quantities, all measurable quantities can be made finite in a renormalizable field theory even if no cutoff is used. The group property of this transformation allows to improve the results obtained in perturbation theory.

In our model there is a well defined cutoff, and similarly to the poor man's scaling approach, this cutoff is used to generate the equivalent, scaled systems. It turns out that in the present model scaling of the physical cutoff generates a multiplicative renormalization analogous to the transformations in field theory, i.e. the model obeys the following scaling relationship:

i/ scale the cutoff $E_o = 2v_F k_o$ to a smaller value $E_o' = 2v_F k_o'$,

ii/ change the couplings g_i $(i=1\shortparallel, 1\perp, 2)$ to g_i' ,

iii/ fix the values of the new couplings from the requirement that the Green's function and vertices of the scaled system preserve the same analytic form as that of the original system, i.e. they should differ in multiplicative factors only which are independent of the energy and momentum variables,

iv/ then the original and new couplings are related by the same multiplicative factors, the new couplings should be independent of the energy and momentum variables and can depend on the ratio of the new and old cutoffs only.

Formulated mathematically this means that

$$d\left(\frac{k}{k_0'}, \frac{\omega}{E_0'}, g_{1\shortparallel}', g_{1\perp}', g_2'\right) =$$

$$= Z\left(\frac{E_0'}{E_0}, g_{1\shortparallel}, g_{1\perp}, g_2\right) d\left(\frac{k}{k_0}, \frac{\omega}{E_0}, g_{1\shortparallel}, g_{1\perp}, g_2\right), \qquad /4.15/$$

$$\tilde{\Gamma}_i\left(\frac{k_j}{k_0'}, \frac{\omega_j}{E_0'}, g_{1\shortparallel}', g_{1\perp}', g_2'\right) =$$

$$= Z_i^{-1}\left(\frac{E_0'}{E_0}, g_{1\shortparallel}, g_{1\perp}, g_2\right) \tilde{\Gamma}_i\left(\frac{k_j}{k_0}, \frac{\omega_j}{E_0}, g_{1\shortparallel}, g_{1\perp}, g_2\right), \quad i = 1\shortparallel, 1\perp, 2, \qquad /4.16/$$

$$g_i' = g_i\, Z^{-2}\left(\frac{E_0'}{E_0}, g_{1\shortparallel}, g_{1\perp}, g_2\right) Z_i\left(\frac{E_0'}{E_0}, g_{1\shortparallel}, g_{1\perp}, g_2\right), \quad i = 1\shortparallel, 1\perp, 2. \qquad /4.17/$$

It follows that the quantity $g_i \tilde{\Gamma}_i d^2$ is invariant, i.e.

$$g_i' \tilde{\Gamma}_i\left(\frac{k_j}{k_0'}, \frac{\omega_j}{E_0'}, g_{1\shortparallel}', g_{1\perp}', g_2'\right) d^2\left(\frac{k}{k_0'}, \frac{\omega}{E_0'}, g_{1\shortparallel}', g_{1\perp}', g_2'\right) =$$

$$= g_i \tilde{\Gamma}_i\left(\frac{k_j}{k_0}, \frac{\omega_j}{E_0}, g_{1\shortparallel}, g_{1\perp}, g_2\right) d^2\left(\frac{k}{k_0}, \frac{\omega}{E_0}, g_{1\shortparallel}, g_{1\perp}, g_2\right). \qquad /4.18/$$

This quantity can be considered as an appropriately renormalized vertex, where the self-energy corrections on the incoming and outgoing lines are also taken into account /Sólyom and Zawadowski 1974/. The invariance of this quantity is required instead of the invariance of the T matrix as in the poor man's scaling.

These scaling relations cannot be satisfied for most theories. For the 1-d Fermi gas model and several other logarithmic problems, however, they are satisfied in perturbation theory at least for the leading and next to leading logarithmic terms. We will assume that they hold for the singular part of the Green's function and vertices in higher orders as well, though they may not be valid for the non-singular part.

Introducing the functions

$$g_i^R\left(\frac{E}{E_0}, g_{1\parallel}, g_{1\perp}, g_2\right) =$$

$$= g_i\, z^{-2}\left(\frac{E}{E_0}, g_{1\parallel}, g_{1\perp}, g_2\right) z_i\left(\frac{E}{E_0}, g_{1\parallel}, g_{1\perp}, g_2\right), \quad i = 1\parallel, 1\perp, 2,$$

/4.19/

the new couplings g_i' are obtained when E is identified with the new cutoff E_0', i.e.

$$g_i' = g_i^R\left(\frac{E_0'}{E_0}, g_{1\parallel}, g_{1\perp}, g_2\right).$$

/4.20/

These quantities are called invariant couplings since they are invariant under the same renormalization transformation

$$g_i^R\left(\frac{E}{E_0'}, g_{1\parallel}', g_{1\perp}', g_2'\right) =$$

$$= g_i^R\left(\frac{E}{E_0'}, g_{1\parallel}^R\left(\frac{E_0'}{E_0}, g_{1\parallel}, g_{1\perp}, g_2\right), g_{1\perp}^R\left(\frac{E_0'}{E_0}, g_{1\parallel}, g_{1\perp}, g_2\right), g_2^R\left(\frac{E_0'}{E_0}, g_{1\parallel}, g_{1\perp}, g_2\right)\right) =$$

$$= g_i^R\left(\frac{E}{E_0}, g_{1\parallel}, g_{1\perp}, g_2\right).$$

/4.21/

This invariance is a consequence of the group property of the renormalization transformation. Starting from the original couplings and scaling the cutoff directly from E_0 to E leads to the same new couplings as to go first from E_0 to E_0', where the new couplings are given by eqn. /4.20/ and then go from E_0' to E.

The scaling equations for d, $\tilde{\Gamma}_i$ and g_i^R can be written in a common form

$$A\left(\frac{\omega}{E_0'}, g_{1\parallel}^R\left(\frac{E_0'}{E_0}, g_{1\parallel}, g_{1\perp}, g_2\right), g_{1\perp}^R\left(\frac{E_0'}{E_0}, g_{1\parallel}, g_{1\perp}, g_2\right), g_2^R\left(\frac{E_0'}{E_0}, g_{1\parallel}, g_{1\perp}, g_2\right)\right) =$$

$$= Z_A\left(\frac{E_0'}{E_0}, g_{1\parallel}, g_{1\perp}, g_2\right) A\left(\frac{\omega}{E_0}, g_{1\parallel}, g_{1\perp}, g_2\right).$$

/4.22/

A single variable is left in these equations. This variable is ω , T or $v_F k$ depending on whether the energy, temperature or momentum dependence is calculated. The further considerations can easily be extended to several variables.

The scaling equations can be written in a differential equation form. Differentiating the logarithm of eqn. /4.22/ with respect to ω/E_0 $(T/E_0$ or $v_F k/E_0)$ and then putting E_0' equal to ω $(T$ or $v_F k)$ we get

$$\frac{d}{dx} \ln A\left(x, g_{1\parallel}, g_{1\perp}, g_2\right) =$$

$$= \frac{1}{x} \frac{d}{d\xi} \ln A\left(\xi, g_{1\parallel}^R(x, g_{1\parallel}, g_{1\perp}, g_2), g_{1\perp}^R(x, g_{1\parallel}, g_{1\perp}, g_2), g_2^R(x, g_{1\parallel}, g_{1\perp}, g_2)\right)\Big|_{\xi=1},$$

/4.23/

where $x = \omega/E_0$ $(T/E_0$ or $v_F k/E_0)$. These are the Lie equations of the renormalization group. The multiplicativ factor drops from this equation since it is independent of the energy, temperature or momentum variables, it depends on the ratio of the new and old cutoffs only. On the other hand when after the differentiation the new cutoff E_0' is put equal to ω $(T$ or $v_F k)$, the invariant couplings in eqn. /4.23/

appear as functions of the energy, temperature or momentum. In this sense it is common to speak about the energy, temperature or momentum dependence of the invariant couplings.

Eqn. /4.23/ tells us that the quantity $\underset{\cdot}{A}$ can be calculated as a function of x if the behaviour of the scaled problem is known near the new cutoff. This renormalization transformation, i.e. scaling the cutoff to the energy variable in question is useful for logarithmic problems since near the new cutoff a perturbational treatment of the right-hand side might be sufficient, provided the invariant couplings are small at x. Unfortunately this is not always the case, as will be seen, but even in this case the scaling properties can give some insight into the behaviour of the system.

In the first step of any calculation the perturbational expressions of the Green's function and vertices have to be determined. Then the renormalization factors and the invariant couplings can be obtained in a perturbational form using the scaling equations /4.15/ - /4.17/. Using this perturbational result the solution of the Lie equation /4.23/ for the invariant couplings gives a summed up expression. Once this expression is known, other quantities can also be determined by solving the respective Lie equations.

According to this programme the invariant couplings are determined in a perturbational form from the self energy and vertex corrections calculated to second order. A straightforward calculation of the self energy and vertex diagrams shown in fig. 10 and 11 gives:

$$d(\omega) = 1 + \frac{1}{8\pi^2 v_F^2}\left(g_{1\parallel}^2 + g_{1\perp}^2 - 2g_{1\parallel}g_2 + 2g_2^2\right)\left(\ln\frac{\omega}{E_0} - \frac{1}{2}i\pi\right) + \dots, \tag{4.24}$$

$$\tilde{\Gamma}_{1\parallel}(\omega) = 1 + \frac{1}{\pi v_F}\frac{g_{1\perp}^2}{g_{1\parallel}}\left(\ln\frac{\omega}{E_0} - \frac{1}{2}i\pi\right) + \frac{1}{\pi^2 v_F^2}g_{1\perp}^2\left(\ln^2\frac{\omega}{E_0} - i\pi\ln\frac{\omega}{E_0} + \dots\right) +$$

$$+ \frac{1}{4\pi^2 v_F^2}\left(-g_{1\parallel}^2 + g_{1\perp}^2 + 2g_{1\parallel}g_2 - 2g_2^2\right)\left(\ln\frac{\omega}{E_0} - \frac{1}{2}i\pi\right) + \dots, \tag{4.25}$$

$$\tilde{\Gamma}_{1\perp}(\omega) = 1 + \frac{1}{\pi v_F}g_{1\parallel}\left(\ln\frac{\omega}{E_0} - \frac{1}{2}i\pi\right) + \frac{1}{2\pi^2 v_F^2}\left(g_{1\parallel}^2 + g_{1\perp}^2\right)\left(\ln^2\frac{\omega}{E_0} - i\pi\ln\frac{\omega}{E_0} + \dots\right) +$$

$$+ \frac{1}{4\pi^2 v_F^2}\left(2g_{1\parallel}g_2 - 2g_2^2\right)\left(\ln\frac{\omega}{E_0} - \frac{1}{2}i\pi\right) + \dots, \tag{4.26}$$

$$\tilde{\Gamma}_2(\omega) = 1 + \frac{1}{2\pi v_F}\frac{g_{1\perp}^2}{g_2}\left(\ln\frac{\omega}{E_0} - \frac{1}{2}i\pi\right) + \frac{1}{2\pi^2 v_F^2}\frac{g_{1\parallel}g_{1\perp}^2}{g_2}\left(\ln^2\frac{\omega}{E_0} - i\pi\ln\frac{\omega}{E_0} + \dots\right) +$$

$$+ \frac{1}{4\pi^2 v_F^2}\left(\frac{g_{1\parallel}g_{1\perp}^2}{g_2} - g_{1\parallel}^2 - g_{1\perp}^2 + 2g_{1\parallel}g_2 - 2g_2^2\right)\left(\ln\frac{\omega}{E_0} - \frac{1}{2}i\pi\right) + \dots \tag{4.27}$$

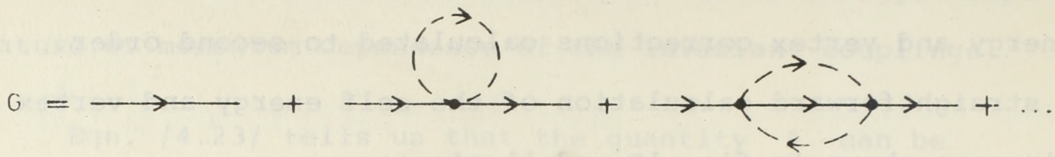

Fig. 10. Low order diagrams for the Green's function. The dots stand for $g_{1\parallel}$, $g_{1\perp}$ and g_2 type couplings.

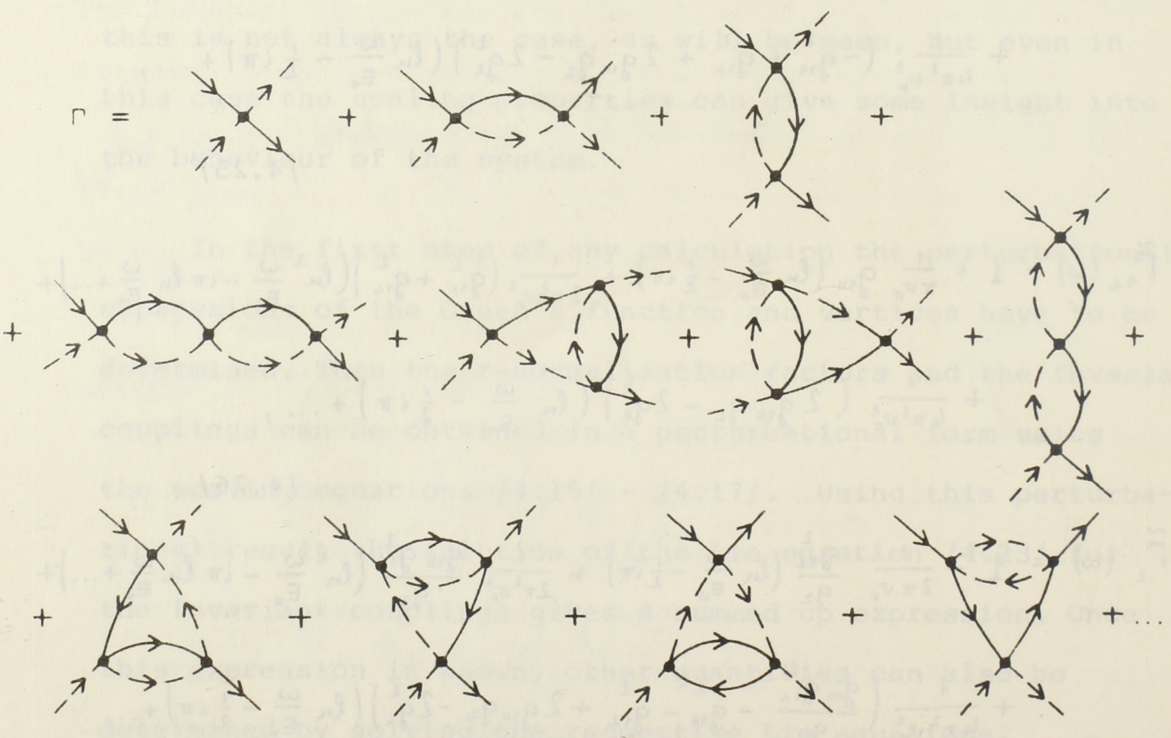

Fig. 11. Low order diagrams for the vertex.

The scaling equations are satisfied if the renormalized couplings have the form:

$$g_{1\parallel}^{R}\left(\frac{E_0'}{E_0}, g_{1\parallel}, g_{1\perp}, g_2\right) = g_{1\parallel} + \frac{g_{1\perp}^2}{\pi v_F} \ln \frac{E_0'}{E_0} + $$

$$+ \frac{1}{\pi^2 v_F^2} g_{1\parallel} g_{1\perp}^2 \left(\ln^2 \frac{E_0'}{E_0} + \frac{1}{2} \ln \frac{E_0'}{E_0}\right) + \ldots ,$$

/4.28/

$$g_{1\perp}^{R}\left(\frac{E_0'}{E_0}, g_{1\parallel}, g_{1\perp}, g_2\right) = g_{1\perp} + \frac{1}{\pi v_F} g_{1\parallel} g_{1\perp} \ln \frac{E_0'}{E_0} + $$

$$+ \frac{1}{2\pi^2 v_F^2}\left(g_{1\parallel}^2 g_{1\perp} + g_{1\perp}^3\right)\left(\ln^2 \frac{E_0'}{E_0} + \frac{1}{2} \ln \frac{E_0'}{E_0}\right) + \ldots ,$$

/4.29/

$$g_2^{R}\left(\frac{E_0'}{E_0}, g_{1\parallel}, g_{1\perp}, g_2\right) = g_2 + \frac{1}{2\pi v_F} g_{1\perp}^2 \ln \frac{E_0'}{E_0} + $$

$$+ \frac{1}{2\pi^2 v_F^2} g_{1\parallel} g_{1\perp}^2 \left(\ln^2 \frac{E_0'}{E_0} + \frac{1}{2} \ln \frac{E_0'}{E_0}\right) + \ldots .$$

/4.30/

The invariant couplings are real as they should be, and they are independent of the variables of the Green's function and vertex /Menyhárd and Sólyom 1975/.

The Lie equations for the invariant couplings are obtained from the perturbational expression in the form

$$\frac{d g_{\parallel}^{R}(x)}{d x} = \frac{1}{x}\left\{ \frac{1}{\pi v_F} g_{1\perp}^{R\,2}(x) + \frac{1}{2\pi^2 v_F^2} g_{1\parallel}^{R}(x)\, g_{1\perp}^{R\,2}(x) + \dots \right\},$$

/4.31/

$$\frac{d g_{1\perp}^{R}(x)}{d x} = \frac{1}{x}\left\{ \frac{1}{\pi v_F} g_{1\parallel}^{R}(x)\, g_{1\perp}^{R}(x) + \right.$$

$$\left. + \frac{1}{4\pi^2 v_F^2}\left[g_{1\parallel}^{R\,2}(x)\, g_{1\perp}^{R}(x) + g_{1\perp}^{R\,3}(x)\right] + \dots \right\}, \quad /4.32/$$

$$\frac{d g_{2}^{R}(x)}{d x} = \frac{1}{x}\left\{ \frac{1}{2\pi v_F} g_{1\perp}^{R\,2}(x) + \frac{1}{4\pi^2 v_F^2} g_{1\parallel}^{R}(x)\, g_{1\perp}^{R\,2}(x) + \dots \right\},$$

/4.33/

where $x = E_o'/E_o$.

In the next order /third order renormalization/ 58 fourth-order vertex diagrams and 2 third-order self-energy diagrams should be considered. Ting /1976/ did the calculation in this order for spin independent couplings and obtained the following result for the invariant couplings:

$$g_1^R\left(\frac{E_o'}{E_o}, g_1, g_2\right) = g_1 + \frac{1}{\pi v_F} g_1^2 \ln\frac{E_o'}{E_o} + \frac{1}{\pi^2 v_F^2} g_1^3 \left(\ln^2\frac{E_o'}{E_o} + \frac{1}{2}\ln\frac{E_o'}{E_o}\right) +$$

$$+ \frac{1}{\pi^3 v_F^3} g_1^4 \left(\ln^3\frac{E_o'}{E_o} + \frac{5}{4}\ln^2\frac{E_o'}{E_o} - \frac{7}{8}\ln\frac{E_o'}{E_o}\right) +$$

$$+ \frac{1}{2\pi^3 v_F^3}\left(g_1^3 g_2 - g_1^2 g_2^2\right)\ln\frac{E_o'}{E_o} + \dots ,$$

/4.34/

$$g_2^R\left(\frac{E_0'}{E_0}, g_1, g_2\right) = g_2 + \frac{1}{2\pi v_F} g_1^2 \ln\frac{E_0'}{E_0} + \frac{1}{2\pi^2 v_F^2} g_1^3\left(\ln^2\frac{E_0'}{E_0} + \frac{1}{2}\ln\frac{E_0'}{E_0}\right) +$$

$$+ \frac{1}{2\pi^3 v_F^3} g_1^4\left(\ln^3\frac{E_0'}{E_0} + \frac{5}{4}\ln^2\frac{E_0'}{E_0} - \frac{7}{8}\ln\frac{E_0'}{E_0}\right) +$$

$$+ \frac{1}{4\pi^3 v_F^3}\left(g_1^3 g_2 - g_1^2 g_2^2\right)\ln\frac{E_0'}{E_0} + \dots \qquad /4.35/$$

The Lie equations in this order are

$$\frac{d g_1^R(x)}{dx} = \frac{1}{x}\left\{\frac{1}{\pi v_F} g_1^{R\,2}(x) + \frac{1}{2\pi^2 v_F^2} g_1^{R\,3}(x) - \frac{7}{8\pi^3 v_F^3} g_1^{R\,4}(x) + \right.$$

$$\left. + \frac{1}{2\pi^3 v_F^3}\left[g_1^{R\,3}(x)\, g_2^R(x) - g_1^{R\,2}(x)\, g_2^{R\,2}(x)\right] + \dots\right\}, \qquad /4.36/$$

$$\frac{d g_2^R(x)}{dx} = \frac{1}{x}\left\{\frac{1}{2\pi v_F} g_1^{R\,2}(x) + \frac{1}{4\pi^2 v_F^2} g_1^{R\,3}(x) - \frac{7}{16\pi^3 v_F^3} g_1^{R\,4}(x) + \right.$$

$$\left. + \frac{1}{4\pi^3 v_F^3}\left[g_1^{R\,3}(x)\, g_2^R(x) - g_1^{R\,2}(x)\, g_2^{R\,2}(x)\right] + \dots\right\}. \qquad /4.37/$$

The result obtained by Ting is actually more complicated. Fowler /1976/ has shown that $g_1^R - 2g_2^R$ is an exact invariant at least for $g_1 - 2g_2 = 0$ and conjectured that it holds for $g_1 - 2g_2 \neq 0$ as well. The approximation leading to eqns. /4.34/ - /4.37/ ensures this invariance.

Keeping the first term of the right-hand side in eqns. /4.31/ - /4.33/, the scaling equations /4.6/ - /4.8/ are recovered. The first order scaling equations led to non-physical singularities for $g_{1\parallel} = g_{1\perp} = g_1 < 0$. We will consider now the effect of higher order corrections.

The best way to analyse these equations is to plot the scaling trajectories, i.e. the lines connecting the equivalent problems in the space of couplings. Using the invariance

$$g_{1\parallel}^R(x) - 2 g_2^R(x) = g_{1\parallel} - 2g_2 , \qquad /4.38/$$

$g_2^R(x)$ is easily obtained from $g_{1\parallel}^R(x)$ and therefore the scaling trajectories are plotted on the $(g_{1\parallel}, g_{1\perp})$ plane. Figs. 12.a and 12.b show the scaling curves in two different approximations, taking into account the second order terms only or the third order terms as well. The arrow on the flow lines represents the direction of scaling when the cutoff is decreased.

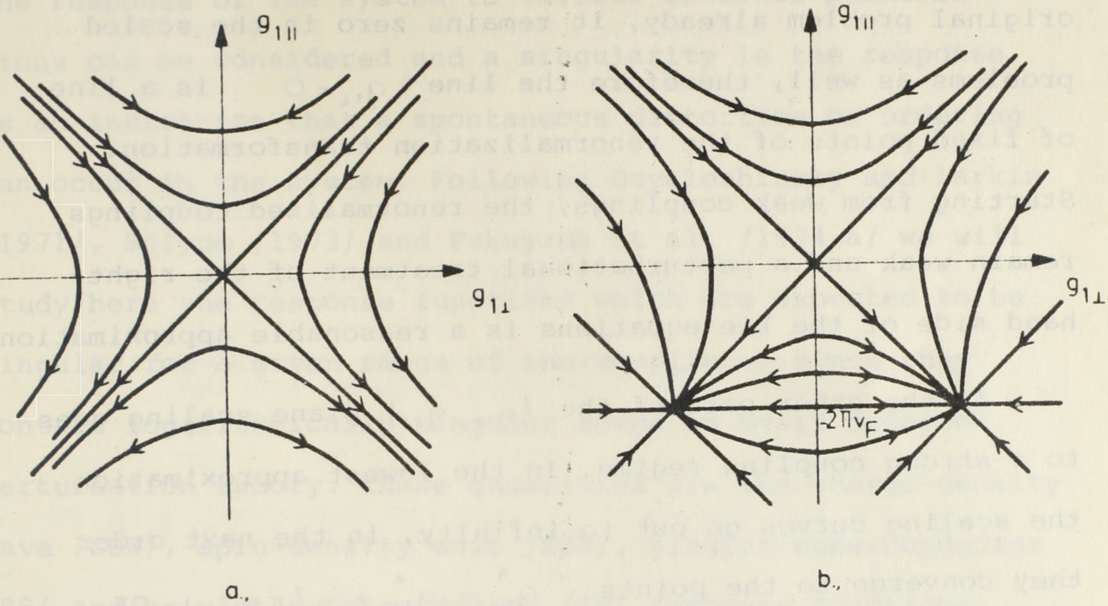

Fig.12. Scaling trajectories in the $(g_{1\parallel}, g_{1\perp})$ plane in /a/ leading and /b/ next to leading logarithmic approximation.

There are two well separated regions in the $(g_{1\parallel}, g_{1\perp})$ plane where the behaviour of the system is expected to be different. For $g_{1\parallel} \geq |g_{1\perp}|$ the scaling trajectories go to the line $g_{1\perp} = 0$, i.e. the problems for which the coupling constants are situated in this region are equivalent to a problem where the backward scattering is unimportant /the $g_{1\parallel}$ processes cannot be distinguished from the forward scattering processes if bandwidth cutoff is used/. If $g_{1\perp}=0$ in the original problem already, it remains zero in the scaled problems as well, therefore the line $g_{1\perp}=0$ is a line of fixed points of the renormalization transformation. Starting from weak couplings, the renormalized couplings remain weak and a perturbational treatment of the right-hand side of the Lie equations is a reasonable approximation.

In the other part of the $(g_{1\parallel}, g_{1\perp})$ plane scaling goes to a strong coupling regime. In the lowest approximation the scaling curves go out to infinity, in the next order they converge to the points $g_{1\parallel}^{*} = g_{1\perp}^{*} = -2\pi v_F$ or $g_{1\parallel}^{*} = -g_{1\perp}^{*} = -2\pi v_F$. These are the two fixed points which are obtained from the zeros of the right-hand side of the Lie equations. In the next order /Ting 1976/ the value of the fixed point is further decreased, instead of $-2\pi v_F$ the value $-0.87\pi v_F$ appears. Since these points are outside of the region of applicability of a series expansion in eqns. /4.31/ - /4.32/, only the tendency, that the problems are equivalent to a strong coupling problem, should be taken seriously.

4.3. Determination of the response functions

As already mentioned, one of the aims in studying this model is to understand what kind of instabilities are likely to occur in this system. The best way to do this is to calculate response functions or generalized susceptibilities. The response of the system to various external perturbations can be considered and a singularity in the response is an indication that a spontaneous distortion or ordering can occur in the system. Following Dzyaloshinsky and Larkin /1971/, Sólyom /1973/ and Fukuyama et al. /1974 a/ we will study here the response functions which are expected to be singular for a given range of the couplings, since they contain logarithmically singular terms in every order of perturbation theory. These quantities are the charge-density wave /CDW/, spin-density wave /SDW/, singlet-superconductor /SS/ and triplet-superconductor /TS/ response functions. An instability in the CDW or SDW response function shows that a CDW or SDW state is formed in the system with that value of the wave vector, where the instability occurs first. This instability is expected to occur at $k = 2k_F$, reflecting the logarithmic singularity in the electron-hole bubble with this wave vector. Singlet or triplet superconductivity is obtained if the response function of singlet or triplet Cooper pairs is singular.

The definitions of these functions are:

$$R(k,\omega) = -i \int dt \, e^{i\omega t} \, \langle T \{ \mathcal{O}(k,t) \, \mathcal{O}^+(k,0) \} \rangle ,$$

/4.39/

where

/I/ for the charge-density response function $N(k,\omega)$

$$O(k,t) = \frac{1}{L^{1/2}} \sum_{k_1} \left[b^{+}_{k_1\uparrow}(t)\, a_{k_1+k\uparrow}(t) + b^{+}_{k_1\downarrow}(t)\, a_{k_1+k\downarrow}(t) \right] \tag{4.40}$$

is the Fourier component of the charge-density for large momentum,

/II/ for the spin-density response function $\chi(k,\omega)$

$$O(k,t) = \frac{1}{L^{1/2}} \sum_{k_1} \left[b^{+}_{k_1\uparrow}(t)\, a_{k_1+k\uparrow}(t) - b^{+}_{k_1\downarrow}(t)\, a_{k_1+k\downarrow}(t) \right], \tag{4.41}$$

or

$$O(k,t) = \frac{1}{L^{1/2}} \sum_{k_1} \left[b^{+}_{k_1\uparrow}(t)\, a_{k_1+k\downarrow}(t) + b^{+}_{k_1\downarrow}(t)\, a_{k_1+k\uparrow}(t) \right] \tag{4.42}$$

are the Fourier components of the longitudinal and transverse spin densities for large momentum, they should be equal in the disordered phase,

/III/ for the singlet-superconductor response $\Delta_s(k,\omega)$

$$\mathcal{O}(k,t) = \frac{1}{L^{1/2}} \sum_{k_1} \left[b_{k_1\uparrow}(t)\, a_{-k_1+k\downarrow}(t) + a_{k_1\uparrow}(t)\, b_{-k_1+k\downarrow}(t) \right]$$

/4.43/

corresponds to singlet Cooper pairs, and

/IV/ for the triplet-superconductor response $\Delta_t(k,\omega)$

$$\mathcal{O}(k,t) = \frac{1}{L^{1/2}} \sum_{k_1} \left[b_{k_1\uparrow}(t)\, a_{-k_1+k\downarrow}(t) - a_{k_1\uparrow}(t)\, b_{-k_1+k\downarrow}(t) \right],$$

/4.44/

or

$$\mathcal{O}(k,t) = \frac{2}{L^{1/2}} \sum_{k_1} b_{k_1\alpha}(t)\, a_{-k_1+k\alpha}(t), \qquad \alpha = \uparrow \text{ or } \downarrow$$

/4.45/

correspond to triplet Cooper pairs. The different forms correspond to different values of the spin projection $(S^z = 0 \text{ and } S^z = \pm 1)$. The wave vectors will be fixed at $k = 2k_F$ in the density responses and to $k = 0$ in the pairing functions.

Throughout the calculation we studied the ω dependence at $T = 0$. The temperature dependence is easily obtained from this result, since in logarithmic approximation $\ln(\omega/E_0)$ should be replaced by $\ln\left[\max(\omega,T)/E_0\right]$.

The perturbational expressions of these quantities are straightforwardly obtained

$$
\begin{aligned}
N(\omega) = \frac{1}{\pi v_F} \ln\frac{\omega}{E_0}\Big[1 &+ \frac{1}{2\pi v_F}\left(g_{1\parallel} + g_{1\perp} - g_2\right)\ln\frac{\omega}{E_0} + \\
&+ \frac{1}{12\pi^2 v_F^2}\left(2g_{1\parallel}^2 + 6g_{1\parallel}g_{1\perp} + 3g_{1\perp}^2 - 2g_{1\parallel}g_2 - 2g_{1\perp}g_2 + 2g_2^2\right)\ln^2\frac{\omega}{E_0} + \\
&+ \frac{1}{8\pi^2 v_F^2}\left(g_{1\parallel}^2 + g_{1\perp}^2 - 2g_{1\parallel}g_2 + 2g_2^2\right)\ln\frac{\omega}{E_0} + \ldots\Big],
\end{aligned}
\qquad /4.46/
$$

$$
\begin{aligned}
\chi(\omega) = \frac{1}{\pi v_F} \ln\frac{\omega}{E_0}\Big[1 &- \frac{1}{2\pi v_F} g_2 \ln\frac{\omega}{E_0} + \frac{1}{12\pi^2 v_F^2}\left(2g_2^2 - g_{1\perp}^2\right)\ln^2\frac{\omega}{E_0} + \\
&+ \frac{1}{8\pi^2 v_F^2}\left(g_{1\parallel}^2 + g_{1\perp}^2 - 2g_{1\parallel}g_2 + 2g_2^2\right)\ln\frac{\omega}{E_0} + \ldots\Big],
\end{aligned}
\qquad /4.47/
$$

$$
\begin{aligned}
\Delta_s(\omega) = \frac{1}{\pi v_F} \ln\frac{\omega}{E_0}\Big[1 &+ \frac{1}{2\pi v_F}\left(g_{1\perp} + g_2\right)\ln\frac{\omega}{E_0} + \\
&+ \frac{1}{12\pi^2 v_F^2}\left(2g_{1\parallel}g_{1\perp} + 3g_{1\perp}^2 + 4g_{1\perp}g_2 + 2g_2^2\right)\ln^2\frac{\omega}{E_0} + \\
&+ \frac{1}{8\pi^2 v_F^2}\left(g_{1\parallel}^2 + g_{1\perp}^2 - 2g_{1\parallel}g_2 + 2g_2^2\right)\ln\frac{\omega}{E_0} + \ldots\Big],
\end{aligned}
\qquad /4.48/
$$

$$
\begin{aligned}
\Delta_t(\omega) = \frac{1}{\pi v_F} \ln\frac{\omega}{E_0}\Big[1 &+ \frac{1}{2\pi v_F}\left(-g_{1\parallel} + g_2\right)\ln\frac{\omega}{E_0} + \\
&+ \frac{1}{12\pi^2 v_F^2}\left(2g_{1\parallel}^2 - 4g_{1\parallel}g_2 - g_{1\perp}^2 + 2g_2^2\right)\ln^2\frac{\omega}{E_0} + \\
&+ \frac{1}{8\pi^2 v_F^2}\left(g_{1\parallel}^2 + g_{1\perp}^2 - 2g_{1\parallel}g_2 + 2g_2^2\right)\ln\frac{\omega}{E_0} + \ldots\Big].
\end{aligned}
\qquad /4.49/
$$

Only the real part has been calculated everywhere.

These susceptibilities do not obey the scaling hypothesis of eqn. /4.22/, they obey, however, the following relation

$$R\left(\frac{\omega}{E_0'}, g_{1\parallel}', g_{1\perp}', g_2'\right) = Z_R\, R\left(\frac{\omega}{E_0}, g_{1\parallel}, g_{1\perp}, g_2\right) + C\left(\frac{E_0'}{E_0}, g_{1\parallel}, g_{1\perp}, g_2\right).$$

/4.50/

The functions are not simply multiplicatively renormalized when the cutoff is scaled, an additional constant appears. This constant disappears when eqn. /4.50/ is differentiated with respect to ω, and the scaling equations can be used. It is more convenient to define the auxiliary functions $\overline{R}$, following a suggestion of Zawadowski,

$$\overline{R}(\omega) = \pi v_F \frac{\partial R(\omega)}{\partial \ln \omega},$$

/4.51/

and to use the scaling equations for these functions. The Lie equations have the form:

$$\frac{d \ln \overline{N}(x)}{dx} = \frac{1}{x}\left\{ \frac{1}{\pi v_F}\left[g_{1\parallel}^R(x) + g_{1\perp}^R(x) - g_2^R(x)\right] + F(x) + \dots \right\},$$

/4.52/

$$\frac{d \ln \bar{\chi}(x)}{dx} = \frac{1}{x}\left\{ -\frac{1}{\pi v_F} g_2^R(x) + F(x) + \dots \right\},$$

/4.53/

$$\frac{d \ln \bar{\Delta}_s(x)}{dx} = \frac{1}{x}\left\{ \frac{1}{\pi v_F}\left[g_{1\perp}^R(x) + g_2^R(x)\right] + F(x) + \dots \right\},$$

/4.54/

$$\frac{d \ln \bar{\Delta}_t(x)}{dx} = \frac{1}{x}\left\{ \frac{1}{\pi v_F}\left[- g_{1\parallel}^R(x) + g_2^R(x)\right] + F(x) + \dots \right\},$$

/4.55/

where

$$F(x) = \frac{1}{4\pi^2 v_F^2}\left[g_{1\parallel}^{R\,2}(x) + g_{1\perp}^{R\,2}(x) - 2 g_{1\parallel}^R(x)\, g_2^R(x) + 2 g_2^{R\,2}(x)\right],$$

/4.56/

and $x = \omega / E_0$, or $x = T/E_0$ if the temperature dependence is studied.

Inserting the invariant couplings into these equations, the energy or temperature dependence of the response functions can be obtained. The invariant couplings are non-singular functions of x if we go beyond the first approximation, therefore any singularity in the response functions can appear at $x = 0$ only, i.e. at $\omega = 0$ and $T = 0$. The leading behaviour at small x is obtained by inserting the invariant couplings at $x=0$, i.e. the fixed point values.

According to the discussion in § 4.2 the scaling curves go to the fixed line $g_{1\perp}^{*}=0$ if $g_{1\parallel} \geq |g_{1\perp}|$ is satisfied for the original couplings. The fixed point value of $g_{1\parallel}$ is not universal, in a first approximation

$$g_{1\parallel}^{*} \approx \sqrt{g_{1\parallel}^{2} - g_{1\perp}^{2}} \quad \text{and} \quad g_{2}^{*} \approx g_{2} - \tfrac{1}{2} g_{1\parallel} + \tfrac{1}{2} \sqrt{g_{1\parallel}^{2} - g_{1\perp}^{2}} . \tag{4.57}$$

Since the fixed point value is small, the higher order corrections in eqns. /4.52/ - /4.55/ can be neglected and the solution of the equations gives for the leading terms

$$N(x) \sim x^{\alpha} , \quad \alpha = \tfrac{1}{2} g_{1\parallel} - g_{2} + \tfrac{1}{2} g_{1\parallel}^{*} + \dots , \tag{4.58}$$

$$\chi(x) \sim x^{\beta} , \quad \beta = \tfrac{1}{2} g_{1\parallel} - g_{2} - \tfrac{1}{2} g_{1\parallel}^{*} + \dots , \tag{4.59}$$

$$\Delta_{s}(x) \sim x^{\gamma} , \quad \gamma = - \tfrac{1}{2} g_{1\parallel} + g_{2} + \tfrac{1}{2} g_{1\parallel}^{*} + \dots , \tag{4.60}$$

$$\Delta_{t}(x) \sim x^{\delta} \quad \delta = - \tfrac{1}{2} g_{1\parallel} + g_{2} - \tfrac{1}{2} g_{1\parallel}^{*} + \dots . \tag{4.61}$$

The result is very simple for spin-independent couplings. In this case $g_{1\parallel}^* = 0$ and depending on the sign of $\frac{1}{2}g_1 - g_2$ either the density response functions or the pairing responses are singular at $\omega = 0$.

The domain of attraction of the fixed point $g_{1\parallel}^* = g_{1\perp}^* = -2\pi v_F$, $g_2^* = g_2 - \frac{1}{2}g_{1\parallel} - \pi v_F$ is $g_{1\parallel} < |g_{1\perp}|$ and $g_{1\perp} < 0$ whereas for the fixed point $g_{1\parallel}^* = -g_{1\perp}^* = -2\pi v_F$, $g_2^* = g_2 - \frac{1}{2}g_{1\parallel} - \pi v_F$ it is $g_{1\perp} > 0$ and $g_{1\parallel} < g_{1\perp}$. Since the fixed point value is not well determined in the approximation considered in § 4.2 and the higher order corrections in eqns. /4.52/ - /4.55/ cannot be neglected, only a rough estimate of the exponents can be obtained. We get in second order scaling

$$N(x) \sim x^{\alpha} \;, \qquad \alpha = \begin{cases} \frac{5}{2} + \frac{1}{\pi v_F}\left(\frac{1}{2}g_{1\parallel} - g_2\right) & \text{if } g_{1\perp} > 0 \\ -\frac{3}{2} + \frac{1}{\pi v_F}\left(\frac{1}{2}g_{1\parallel} - g_2\right) & \text{if } g_{1\perp} < 0 \end{cases}$$

/4.62/

$$\chi(x) \sim x^{\beta} \;, \qquad \beta = \frac{5}{2} + \frac{1}{\pi v_F}\left(\frac{1}{2}g_{1\parallel} - g_2\right)$$

/4.63/

$$\Delta_s(x) \sim x^{\gamma} \;, \qquad \gamma = \begin{cases} \frac{5}{2} - \frac{1}{\pi v_F}\left(\frac{1}{2}g_{1\parallel} - g_2\right) & \text{if } g_{1\perp} > 0 \\ -\frac{3}{2} - \frac{1}{\pi v_F}\left(\frac{1}{2}g_{1\parallel} - g_2\right) & \text{if } g_{1\perp} < 0 \end{cases}$$

/4.64/

$$\Delta_t(x) \sim x^{\delta} \quad , \qquad \delta = \frac{5}{2} - \frac{1}{\pi v_F}\left(\frac{1}{2} g_{1\parallel} - g_2\right). \tag{4.65}$$

Singularity can appear in the charge-density response function and in the singlet-superconductor response if $g_{1\perp} < 0$. The exponents are certainly modified by higher order corrections, in the next approximation /Ting 1976/ the exponent $-3/2$ is changed to -1.02 . It can be assumed, however, that the tendency is correctly obtained, only the charge-density and singlet-superconductor responses are divergent for $g_{1\perp} < 0$. The singularity in the singlet-superconductor response is stronger if $\frac{1}{2} g_{1\parallel} - g_2 > 0$, on the other hand for $\frac{1}{2} g_{1\parallel} - g_2 < 0$ the charge-density response is more strongly divergent.

The results are summarized in fig. 13, where the phase diagram is shown, i.e. the possible ground states of the system inferred from the singularity of the response function are indicated for spin-independent couplings in the (g_1, g_2) plane. The solution is probably quite good in the upper half-plane $(g_1 > 0)$, there may be, however, doubts about the results obtained for $g_1 < 0$. In the following paragraphs other approaches will be considered, which - combined with scaling argument - can provide a better treatment of the model.

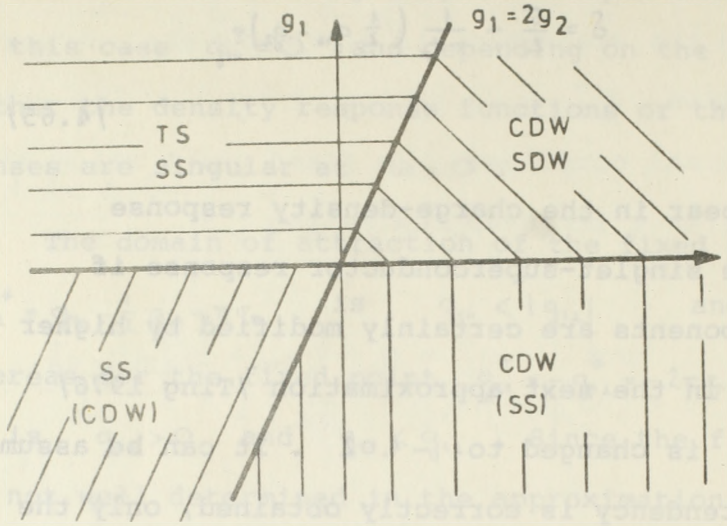

Fig. 13. Phase diagram of the 1-d Fermi gas obtained in the second order scaling approximation. The response functions corresponding to the phases indicated in bracket have a lower degree of divergence than those without bracket.

There is yet another response function which is of interest when comparison is made with the behaviour of quasi-1-d materials. This is the $4k_F$ response function $\Pi(4k_F, \omega)$ calculated by Lee et al. /1977/ using the renormalization group approach. Without speaking about the mechanism how the $4k_F$ fluctuations are excited by two-particle interactions, we consider the response function $\Pi(4k_F, \omega)$

which is defined through eqn. /4.39/ with

$$\mathcal{O}(4k_F, t) = \frac{1}{L^{3/2}} \sum_{k_1,k_2,k_3} b^+_{k_1\uparrow}(t)\, b^+_{k_2\downarrow}(t)\, a_{k_3\downarrow}(t)\, a_{k_1+k_2-k_3+4k_F\uparrow}(t). \qquad /4.66/$$

A straightforward perturbational calculation gives

$$\Pi(\omega) = \mathrm{const} + \frac{1}{32\pi^3 v_F^3}\,\omega^2 \ln\frac{\omega}{E_0}\left[1 + \frac{1}{\pi v_F}(g_{1\parallel} - 2g_2)\ln\frac{\omega}{E_0} + \dots\right]. \qquad /4.67/$$

Since the factor ω^2 appears in the contribution of all diagrams, and the remaining logarithms can give an extra power, the function

$$\Pi'(\omega) = \frac{1}{\pi v_F} \ln\frac{\omega}{E_0}\left[1 + \frac{1}{\pi v_F}(g_{1\parallel} - 2g_2)\ln\frac{\omega}{E_0} + \dots\right] \qquad /4.68/$$

can be studied. Assuming that the function $\overline{\Pi'}$ defined by eq. /4.51/ from the truncated function satisfies the scaling equation, we get

$$\frac{d \ln \overline{\Pi'}(x)}{dx} = \frac{1}{x}\left\{\frac{2}{\pi v_F}\left[g^R_{1\parallel}(x) - 2g^R_2(x)\right] + \dots\right\}. \qquad /4.69/$$

Since the combination $g_{1\parallel}-2g_2$ is invariant under scaling, we obtain for the $4k_F$ response function when the factor ω^2 is taken into account

$$\Pi(\omega) \sim \omega^{2+\frac{2}{\pi v_F}(g_{1\parallel}-2g_2)+\dots} \quad . \tag{4.70}$$

A singularity can appear provided that $g_{1\parallel}-2g_2$ has a large negative value. At this value the higher order corrections are not negligible. As it will be seen later an exact solution of this problem can be obtained. It will be still valid that $\Pi(\omega)$ is singular for large negative $g_{1\parallel}-2g_2$.

§ 5. EXACT SOLUTION OF THE TOMONAGA-LUTTINGER MODEL

The Tomonaga model /Tomonaga 1950/ is that particular case of the Fermi gas model in which the large momentum transfer interactions are neglected. These are the $g_{1\perp}$ and g_3 terms in the Hamiltonian. In the model with bandwidth cutoff, $g_{4\parallel}$ gives no contribution and the effect of $g_{1\parallel}$ is indistinguishable from that of $g_{2\parallel}$, therefore three couplings, $g_{2\parallel}$, $g_{2\perp}$ and $g_{4\perp}$ remain. It follows from the scaling equations /4.31/ - /4.33/ and the more detailed treatment of § 7 where all the couplings are considered, that the couplings do not get renormalized in the case when $g_{1\perp} = g_3 = 0$, i.e. the invariant couplings are the same as the bare couplings. Consequently $g_{1\perp}$ itself remains zero in the equivalent models. This model is of particular interest since - as it will be seen - models with $g_{1\perp} \neq 0$ and $g_3 \neq 0$ can be scaled onto it if $g_{1\parallel} \geq |g_{1\perp}|$ and $g_{1\parallel} - 2g_2 \geq |g_3|$. It turns out that the Tomonaga model can be solved exactly, thereby the solution of the Fermi gas model is known for the models which belong to the domain of attraction of the Tomonaga fixed points.

In the usual treatments of the Tomonaga model the dispersion relation is that shown in fig. 3 and a momentum transfer cutoff is used instead of the bandwidth cutoff.

In this case $g_{4\parallel}$ gives a finite contribution coming from the first order self-energy corrections shown in fig. 14.

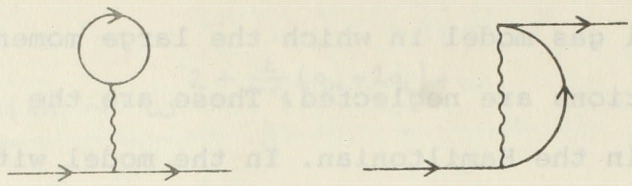

Fig. 14. First order self-energy diagrams.

The contributions of the two processes cancel each other in the model with bandwidth cutoff; with momentum transfer cutoff, however, this cancellation is not complete and a contribution proportional to the incoming momentum is obtained, which leads to a Fermi velocity renormalization. In the higher order diagrams the difference between the two kinds of cutoff is not important and either can be used. Another complication can arise from the $g_{1\parallel}$ and $g_{2\parallel}$ processes which become inequivalent in the model with momentum transfer cutoff. The problem of cutoffs will be considered in § 9. In this section momentum transfer cutoff will be used and the coupling $g_{2\parallel}$ will be considered together with $g_{2\perp}$, $g_{4\parallel}$ and $g_{4\perp}$.

A similar model has been proposed by Luttinger /1963/. The free particle spectrum is linear as shown in fig. 4.

Compared to the Tomonaga model new states are introduced far from the Fermi points. These new states are all filled in the ground state. Since it is usually assumed that only electrons and holes lying in the neighbourhood of the Fermi surface are important in physical processes, the Luttinger and Tomonaga models are equivalent. Mathematically it is more convenient to work with the Luttinger model. The results - at least in the weak coupling case when the states with electrons and holes far from the Fermi surface can be neglected - can be applied to the Tomonaga model, too.

5.1. Green's function in the Tomonaga-Luttinger model

Dzyaloshinsky and Larkin /1973/ have shown that the Green's function of the Tomonaga model can be calculated exactly in a simple form in real space and time representation. This is the consequence of two essential features of the model, namely that the dispersion relation is linear, and the number of particles in each branch and for each spin direction is conserved in every scattering process. They lead to the cancellation of many contributions and to a simple Ward identity and allow an exact summation of all the contributions.

The Green's functions of the unperturbed system are denoted by $G_+^{(0)}(k,\omega)$ and $G_-^{(0)}(k,\omega)$ corresponding to electrons on the two different branches,

$$G_+^{(0)}(k,\omega) = \frac{1}{\omega - v_F(k-k_F) + i\delta\, \mathrm{sign}\,(k-k_F)}$$

$$G_-^{(0)}(k,\omega) = \frac{1}{\omega - v_F(-k-k_F) + i\delta\, \mathrm{sign}\,(-k-k_F)}\,. \tag{5.1}$$

The low order self-energy diagrams are shown in fig. 15. The diagrams contain one solid line connecting the incoming and outgoing lines and may contain several loops, but solid and dashed lines never mix in the loops. This is due to the neglect of backward scattering processes. In the calculation of the contributions, the relations

$$G_+^{(0)}(k+p,\omega+\varepsilon)\,G_+^{(0)}(k,\omega) = \frac{1}{\varepsilon - v_F p}\left[G_+^{(0)}(k,\omega) - G_+^{(0)}(k+p,\omega+\varepsilon)\right], \tag{5.2}$$

$$G_-^{(0)}(k+p,\omega+\varepsilon)\,G_-^{(0)}(k,\omega) = \frac{1}{\varepsilon + v_F p}\left[G_-^{(0)}(k,\omega) - G_-^{(0)}(k+p,\omega+\varepsilon)\right] \tag{5.3}$$

can be used, which are the consequences of the linear dispersion. It turns out that all the diagrams which contain loops with more than two interaction vertices are cancelled. Such mutually cancelling diagrams are shown in fig. 16.

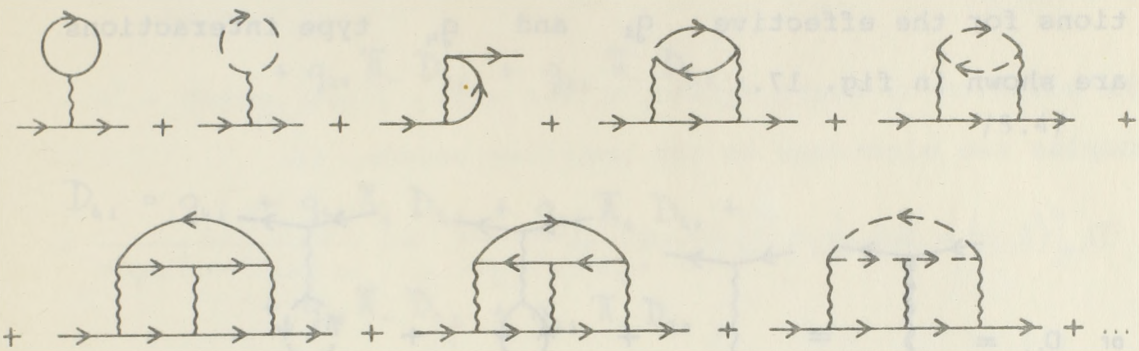

Fig. 15. Low order self-energy diagrams in the Tomonaga model for electrons on the right going branch.

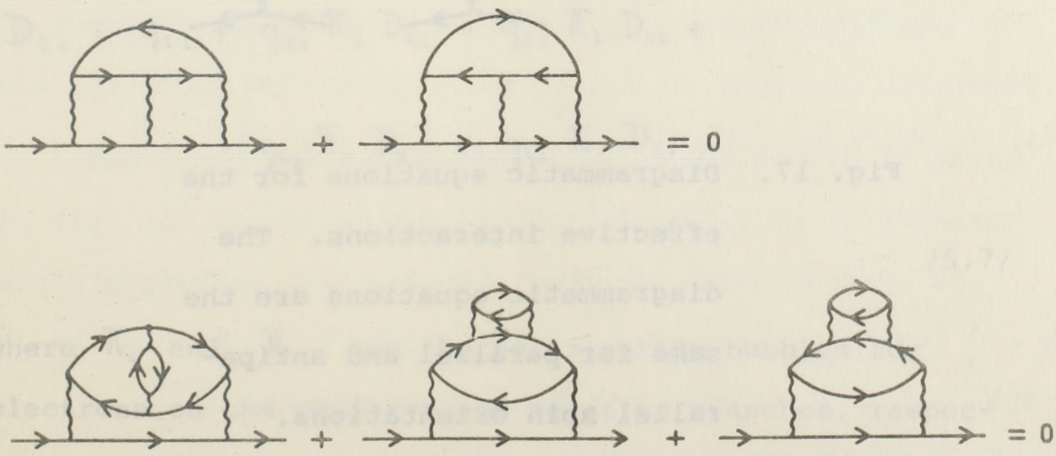

Fig. 16. Self-energy diagrams with cancelling contribution.

As a consequence of this cancellation the remaining self-energy diagrams may contain bubbles and series of bubbles only. The series of bubbles can easily be summed leading to effective interactions. The diagrammatic equations for the effective g_2 and g_4 type interactions are shown in fig. 17.

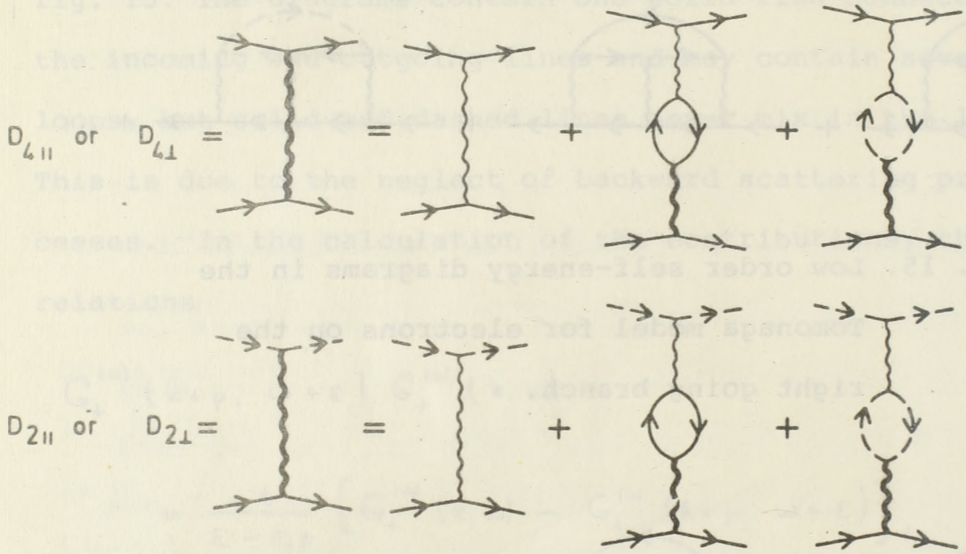

Fig. 17. Diagrammatic equations for the effective interactions. The diagrammatic equations are the same for parallel and antiparallel spin orientations.

Denoting the effective couplings by $D_{2\parallel}$, $D_{2\perp}$, $D_{4\parallel}$ and $D_{4\perp}$, these equations are simple algebraic equations:

$$D_{4\parallel} = g_{4\parallel} + g_{4\parallel}\Pi_+ D_{4\parallel} + g_{4\perp}\Pi_+ D_{4\perp} + $$
$$+ g_{2\parallel}\Pi_- D_{2\parallel} + g_{2\perp}\Pi_- D_{2\perp}\,, \qquad /5.4/$$

$$D_{4\perp} = g_{4\perp} + g_{4\parallel}\Pi_+ D_{4\perp} + g_{4\perp}\Pi_+ D_{4\parallel} + $$
$$+ g_{2\parallel}\Pi_- D_{2\perp} + g_{2\perp}\Pi_- D_{2\parallel}\,, \qquad /5.5/$$

$$D_{2\parallel} = g_{2\parallel} + g_{2\parallel}\Pi_+ D_{4\parallel} + g_{2\perp}\Pi_+ D_{4\perp} + $$
$$+ g_{4\parallel}\Pi_- D_{2\parallel} + g_{4\perp}\Pi_- D_{2\perp}\,, \qquad /5.6/$$

$$D_{2\perp} = g_{2\perp} + g_{2\parallel}\Pi_+ D_{4\perp} + g_{2\perp}\Pi_+ D_{4\parallel} + $$
$$+ g_{4\parallel}\Pi_- D_{2\perp} + g_{4\perp}\Pi_- D_{2\parallel}\,, \qquad /5.7/$$

where Π_+ and Π_- are the polarization bubbles for electrons on the positive and negative branches, respectively, for one spin orientation.

Using the expressions

$$\Pi_+(k,\omega) = \frac{k}{2\pi(\omega - v_F k)}, \quad \Pi_-(k,\omega) = -\frac{k}{2\pi(\omega + v_F k)},$$

/5.8/

the solution of these equations for e.g. $D_{4\parallel}$ which couples two electrons on the positive branch, is

$$D_{4\parallel}(k,\omega) = (\omega - v_F k)\left[\frac{A}{\omega - u_\sigma k + i\delta\,\mathrm{sign}\,k} + \frac{B}{\omega + u_\sigma k - i\delta\,\mathrm{sign}\,k} + \right.$$

$$\left. + \frac{C}{\omega - u_\rho k + i\delta\,\mathrm{sign}\,k} + \frac{D}{\omega + u_\rho k - i\delta\,\mathrm{sign}\,k}\right],$$

/5.9/

where

$$u_\sigma^2 = \left[v_F + \frac{1}{2\pi}(g_{4\parallel} - g_{4\perp})\right]^2 - \left[\frac{1}{2\pi}(g_{2\parallel} - g_{2\perp})\right]^2,$$

/5.10/

$$u_\rho^2 = \left[v_F + \frac{1}{2\pi}(g_{4\parallel} + g_{4\perp})\right]^2 - \left[\frac{1}{2\pi}(g_{2\parallel} + g_{2\perp})\right]^2,$$

/5.11/

$$A = \frac{1}{4}(g_{4\parallel} - g_{4\perp}) +$$

$$+ \frac{1}{4u_\sigma}\left[v_F(g_{4\parallel} - g_{4\perp}) + \frac{1}{2\pi}(g_{4\parallel} - g_{4\perp})^2 - \frac{1}{2\pi}(g_{2\parallel} - g_{2\perp})^2\right],$$

/5.12/

$$B = \frac{1}{4}\left(g_{4\parallel} - g_{4\perp}\right) - \frac{1}{4u_\sigma}\left[v_F\left(g_{4\parallel} - g_{4\perp}\right) + \frac{1}{2\pi}\left(g_{4\parallel} - g_{4\perp}\right)^2 - \frac{1}{2\pi}\left(g_{2\parallel} - g_{2\perp}\right)^2\right], \qquad /5.13/$$

$$C = \frac{1}{4}\left(g_{4\parallel} + g_{4\perp}\right) + \frac{1}{4u_\rho}\left[v_F\left(g_{4\parallel} + g_{4\perp}\right) + \frac{1}{2\pi}\left(g_{4\parallel} + g_{4\perp}\right)^2 - \frac{1}{2\pi}\left(g_{2\parallel} + g_{2\perp}\right)^2\right], \qquad /5.14/$$

$$D = \frac{1}{4}\left(g_{4\parallel} + g_{4\perp}\right) - \frac{1}{4u_\rho}\left[v_F\left(g_{4\parallel} + g_{4\perp}\right) + \frac{1}{2\pi}\left(g_{4\parallel} + g_{4\perp}\right)^2 - \frac{1}{2\pi}\left(g_{2\parallel} + g_{2\perp}\right)^2\right]. \qquad /5.15/$$

It should be noted that the effective coupling between two electrons on the negative branch is somewhat different.

The Dyson equation for the Green's function is given in fig. 18. The three-leg vertex appearing in it contains again effective interactions only, as shown in fig. 19. It is denoted by $\Gamma_+(p, \varepsilon, k, \omega)$. A similar vertex $\Gamma_-(p, \varepsilon, k, \omega)$ can be introduced for electrons on the negative branch.

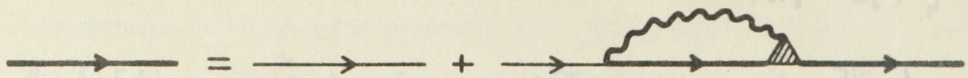

Fig. 18. Dyson equation for the Green's function.

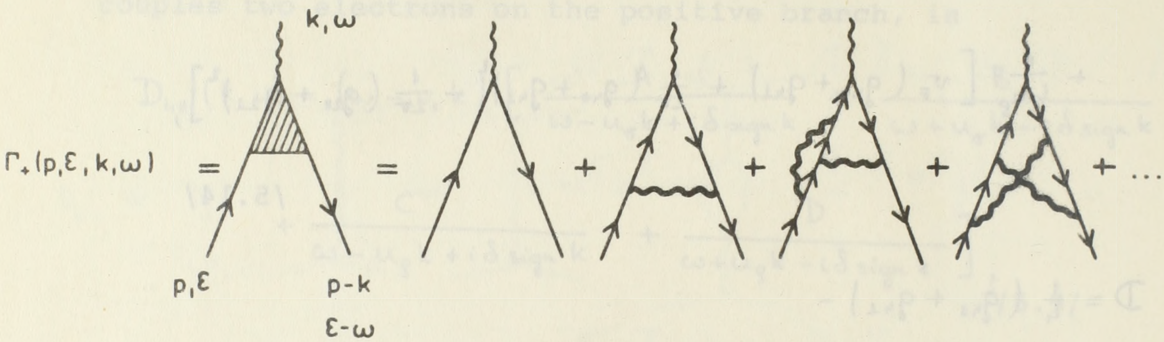

Fig. 19. Three-leg vertex appearing in the Dyson equation.

Using eqns. /5.2/ and /5.3/ a Ward identity can be derived,

$$\Gamma_+(p, \varepsilon, k, \omega) = \frac{G_+^{-1}(p, \varepsilon) - G_+^{-1}(p-k, \varepsilon-\omega)}{\omega - v_F k}$$

/5.16/

and analogously

$$\Gamma_-(p, \varepsilon, k, \omega) = \frac{G_-^{-1}(p, \varepsilon) - G_-^{-1}(p-k, \varepsilon-\omega)}{\omega + v_F k}$$

/5.17/

It should be emphasized that the Ward identities are direct consequences of the conservation of the number of particles in each branch for each spin orientation and of the linear dispersion relation. They can also be obtained by writing the equation of motion for the vertex $\Gamma_{\pm}$ /Everts and Schulz 1974/. The Ward identity allows to write the Dyson equation as a closed integral equation

$$\left[\varepsilon - v_F (p-k_F)\right] G_+ (p, \varepsilon) = 1 + \frac{i}{4\pi^2} \int dk\, d\omega \, \frac{D_{4\parallel}(k,\omega)}{\omega - v_F k} G_+ (p-k, \varepsilon - \omega) \qquad /5.18/$$

where the term leading to Fermi energy renormalization has been neglected.

This equation can be solved in a simple form by transforming it to real space and time and the following expression is obtained

$$G_+ (x,t) = \frac{1}{2\pi} \frac{1}{x - v_F t + i\delta(t)} \frac{x - v_F t + i/\Lambda(t)}{\left[x - u_\sigma t + i/\Lambda(t)\right]^{1/2} \left[x - u_\rho t + i/\Lambda(t)\right]^{1/2}} \times$$
$$\times \left[\Lambda^2 \left(x - u_\sigma t + i/\Lambda(t)\right)\left(x + u_\sigma t - i/\Lambda(t)\right)\right]^{-\alpha_\sigma} \times$$
$$\times \left[\Lambda^2 \left(x - u_\rho t + i/\Lambda(t)\right)\left(x + u_\rho t - i/\Lambda(t)\right)\right]^{-\alpha_\rho} , \qquad /5.19/$$

with

$$\alpha_\sigma = \frac{1}{4u_\sigma}\left[v_F + \frac{1}{2\pi}\left(g_{4\parallel} - g_{4\perp}\right) - u_\sigma\right],$$

/5.20/

$$\alpha_\rho = \frac{1}{4u_\rho}\left[v_F + \frac{1}{2\pi}\left(g_{4\parallel} + g_{4\perp}\right) - u_\rho\right].$$

/5.21/

In the calculation a smooth cutoff is used for the momentum transfer, i.e. a factor $\exp(-|k|/\Lambda)$ is introduced in the effective interaction and $\Lambda(t) = \Lambda$ sign t. A bandwidth cutoff $\delta \sim 1/k_F$ $(\delta(t) = \delta\ \mathrm{sign}\, t)$ should also appear in the free Green's function.

The analytic expression of the Green's function is complicated in momentum space. The interesting feature of the solution is that the Green's function has a branch cut instead of the usual pole structure. The cut is between $\varepsilon = u_\sigma(p-k_F)$ and $\varepsilon = u_\rho(p-k_F)$. The influence of the branch cut is negligible for electrons far from the Fermi surface. Near the Fermi surface, however, a drastic change occurs compared to normal Fermi systems. As a consequence of the branch cut, the momentum distribution $n(p)$ /Luttinger 1963, Gutfreund and Schick 1968/ has a power law type behaviour at the Fermi points

$$n(p) = \frac{1}{2} - \text{const}\ |p \mp k_F|^{2\alpha_\sigma + 2\alpha_\varrho} \operatorname{sign}(\pm p - k_F), \qquad /5.22/$$

with infinite slope at $p = \pm k_F$, since both α_σ and α_ϱ are small positive numbers in the weak coupling case. This has to be contrasted with the jump discontinuity in normal Fermi systems. The difference in the distribution function is an indication that the usual quasi-particle picture breaks down in the Tomonaga model, the excitations of the system are collective excitations. On the other hand the effective interactions are very similar to boson propagators. Eqn. /5.9/ can be interpreted as describing the propagation of two bosons, one with velocity u_σ the other with u_ϱ. These bosons are the density fluctuations of the fermion system. It will be shown in § 5.3 that the Tomonaga model can in fact be solved in terms of boson fields, related to the densities.

5.2. Response functions of the Tomonaga-Luttinger model

Arguments similar to those applied in the calculation of the Green's function can be used for the response functions. The same response functions will be studied as in § 4.3. It will be seen that similarly as before they have a simple form in real space and time representation. These charge-density and spin-density response functions correspond to large mementum transfer. The same response functions can also be calculated for small k and ω in momentum representation. In this case

$$O(k,t) = \frac{1}{L^{1/2}} \sum_{k_1} \left[a^+_{k_1\uparrow}(t) a_{k_1+k\uparrow}(t) + b^+_{k_1\uparrow}(t) b_{k_1+k\uparrow}(t) + a^+_{k_1\downarrow}(t) a_{k_1+k\downarrow}(t) + b^+_{k_1\downarrow}(t) b_{k_1+k\downarrow}(t) \right] \qquad /5.23/$$

for the charge-density response, and

$$O(k,t) = \frac{1}{L^{1/2}} \sum_{k_1} \left[a^+_{k_1\uparrow}(t) a_{k_1+k\uparrow}(t) + b^+_{k_1\uparrow}(t) b_{k_1+k\uparrow}(t) - a^+_{k_1\downarrow}(t) a_{k_1+k\downarrow}(t) - b^+_{k_1\downarrow}(t) b_{k_1+k\downarrow}(t) \right] \qquad /5.24/$$

for the spin-density response. The diagrams which contribute to these response functions are the bubble series diagrams, as in the effective couplings, i.e. the external vertices are connected with a bubble series. We get

$$N(k,\omega) = \frac{2}{\pi} \gamma_\rho \frac{u_\rho k^2}{\omega^2 - u_\rho^2 k^2} , \qquad \chi(k,\omega) = \frac{2}{\pi} \gamma_\sigma \frac{u_\sigma k^2}{\omega^2 - u_\sigma^2 k^2} , \qquad /5.25/$$

where

$$\gamma_\sigma = \left[\frac{1 + \frac{1}{2\pi v_F}(g_{4\parallel} - g_{4\perp}) - \frac{1}{2\pi v_F}(g_{2\parallel} - g_{2\perp})}{1 + \frac{1}{2\pi v_F}(g_{4\parallel} - g_{4\perp}) + \frac{1}{2\pi v_F}(g_{2\parallel} - g_{2\perp})} \right]^{1/2} \tag{5.26}$$

$$\gamma_\varrho = \left[\frac{1 + \frac{1}{2\pi v_F}(g_{4\parallel} + g_{4\perp}) - \frac{1}{2\pi v_F}(g_{2\parallel} + g_{2\perp})}{1 + \frac{1}{2\pi v_F}(g_{4\parallel} + g_{4\perp}) + \frac{1}{2\pi v_F}(g_{2\parallel} + g_{2\perp})} \right]^{1/2} \tag{5.27}$$

This result shows that the density fluctuations with velocity u_σ are in fact spin-density fluctuations, whereas those with velocity u_ϱ are charge-density fluctuations.

Turning now to the density responses at $2k_F$ and to the pairing functions, it is still valid that apart from the two lines connecting the external vertices of the response functions, only simple bubbles should appear. A new vertex Γ_{+-} can be introduced which corresponds to large momentum transfer, the incoming and outgoing electrons are on different branches. This vertex is shown in fig. 20. The charge-density and spin-density response functions can be written in terms of this vertex as shown diagrammatically in fig. 21.

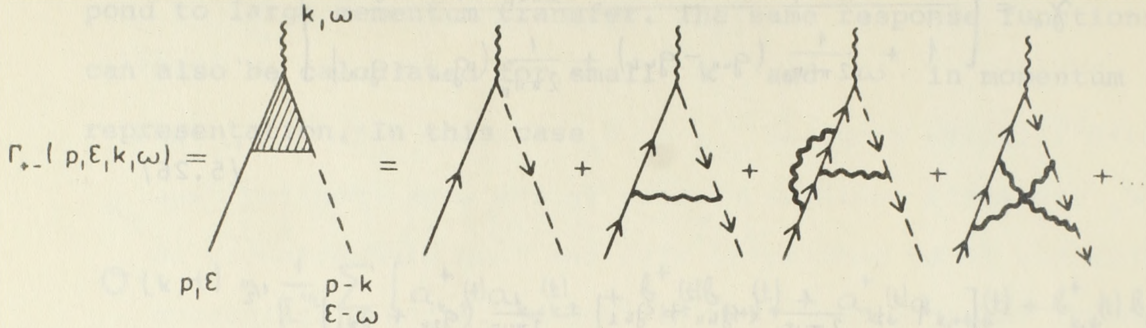

Fig. 20. Vertex with large momentum transfer.

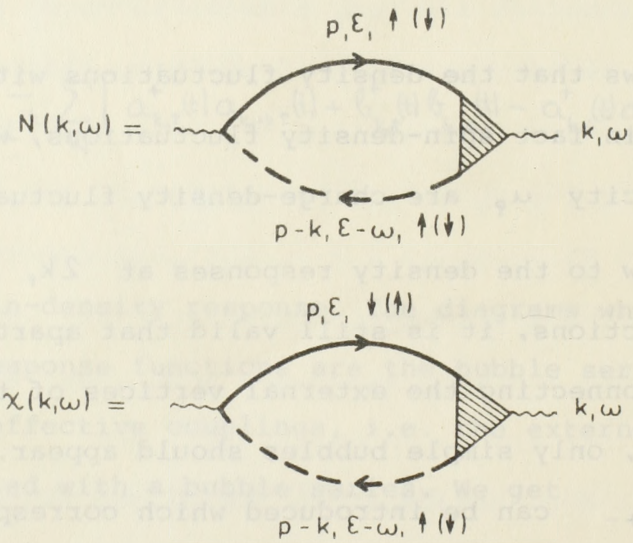

Fig. 21. Diagrammatic representation of the charge-

-density and transverse spin-density res-

ponse functions.

The vertex Γ_{+-} is not related to the Green's functions by a simple Ward identity. A generalized Ward identity can, however, be derived which relates Γ_{+-} to vertices with two electron lines and two interaction lines. Let us define a vertex in which the incoming and outgoing electrons belong to different branches, one interaction line corresponds to the coupling to the external field with large momentum transfer, the other interaction line couples to electrons on the positive branch. This vertex is shown in fig. 22. Using eqn. /5.2/ or the Ward identity in eqn. /5.16/, this vertex can be expressed in terms of Γ_{+-}, as also shown in fig. 22. A similar four-leg vertex can be introduced in which the interaction line couples to electrons on the negative branch. A generalized Ward identity can be derived for this vertex, too, as given in fig. 23. These relations are valid only if the self-energy corrections on the external Green's function legs are also taken into account.

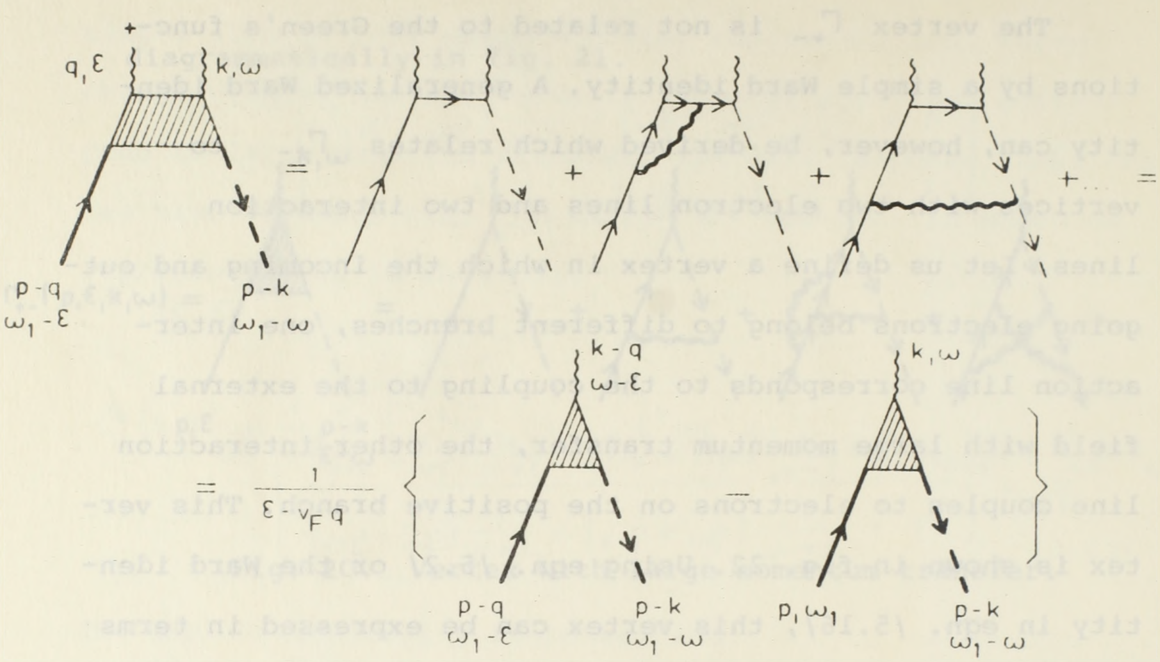

Fig. 22. Four-leg vertex with a large momentum transfer interaction and with a small momentum transfer interaction coupling to electrons on the positive branch.

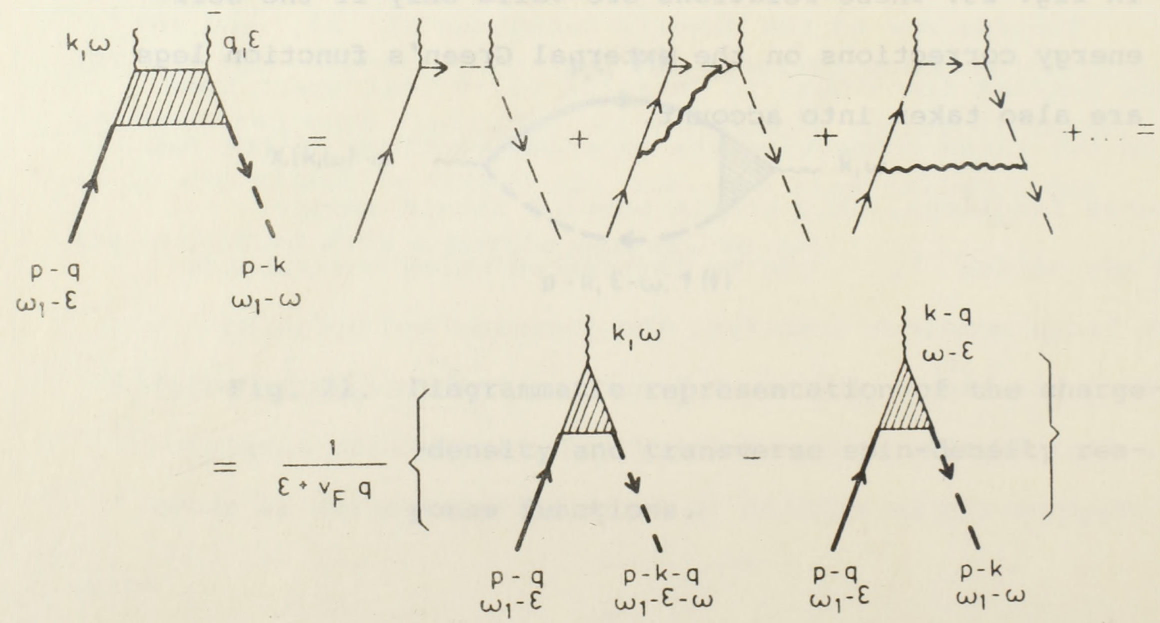

Fig. 23. Four-leg vertex with a large momentum transfer interaction and with a small momentum transfer interaction coupling to electrons on the negative branch.

The response functions can be expressed with the help of the four-leg vertices, as shown e.g. for $N(k,\omega)$ in fig. 24. The spin-density response $\chi(k,\omega)$ can be written similarly, with $D_{2\perp}$ instead of $D_{2\parallel}$. Comparing the two representations of the response functions in figs. 21 and 24 and using the relations given in figs. 22 and 23 the vertex Γ_{+-} and the response functions can be calculated. The solution of the integral equation in real space and time gives:

$$N(x,t) = -2i\, G_+(x,t)\, G_-(-x,-t)\cdot$$
$$\cdot\left[\Lambda^2\left(x-u_\sigma t+i/\Lambda(t)\right)\left(x+u_\sigma t-i/\Lambda(t)\right)\right]^{\beta_\sigma}\cdot$$
$$\cdot\left[\Lambda^2\left(x-u_\rho t+i/\Lambda(t)\right)\left(x+u_\rho t-i/\Lambda(t)\right)\right]^{\beta_\rho}$$

/5.28/

where

$$\beta_\sigma = \frac{1}{4\pi u_\sigma}\left(g_{2\parallel}-g_{2\perp}\right) \qquad \beta_\rho = \frac{1}{4\pi u_\rho}\left(g_{2\parallel}+g_{2\perp}\right).$$

/5.29/

$\chi(x,t)$ has the same structure as $N(x,t)$, only β_σ in the exponent is replaced by $-\beta_\sigma$.

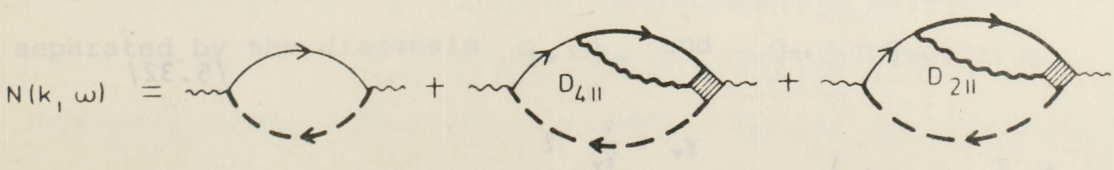

Fig. 24. Diagrammatic representation of the charge-density response in terms of the four-leg vertices of figs. 22 and 23.

A similar calculation can be performed for the pairing functions and we obtain:

$$\Delta_s(x,t) = 2i\, G_+(x,t)\, G_-(x,t) \cdot \left[\Lambda^2 \left(x - u_\sigma t + i/\Lambda(t)\right)\left(x + u_\sigma t - i/\Lambda(t)\right)\right]^{\beta_\sigma} \cdot \left[\Lambda^2 \left(x - u_\rho t + i/\Lambda(t)\right)\left(x + u_\rho t - i/\Lambda(t)\right)\right]^{-\beta_\rho} \qquad /5.30/$$

The triplet-superconductor response is obtained by replacing β_σ by $-\beta_\sigma$. In the case of spin-independent couplings these expressions reduce to those obtained by Fogedby /1976/ with a different method.

The analytic expressions of the response functions in (k,ω) representation are quite involved. Their asymptotic form can however be obtained from dimensionality arguments when ω and $v_F(k-2k_F)$ or $v_F k$ are small and of the same order of magnitude:

$$N\left(\omega \sim v_F(k-2k_F)\right) \sim \omega^{\gamma_\sigma + \gamma_\rho - 2}, \qquad /5.31/$$

$$\chi\left(\omega \sim v_F(k-2k_F)\right) \sim \omega^{\frac{1}{\gamma_\sigma} + \gamma_\rho - 2}, \qquad /5.32/$$

$$\Delta_s\left(\omega \sim v_F k\right) \sim \omega^{\gamma_\sigma + \frac{1}{\gamma_\rho} - 2}, \qquad /5.33/$$

$$\Delta_t (\omega - v_F k) \sim \omega^{\frac{1}{\gamma_\sigma} + \frac{1}{\gamma_\rho} - 2} , \qquad /5.34/$$

where γ_σ and γ_ρ are given in eqns. /5.26/ and /5.27/. Expanding the exponents in powers of the couplings, neglecting g_4 and taking into account that the choice of the couplings in § 4.3 corresponds to $g_{2\parallel} \to g_2 - g_{1\parallel}$ and $g_{2\perp} \to g_2$, it is seen from eqns. /4.58/ - /4.61/, that the renormalization group treatment in its first approximation gives the first terms in the exponents. Higher order terms in the Lie equations could produce the higher order terms in the exponents. The branch cut in the Green's function is not obtained by this simple scaling procedure, since it comes from non-logarithmic terms which have been neglected in the scaling.

The phase diagram analogous to the one shown in fig. 13 is given in fig. 25 in the $(g_{2\parallel}, g_{2\perp})$ plane. There are such regions in the plane of couplings where two response functions diverge. Assuming that the leading singularity will determine the structure of the system, the phase diagram is a very simple one: the different phases are separated by the diagonals $g_{2\parallel} = g_{2\perp}$ and $g_{2\parallel} = -g_{2\perp}$.

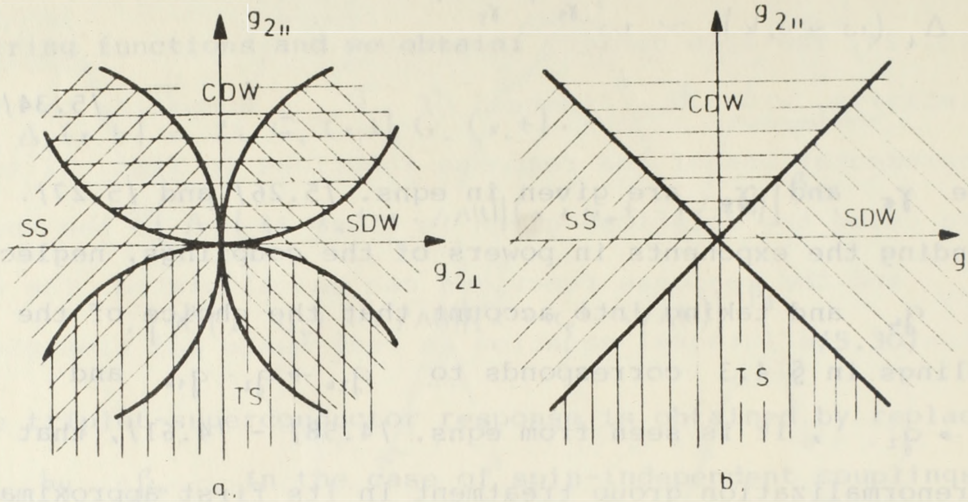

Fig. 25. Phase diagram of the spin dependent Tomonaga model.
Fig. /a/ shows the regions where the various response functions are singular. Fig. /b/ shows the phase diagram obtained from the dominant singularity.

Generalized Ward identities can be used also in the calculation of the $4k_F$ response function. A straightforward but tedious calculation of the Fourier transform of $\Pi(k-4k_F, \omega)$ defined by eqns. /4.39/ and /4.66/ leads to the result

$$\Pi(x,t) = G_+(x,t)\, G_+(x,t)\, G_-(-x,-t)\, G_-(-x,-t) \times$$
$$\cdot \frac{(x-u_\sigma t + i/\Lambda(t))(x+u_\sigma t - i/\Lambda(t))}{(x-u_\varrho t + i/\Lambda(t))(x+u_\varrho t - i/\Lambda(t))} \times$$
$$\cdot \left[\Lambda^2(x-u_\sigma t + i/\Lambda(t))(x+u_\sigma t - i/\Lambda(t))\right]^{4\alpha_\sigma} \times \qquad /5.25/$$
$$\cdot \left[\Lambda^2(x-u_\varrho t + i/\Lambda(t))(x+u_\varrho t - i/\Lambda(t))\right]^{-4\alpha_\varrho+4\beta_\varrho},$$

where α and β are given in eqns. /5.20/, /5.21/ and /5.29/. Inserting the expressions for the Green's functions we get

$$\Pi(x,t) = \left[\frac{(x - v_F t + i/\Lambda(t))(x + v_F t - i/\Lambda(t))}{(x - v_F t + i\delta(t))(x + v_F t - i\delta(t))}\right]^{2} \times$$
$$\times \left[\Lambda^2 (x - u_\rho t + i/\Lambda(t))(x + u_\rho t - i/\Lambda(t))\right]^{-2\gamma_\rho} , \qquad /5.36/$$

where γ_ρ is given in eqn. /5.27/. It is interesting to note that this response function depends on the charge-density modes only as noticed by Emery /1976a/. In the spin-independent case this result agrees with his result and with the result obtained by Klemm and Larkin /1978/.

5.3. Boson representation of the Tomonaga-Luttinger Hamiltonian

We have seen in the preceding paragraph that two kinds of boson-like collective excitations appear in the Tomonaga-Luttinger-model, one of them corresponds to long wave length charge-density oscillations, the other to spin-density oscillations. Mattis and Lieb /1965/ have in fact shown that the Luttinger model Hamiltonian can be written in terms of boson operators in a diagonal form and the exact spectrum is easily obtained.

Let us define the density operators for the two branches by

$$\rho_{1\alpha}(p) = \sum_k a^+_{k+p\,\alpha}\, a_{k\alpha} \,, \qquad \rho_{1\alpha}(-p) = \sum_k a^+_{k\alpha}\, a_{k+p\,\alpha} \,,$$

$$\rho_{2\alpha}(p) = \sum_k b^+_{k+p\,\alpha}\, b_{k\alpha} \,, \qquad \rho_{2\alpha}(-p) = \sum_k b^+_{k\alpha}\, b_{k+p\,\alpha} \,, \tag{5.35}$$

where $p>0$. The summation over k goes for the whole branch in the Luttinger model; in the Tomonaga model, however, the momenta k and $k+p$ should belong to the same branch, e.g. in $\rho_{2\alpha}(p)$ the summation goes for $k<-p$. These operators obey boson-like commutation relations:

$$\left[\rho_{1\alpha}(-p), \rho_{1\beta}(p')\right] = \left[\rho_{2\alpha}(p), \rho_{2\beta}(-p')\right] = \frac{pL}{2\pi}\,\delta_{\alpha\beta}\,\delta_{pp'} \,,$$

$$\left[\rho_{1\alpha}(p), \rho_{2\beta}(p')\right] = 0 \,. \tag{5.36}$$

They are exact relations in the Luttinger model. The same commutation relations are obtained in the Tomonaga model, but here they are obeyed approximately only. The non-vanishing contribution in the commutators comes from particles near $k=0$, i.e. from a region far from the Fermi points $\pm k_F$. Eqns. /5.36/ are valid for states in which no holes are excited in that region. This is the situation in the weak coupling case. In the strong coupling limit, when states far from the Fermi points are also excited, the use of a linearized dispersion relation cannot be justified and a different treatment of the electron gas model should be made /see § 11.4/.

Introducing the charge and spin densities by the definitions

$$\rho_i(p) = \frac{1}{\sqrt{2}}\left[\rho_{i\uparrow}(p) + \rho_{i\downarrow}(p)\right] , \qquad i = 1,2$$

/5.37/

$$\sigma_i(p) = \frac{1}{\sqrt{2}}\left[\rho_{i\uparrow}(p) - \rho_{i\downarrow}(p)\right] , \qquad i = 1,2$$

/5.38/

they obey the same commutation relations

$$\left[\rho_1(-p), \rho_1(p')\right] = \left[\rho_2(p), \rho_2(-p')\right] = \frac{pL}{2\pi}\delta_{pp'} ,$$

$$\left[\rho_1(p), \rho_2(p')\right] = 0$$

/5.39/

and similar relations hold for the spin-density operators. The charge and spin-density operators commute.

The interaction part of the Hamiltonian in the Tomonaga-Luttinger model can straightforwardly be expressed in terms of the charge and spin-density operators

$$H_{int} = \frac{1}{L}\sum_{p>0}(g_{2\parallel}+g_{2\perp})\left[\rho_1(p)\rho_2(-p)+\rho_1(-p)\rho_2(p)\right]+$$

$$+\frac{1}{L}\sum_{p>0}(g_{2\parallel}-g_{2\perp})\left[\sigma_1(p)\sigma_2(-p)+\sigma_1(-p)\sigma_2(p)\right]+$$

$$+\frac{1}{L}\sum_{p>0}(g_{4\parallel}+g_{4\perp})\left[\rho_1(p)\rho_1(-p)+\rho_2(-p)\rho_2(p)\right]+$$

$$+\frac{1}{L}\sum_{p>0}(g_{4\parallel}-g_{4\perp})\left[\sigma_1(p)\sigma_1(-p)+\sigma_2(-p)\sigma_2(p)\right]. \qquad /5.40/$$

The interaction Hamiltonian is bilinear in the charge and spin-density operators. This is a consequence of the neglect of the backward scattering terms.

As a consequence of the linear dispersion relation, the free Hamiltonian of eqn. /2.3/ too can be written in terms of the density operators in a bilinear form:

$$H_o = \frac{2\pi v_F}{L}\sum_{p>0}\left[\rho_1(p)\rho_1(-p)+\rho_2(-p)\rho_2(p)\right]$$

$$+\frac{2\pi v_F}{L}\sum_{p>0}\left[\sigma_1(p)\sigma_1(-p)+\sigma_2(-p)\sigma_2(p)\right]. \qquad /5.41/$$

This form of the Hamiltonian follows from the facts that

$$H_1 = \sum_{k,\alpha} v_F (k - k_F) a^+_{k\alpha} a_{k\alpha} + \sum_{k,\alpha} v_F(-k \; k_F) b^+_{k\alpha} b_{k\alpha} -$$

$$- \frac{2\pi v_F}{L} \sum_{p>0} \left[\rho_1(p) \rho_1(-p) + \rho_2(-p) \rho_2(p) \right] -$$

$$- \frac{2\pi v_F}{L} \sum_{p>0} \left[\sigma_1(p) \sigma_1(-p) + \sigma_2(-p) \sigma_2(p) \right] \qquad /5.42/$$

commutes with

$$H_2 = \frac{2\pi v_F}{L} \sum_{p>0} \left[\rho_1(p) \rho_1(-p) + \rho_2(-p) \rho_2(p) \right] +$$

$$+ \frac{2\pi v_F}{L} \sum_{p>0} \left[\sigma_1(p) \sigma_1(-p) + \sigma_2(-p) \sigma_2(p) \right] + H_{int} \qquad /5.43/$$

and the commutators of H_o with the density operators are the same irrespective whether eqn. /2.3/ or /5.41/ is used. As it will be seen, H_2 can be exactly diagonalized. The eigenstates of H_2 which are constructed with the help of ρ_i and σ_i are at the same time eigenstates of H_1 with eigenvalues depending only on the filling of the Fermi sea. For a fixed number of particles in the Fermi system H_1 gives a ground state energy renormalization only and will be neglected. H_2 can be considered as the Hamiltonian of the system and it will simply be denoted by H.

The total Hamiltonian given in eqns. /5.43/ and /5.40/ is conveniently split into a charge-density H_ρ and spin--density H_σ part. Both of them can be diagonalized by a canonical transformation

$$\tilde{H} = e^{iS} H e^{-iS} , \qquad /5.44/$$

where

$$S = \frac{2\pi i}{L} \sum_{p>0} \frac{\varphi(p)}{p} \left[\rho_1(p)\rho_2(-p) - \rho_1(-p)\rho_2(p) \right] \qquad /5.45/$$

for the charge-density part and

$$S = \frac{2\pi i}{L} \sum_{p>0} \frac{\psi(p)}{p} \left[\sigma_1(p)\sigma_2(-p) - \sigma_1(-p)\sigma_2(p) \right] \qquad /5.46/$$

for the spin-density part. Using the relations

$$e^{iS} \rho_1(p) e^{-iS} = \rho_1(p) \cosh\varphi(p) + \rho_2(p) \sinh\varphi(p) , \qquad /5.47/$$

$$e^{iS} \rho_2(p) e^{-iS} = \rho_2(p) \cosh\varphi(p) + \rho_1(p) \sinh\varphi(p) , \qquad /5.48/$$

and similar relations for the spin-density operators, which are valid for all p, diagonalization is achieved if

$$\tanh 2\varphi(p) = -\frac{g_{2\parallel} + g_{2\perp}}{2\pi v_F + g_{4\parallel} + g_{4\perp}}, \qquad \tanh 2\psi(p) = -\frac{g_{2\parallel} - g_{2\perp}}{2\pi v_F + g_{4\parallel} - g_{4\perp}}.$$

/5.49/

The dependence of φ and ψ on p comes from the dependence of the couplings on the momentum transfer. Finally we get

$$\tilde{H} = \tilde{H}_\sigma + \tilde{H}_\rho,$$

/5.50/

$$\tilde{H}_\sigma = \frac{2\pi u_\sigma}{L} \sum_{p>0} \left[\sigma_1(p)\,\sigma_1(-p) + \sigma_2(-p)\,\sigma_2(p) \right],$$

/5.51/

$$\tilde{H}_\rho = \frac{2\pi u_\rho}{L} \sum_{p>0} \left[\rho_1(p)\,\rho_1(-p) + \rho_2(-p)\,\rho_2(p) \right],$$

/5.52/

where u_σ and u_ρ are given in eqns. /5.10/ - /5.11/.

After a suitable normalization, boson operators can be defined

$$\alpha^+(p) = \left(\frac{2\pi}{pL}\right)^{1/2} \rho_1(p) \, , \qquad \alpha(p) = \left(\frac{2\pi}{pL}\right)^{1/2} \rho_1(-p) \, ,$$

$$\beta^+(p) = \left(\frac{2\pi}{pL}\right)^{1/2} \rho_2(-p) \, , \qquad \beta(p) = \left(\frac{2\pi}{pL}\right)^{1/2} \rho_2(p) \, ,$$

/5.53/

and similar operators for the spin-density part, which satisfy the canonical commutation relations. Written in terms of these boson operators, the Hamiltonian is the sum of free boson field Hamiltonians, the dispersion relation is $\varepsilon_k = u_\sigma k$ for the spin-density part and $\varepsilon_k = u_\rho k$ for the charge-density part, in agreement with the conclusions of the preceding paragraph.

Using the bosonized Hamiltonian, all quantities which are expressed in terms of the density operators can be easily calculated. This is the case for the long wavelength susceptibilities. The single-particle Green's function, the CDW and SDW response functions at $k = 2k_F$ and the superconductor responses are not trivially expressed in terms of the density operators. Theumann /1967/ succeeded in calculating the Green's function after very lengthy algebra. The advantage of the bosonized Hamiltonian became evident later, when Luther and Peschel /1974a/ and Mattis /1974a/ have shown independently that an operator identity can be used to calculate matrix elements of fermion operators in boson representation. The identities

$$\psi_{1s}(x) = \frac{1}{L^{1/2}} \sum_k a_{ks}\, e^{ikx} \Longleftrightarrow \frac{e^{ik_F x}}{L^{1/2}}\, e^{i\phi_{1s}} \exp\left[\frac{2\pi}{L} \sum_{p>0} \frac{e^{-\frac{\alpha}{2}p - ipx}}{p} \rho_{1s}(p)\right] \times$$

$$\times \exp\left[-\frac{2\pi}{L} \sum_{p>0} \frac{e^{-\frac{\alpha}{2}p + ipx}}{p} \rho_{1s}(-p)\right] =$$

$$= \frac{e^{ik_F x}}{(2\pi\alpha)^{1/2}}\, e^{i\phi_{1s}} \exp\left[\frac{2\pi}{L} \sum_p \frac{e^{-\frac{\alpha}{2}|p| - ipx}}{p} \rho_{1s}(p)\right], \qquad /5.54/$$

$$\psi_{2s}(x) = \frac{1}{L^{1/2}} \sum_k b_{ks}\, e^{ikx} \Longleftrightarrow \frac{e^{-ik_F x}}{L^{1/2}}\, e^{i\phi_{2s}} \exp\left[-\frac{2\pi}{L} \sum_{p>0} \frac{e^{-\frac{\alpha}{2}p - ipx}}{p} \rho_{2s}(p)\right] \times$$

$$\times \exp\left[\frac{2\pi}{L} \sum_{p>0} \frac{e^{-\frac{\alpha}{2}p + ipx}}{p} \rho_{2s}(-p)\right] =$$

$$= \frac{e^{-ik_F x}}{(2\pi\alpha)^{1/2}}\, e^{i\phi_{2s}} \exp\left[-\frac{2\pi}{L} \sum_p \frac{e^{-\frac{\alpha}{2}|p| - ipx}}{p} \rho_{2s}(p)\right] \qquad /5.55/$$

are operator identities in the sense that the matrix elements are the same for properly chosen states, i.e. since $\psi_{1s}(x)$ decreases the number of particles by one

$$\psi_{1s}(x)\, \Omega_1 \,|F, N+1\rangle =$$

$$= \frac{e^{ik_F x}}{(2\pi\alpha)^{1/2}}\, e^{i\phi_{1s}} \exp\left[\frac{2\pi}{L} \sum_p \frac{e^{-\frac{\alpha}{2}|p| - ipx}}{p} \rho_{1s}(p)\right] \Omega_1 \,|F, N\rangle, \qquad /5.56/$$

where $|F, N\rangle$ is the ground state having N particles on the positive branch and Ω_1 is an arbitrary function of ρ_{1s}. The parameter α is introduced to make the summation over p convergent, but the identity is in fact correct, the anticommutation relation of ψ_{1s} and ψ_{1s}^+ is satisfied, only if the limit $\alpha \to 0$ is taken at the end of the calculation. The anticommutation relations of ψ_{js} with different j and s are ensured by the phase factors

$$\phi_{1\uparrow} = 0 \,, \qquad \phi_{1\downarrow} = \pi \int dx \, \psi_{1\uparrow}^+(x) \, \psi_{1\uparrow}(x) \,,$$

$$\phi_{2\uparrow} = \pi \sum_s \int dx \, \psi_{1s}^+(x) \, \psi_{1s}(x) \,,$$

$$\phi_{2\downarrow} = \phi_{2\uparrow} + \pi \int dx \, \psi_{2\uparrow}^+(x) \, \psi_{2\uparrow}(x) \,. \qquad /5.57/$$

With this transformation from fermion to boson operators the Green's function and the response functions can be calculated /Luther and Peschel 1974a/. The results are the same as obtained in the preceding paragraphs. In fact Luther and Peschel did the calculation for the spinless Tomonaga model, which can be obtained from our earlier treatment when putting $g_{2\perp} = g_{4\perp} = 0$. As we have seen earlier in a consequent treatment of the Tomonaga model a cutoff is needed to avoid the non-physical divergences. This cutoff is usually taken in the momentum transfer. A consistent treatment of the

Tomonaga model in the bosonized representation would be to introduce a momentum transfer cutoff Λ and the parameter α and take the limit $\alpha \to 0$. Instead of this procedure it is common to keep α finite in which case no further cutoff is needed and α plays the role of the cutoff. This defines a particular model. The relationship between this model and others with different choices of the cutoff will be discussed in § 9.

§ 6. THE LUTHER-EMERY SOLUTION OF THE BACKWARD SCATTERING PROBLEM

The exact solution of the Tomonaga-Luttinger model is rendered possible by the neglect of the large momentum transfer interactions, $g_{1\perp}$ and g_3 . When these interactions are present, the Ward identities cannot be used and one cannot do better than to apply the renormalization group approach to sum a subset of diagrams, if the fermion representation of the Hamiltonian is used. Alternatively one can try to apply the bosonization transformation and explore the properties of the model in this way. Luther and Emery /1974/ realized using the bosonized Hamiltonian that for particular values of the couplings the backward scattering problem can be solved again exactly.

6.1. Calculation of the energy spectrum on the Luther-Emery line

We consider now the total Hamiltonian as given in eqns. /2.3/ - /2.4/, and use the boson representation of fermion operators. Writing the forward scattering processes in terms of the densities as in § 5.3 implies the use of a momentum transfer cutoff. As mentioned already, the terms with $g_{1\parallel}$ and $g_{2\parallel}$ are equivalent - apart from a sign difference - in the model with bandwidth cutoff, they are, however, not strictly equivalent if a momentum transfer cutoff is used. The difference is not significant if the contribution of states far from the Fermi surface can be neglected. It is therefore usual to assume the same boson representation for the $g_{1\parallel}$ term as that for the $g_{2\parallel}$ term and to use $g_{1\parallel}$ and g_2 as independent couplings. This means that in the expressions obtained for the Tomonaga model $g_{2\parallel}$ is replaced by $g_2 - g_{1\parallel}$ and $g_{2\perp}$ is replaced by g_2 .

The transformation of the large momentum transfer terms is more delicate. Let us study first the backward scattering term $g_{1\perp}$. Transforming this term from the momentum representation to real space, a non-local interaction appears:

$$\sum_s \int dx_1\, dx_2\, dx_3\, dx_4\; g_{1\perp}(x_1, x_2, x_3 - x_4)\, \psi^+_{1s}(x_1+x_3)\, \psi^+_{2,-s}(x_2+x_4)\, \psi_{1,-s}(x_4)\, \psi_{2s}(x_3) \tag{6.1}$$

The non-locality, the functional form of $g_{1\perp}$ depends on the choice of cutoff in momentum space. With any reasonable choice this interaction is of short range. Approximating this expression by a local interaction $g_{1\perp}\, \delta(x_1)\, \delta(x_2)\, \delta(x_3 - x_4)$, which is equivalent to having no cutoff in momentum space, the expression

$$\sum_s g_{1\perp} \int dx\, \psi^+_{1s}(x)\, \psi^+_{2,-s}(x)\, \psi_{1,-s}(x)\, \psi_{2s}(x)$$

/6.2/

can be written in boson representation in the following form

$$\frac{1}{(2\pi\alpha)^2}\, g_{1\perp} \int dx \left\{ \exp\left[\frac{2\pi}{L} \sum_p \frac{e^{-\frac{\alpha}{2}|p| - ipx}}{p} \sqrt{2}\left(\sigma_1(p) + \sigma_2(p)\right)\right] + h.c. \right\} =$$

$$= \frac{2}{(2\pi\alpha)^2}\, g_{1\perp} \int dx\, \cosh\left[\frac{2\pi}{L} \sum_p \frac{e^{-\frac{\alpha}{2}|p| - ipx}}{p} \sqrt{2}\left(\sigma_1(p) + \sigma_2(p)\right)\right].$$

/6.3/

The backward scattering term could be expressed entirely through the spin-density operators, though the expression is highly non-linear.

The same bosonization transformation can be performed on the umklapp terms in eq. /2.4/ leading to

$$\frac{1}{(2\pi\alpha)^2}\, g_3 \int dx \left\{ \exp\left[\frac{2\pi}{L} \sum_p \frac{e^{-\frac{\alpha}{2}|p| - ipx}}{p} \sqrt{2}\left(\rho_1(p) + \rho_2(p)\right) + ix\left(G - 4k_F\right)\right] + h.c. \right\}$$

/6.4/

The umklapp processes may play important role in a system with half-filled band. In this case $G = 4k_F$ and eqn. /6.4/

can be written as

$$\frac{2}{(2\pi\alpha)^2}\, g_3 \int dx \, \cosh\left[\frac{2\pi}{L}\sum_p \frac{e^{-\frac{\alpha}{2}|p|-ipx}}{p}\sqrt{2}\left(\rho_1(p)+\rho_2(p)\right)\right].$$

/6.5/

The umklapp term is expressed entirely in terms of the charge-density operators.

Using the results of § 5.3 and eqns. /6.3/ and /6.4/, the total Hamiltonian can be separated into two parts, one containing the spin-density degrees of freedom, the other the charge-density degrees of freedom. Remembering that our choice of the couplings corresponds to replacing $g_{2\parallel}$ by $g_2 - g_{1\parallel}$ and $g_{2\perp}$ by g_2, we get

$$H = H_\sigma + H_\rho ,$$

/6.6/

where the spin-density part is

$$H_\sigma = \frac{2\pi v_\sigma}{L}\sum_{p>0}\left[\sigma_1(p)\sigma_1(-p) + \sigma_2(-p)\sigma_2(p)\right] -$$

$$-\frac{g_{1\parallel}}{L}\sum_{p>0}\left[\sigma_1(p)\sigma_2(-p) + \sigma_1(-p)\sigma_2(p)\right] +$$

$$+\frac{g_{1\perp}}{(2\pi\alpha)^2}\int dx\left\{\exp\left[\frac{2\pi}{L}\sum_p\frac{e^{-\frac{\alpha}{2}|p|-ipx}}{p}\sqrt{2}\left(\sigma_1(p)+\sigma_2(p)\right)\right] + h.c.\right\},$$

/6.7/

and the charge-density part is

$$H_\rho = \frac{2\pi v_\rho}{L} \sum_{p>0} \left[\rho_1(p)\rho_1(-p) + \rho_2(-p)\rho_2(p) \right] +$$

$$+ \frac{2g_2 - g_{1\parallel}}{L} \sum_{p>0} \left[\rho_1(p)\rho_2(-p) + \rho_1(-p)\rho_2(p) \right] +$$

$$+ \frac{g_3}{(2\pi\alpha)^2} \int dx \left\{ \exp\left[\frac{2\pi}{L} \sum_p \frac{e^{-\frac{\alpha}{2}|p| - ipx}}{p} \left(\rho_1(p) + \rho_2(p)\right) + ix(G - 4k_F) \right] + \right.$$

$$\left. + h.c. \right\},$$

/6.8/

with

$$v_\sigma = v_F + \frac{1}{2\pi}(g_{4\parallel} - g_{4\perp}), \qquad v_\rho = v_F + \frac{1}{2\pi}(g_{4\parallel} + g_{4\perp}).$$

/6.9/

Before going on to solve the model, a few remarks have to be made about the approximations involved in this transformed Hamiltonian. One approximation is to neglect the difference between the $g_{1\parallel}$ and $g_{2\parallel}$ processes. This difference disappears if a bandwidth cutoff or no cutoff at all is used. The other approximation is in the bosonization of the large momentum transfer terms. If the limit $\alpha \to 0$ is taken, as it should be, these terms contain no cutoff. We know, however, that without cutoff non-physical divergences appear and a cutoff should necessarily be introduced. Since in the Tomonaga model the procedure to identify α with the cutoff and to keep it finite leads to results which are very similar to the ones obtained by using a momentum transfer cutoff, it is hoped that the same is true for the backward scattering model.

In the case of half-filled band the two parts of the Hamiltonian have exactly the same structure. For non-half--filled band the charge-density part reduces to a Tomonaga Hamiltonian for which the solution is already known. The further discussion relates to the spin-density part, but all results can trivially be extended to the charge-density part as well, if umklapp processes are important.

We have seen in § 5.3 that the bilinear part of H_σ can be diagonalized by a canonical transformation. Using this transformation for the whole H_σ, it affects the backward scattering term in a simple way, the exponent is multiplied by e^{ψ} /see eqn. /5.49/ for ψ / and

$$\tilde{H}_\sigma = \frac{2\pi u_\sigma}{L} \sum_{p>0} \left[\sigma_1(p)\,\sigma_1(-p) + \sigma_2(-p)\,\sigma_2(p) \right]$$

$$+ \frac{g_{1\perp}}{(2\pi\alpha)^2} \int dx \left\{ \exp\left[\frac{2\pi}{L} \sum_p \frac{e^{-\frac{\alpha}{2}|p| - ipx}}{p} \sqrt{2}\, e^{\psi} \left(\sigma_1(p) + \sigma_2(p) \right) \right] + \right.$$

$$\left. + h.c. \right\}, \qquad /6.10/$$

$$c_{k+k_F} = \cos\Theta_k\ \alpha_1(k+k_F) - \sin\Theta_k\ \alpha_2(k-k_F),$$

$$d_{k-k_F} = \sin\Theta_k\ \alpha_1(k+k_F) + \cos\Theta_k\ \alpha_2(k-k_F)$$

/6.16/

is performed to new operators $\alpha_1(k)$ and $\alpha_2(k)$. The transformed Hamiltonian is diagonal if

$$\tan 2\Theta_k = \frac{g_{1\perp}}{2\pi\alpha u_\sigma k},$$

/6.17/

and the spectrum for the two particles associated with $\alpha_1(k)$ and $\alpha_2(k)$ is

$$\varepsilon_1(k) = u_\sigma k_F + \mathrm{sign}\,(k-k_F)\left[u_\sigma^2 (k-k_F)^2 + \Delta_\sigma^2\right]^{1/2} \qquad k\sim k_F$$

$$\varepsilon_2(k) = u_\sigma k_F - \mathrm{sign}\,(k+k_F)\left[u_\sigma^2 (k+k_F)^2 + \Delta_\sigma^2\right]^{1/2} \qquad k\sim -k_F$$

/6.18/

where

$$\Delta_\sigma = \frac{|g_{1\perp}|}{2\pi\alpha}.$$

/6.19/

At the particular value of the couplings given in eqn. /6.15/ a gap is present in the excitation spectrum of the spin-

-density degrees of freedom. This has serious consequences for the response functions.

In the case of a half-filled band, when the charge-density part of the Hamiltonian has a form similar to the spin-density part, an analogous calculation /Emery et al. 1976/ gives that the problem with umklapp scattering is soluble for

$$\frac{2g_2 - g_{1\parallel}}{2\pi v_\rho} = \frac{2g_2 - g_{1\parallel}}{2\pi v_F + g_{4\parallel} + g_{4\perp}} = \frac{3}{5} . \qquad /6.20/$$

The charge-density excitations have a gap which is given by

$$\Delta_\rho = \frac{|g_3|}{2\pi\alpha} \qquad /6.21/$$

There is no gap if the band is not half-filled.

6.2 Response functions of the Luther-Emery model

The response functions have already been considered in § 4.3 in the framework of the renormalization group approach and in § 5.2 where exact results were given for the Tomonaga-Luttinger model. The same response functions will be treated here for the particular values of the couplings for which the Luther-Emery solution applies.

The response function defined by eqns. /4.40/ - /4.45/ can be expressed in terms of the ρ_i and σ_i operators using the bosonization transformations /5.54/ - /5.55/. To simplify the discussion only one term in $\sigma(k,t)$ will be considered. This will reflect the correct analytic behaviour of the whole response function. In real space and time representation the response functions

$$N(x,t) = -i < T\left\{ \psi^+_{2\uparrow}(x,t)\, \psi_{1\uparrow}(x,t)\, \psi^+_{1\uparrow}(0,0)\, \psi_{2\uparrow}(0,0) \right\} >, \tag{6.22}$$

$$\chi(x,t) = -i < T\left\{ \psi^+_{2\downarrow}(x,t)\, \psi_{1\uparrow}(x,t)\, \psi^+_{1\uparrow}(0,0)\, \psi_{2\downarrow}(0,0) \right\} >, \tag{6.23}$$

$$\Delta_s(x,t) = -i < T\left\{ \psi_{2\downarrow}(x,t)\, \psi_{1\uparrow}(x,t)\, \psi^+_{1\uparrow}(0,0)\, \psi^+_{2\downarrow}(0,0) \right\} >, \tag{6.24}$$

$$\Delta_t(x,t) = -i < T\left\{ \psi_{2\uparrow}(x,t)\, \psi_{1\uparrow}(x,t)\, \psi^+_{1\uparrow}(0,0)\, \psi^+_{2\uparrow}(0,0) \right\} >, \tag{6.25}$$

when written in terms of the ϱ_i and σ_i operators have the following form:

$$N(x,t) = -i \frac{e^{2ik_F x}}{(2\pi\alpha)^2} S_\varrho^+(x,t)\, S_\sigma^+(x,t)\,, \tag{6.26}$$

$$\chi(x,t) = -i \frac{e^{2ik_F x}}{(2\pi\alpha)^2} S_\varrho^+(x,t)\, S_\sigma^-(x,t)\,, \tag{6.27}$$

$$\Delta_s(x,t) = -i \frac{1}{(2\pi\alpha)^2} S_\varrho^-(x,t)\, S_\sigma^+(x,t)\,, \tag{6.28}$$

$$\Delta_t(x,t) = -i \frac{1}{(2\pi\alpha)^2} S_\varrho^-(x,t)\, S_\sigma^-(x,t)\,, \tag{6.29}$$

where

$$S_\varrho^\pm(x,t) = \Big\langle T\Big\{ e^{itH_\varrho} \exp\Big[\frac{2\pi}{L}\sum_p \frac{e^{-\frac{\alpha}{2}|p|-ipx}}{p} \frac{1}{\sqrt{2}}\big(\varrho_1(p) \pm \varrho_2(p)\big)\Big] e^{-itH_\varrho} \times$$
$$\times \exp\Big[-\frac{2\pi}{L}\sum_p \frac{e^{-\frac{\alpha}{2}|p|}}{p} \frac{1}{\sqrt{2}}\big(\varrho_1(p) \pm \varrho_2(p)\big)\Big]\Big\}\Big\rangle_\varrho\,, \tag{6.30}$$

and

$$S_\sigma^{\pm}(x,t) = \langle T\left\{ e^{itH_\sigma} \exp\left[\frac{2\pi}{L}\sum_p \frac{e^{-\frac{\alpha}{2}|p| - ipx}}{p} \frac{1}{\sqrt{2}}\left(\sigma_1(p) \pm \sigma_2(p)\right)\right] e^{-itH_\sigma} \times \right.$$

$$\left. \times \exp\left[-\frac{2\pi}{L}\sum_p \frac{e^{-\frac{\alpha}{2}|p|}}{p} \frac{1}{\sqrt{2}}\left(\sigma_1(p) \pm \sigma_2(p)\right)\right]\right\}\rangle_\sigma .$$

/6.31/

The ρ and σ parts of the response functions can be calculated independently. For a non-half-filled band the large momentum transfer terms affect H_σ only and H_ρ is the same as in the Tomonaga model. Therefore $S_\rho^{\pm}$ is easily obtained from the charge-density part of the response functions of the Tomonaga model in eqns. /5.28/ and /5.30/ by identifying the cutoff Λ with $1/\alpha$

$$S_\rho^{\pm}(x,t) = \left[(x - u_\rho t + i\alpha \operatorname{sign} t)(x + u_\rho t - i\alpha \operatorname{sign} t)\right]^{\gamma_\rho^{\pm 1}},$$

/6.32/

where γ_ρ is given in eqn./5.27/. With our present choice of the couplings $g_{2\parallel} \to g_2 - g_{1\parallel}$, $g_{2\perp} \to g_2$ and remembering that the Luther-Emery solution is obtained for $g_{1\parallel} = -6/5\,\pi v_F = -3/5\,(2\pi v_F + g_{4\parallel} - g_{4\perp})$,

$$\gamma_\rho = \left[\frac{1 + \frac{1}{2\pi v_F}(g_{4\parallel} + g_{4\perp}) - \frac{1}{2\pi v_F}(2g_2 - g_{1\parallel})}{1 + \frac{1}{2\pi v_F}(g_{4\parallel} + g_{4\perp}) + \frac{1}{2\pi v_F}(2g_2 - g_{1\parallel})}\right]^{1/2} =$$

$$= \left[\frac{1 + \frac{1}{2\pi v_F}(g_{4\parallel} + 4g_{4\perp}) - \frac{5g_2}{2\pi v_F}}{4 + \frac{1}{2\pi v_F}(4g_{4\parallel} + g_{4\perp}) + \frac{5g_2}{2\pi v_F}}\right]^{1/2} .$$

/6.33/

In calculating the σ part of the response functions the canonical transformation /5.44/ diagonalizing the bilinear part of the Hamiltonian is performed first leading to

$$S_{\sigma}^{\pm}(x,t) = \langle T\Big\{ e^{it\tilde{H}_{\sigma}} \exp\Big[\frac{2\pi}{L}\sum_{p}\frac{e^{-\frac{\alpha}{2}|p|-ipx}}{p}\frac{1}{\sqrt{2}}e^{\pm\varphi}\big(\sigma_1(p)\pm\sigma_2(p)\big)\Big] e^{-it\tilde{H}_{\sigma}} \times$$

$$\times \exp\Big[-\frac{2\pi}{L}\sum_{p}\frac{e^{-\frac{\alpha}{2}|p|}}{p}\frac{1}{\sqrt{2}}e^{\pm\varphi}\big(\sigma_1(p)\pm\sigma_2(p)\big)\Big]\Big\}\rangle_{\sigma} .$$

/6.34/

On the Luther-Emery line, where $\sqrt{2}e^{\varphi}=1$, $S_{\sigma}^{\pm}$ can be expressed in terms of the spinless fermion fields introduced in eqns. /6.12/ - /6.13/ and we get

$$S_{\sigma}^{+}(x,t) = 2\pi\alpha \langle T\Big\{\big[\psi_1(x,t)\psi_2^{+}(x,t)e^{-2ik_F x}\big]^{1/2}\big[\psi_2(0,0)\psi_1^{+}(0,0)\big]^{1/2}\Big\}\rangle ,$$

/6.35/

$$S_{\sigma}^{-}(x,t) = (2\pi\alpha)^2 \langle T\big\{\psi_1(x,t)\psi_2(x,t)\psi_2^{+}(0,0)\psi_1^{+}(0,0)\big\}\rangle$$

/6.36/

These expressions have been studied by Lee /1975/, Gutfreund and Klemm /1976/ and Finkelstein /1977/. The low frequency behaviour of the response functions is determined by the behaviour of $S_\sigma^\pm$ at large x and t. The most important difference in the behaviour of S_σ^+ and S_σ^- comes from the fact that S_σ^+ corresponds to the propagation of a particle-hole pair in the spinless fermion representation, whereas S_σ^- describes the propagation of a pair of particles. In this respect the square root in S_σ^+ is of minor importance. Since

$$\langle \left[\psi_2(0,0)\, \psi_1^+(0,0) \right]^{1/2} \rangle \neq 0 \,, \tag{6.37}$$

but

$$\langle \psi_2^+(0,0)\, \psi_1^+(0,0) \rangle = 0 \tag{6.38}$$

therefore in the limit $x, t \to \infty$, when $\langle A(x,t)\, B \rangle \approx \langle A(x,t) \rangle \langle B \rangle$, $S_\sigma^+(x,t)$ goes to a constant, while $S_\sigma^-(x,t)$ vanishes. Thus the low frequency behaviour of the charge-density and singlet-superconductor responses which contain S_σ^+ is completely determined by the ρ part and after Fourier transforming $S_\rho^\pm$ of eqn. /6.32/ we get

$$N(\omega) \sim \omega^{\gamma_\rho - 2} \,, \tag{6.39}$$

$$\Delta_s(\omega) \sim \omega^{\frac{1}{\gamma_\rho} - 2} \,. \tag{6.40}$$

Using eqn. /6.33/ for γ_ρ, the charge-density response function is divergent if

$$g_2 > -\frac{6}{5}\pi v_F - \frac{3}{5} g_{4\parallel} = -\frac{3}{5}\pi\left(v_\rho + v_\sigma\right), \qquad /6.41/$$

and the singlet-superconductor response is divergent if

$$g_2 < \frac{3}{5} g_{4\perp} = \frac{3}{5}\pi\left(v_\rho - v_\sigma\right). \qquad /6.42/$$

The point $g_2 = -\frac{3}{5}\pi v_F$ divides the regions in which one or the other response function is more divergent.

The behaviour of the spin-density and triplet-superconductor responses is different. $S_\sigma^-(x,t)$ is an oscillatory function of t and vanishes when $x, t \to \infty$. In Fourier representation $S_\sigma^-(k,\omega)$ has Fourier components only for $|\omega| > 2\Delta_\sigma$ corresponding to the fact that the minimum energy needed for the excitation of a pair of spinless fermions is $2\Delta_\sigma$. The response functions are convolutions of $S_\sigma^-(k_1,\omega_1)$ with $S_\rho^\pm(k-k_1, \omega-\omega_1)$. Due to the existence of a gap in $S_\sigma^-(k_1,\omega_1)$, the divergence of $S_\rho^\pm$ will not be present in the total response function at small ω. Gutfreund and Klemm /1976/ have also shown that these response function are not divergent at the gap edge either.

The calculations can similarly be done for a half-filled model. In this case the analysis given for $S_\sigma^\pm$ will be valid for $S_\rho^\pm$ as well. The response functions containing either S_σ^- or S_ρ^- cannot be divergent. The only divergence is in $N(k,\omega)$. Since both S_σ^+ and S_ρ^+ go to constant for large x and t, Fourier transformation for small k and ω gives

$$N(\omega) \sim \omega^{-2}. \qquad /6.43/$$

The $4k_F$ response function depends entirely on the charge-density degrees of freedom, therefore the same result is obtained as in the Tomonaga model, if the band is not half-filled.

§ 7. SCALING TO THE EXACTLY SOLUBLE MODELS

We have seen in § 5 and 6 that the Fermi gas model can be solved exactly for particular values of the couplings, namely for $g_{1\perp} = g_3 = 0$ /Tomonaga-Luttinger model/ and for $g_{1\parallel} = -\frac{3}{5}(2\pi v_F + g_{4\parallel} - g_{4\perp})$ or $-2g_2 + g_{1\parallel} = -\frac{3}{5}(2\pi v_F + g_{4\parallel} + g_{4\perp})$. A solution for other values of the couplings is not an easy task. A calculation of the eigenvalues of the Hamiltonian in the vicinity of the Luther-Emery limit was attempted by Schlottmann /1977a, 1977b/. He found that the gap exists in H_σ for $g_{1\parallel} < 0$, though the derivation of this result contains approximations, which restrict the validity of the calculation to the neighbourhood of the Luther-Emery line.

The renormalization group and scaling arguments presented in § 4 establish a relationship between the original problem and a set of problems in which the coupling constants have a somewhat different value. If an exactly soluble model appears among these equivalent systems, the physical behaviour of the original model can be predicted using the scaling arguments. The scaling relations were derived in § 4 neglecting g_3 and g_4 . We study now the scaling curves for the most general case with five coupling constants. As before, a bandwidth cutoff will be used and the couplings are $g_{1\parallel}$, $g_{1\perp}$, $g_2 = g_{2\parallel} = g_{2\perp}$, g_3 and $g_4 = g_{4\perp}$; in this case $g_{4\parallel}$ gives no contribution.

The scaling relations are analogous to those given in eqns. /4.12/ - /4.18/, they have to be complemented by relations for the vertices Γ_3 and Γ_4 shown in figs. 26. and 27. A lengthy but straightforward calculation /Kimura 1975, Sólyom 1975/ leads to the following equations for the invariant couplings /the superscript R is neglected for convenience/:

$$\frac{d g_{1\parallel}}{dx} = \frac{1}{x}\left\{ \frac{1}{\pi v_F} g_{1\perp}^2 + \frac{1}{2\pi^2 v_F^2}\left(g_{1\parallel} g_{1\perp}^2 + g_{1\perp}^2 g_4 \right) + \dots \right\}, \qquad /7.1/$$

$$\frac{d g_{1\perp}}{dx} = \frac{1}{x}\left\{ \frac{1}{\pi v_F} g_{1\parallel} g_{1\perp} + \frac{1}{4\pi^2 v_F^2}\left(g_{1\parallel}^2 g_{1\perp} + g_{1\perp}^3 + 2 g_{1\parallel} g_{1\perp} g_4 \right) + \dots \right\}, \qquad /7.2/$$

$$\frac{d\left(g_{1\parallel} - 2g_2\right)}{dx} = \frac{1}{x}\left\{ \frac{1}{\pi v_F} g_3^2 + \frac{1}{2\pi^2 v_F^2}\left[\left(g_{1\parallel} - 2g_2\right) g_3^2 - g_3^2 g_4 \right] + \dots \right\}, \qquad /7.3/$$

$$\frac{d g_3}{dx} = \frac{1}{x}\left\{ \frac{1}{\pi v_F}\left(g_{1\parallel} - 2g_2\right) g_3 + \frac{1}{4\pi^2 v_F^2}\left[\left(g_{1\parallel} - 2g_2\right)^2 g_3 + g_3^3 - 2\left(g_{1\parallel} - 2g_2\right) g_3 g_4 \right] + \dots \right\}, \qquad /7.4/$$

$$\frac{d g_4}{dx} = \frac{1}{x}\left\{ \frac{1}{4\pi^2 v_F^2}\left[-3 g_{1\parallel} g_{1\perp}^2 + 3\left(g_{1\parallel} - 2g_2\right) g_3^2 \right] + \dots \right\}, \qquad /7.5/$$

where x is the relative change of the cutoff. The combination $g_{1\parallel}-2g_2$ is used instead of g_2 for later convenience.

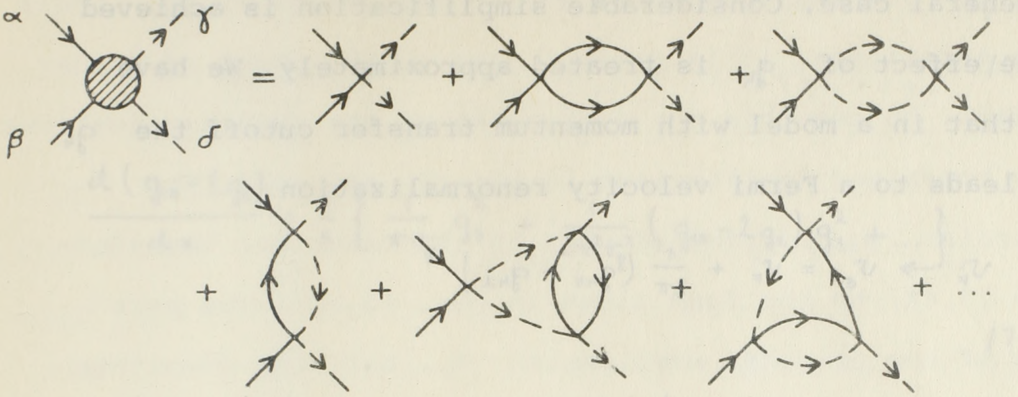

Fig. 26. Low order diagrams of the vertex $\Gamma_3 = g_3\tilde{\Gamma}_3\left(\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma}\right)$.

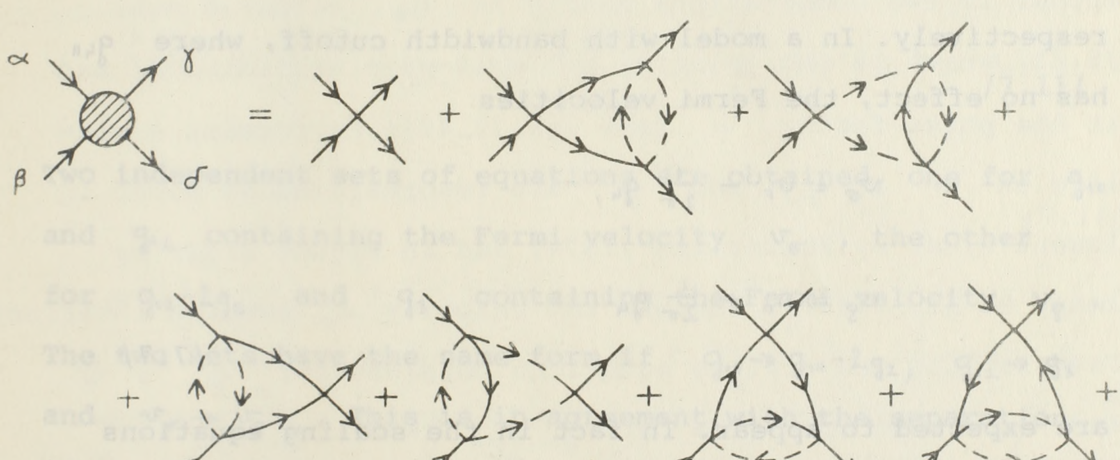

Fig. 27. Low order diagrams of the vertex $\Gamma_4 = g_4\tilde{\Gamma}_4\left(\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma}\right)$. All the third order diagrams are logarithmically divergent.

The scaling equations form a complicated set of coupled equations. No analytic solution is possible in the general case. Considerable simplification is achieved if the effect of g_4 is treated approximately. We have seen that in a model with momentum transfer cutoff the g_4 term leads to a Fermi velocity renormalization

$$v_F \rightarrow v_\sigma = v_F + \frac{1}{2\pi}\left(g_{4\parallel} - g_{4\perp}\right)$$

or

$$v_F \rightarrow v_\rho = v_F + \frac{1}{2\pi}\left(g_{4\parallel} + g_{4\perp}\right) \qquad /7.6/$$

in the spin- and charge-density part of the Hamiltonian, respectively. In a model with bandwidth cutoff, where $g_{4\parallel}$ has no effect, the Fermi velocities

$$v_\sigma = v_F - \frac{1}{2\pi} g_4 ,$$

$$v_\rho = v_F + \frac{1}{2\pi} g_4 \qquad /7.7/$$

are expected to appear. In fact in the scaling equations the terms containing g_4 can be incorporated in the calculated order into the renormalized Fermi velocities leading to

$$\frac{d g_{1\parallel}}{dx} = \frac{1}{x}\left\{ \frac{1}{\pi v_\sigma} g_{1\perp}^2 + \frac{1}{2\pi^2 v_\sigma^2} g_{1\parallel} g_{1\perp}^2 + \dots \right\} , \qquad /7.8/$$

$$\frac{dg_{1\perp}}{dx} = \frac{1}{x}\left\{ \frac{1}{\pi v_\sigma} g_{1\parallel} g_{1\perp} + \frac{1}{4\pi^2 v_\sigma^2}\left(g_{1\parallel}^2 g_{1\perp} + g_{1\perp}^3 \right) + \dots \right\}, \qquad /7.9/$$

$$\frac{d(g_{1\parallel} - 2g_2)}{dx} = \frac{1}{x}\left\{ \frac{1}{\pi v_\rho} g_3^2 + \frac{1}{2\pi^2 v_\rho^2}\left(g_{1\parallel} - 2g_2 \right) g_3^2 + \dots \right\}, \qquad /7.10/$$

$$\frac{dg_3}{dx} = \frac{1}{x}\left\{ \frac{1}{\pi v_\rho}(g_{1\parallel} - 2g_2) g_3 + \frac{1}{4\pi^2 v_\rho^2}\left[(g_{1\parallel} - 2g_2)^2 g_3 + g_3^3 \right] + \dots \right\}. \qquad /7.11/$$

Two independent sets of equations are obtained, one for $g_{1\parallel}$ and $g_{1\perp}$ containing the Fermi velocity v_σ, the other for $g_{1\parallel} - 2g_2$ and g_3 containing the Fermi velocity v_ρ. The two sets have the same form if $g_{1\parallel} \to g_{1\parallel} - 2g_2$, $g_{1\perp} \to g_3$ and $v_\sigma \to v_\rho$. This is in agreement with the separation of the Hamiltonian into H_σ and H_ρ in the bosonized form. These equations agree with the scaling equations derived by Emery et al. /1976/, using a somewhat different scaling approach by analogy with the scaling treatment of the Kondo problem by Anderson et al. /1970/. The cutoff α is the scaling parameter in this approach.

In disagreement, however, with the separation of H_σ and H_ρ , the scaling equation /7.5/ for g_4 couples all the other coupling constants. A renormalization of g_4 itself is not surprising, it is present also in the model of Luther and Emery. To see this the derivation of the scaling equations of the bosonized Hamiltonians is briefly sketched. The Hamiltonian H_σ is transformed by the canonical transformation /5.44/ to the form given in eqn. /6.10/. The parameters of the original Hamiltonian H_σ are: the couplings $g_{1\parallel}$, $g_{1\perp}$ and $g_{4\parallel}-g_{4\perp}$ as well as the cutoff α . After the canonical transformation the couplings $g_{1\parallel}$ and $g_{4\parallel}-g_{4\perp}$ appear in two combinations namely in u_σ given in eqn. /6.11/ /which is the velocity of spin-wave excitations/ and in the phase factor ψ /see eqn. /5.49//. Performing a scaling on α , $g_{1\perp}$ and ψ are renormalized as given in lowest order by Kosterlitz /1974/ and in higher order by Emery et al. /1976/, whereas the velocity u_σ remains invariant. Transforming the scaling equations obtained for $g_{1\perp}$, ψ and u_σ into equations for $g_{1\parallel}$, $g_{1\perp}$ and $g_{4\parallel}-g_{4\perp}$, eqns. /7.8/ and /7.9/ are recovered for $g_{1\parallel}$ and $g_{1\perp}$ with u_σ instead of v_σ . In the calculated order u_σ can be approximated by v_σ and the two methods lead to the same results. For $g_{4\parallel}-g_{4\perp}$, however, from the condition of the invariance of u_σ we get

$$\frac{d\,(g_{4\parallel}-g_{4\perp})}{dx} = \frac{g_{1\parallel}}{2\pi v_F + g_{4\parallel}-g_{4\perp}}\,\frac{d\,g_{1\parallel}}{dx} =$$

$$= \frac{1}{x}\left\{ \frac{1}{2\pi^2 v_F^2}\, g_{1\parallel}\, g_{1\perp}^2 + \dots \right\}. \qquad /7.12/$$

Similarly, scaling of α in H_ρ leads to a scaling relation for $g_{4\parallel}+g_{4\perp}$,

$$\frac{d\,(g_{4\parallel}+g_{4\perp})}{d\,x} = \frac{g_{1\parallel}-2g_2}{2\pi v_F + g_{4\parallel}+g_{4\perp}} \; \frac{d\,(g_{1\parallel}-2g_2)}{d\,x} =$$

$$= \frac{1}{x}\left\{ \frac{1}{2\pi^2 v_F^2}\,(g_{1\parallel}-2g_2)\,g_3^2 + \dots \right\} . \qquad /7.13/$$

From these two equations a scaling equation

$$\frac{d\,g_{4\perp}}{d\,x} = \frac{1}{x}\left\{ -\frac{1}{4\pi^2 v_F^2}\,g_{1\parallel}\,g_{1\perp}^2 + \frac{1}{4\pi^2 v_F^2}\,(g_{1\parallel}-2g_2)\,g_3^2 + \right\} \qquad /7.14/$$

is obtained for $g_{4\perp}$. This equation is very similar to eqn. /7.5/ apart from a factor of 3. There is, however, a fundamental difference. In the bosonized Hamiltonian $g_{4\parallel}$ is also present and the separation of H_ρ and H_σ is complete if the combinations $g_{4\parallel} \pm g_{4\perp}$ are used. In the model with bandwidth cutoff the $g_{4\parallel}$ coupling gives no contribution and the scaling equations are all coupled via $g_{4\perp}$. The difference comes from the different choice of the cutoff. This problem will be further discussed in § 9.

Neglecting now the renormalization of g_4 as a higher order correction, the renormalization group equations for $g_{1\parallel}$ and $g_{1\perp}$ are identical to the equations /4.31/ - /4.32/. The flow lines in the $(g_{1\parallel}, g_{1\perp})$ plane are reproduced in fig. 28. The Tomonaga-Luttinger model and the Luther-Emery model correspond to two crossing lines, the vertical line $g_{1\perp} = 0$ and the horizontal line $g_{1\parallel} = -\frac{6}{5}\pi v_\sigma$, respectively. Similar scaling curves are obtained in the $(g_{1\parallel} - 2g_2, g_3)$ plane for the dimensionless couplings $(g_{1\parallel} - 2g_2)/2\pi v_\rho$ and $g_3/2\pi v_\rho$. The models for which $g_{1\parallel} \geq |g_{1\perp}|$ or $g_{1\parallel} - 2g_2 \geq |g_3|$ are equivalent to the Tomonaga model at least in their σ or ρ parts and the results of the Tomonaga model can be used to determine their behaviour.

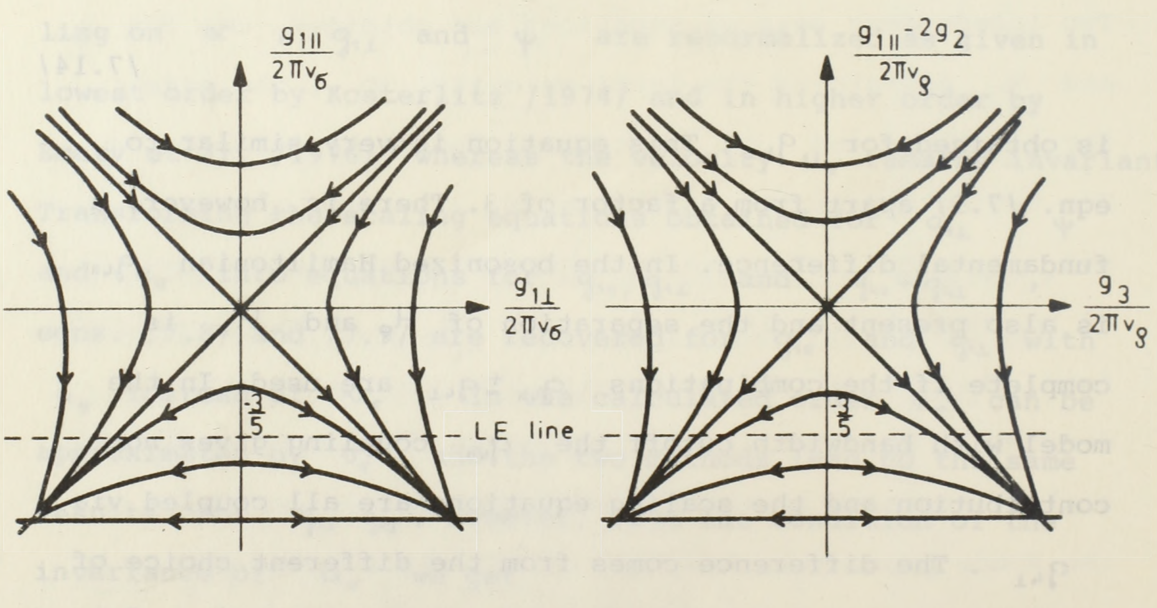

Fig. 28. Flow lines of scaling in the $(g_{1\parallel}, g_{1\perp})$ and $(g_{1\parallel}-2g_2, g_3)$ planes. The dotted line is the Luther-Emery line.

The models for which $g_{1\parallel} < |g_{1\perp}|$ or $g_{1\parallel} - 2g_2 < |g_{3\perp}|$ scale to a strong coupling fixed point, but before reaching this fixed point they pass through the Luther-Emery line. Thus the behaviour of these models can be inferred from the Luther-Emery solution, not relying on the approximations of § 4.3.

§ 8. PHYSICAL PROPERTIES OF THE MODEL

It was shown in the preceding paragraph that the weak coupling Fermi gas model is equivalent to either the Tomonaga-Luttinger model or the Luther-Emery model. Thus the properties of the model can be obtained from the known behaviour of these models. First the possible ground states will be discussed, then various quantities such as the susceptibility, specific heat, conductivity will be considered.

8.1. Phase diagram of the Fermi gas model

Let us consider first a simple situation when the couplings are spin-independent, $g_{1\parallel} = g_{1\perp} = g_1$, the band is not half-filled, i.e. g_3 gives no contribution and the Fermi velocity renormalization due to g_4 is neglected. The domain of attraction of the Tomonaga fixed point $(g_1^* = 0)$ is $g_1 \geq 0$. According to Fowler /1976/ $g_1 - 2g_2$ is an exact invariant of the renormalization transformation, therefore the fixed point value of g_2 is $g_2^* = g_2 - \frac{1}{2} g_1$. It follows from the scaling relations for the response functions given in eqn. /4.50/ that the ω dependence of the response functions of the original model with couplings g_1 and g_2 is the same as that of the Tomonaga model with the fixed point value $g_2^* = g_2 - \frac{1}{2} g_1$. Inserting this value into the exponents in eqns. /5.31/ - /5.34/, a singularity is obtained for small ω in the density responses if $g_2 - \frac{1}{2} g_1 > 0$ and in the pairing responses if $g_2 - \frac{1}{2} g_1 < 0$.

The behaviour of the system if $g_1 < 0$ can be determined by scaling to the Luther-Emery /LE/ line. The spin-density and triplet-superconductor responses are not singular on the LE line, so they are non-singular for arbitrary $g_1 < 0$. The ω dependence of the charge-density and singlet-superconductor responses is governed by the exponent γ_ρ /see eqns. /6.39/ and /6.40//. The value of γ_ρ given on the LE line in eqn. /6.33/ can be easily extended by using the invariance of $g_1 - 2g_2$ to give for $g_4 = 0$

$$\gamma_\rho = \left[\frac{1 - \frac{1}{2\pi v_F}(2g_2 - g_1)}{1 + \frac{1}{2\pi v_F}(2g_2 - g_1)} \right]^{-1/2}$$

/8.1/

Inserting this value into eqns. /6.39/ and /6.40/, the charge-density response function is divergent if

$$2g_2 - g_1 > \frac{6}{5}\pi v_F ,$$

/8.2/

and the singlet-superconductor response is divergent if

$$2g_2 - g_1 < \frac{6}{5}\pi v_F . \tag{8.3}$$

The line $2g_2 - g_1 = 0$ divides the regions in which one or the other response function is more divergent. The phase diagram is shown in fig. 29 /Lee 1975/. For weak couplings it coincides with the phase diagram obtained from the renormalization group treatment of § 4.3.

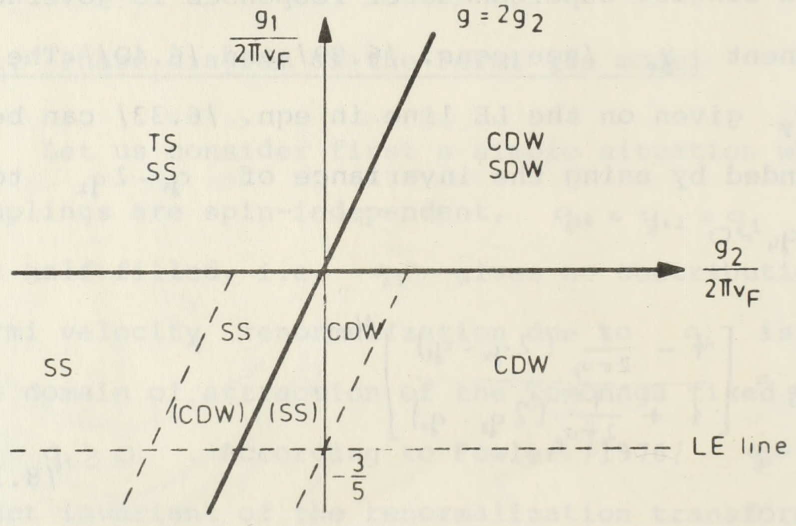

Fig. 29. Phase diagram obtained by scaling to the exact solutions of the Tomonaga-Luttinger and Luther-Emery models. The response functions corresponding to the phases indicated in bracket have a lower degree of divergence than those without bracket.

The phase diagram becomes much more complicated if the couplings are spin-dependent or if the band is half--filled and umklapp processes are important. There are four different regions in the space of couplings limited by the demarkation planes $g_{1\parallel} - 2g_2 = |g_3|$ and $g_{1\parallel} = |g_{1\perp}|$. In region I, where $g_{1\parallel} \geq |g_{1\perp}|$ and $g_{1\parallel} - 2g_2 \geq |g_3|$, there is no gap in either the ϱ or σ -part of the Hamiltonian; both H_ϱ and H_σ are equivalent to the Tomonaga Hamiltonian. The scaling equations can be solved approximately only leading to the fixed point values

$$g_{1\parallel}^* \approx \left(g_{1\parallel}^2 - g_{1\perp}^2\right)^{1/2}, \quad g_{1\perp}^* = 0, \quad g_{1\parallel}^* - 2g_2^* \approx \left[\left(g_{1\parallel} - 2g_2\right)^2 - g_3^2\right]^{1/2}, \quad g_3^* = 0.$$

These values are to be used in the expressions for the Tomonaga model. All the four response functions have a power law behaviour for small ω as given in eqns. /5.31/ - /5.34/ with exponents

$$\gamma_\sigma = \left[\frac{v_\sigma + \frac{1}{2\pi} g_{1\parallel}^*}{v_\sigma - \frac{1}{2\pi} g_{1\parallel}^*}\right]^{1/2} \approx \left[\frac{v_F - \frac{1}{2\pi} g_4 + \frac{1}{2\pi}\left(g_{1\parallel}^2 - g_{1\perp}^2\right)^{1/2}}{v_F - \frac{1}{2\pi} g_4 - \frac{1}{2\pi}\left(g_{1\parallel}^2 - g_{1\perp}^2\right)^{1/2}}\right]^{1/2} \qquad /8.4/$$

$$\gamma_\varrho = \left[\frac{v_\varrho + \frac{1}{2\pi}\left(g_{1\parallel}^* - 2g_2^*\right)}{v_\varrho - \frac{1}{2\pi}\left(g_{1\parallel}^* - 2g_2^*\right)}\right] \approx \left[\frac{v_F + \frac{1}{2\pi} g_4 + \frac{1}{2\pi}\left[\left(g_{1\parallel} - 2g_2\right)^2 - g_3^2\right]^{1/2}}{v_F + \frac{1}{2\pi} g_4 - \frac{1}{2\pi}\left[\left(g_{1\parallel} - 2g_2\right)^2 - g_3^2\right]^{1/2}}\right]^{1/2} \qquad /8.5/$$

Since both γ_σ and γ_ρ are larger than unity, the charge-density response function is not singular. The dominant singularity is in the triplet-superconductor response, though the spin-density and singlet-superconductor responses can also be divergent.

In region II, where $g_{1\parallel} < |g_{1\perp}|$ but $g_{1\parallel} - 2g_2 \geq |g_3|$, the models are equivalent to the Luther-Emery model. The σ -part of the Hamiltonian scales to the LE line, there is a gap in the σ -part of the excitation spectrum. The ρ -part scales to a Tomonaga model, the umklapp processes have no influence. Analogously to the situation discussed in § 6.2 only the singlet-superconductor and charge-density wave responses can be divergent. Their behaviour for small ω is given in eqns. /6.39/ and /6.40/ with γ_ρ given in eqn. /8.5/. The dominant singularity is in the singlet-superconductor response.

In region III, where $g_{1\parallel} \geq |g_{1\perp}|$ but $g_{1\parallel} - 2g_2 < |g_3|$, the situation is inversed concerning the σ and ρ -parts of the Hamiltonian. The gap is in the ρ -part and therefore only the charge- and spin-density responses containing S_ρ^+ can be divergent. The exponents are $\gamma_\sigma - 2$ and $1/\gamma_\sigma - 2$ for the charge and spin-density responses, respectively, with γ_σ given in eqn. /8.4/. The dominant singularity is in the spin-density response.

Finally in region IV, where $g_{1\parallel} < |g_{1\perp}|$ and $g_{1\parallel} - 2g_2 < |g_3|$, there is a gap in both the σ and ρ -parts of the Hamiltonian. As discussed in § 6.2 only the charge-density response function is divergent with an exponent -2.

The phase diagram has been constructed by Prigodin and Firsov /1976/ for this most general case. It is shown in fig. 30.

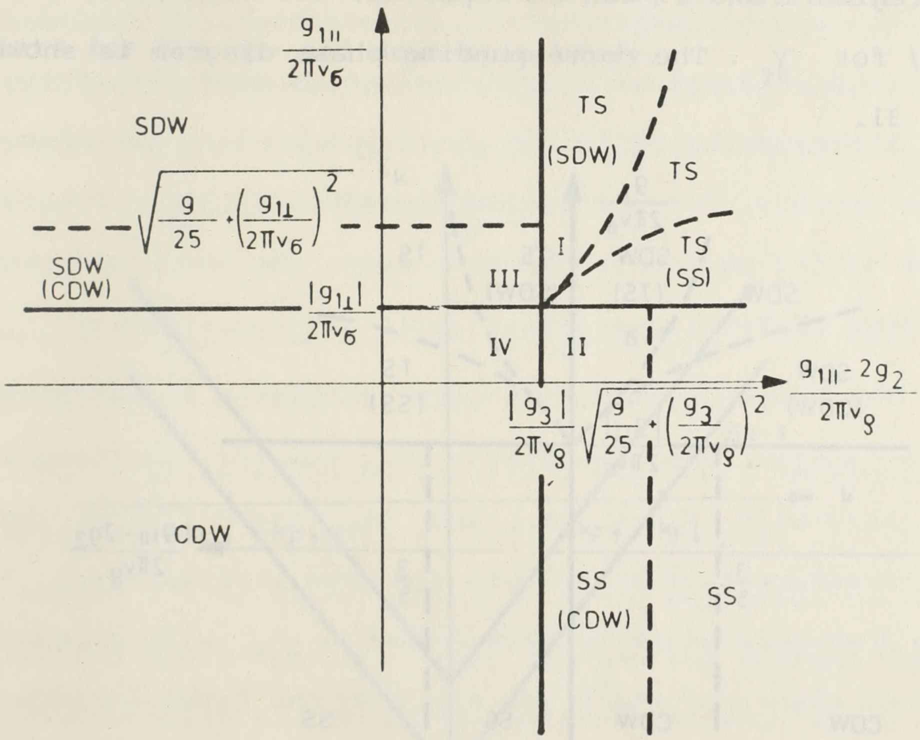

Fig. 30. Phase diagram of the Fermi gas model for a half-filled band with spin-dependent couplings. The response functions corresponding to the phases indicated in bracket have a lower degree of divergence.

The situation is somewhat different if the band is not half-filled. In this case the ϱ part is always equivalent to a Tomonaga model, $g_{1\parallel} - 2g_2$ is invariant and γ_ϱ is given by

$$\gamma_\varrho = \left[\frac{v_\varrho + \frac{1}{2\pi}(g_{1\parallel} - 2g_2)}{v_\varrho - \frac{1}{2\pi}(g_{1\parallel} - 2g_2)}\right]^{1/2} = \left[\frac{v_F + \frac{1}{2\pi}g_4 + \frac{1}{2\pi}(g_{1\parallel} - 2g_2)}{v_F + \frac{1}{2\pi}g_4 - \frac{1}{2\pi}(g_{1\parallel} - 2g_2)}\right]^{1/2} . \qquad /8.6/$$

Depending on whether $g_{1\parallel} \geq |g_{1\perp}|$ or $g_{1\parallel} < |g_{1\perp}|$ the analysis given for region I and II can be repeated, but with /8.6/ instead of /8.5/ for γ_ϱ . The corresponding phase diagram is shown in fig. 31.

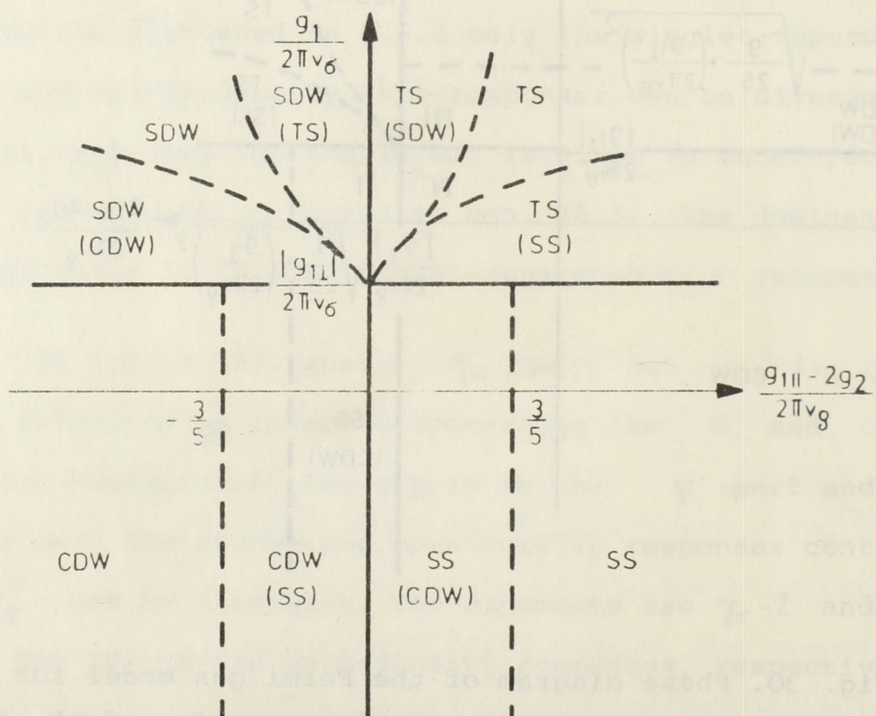

Fig. 31. Phase diagram of the Fermi gas model for not half--filled band with spin-dependent coupling. The response functions corresponding to the phases indicated in bracket have a lower degree of divergence.

Gurgenishvili et al. /1977/ studied the phase diagram in magnetic field. The energy spectrum of free electrons is modified to

$$\varepsilon_\sigma(k) = \varepsilon_k + \sigma\mu H = v_F(|k| - k_{F\sigma}), \quad \sigma = \pm 1. \qquad /8.7/$$

The energies are shifted differently for the two spin orientations, as shown in fig. 32, resulting in different Fermi points for the up- and down-spin bands.

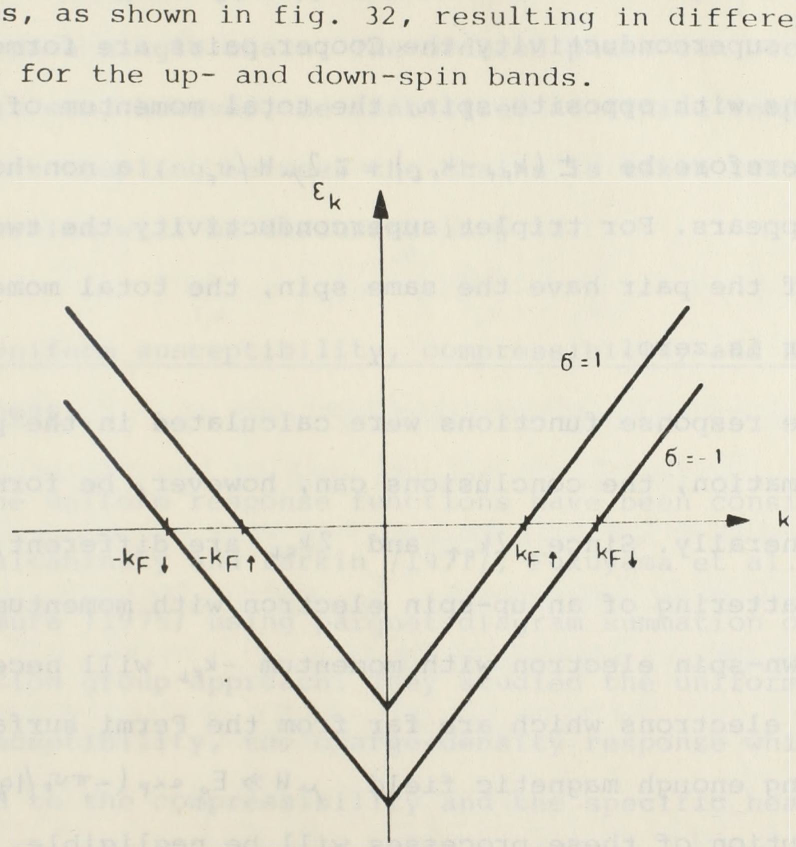

Fig. 32. Dispersion relation of the Fermi gas model in external magnetic field.

The charge-density response function corresponds to the propagation of an electron-hole pair with the same spin. The singularity will appear at $2k_{F\uparrow}$ or $2k_{F\downarrow}$ i.e. at $k = 2k_F \mp 2\mu H/v_F$. The charge-density wave state will therefore be a superposition of two charge-density waves. The spin-density response corresponds to the propagation of an electron-hole pair with different spin orientation, and the singularity appears at $k_{F\uparrow} + k_{F\downarrow} = 2k_F$. In the singlet superconductivity the Cooper pairs are formed from electrons with opposite spin, the total momentum of the pair will therefore be $\pm(k_{F\uparrow} - k_{F\downarrow}) = \mp 2\mu H/v_F$, a non-homogeneous state appears. For triplet superconductivity the two electrons of the pair have the same spin, the total momentum of the pair is zero.

The response functions were calculated in the parquet approximation, the conclusions can, however, be formulated more generally. Since $2k_{F\uparrow}$ and $2k_{F\downarrow}$ are different, backward scattering of an up-spin electron with momentum $+k_{F\uparrow}$ on a down-spin electron with momentum $-k_{F\downarrow}$ will necessarily produce electrons which are far from the Fermi surface. In strong enough magnetic field $\mu H \gg E_0 \exp(-\pi v_F/|g_1|)$ the contribution of these processes will be negligible, the backward scattering processes are frozen in. The forward scattering terms are not affected by the external field, therefore the model becomes equivalent to the Tomonaga model and the phase diagram is the same as in fig. 25.

The $4k_F$ response function has been studied in § 4.3 and 5.2. It depends on the charge-density degrees of freedom only, the results obtained in the Tomonaga model are therefore valid in the backward scattering model, provided the band is not half-filled. Singularity can appear for large negative values of $g_1 - 2g_2$.

This completes the determination of the possible ground states of a single chain. The ordered phase can occur at $T=0$ only, it can, however, be stabilized at finite temperatures if a weak coupling between the chains is taken into account. This problem will be discussed in § 12.

8.2. Uniform susceptibility, compressibility and specific heat.

The uniform response functions have been considered by Dzyaloshinsky and Larkin /1971/, Fukuyama et al. /1974a/ and Kimura /1975/ using parquet diagram summation or renormalization group approach. They studied the uniform magnetic susceptibility, the charge-density response which is related to the compressibility and the specific heat. The existence of a gap in the excitation spectrum of the charge- or spin-density degrees of freedom will manifest itself in these quantities and may lead to unusual temperature dependence.

The long wavelength susceptibilities of the Tomonaga model were obtained in eqn. /5.25/. In the so-called k limit, i.e. for $k \to 0$ and $\omega / k \to 0$ they go to a finite value. This is the situation for $g_{1\parallel} \geq |g_{1\perp}|$ and $g_{1\parallel} - 2g_2 \geq |g_3|$. Since $N(0,0)$ is related to the compressibility, both the magnetic susceptibility and the compressibility are finite in this region.

In the case when $g_{1\parallel} < |g_{1\perp}|$, a gap appears in the spectrum of the spin degrees of freedom, and as a consequence the magnetic susceptibility has an activated behaviour /Luther and Emery 1974/.

$$\chi(0,0) = \frac{1}{\pi u_\sigma} \left(\frac{2\pi \Delta_\sigma}{k_B T} \right)^{1/2} \exp\left[- \frac{\Delta_\sigma}{k_B T} \right] \qquad /8.8/$$

The value of the gap has been determined in § 6.1 for particular couplings only. Its calculation for arbitrary couplings is a delicate problem, we will return to this problem in § 10. The magnetic susceptibility vanishes at $T=0$ and has a maximum at $k_B T = 2\Delta_\sigma$.

The compressibility has an analogous activated behaviour for $g_{1\parallel} - 2g_2 < |g_3|$.

The activated behaviour in the specific heat appears for half-filled band only in the region $g_{1\parallel} < |g_{1\perp}|$ and $g_{1\parallel} - 2g_2 < |g_3|$. Otherwise there are always excitations which have no gap. The specific heat is then linear in temperature at low temperatures, the singularity of the Green's function does not show up in the specific heat.

8.3. Temperature dependence of the conductivity

In the Fermi gas model as described in § 2 the only way to get momentum relaxation is by umklapp processes. In that region of temperatures where the kinetic equation can be used $(\tau T \gg 1)$, the transport relaxation time coming from the electron-electron scattering /Gorkov and Dzyaloshinsky 1973/ is

$$\tau_{ee}^{-1} \sim T \, |\Gamma_3(T)|^2 , \tag{8.9}$$

where $\Gamma_3(T)$ is the renormalized vertex for umklapp scattering defined in fig. 26. The prefactor T is specific for the 1-d case and follows from phase space arguments in contrast to the T^2 dependence in higher dimensions. The vertex Γ_3 can be calculated by the use of the renormalization group. The Lie equation for Γ_3 is

$$\frac{d \ln \Gamma_3(x)}{dx} = \frac{1}{x}\left\{ \frac{1}{\pi v_F}\left(g_{1\parallel} - 2g_2\right) - \frac{i\pi}{2\pi^2 v_F^2}\left[\left(g_{1\parallel} - 2g_2\right)^2 + g_3^2\right] + \right.$$
$$\left. + \frac{1}{4\pi^2 v_F^2}\left(-g_{1\perp}^2 - 2g_{1\parallel}g_2 + 2g_2^2\right) + \ldots \right\} , \tag{8.10}$$

where $x = T/E_0$. Depending on the sign of $g_{1\parallel} - 2g_2$, the resistivity is increased or decreased compared to the linear dependence. The conductivity can be a monotonous function of temperature or can have a maximum, though the results obtained from eqns. /8.9/ and /8.10/ cannot be taken seriously if $|\Gamma_3(T)| > 1$.

Unfortunately no better theory is available for the conductivity due to electron-electron scattering. One of the reasons is that the dominant contribution to the resistivity in one-dimensional conductors is probably coming from other scattering mechanisms, such as electron-phonon scattering or scattering on impurities; moreover the effect of disorder can be very important. A detailed discussion of these effects is outside the scope of the present review. A detailed study of the resistivity caused by impurities is presented in the review by Abrikosov and Ryzhkin /1978/. Here only a few remarks will be made.

It is known from the work of Mott and Twose /1961/ that a random potential, however weak it is, leads to localization of the states in 1-d systems, Berezinsky /1973/ made a detailed calculation of the conductivity of a 1-d electron gas taking into account the effect of impurities exactly. The electron-electron interaction was neglected. He has shown that the dc conductivity vanishes, indicating that the electron states are in fact localized. The conductivity at finite frequencies depends on the amplitude of backward scattering on the impurity and has a maximum at $\omega \sim 1/\tau_2$ where τ_2 is the corresponding collision time. This conductivity is roughly temperature independent.

The interaction between electrons can give rise to an essential temperature dependence of the conductivity. Luther and Peschel /1974b/ and Mattis /1974b/ have shown that treating the impurity scattering in Born approximation, but considering the electron-electron interaction exactly, the conductivity of the Tomonaga-Luttinger model is strongly temperature dependent and can be enhanced compared to the non-interacting case. It can be divergent at $T = 0$ for attractive g_2 interactions and its temperature dependence is

$$\sigma(T) \sim \sigma_0(T) \cdot T^{\gamma} , \qquad \gamma = \frac{\frac{g_2}{2\pi v_F}}{1 + \frac{g_2}{\pi v_F}} , \tag{8.11}$$

where $\sigma_0(T)$ is the conductivity for impurity scattering calculated in Born approximation for the non-interacting system. In deriving this expression, Luther and Peschel used a relationship obtained by Götze and Wölfe /1972/ between the momentum relaxation time and charge-density response function

$$\tau^{-1} = \lim_{\omega \to 0} \frac{c}{\omega} U^2 v_F^2 \sum_{k,k'} (k-k')^2 \, \mathrm{Im}\, N(k-k', \omega) , \tag{8.12}$$

where c is the impurity concentration and $\mathcal{U}$ is the impurity potential. The main contribution comes from the $2k_F$ part of the charge-density response. The results have been extended by Gorkov and Dzyaloshinsky /1973/ and by Fukuyama et al. /1974b/ for the backward scattering model. Since the impurity scattering is treated in Born approximation, the localization effects are neglected. Therefore these calculations cannot predict the correct low temperature behaviour of the conductivity.

§ 9. DIFFERENT CHOICES OF THE CUTOFF

It was emphasized in § 2 where the Fermi gas model was introduced that the cutoff has to be properly given in order to have a well defined theory. It was shown that there are two usual choices of the cutoff with different physical interpretation. In one case, i.e. in the model with bandwidth cutoff, all the momenta are restricted to a band which is symmetric around the Fermi points. This cutoff procedure was used in the renormalization group treatment. The other choice is to have a momentum transfer cutoff, i.e. to limit the momentum p in eqn. /2.4/ to small values. In fact the transferred momentum is small in the g_2 and g_4 terms, whereas it is around $2k_F$ for the g_1 and g_3 terms. These latter processes can be interpreted as coming from phonon mediated effective electron-electron scattering processes in which a phonon with momentum $2k_F$ is exchanged. Physically it would be more appropriate to have a cutoff on the energy transfer, but this model is less easy to treat.

In the Tomonaga model where the large momentum transfer terms are neglected, a cutoff $k_c \sim \Lambda$ on p allows a consistent treatment as shown in § 5. It was pointed out by Chui et al. /1974/ that in a model with phonon mediated electron-electron interactions where the backward scattering terms are important two cutoffs should appear: a bandwidth cutoff E_0 and an energy transfer cutoff ω_D. While a model with bandwidth cutoff is properly defined, a model with energy transfer cutoff is not well defined if backward scattering terms are present and a bandwidth cutoff should necessarily be introduced. The energy transfer cutoff will be replaced for simplicity by a momentum transfer cutoff k_c, where $\omega_D = 2v_F k_c$. The results obtained for models with two characteristic energies will be reviewed in this section.

A different cutoff procedure is used if the bosonized Hamiltonian is considered. In this case the parameter α is kept finite and identified with the cutoff. Following Grest /1976/ and Theumann /1977/ the problem whether α can be taken as a physical cutoff will be discussed in § 9.2.

9.1. Scaling theories with two cutoffs

The phonon mediated attractive electron-electron interaction which is used in the BCS theory of superconductivity is usually taken in an energy range of width $2\omega_D$ around the Fermi energy, where ω_D is the Debye energy. This cutoff is usually smaller than the real bandwidth, this latter plays, however, no role in a 3-d system where only the Cooper-pair diagrams are singular. In a 1-d system, when electron-hole bubbles should also be taken into account on an equal footing, both cutoffs appear in logarithmic contributions. Fig. 33 shows two typical diagrams. In the Cooper pair diagram of fig. 33.a the momenta in the intermediate state are limited to the neighbourhood of the incoming momenta if the momentum transfer is restricted to a range $2k_c$ around $2k_F$ and the analytic contribution is

$$- \frac{g_1^2}{2\pi v_F} \left(\ln \frac{\omega}{\omega_D} - \frac{i\pi}{2} \right) \delta_{\alpha\delta} \delta_{\beta\gamma} . \qquad /9.1/$$

The transferred momentum of the electron-hole bubble of fig. 33.b does not appear in the integration, the momenta in the intermediate state are not limited by the transfer cutoff and therefore a bandwidth cutoff has to be used leading to

$$\frac{g_1^2}{\pi v_F} \left(\ln \frac{\omega}{E_0} - \frac{i\pi}{2} \right) \delta_{\alpha\gamma} \delta_{\beta\delta} . \qquad /9.2/$$

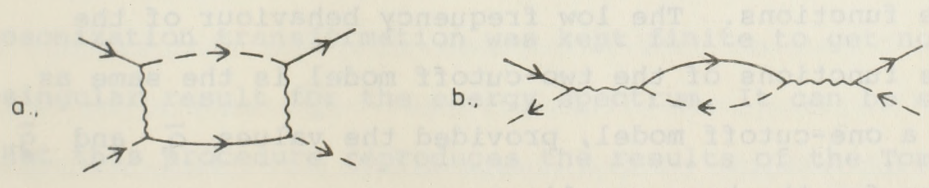

Fig. 33. Cooper pair /a/ and electron-hole polarization /b/ diagrams in lowest order. The wavy line represents a phonon exchange.

The model with two cutoffs, a cutoff k_c for the momentum transfer and another cutoff E_0 for the bandwidth, has been studied by Grest et al. /1976/ using the renormalization group. These authors show, extending a suggestion by Chui et al. /1974/ that this problem can be mapped onto a one-cutoff problem by eliminating the phase space between the energies ω_D and E_0. The new couplings of the one--cutoff problem are

$$\bar{g}_1 = \frac{g_1}{1 - \frac{g_1}{\pi v} \ln \frac{\omega_D}{E_0}} , \qquad \bar{g}_2 = g_2 , \tag{9.3}$$

and the Fermi velocity v_F is replaced by

$$v = v_F - \frac{g_1}{2\pi} - \frac{g_1^2}{2\pi^2 v} \ln \frac{\omega_D}{E_0} + \ldots \tag{9.4}$$

The apparent divergence of $\bar{g}_1$ is cured if this Fermi velocity renormalization is taken into account. The scaling equations derived in § 4.3 can be used to calculate the response functions. The low frequency behaviour of the response functions of the two-cutoff model is the same as that of a one-cutoff model, provided the values $\bar{g}_1$ and $\bar{g}_2$ are taken for the bare couplings.

An alternative two-cutoff model has been considered by Sólyom and Szabó /1977/. Since the momentum transfer cutoff is only an approximation for the energy transfer cutoff in the phonon mediated electron-electron coupling, they made a different approximation by using a bandwidth cutoff at ω_D for this interaction. The direct electron-electron interaction is also considered with a bandwidth cutoff at E_0. It is shown that similarly as in the model of Grest et al., this model can also be mapped onto a one-cutoff model with effective $\bar{g}_1$ and $\bar{g}_2$ couplings. Therefore the physical behaviour of the system is the same as discussed in § 8.

9.2. Relationship between the physical cutoffs and the cutoff of the bosonized Hamiltonian

The overall behaviour of the Fermi gas model is similar either a bandwidth cutoff or a momentum transfer cutoff is used. There are certainly differences in detail which were already mentioned in § 2 and in § 9.1. In the solution of the backward scattering model in § 6 a different cutoff procedure was used, namely the parameter α of the bosonization transformation was kept finite to get non-singular result for the energy spectrum. It can be shown that this procedure reproduces the results of the Tomonaga model with only a slight difference. In the expressions /5.19/, /5.28/ and /5.30/ obtained for the Green's function and response functions by direct diagram summation, the free Green's function contains a bandwidth cutoff $\delta \sim 1/k_o$, whereas the correction terms contain only $\Lambda \sim k_c$. Calculating the same quantities in the boson representation, the bosonization of the Tomonaga Hamiltonian can be done without using eqns. /5.54/ - /5.55/. These operator identities are used only to express the fermion operators of the Green's function and response functions in terms of the boson operators. Finite α leads to an expression for $G_+(x,t)$ similar to eqn. /5.19/ where α appears instead of both δ and Λ^{-1} . The same is true for the response functions. This shows that in the Tomonaga model, where exact results are available, α does not correspond exactly either to a bandwidth cutoff or to momentum transfer cutoff, α appears at the place of both of them.

The situation is more complicated in the backward scattering model. Here no exact solutions are available without the bosonization transformation, and therefore no definite statement can be made. Calculations made by Grest /1976/ and Theumann /1977/ indicate, however, that keeping α finite poses some problems. Grest compared the perturbational expressions of the charge-density response function obtained in a model with bandwidth cutoff /see § 4.3/, using momentum transfer and bandwidth cutoff /see § 9.1/

and using the bosonization transformation /see § 6.2/. These are denoted by N_S /Sólyom 1973/, N_G /Grest et al. 1976/ and N_{LE} /Luther and Emery 1974/, respectively. The results are as follows:

$$N_S(\omega) = \frac{1}{\pi v_F} \ln \frac{\omega}{E_0} \left[1 + \frac{1}{2\pi v_F} \left(g_{1\parallel} + g_{1\perp} - g_2 \right) \ln \frac{\omega}{E_0} + \dots \right], \tag{9.5}$$

$$N_G(\omega) = \frac{1}{\pi v_F} \ln \frac{\omega}{E_0} \left[1 + \frac{1}{2\pi v_F} \left(g_{1\parallel} + g_{1\perp} \right) \ln \frac{\omega}{E_0} + \dots \right] - \frac{1}{2\pi^2 v_F^2} g_2 \ln^2 \frac{\omega}{\omega_D} + \dots , \tag{9.6}$$

$$N_{LE}(\omega) = \frac{1}{\pi v_F} \ln \frac{\omega \alpha}{v_F} \left[1 + \frac{g_{1\perp}}{2\pi v_F} \ln \frac{\omega \alpha}{v_F} \right] + \frac{1}{2\pi^2 v_F^2} \left(g_{1\parallel} - g_2 \right) \ln^2 \frac{\omega \alpha}{2 v_F} + \dots \tag{9.7}$$

The terms proportional to $g_{1\parallel}$ and $g_{1\perp}$ correspond to the same diagram, they differ only in the spin orientations on the lines. The corresponding contribution should be the same and it should be proportional to the square of the contribution of the simple electron-hole bubble. This is not true in the Luther-Emery solution. The extra factor of 2 in the argument of $\ln^2 (\omega\alpha / 2v_F)$ indicates that the use of α may lead to nonphysical results in the next to leading logarithmic contributions already. Similar situation has been noticed by Wilson /1975/ by comparing his exact results with the results obtained with the bosonization transformation in the Kondo problem.

Theumann /1977/ studied the problem under which condition the transformations which allow to obtain the Luther-Emery solution, can be performed. She showed that the canonical transformation /5.44/ preserves the anticommutation relation of the fermion fields only if the limit $\alpha \to 0$ is taken first. The momentum transfer cutoff Λ appearing in the canonical transformation through $\varphi(p)$ and $\psi(p)$ cannot be identified with $1/\alpha$, since in this case normalization factors would appear in the anticommutators. On the other hand, the backward scattering term can be written as a bilinear expression of spinless fermions for the particular value $g_{1\parallel} = -\frac{3}{5}\left(2\pi v_F + g_{4\parallel} - g_{4\perp}\right)$ only if the limiting process is inverted as in the Luther-Emery solution and $\Lambda \to \infty$ first while keeping α finite.

There are, however, other physical arguments which suggest that the choice of a finite α may be a reasonable cutoff procedure. Lee /1975/ pointed out, that the binding energy of a singlet pair in a model with linear dispersion and with a bandwidth cutoff at $E_o = v_F/\alpha$ is

$$2\Delta = \frac{v_F}{\alpha}\left[\exp\left(\frac{\pi v_F}{g}\right) - 1\right]^{-1} . \qquad /9.8/$$

In the limit of strong coupling Δ reduces to the value obtained by Luther and Emery /1974/ supporting that working with α leads to physically correct results. Without making strong statements the only conclusion can be that the results obtained for the bosonized Hamiltonian should always be considered with some caution.

§ 10. SOLUTION OF THE MODEL BELOW THE LUTHER-EMERY LINE

The strategy of this paper until now was to find exact solutions for particular cases of the Fermi gas model and to use scaling arguments to extend these results for arbitrary values of the couplings. Luther /1976, 1977/ has shown that an alternative approach is possible. He pointed out that an exactly soluble model, the 1-d anisotropic Heisenberg model can be transformed to a form equivalent with the backward scattering model. Knowing the energy spectrum of the 1-d spin model /Johnson et al. 1973/, that of the 1-d Fermi gas can also be obtained. New features will appear below the Luther-Emery line and here we will mostly be interested in this region.

Luther /1977/ started from the boson representation of the Fermi gas Hamiltonian. Similarly as in § 6.1, two transformations are performed on H_σ, first a canonical transformation given in eqns. /5.44/ and /5.46/, then a transformation to the spinless fermions introduced in eqns. /6.12/ and /6.13/. Leaving ψ undetermined for the moment, the first transformation leads to

$$\tilde{H}_\sigma = \frac{2\pi v_F}{L} \sum_{p>0} \left\{ \left[1 + \frac{1}{2\pi v_F}(g_{4\parallel} - g_{4\perp})\right] \cosh 2\psi(p) - \frac{1}{2\pi v_F} g_{1\parallel} \sinh 2\psi(p) \right\} \times$$

$$\times \left[\sigma_1(p)\, \sigma_1(-p) + \sigma_2(-p)\, \sigma_2(p) \right] +$$

$$+ \frac{2\pi v_F}{L} \sum_{p>0} \left\{ \left[1 + \frac{1}{2\pi v_F}(g_{4\parallel} - g_{4\perp})\right] \sinh 2\psi(p) - \frac{1}{2\pi v_F} g_{1\parallel} \cosh 2\psi(p) \right\} \times$$

$$\times \left[\sigma_1(p)\, \sigma_2(-p) + \sigma_1(-p)\, \sigma_2(p) \right] +$$

$$+ \frac{g_{1\perp}}{(2\pi\alpha)^2} \int dx \left\{ \exp\left[\frac{2\pi}{L} \sum_p \frac{e^{-\frac{\alpha}{2}|p| - ipx}}{p} \sqrt{2}\, e^{\psi} \left(\sigma_1(p) + \sigma_2(p)\right)\right] + h.c. \right\} .$$

/10.1/

The spinless fermion representation can be used for the bacward scattering term if $\sqrt{2}\,e^{\psi} = 1$. Fixing ψ at this value, the Hamiltonian can be written in terms of c_k and d_k introduced in eqns. /6.12/ and /6.13/:

$$\tilde{H}_\sigma = \frac{2\pi v'}{L} \sum_{k>0} \left[\sigma_1(k)\, \sigma_1(-k) + \sigma_2(-k)\, \sigma_2(k) \right] -$$

$$- \frac{1}{L} \sum_k \left\{ \frac{5}{4} g_{1\parallel} + \frac{3}{2}\pi v_F \left[1 + \frac{1}{2\pi v_F}(g_{4\parallel} - g_{4\perp})\right] \right\} \sigma_1(k)\, \sigma_2(-k) +$$

$$+ \frac{g_{1\perp}}{2\pi\alpha} \sum_k \left(c^+_{k+k_F}\, d_{k-k_F} + d^+_{k-k_F}\, c_{k+k_F} \right),$$

/10.2/

where

$$\sigma_1(k) = \sum_p c^+_{p+k}\, c_p \, , \qquad \sigma_2(k) = \sum_p d^+_{p+k}\, d_p \, ,$$

/10.3/

and

$$v' = \frac{5}{4}\left[v_F + \frac{1}{2\pi}(g_{4\parallel} - g_{4\perp})\right] + \frac{3}{8\pi} g_{1\parallel} .$$

/10.4/

This Hamiltonian corresponds to a one-dimensional interacting spinless fermion system. The interaction vanishes on the Luther-Emery line, and the problem can be solved as shown by Luther and Emery /1974/. Apart from an attempt by Schlottmann /1977a,b/ no direct solution is known for other values of the couplings. However Luther produced a remarkable solution by noticing the analogy of this model to the spin-$1/2$ 1-d anisotropic Heisenberg model.

The Hamiltonian of an anisotropic Heisenberg model on a chain with N sites is

$$H_{XYZ} = - \sum_{\substack{j=1 \\ \alpha}}^{N} J_\alpha S_j^\alpha S_{j+1}^\alpha ,$$

/10.5/

where j denotes the lattice sites on the chain, α stands for the component index x, y and z and the exchange coupling J_α can be different for the three components. This model is also called X-Y-Z model. The $S=1/2$ spin problem is conveniently transformed to a spinless fermion problem by using the Jordan-Wigner transformation:

$$S_j^+ = S_j^x + i S_j^y = a_j^+ \prod_{l=1}^{j-1} \exp\left(i\pi a_l^+ a_l\right),$$

$$S_j^- = S_j^x - i S_j^y = a_j \prod_{l=1}^{j-1} \exp\left(-i\pi a_l^+ a_l\right)$$

$$S_j^z = a_j^+ a_j - \frac{1}{2} .$$

/10.6/

The spin operators expressed in terms of spinless fermions obey the correct commutation relations.

With a cyclic boundary condition for the chain, i.e. taking $S_{N+1} = S_1$, we get

$$\begin{aligned} H_{XYZ} = & -\frac{1}{4}\left(J_x + J_y\right)\sum_{j=1}^{N-1}\left(a_j^+ a_{j+1} + a_{j+1}^+ a_j\right) - \\ & -\frac{1}{4}\left(J_x - J_y\right)\sum_{j=1}^{N-1}\left(a_j^+ a_{j+1}^+ + a_{j+1} a_j\right) - \\ & - J_z \sum_{j=1}^{N}\left(a_j^+ a_j - \frac{1}{2}\right)\left(a_{j+1}^+ a_{j+1} - \frac{1}{2}\right) - \\ & -\frac{1}{4}\left(J_x + J_y\right)\left(a_N^+ a_1 + a_1^+ a_N\right)\exp\left(i\pi\sum_{l=1}^{N} a_l^+ a_l\right) - \\ & -\frac{1}{4}\left(J_x - J_y\right)\left(a_N^+ a_1^+ + a_1 a_N\right)\exp\left(i\pi\sum_{l=1}^{N} a_l^+ a_l\right). \end{aligned}$$

/10.7/

The last two terms can be merged into the first two terms by extending the summation up to N if $\mathcal{N} = \sum_{l=1}^{N} a_l^+ a_l$ is an even number. If it is odd, the Hamiltonian should be treated somewhat differently. We will forget about this complication for a while, but will return to its consequences later on.

Introducing the Fourier transformed operators with

$$a_k = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{ikja} a_j \qquad -\frac{\pi}{a} \leq k \leq \frac{\pi}{a}, \tag{10.8}$$

where a is the lattice constant, the Hamiltonian can be written in the form

$$H_{XYZ} = -\frac{1}{2}\left(J_x + J_y\right) \sum_k \cos(ka)\, a_k^+ a_k - \\ - \frac{J_z}{N} \sum_k \cos(ka)\, \varrho(k)\, \varrho(-k) + \\ + \frac{1}{4}\left(J_x - J_y\right) \sum_k \left(e^{ika} a_k^+ a_{-k}^+ + e^{-ika} a_{-k} a_k\right), \tag{10.9}$$

where

$$\varrho(k) = \sum_p a_{p+k}^+ a_p . \tag{10.10}$$

In zero magnetic field the total magnetization $\langle S^z \rangle = \sum_{j=1}^{N} \langle S_j^z \rangle = 0$. Using eqn. /10.6/ this means that the number of fermions $\mathcal{N} = \sum_{j=1}^{N} a_j^+ a_j$ is $N/2$, corresponding to a half-filled band. The Fermi points are at $k_F = \pm \pi/2a$. The dispersion relation of the free fermions is the same as that shown in fig. 1. Assuming that particles lying near the Fermi surface play important role, the wave vectors in the region $k \sim \pm k_F$ contribute in the first and third terms, while the regions around 0 and $2k_F$ should be considered in the second term of eqn. /10.9/. Approximating the dispersion relation by two linear spectra as given in fig. 2, the particles corresponding to the two branches will be distinguished by the index 1 and 2 referring to the regions around $+k_F$ and $-k_F$, respectively. Further approximations are made in the second and third terms, where $\cos(ka)$ and e^{ika} are approximated by their first terms in the expansion around $k=0$ and $k=2k_F$ and $k=\pm k_F$, respectively. The Hamiltonian, when written in terms of the operators $a_{1,k+k_F}$, $a_{2,k-k_F}$ and $\rho_1(k) = \sum_p a_{1,p+k}^+ a_{1,p}$, $\rho_2(k) = \sum_p a_{2,p+k}^+ a_{2,p}$ has the form

$$
\begin{aligned}
H_{XYZ} = {} & \frac{1}{2}(J_x + J_y)\, a \sum_k k \left(a_{1,k+k_F}^+ a_{1,k+k_F} - a_{2,k-k_F}^+ a_{2,k-k_F} \right) - \\
& - \frac{2J_z}{N} \sum_{k>0} \left[\rho_1(k)\rho_1(-k) + \rho_2(-k)\rho_2(k) \right] - \\
& - \frac{4J_z}{N} \sum_k \rho_1(k)\rho_2(-k) + \\
& + i\,\frac{1}{2}(J_x - J_y) \sum_k \left(a_{1,k+k_F}^+ a_{2,-k-k_F}^+ + a_{1,k+k_F} a_{2,-k-k_F} \right).
\end{aligned}
$$

/10.11/

The momentum summation is cut off at $1/\alpha$. After a further transformation $a_{2,k-k_F} \to i a^{+}_{2,-k-k_F}$ and using that the kinetic energy term can be written in terms of ρ as in the Tomonaga-Luttinger model, we have

$$H_{XYZ} = \alpha \Big\{ \frac{2\pi}{L} \Big[\frac{1}{2}(J_x + J_y) - \frac{1}{\pi} J_z \Big] \sum_{k>0} \Big[\rho_1(k)\, \rho_1(-k) + \rho_2(-k)\, \rho_2(k) \Big] +$$

$$+ \frac{4 J_z}{L} \sum_k \rho_1(k)\, \rho_2(-k) +$$

$$+ \frac{1}{2\alpha} (J_x - J_y) \sum_k \Big(a^{+}_{1,k+k_F}\, a_{2,k-k_F} + a^{+}_{2,k-k_F}\, a_{1,k+k_F} \Big) \Big\} . \qquad /10.12/$$

This Hamiltonian is the same as /10.2/, if measured in units of α . The correspondence between the coefficients is

$$\alpha \Big[\frac{1}{2}(J_x + J_y) - \frac{1}{\pi} J_z \Big] = \frac{5}{4} \Big[v_F + \frac{1}{2\pi}(g_{4\parallel} - g_{4\perp}) \Big] + \frac{3}{8\pi}\, g_{1\parallel} , \qquad /10.13/$$

$$\alpha\, 4 J_z = - \frac{5}{4}\, g_{1\parallel} - \frac{3}{2} \pi \Big[v_F + \frac{1}{2\pi} (g_{4\parallel} - g_{4\perp}) \Big] , \qquad /10.14/$$

$$\frac{1}{2} (J_x - J_y) = \frac{g_{1\perp}}{2\pi\alpha} . \qquad /10.15/$$

The two soluble cases of the Fermi gas model correspond to simple situations in the X-Y-Z model. In the Tomonaga model $g_{1\perp}=0$. This corresponds to $J_x = J_y$ which is the Heisenberg-Ising or X-X-Z model. The r.h.s. of /10.14/ vanishes on the Luther-Emery line, i.e. the Luther-Emery model corresponds to $J_z = 0$ which is the X-Y model.

The low-lying excited states of the X-Y-Z model have been determined exactly by Johnson et al. /1973/ using Baxter's relation /Baxter 1971, 1972/ between the energy levels of the X-Y-Z Hamiltonian and the eigenvalues of the eight-vertex transfer matrix. Using the correspondence in eqns. /10.13/ - /10.15/, the energy spectrum of the 1-d Fermi gas model can also be obtained.

Before writing down the results, it is appropriate to comment on the exactness of this correspondence. In the course of transformations leading to eqn. /10.12/, two approximations were made. First we simplified the Hamiltonian by neglecting in eqn. /10.7/ the phase factors $\exp\left(i\pi \sum_{\ell=1}^{N} a_\ell^+ a_\ell\right)$. This factor is 1 if the number of particles is even. It is equal to -1 if the number of particles is odd. The Hamiltonian has to be treated separately in the two subspaces. Since the evenness and oddness of the number of particles is invariant under the further transformations, the Hamiltonian has to be treated separately in the subspaces in which the number of spin-wave quasiparticles is even or odd. As a consequence, the excited states of the system are obtained by adding pairs of quasiparticles to the ground state. The gap in the excitation spectrum is twice as big as the gap in the single spin-wave energy, but states with a single spin-wave are not eigenstates of the X-Y-Z Hamiltonian. The equivalence with the Fermi gas model was obtained for the simplified Hamiltonian, therefore there is no restriction in the Fermi gas model that only pairs of quasiparticles can be excited.

An important approximation was made in eqn. /10.9/ when $\cos(ka)$ and e^{ika} were expanded and only the first terms were considered. Luther /1976/ argues that a continuum version of the X-Y-Z model can be introduced in the limit $a \to 0$, and the steps leading to eqn. /10.12/ are exact in the continuum limit. The continuum version of the Jordan-Wigner transformation was constructed by Luther and Peschel /1975/. There are of course important renormalizations of the coupling constants and it is in general a hard problem to consider this renormalization explicitly. Luther suggested a smart procedure to avoid this problem. As mentioned before, the X-X-Z model $(J_x = J_y)$ corresponds to the Tomonaga model. The correlation functions can be calculated exactly for the X-X-Z model /Johnson et al. 1973/.

On the other hand these quantities are known for the Tomonaga model as well. The corresponding problems should have the same exponent. The exponents appearing in the calculation for the Tomonaga model are expressed in terms of γ_σ given in eqn. /5.26/, which can be written as

$$\gamma_\sigma = \left[\frac{v_\sigma + \frac{1}{2\pi} g_{1\parallel}}{v_\sigma - \frac{1}{2\pi} g_{1\parallel}}\right]^{1/2} = \left[\frac{v_F + \frac{1}{2\pi}\left(g_{4\parallel} - g_{4\perp}\right) + \frac{1}{2\pi} g_{1\parallel}}{v_F + \frac{1}{2\pi}\left(g_{4\parallel} - g_{4\perp}\right) - \frac{1}{2\pi} g_{1\parallel}}\right]^{1/2} \qquad /10.16/$$

if $g_{1\parallel}$ and g_{2} are used instead of $g_{2\parallel}$ and $g_{2\perp}$. Calculating the response functions for the Heisenberg-Ising model, the corresponding exponents are obtained in the form

$$\gamma_\sigma = 1 - \frac{\mu}{\pi} = 1 - \frac{1}{\pi} \arccos\left(-\frac{J_z}{J_x}\right). \qquad /10.17/$$

Using the identification given in eqns. /10.13/ and /10.14/ for $J_x = J_y$, the two expressions are not equivalent. This apparent contradiction is resolved if we take into account that eqns. /10.13/ - /10.15/ were obtained without performing the renormalization necessary in the continuum limit. In this limit, when $a \to 0$, the renormalization should lead to identical exponents. Thus the identification of the renormalized parameters of the spin model in the continuum limit with the parameters of the Fermi gas should be done by equating /10.16/ and /10.17/. It should be noted, that near the LE line, where J_z is small, the expansions of /10.16/ and /10.17/ agree to first order, indicating that in the weak coupling limit of the spinless fermion problem states lying far from the Fermi surface, in a region where the dispersion is not linear, do not contribute essentially.

A further consequence of the continuum limit is that the anisotropy $(J_x - J_y)/(J_x + J_y)$ should go to zero as $a \to 0$. One can therefore use the excitation spectrum valid in the limit of small anisotropy; $\ell^2 = (J_x^2 - J_y^2)/(J_x^2 - J_z^2) \to 0$. In this limit Johnson et al. /1973/ find that in the lattice theory there is a gap in the spin wave spectrum with

$$\Delta = 4\pi \frac{\sin\mu}{\mu} \, |J_x| \left(\frac{\ell}{4}\right)^{\pi/\mu} ,$$

/10.18/

where μ is given in eqn. /10.17/. In the continuum limit J_x and ℓ should be replaced by their renormalized values. The renormalized value of J_x diverges as $1/a$. At the same time ℓ should go to zero as $a^{\mu/\pi}$ to have a finite gap as $\alpha \to 0$. It follows from eqn. /10.15/ that the quantity corresponding to the anisotropy $J_x - J_y$ is $g_{1\perp}$, therefore Luther /1977/ concludes that the value of the gap in the Fermi gas problem is

$$\Delta = \frac{4\pi v_\sigma}{\alpha} \frac{\sin\mu}{\mu} \left(\frac{g_{1\perp}}{8\pi v_\sigma} \right)^{\pi/2\mu} ,$$

/10.19/

where now $\mu = \pi(1-\gamma_\sigma)$ is given by eqn. /10.16/. The identification of μ and ℓ with the corresponding expressions in the Fermi gas model is not rigorous in a mathematical sense. It is, however, very reasonable physically to use this identification, though certain problems arise. The exponent $\pi/2\mu$ diverges when $g_{1\parallel} \to 0$, i.e. the gap vanishes on the line $g_{1\parallel} = 0$ if $|g_{1\perp}| < 8\pi v_\sigma$. This contradicts the result obtained in the renormalization group treatment according to which the scaling trajectories go through the $g_{1\parallel} = 0$ line as shown in fig. 28. This discrepancy is either the consequence of treating the cutoff in a particular way or is due to the ambiguities in taking the continuum limit. The results are probably correct in the vicinity of the LE line where the renormalizations are unimportant.

Though the expression of μ is terms of the Fermi gas parameters depends on the cutoff prescription, it is expected that the analogy with the X-Y-Z model allows a qualitatively correct behaviour. There is a gap in the excitation spectrum for any $g_{1\parallel} < 0$. On the LE line itself the gap determined in § 6.1 is reproduced. Here we have to take into account that, as discussed earlier in this paragraph, the gap of the X-Y-Z model is twice the single particle gap.

Completely new features appear below the Luther-Emery line, i.e. for $g_{1\parallel} < -\frac{6}{5}\pi v_\sigma$. In this region, which corresponds to $J_z > 0$ in the X-Y-Z model, bound states appear in addition to the free state solutions. In the limit of weak anisotropy, the energies of these excitations are given by

$$\Delta_n^2(k) = \Delta_n^2 + c^2 k^2 , \tag{10.20}$$

where

$$\Delta_n = \Delta \cdot \sin\left(\frac{n\pi}{2} \frac{\pi-\mu}{\mu}\right) , \qquad n = 1, 2, \ldots, < \frac{\mu}{\pi-\mu} . \tag{10.21}$$

These new states appear first at $g_{1\parallel} = -\frac{6}{5}\pi v_\sigma$ where $\mu = \pi/2$. In the X-Y-Z model these are bound spin wave states, since for a little bit stronger coupling, their energy is lower than the energy of two free spin waves. The binding energy increases with stronger coupling. At $\mu = \frac{2}{3}\pi$, then at $\mu = \frac{3}{4}\pi$ etc., further bound states will appear.

In the Fermi gas model where one should not distinguish between states with even or odd number of particles, the bound states appear at twice the single particle gap, but can descend below it for strong enough coupling.

The appearance of bound states in the strong coupling regime will give rise to significant modifications in the response functions if compared with a model in which only free states are present. It is expected that the correlation function critical exponents will be modified. In the scaling treatment in § 7 where scaling to the LE line was used to get information about the region above the LE line, there was no indication that scaling cannot be continued to couplings below this line. Nevertheless, since the appearance of bound states is not compatible with the simple scaling ideas suggested for this model in § 4.2 , it is believed that this scaling does not hold any more in the strong coupling regime.

One should not, however, overemphasize the results obtained for the Fermi gas model using this analogy with the X-Y-Z model. This latter model is related directly to the spinless fermion model. It was mentioned already that the transformations leading from the original Fermi gas model to the spinless fermion model are not exact if α in the bosonization transformation is identified with the cutoff and kept finite.

§ 11. RELATIONSHIP BETWEEN THE FERMI GAS MODEL AND OTHER MODELS

As we have seen in the preceding section the spin-1/2 X-Y-Z chain is equivalent to the Fermi gas model and the known solution of the spin model can help to construct the solution of the Fermi gas model. A remarkable feature of this particular model is that equivalence to several other models can be found. 1-d and 2-d spin models, field theoretical models and the two-dimensional Coulomb gas are among these equivalent models. Using these relationships a large class of models can be studied, most of which are of interest on their own right.

11.1. The two-dimensional Coulomb gas

Chui and Lee /1975/ recognized that the 1-d Fermi gas model is equivalent to the 2-d classical plasma with a long range logarithmic Coulomb potential. They started from the bosonized form of the Hamiltonian as given in eqns. /6.6/ - /6.8/. For a non half-filled band the charge-density part is a Tomonaga model Hamiltonian and only the spin-density part will be considered. Similar equivalence holds for H_ρ if umklapp processes are not negligible.

After a canonical transformation $\tilde{H}_\sigma$ takes the form given in eqn. /6.10/. If the backward scattering term is treated as a perturbation, the partition function $Z = \langle \exp(-\beta \tilde{H}_\sigma) \rangle$ can be written as $Z = Z_0 \cdot \Delta Z$ where Z_0 is the partition function of the system without backward scattering and

$$\Delta Z = \sum_n \left[\frac{g_{1\perp}}{(2\pi\alpha)^2} \right]^{2n} \int_0^\beta d\tau_{2n} \dots \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} d\tau_1 \times$$

$$\times \int_0^L \left(\prod_{i=1}^{2n} dx_i \right) \sum_Q \left\langle \prod_{i=1}^{2n} \exp\left\{ s_Q(i) \sqrt{2}\, e^{\psi} \left[\varphi_1(\tau_i, x_i) + \varphi_2(\tau_i, x_i) \right] \right\} \right\rangle_{H_0}$$

/11.1/

where

$$\varphi_j(x) = \frac{2\pi}{L} \sum_k \frac{e^{-\frac{\alpha}{2}|k| - ikx}}{k} \sigma_j(k)$$

/11.2/

and the time dependence is defined in the usual way. $s_Q = \pm 1$ depending on whether the term written out in eq. /6.10/ or its complex conjugate is taken. The summation over Q goes over all possibilities in taking one of the two terms. The average over the noninteracting Hamiltonian leads to /Schulz 1977/

$$\Delta Z = \sum_n \left[\frac{g_{11}}{(2\pi\alpha)^2}\right]^{2n} \int_0^\beta d\tau_{2n} \cdots \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} d\tau_1 \;\times$$

$$\times \int_0^L \left(\prod_{i=1}^{2n} dx_i\right) \sum_Q \exp\left\{-2e^{2\psi} \sum_{i>j=1}^{2n} s_Q(i) s_Q(j)\, g\left(x_i - x_j, u_\sigma\tau_i - u_\sigma\tau_j\right)\right\},$$

/11.3/

with

$$g(x,y) = \int_0^\infty dq \, \frac{e^{-\alpha q}}{q} \left[(1+n_q)\, e^{iqx - q|y|} + n_q\, e^{iqx + q|y|} - 1 - 2n_q \right] + h.c.$$

/11.4/

where

$$n_q = \left[e^{\beta u_\sigma q} - 1 \right]^{-1}.$$

/11.5/

Using the symmetry properties of the g function and taking into account that the number of possible combinations in the summation over Q is $(2n)!\,/\,(n!)^2$, since an equal number of terms with $s_Q(i) = 1$ and $s_Q(i) = -1$ should be present, we get

$$\Delta Z = \sum_n \frac{1}{(n!)^2} \left[\frac{g_{11}}{u_\sigma (2\pi\alpha)^2}\right]^{2n} \int_0^L \int_0^{u_\sigma\beta} \left(\prod_{i=1}^{2n} dx_i\, dy_i\right) \times$$

$$\times \exp\left\{-2e^{2\psi} \sum_{i>j=1}^{2n} s_i s_j \, g(x_i - x_j,\, y_i - y_j)\right\}.$$

/11.6/

It is possible to arrange the terms with $s_i = \pm 1$, that $s_i = 1$ for $i=1$ to n and $s_i = -1$ for $i=n+1$ to $2n$.

Following Chui and Lee /1975/ one can recognize in eqn. /11.6/ the grand partition function of a classical real gas of two kinds of particles with equal masses but opposite charges moving in a two-dimensional area $L u_\sigma \beta$ and interacting via a potential proportional to $g(x,y)$. In the limit $\alpha \to 0$, this potential is the two-dimensional Coulomb potential on the surface of a cylinder of length L and circumference $u_\sigma \beta$. In the zero temperature limit the usual logarithmic 2-d Coulomb potential with soft core cutoff is recovered

$$\frac{1}{2}\, g(x,y) \to -\frac{1}{2} \ln \frac{(\alpha + |y|)^2 + x^2}{\alpha^2} \, . \qquad /11.7/$$

From the comparison of eqn. /11.6/ with the grand partition function of the 2-d Coulomb plasma, the fugacity $z = \exp(-\beta_{pl}\mu)$, where μ is the chemical potential and β_{pl} is the inverse of the plasma temperature, corresponds to

$$e^{-\beta_{pl}\mu} = \frac{g_{1\perp}}{4\pi^2 u_\sigma} \, , \qquad /11.8/$$

and the charge e is related to ψ by the relation

$$\beta_{pe} e^2 = 4 e^{2\psi} = 4 \left[\frac{v_\sigma + \frac{g_{\parallel}}{2\pi}}{v_\sigma - \frac{g_{\parallel}}{2\pi}} \right]^{1/2} \qquad /11.9/$$

With this identification all that we know about the Coulomb plasma can be translated to the Fermi gas model and vice versa.

The properties of the 2-d Coulomb gas have been extensively studied recently. Hauge and Hemmer /1971/ and Kosterlitz and Thouless /1973/ pointed out that this system has a metal-insulator transition at a finite temperature. It is interesting to remark that according to Chui and Weeks /1976/ and Zittartz /1978/ the neutral Coulomb gas in more than two dimensions is a conductor at any temperature, whereas it is insulating at any temperature in less than two dimensions. Phase transition occurs in two-dimensional system only, at low temperatures particles of opposite charges are bound into pairs, but at high temperatures the system shows the usual properties of a free particle plasma. Contrary to this result Kosterlitz /1977/ found a metal-insulator transition for $d \leq 2$. This difference arises from different formulation of the renormalization transformation.

Zittartz and Huberman /1976/ calculated the wave vector and temperature dependent dielectric function $\varepsilon(q, T)$ in the low density limit. Instead of a metallic and an insulating phase they find three different temperature regimes:

/i/ At low temperatures, i.e. for $\beta_{pe} e^2 > 4$ $(T < e^2/4k_B)$, the system behaves like an insulator, $\varepsilon(q, T)$ has a finite value at $q \to 0$. All charges are bound into pairs.

/ii/ In an intermediate temperature range $2 < \beta_{pe} e^2 < 4$ $(e^2/4k_B < T < e^2/2k_B)$ the pairs start to break up, the system goes continuously into a metallic-like state, the dielectric function is singular for small q $/q\alpha \ll 1$, where α is a cutoff i.e. the minimal distance between the particles/. According to a suggestion of Everts and Koch /1977/ which was verified by Grinstein et al. /1978/, the behaviour of $\varepsilon(q, T)$ is different for $q\xi \ll 1$ and $q\xi \gg 1$, where ξ is a correlation length given by $\xi = \alpha\left(4\pi z \beta_{pe} e^2\right)^{-\nu}$, with $\nu = 2/(4 - \beta_{pe} e^2)$,

$$\varepsilon(q, T) = \begin{cases} 1 + \dfrac{C(T)}{(q\xi)^2} & \text{if} \quad q\xi \ll 1 \\[2ex] 1 + \dfrac{D(T)}{(q\xi)^{2/\nu}} & \text{if} \quad q\xi \gg 1 . \end{cases}$$

/11.10/

/iii/ In the high temperature region, i.e. for $\beta_{pe} e^2 < 2$ ($T > e^2/2k_B$) all the charges are free, the fully screened Debye-Hückel dielectric function of a plasma is recovered,

$$\varepsilon(q, T) = 1 + \frac{C(T)}{(q\xi)^2} \; . \tag{11.11}$$

The transitions between the three regions are continuous. The metallic screening characteristic for the high temperature region is preserved in the intermediate temperature range for very large distances /very small q/. As the temperature approaches $e^2/4k_B$ ν goes to infinity and in the low density limit the correlation length diverges as well. No metallic screening can be seen, the system becomes insulating. This result shows that the metal-insulator transition is a smooth one, the phase transition is of continuous order /Müller-Hartmann and Zittartz 1974, 1975/.

These three temperature regimes correspond to three regions in the space of couplings of the 1-d Fermi gas. Since the investigation of the Coulomb plasma is possible in the low density limit, i.e. for $z \to 0$, the corresponding results in the 1-d Fermi gas are related to the neighbourhood of the $g_{1\perp} = 0$ line. Using the equivalence expressed in eqn.

/11.9/, the low temperature region corresponds to $g_{1\parallel} > 0$, the high temperature region corresponds to $g_{1\parallel} < -\frac{6}{5}\pi v_\sigma$ and the intermediate temperature region to $-\frac{6}{5}\pi v_\sigma < g_{1\parallel} < 0$. We can recognize in these values the boundaries obtained earlier in the 1-d Fermi gas. The region $g_{1\parallel} > 0$ $(g_{1\perp} \approx 0)$ scales to the Tomonaga model, the correlation functions have a simple power law behaviour with continuously varying exponent. In the 2-d Coulomb gas the corresponding critical exponents vary continuously with temperature.

The gap in the region $g_{1\parallel} < 0$ is related to the screening length in the plasma, which in turn is inversely proportional to the number of free charges.The Debye-Hückel theory of the plasma can be applied in this region. In the usual approximations of this theory the free energy can be calculated leading to /Schulz 1977/:

$$F = E_0 - \frac{L}{u_\sigma} \frac{4 - \beta_{pe} e^2}{8\pi \beta_{pe} e^2} \Delta^2 - \frac{L}{u_\sigma} \sqrt{\frac{2}{\pi}}\, T^{3/2} \Delta^{1/2} e^{-\Delta/T},$$

/11.12/

where the gap Δ is given by

$$\Delta \sim \frac{1}{\alpha} \left(\frac{g_{1\perp}}{2\pi v_F} \right)^{\frac{2}{4-\beta_{pe} e^2}} .$$

/11.13/

The exponent in this expression agrees with the one in eqn. /10.19/ if the equivalence in /11.9/ is used; the numerical factors are somewhat different.

The analogy shows clearly that the behaviour of the 1-d Fermi gas is different below and above the LE line $g_{1\parallel} = -\frac{6}{5}\pi v_\sigma$. Unfortunately this analogy does not indicate the existence of a set of bound states for $g_{1\parallel} < -\frac{6}{5}\pi v_\sigma$.

The analogy with the 2-d Coulomb gas allows also to calculate the response functions of the Fermi gas. Chui and Lee /1975/ have shown that e.g. the spin-density part $S_\sigma^+(x,t)$ of the singlet-superconductor response $\Delta_s(x,t)$ /see eq./6.28// is related to the screened interaction of two charges $\pm\frac{1}{2}e$ imbedded into the Coulomb gas. This interaction can be determined from the knowledge of the dielectric constant in the low density limit leading to the already known result for the response function /Gutfreund and Huberman 1977/. The magnetic susceptibility has been studied by Everts and Koch /1977/. Unfortunately the temperature dependence of this quantity does not produce the dependence expected for the 1-d Fermi system, showing that the analogy to the Coulomb gas has limited usefulness in the study of the behaviour of the 1-d Fermi gas.

11.2. Spin models

We have seen in § 10 that the spin-1/2 X-Y-Z chain problem can be transformed by a Jordan-Wigner transformation to a spinless fermion problem which in the continuum limit is equivalent to the spinless fermion representation of the spin part of the backward scattering problem. This analogy has been used to obtain the solution of this latter problem for $g_{1\parallel} < 0$.

It is remarkable that the Fermi gas model is related not only to the spin chain problem but also to two-dimensional spin problems. Unfortunately this relationship is rather controversial, various approximate treatments lead to completely different mappings between the 2-d X-Y model and the 1-d Fermi gas. Following Kosterlitz and Thouless /1973/, Villain /1975/, Zittartz /1976a, 1976b/ and José et al. /1977/ used the analogy between the X-Y model and the 2-d Coulomb gas to describe the low temperature behaviour of the X-Y model. On the other hand Luther and Scalapino /1977/ mapped the 2-d X-Y model directly onto the 1-d Fermi gas and transcribed the results known for the Fermi gas to the corresponding quantities in the X-Y model. The main ideas of the various mappings are briefly sketched.

The low temperature behaviour of the classical planar rotator model /X-Y model/ defined by the Hamiltonian

$$H = -\sum_{ij} J_{ij}\, \vec{S}_i \vec{S}_j = -\sum_{ij} J_{ij} \cos(\varphi_i - \varphi_j) , \qquad /11.14/$$

where $\vec{S}_i$ is a 2-d classical vector of unit length at the lattice site R_i and φ_i is the polar angle of this vector, has been studied by Wegner /1967/ and Berezinsky /1970/. They used a harmonic approximation, i.e. replaced $\cos\varphi$ by $1-\frac{1}{2}\varphi^2$ and studied the properties of this model, which can be called harmonic rotator model. This is equivalent to taking spin wave excitations only and the correlation functions can be calculated giving

$$\langle \vec{S}_i \vec{S}_j \rangle \sim |R_i - R_j|^{-\eta}, \quad \eta = \frac{T}{4\pi J} \tag{11.15}$$

if first neighbour interaction is assumed. The magnetization in an external magnetic field h goes as

$$\langle m \rangle \sim h^{\frac{\eta}{4-\eta}} \tag{11.16}$$

and the zero field susceptibility diverges for $T < 8\pi J$.

Berezinsky /1970/ emphasized that these results cannot be easily extended to high temperatures. Here such spin--configurations may be important in which the spins make a full turn of 2π and the neglect of the 2π periodicity of the original interaction in the harmonic approximation can have serious consequences. Kosterlitz and Thouless /1973/ pointed out that these spin-configurations /called vortex configurations/ are metastable and should be present at any finite temperature in a 2-d X-Y model. The energy of a vortex configuration is a logarithmic function of its size.

The interaction energy between two vortices is also logarithmic. Assuming that the spin waves and the vortices do not interact, Kosterlitz and Thouless proposed in a rather intuitive way a Hamiltonian

$$H = J\int d^2r\,(\nabla\varphi(r))^2 - 2\pi J\sum_{i\neq j} q_i q_j \ln\left|\frac{r_i - r_j}{a}\right| + \pi^2 J\sum_i q_i^2 \,, \qquad /11.17/$$

where the first term corresponds to the spin wave excitations, $\varphi(r)$ is the phase at the point r, the second term describes the interaction of vortices situated at r_i and r_j, q_i is the vorticity and a is the lattice constant, finally the last term corresponds to the chemical potential.

It can be seen that the vortex part of the Hamiltonian is equivalent to the 2-d Coulomb gas; the same logarithmic potential appears here. It is a plausible assumption that only vortices of unit strength are present since the creation of vortices with $|q_i| > 1$ is energetically not favourable. With this assumption the identification of the parameters gives

$$4\pi\beta_{XY} J = \beta_{pe} e^2 \,, \qquad \pi^2 J = \mu \,, \qquad /11.18/$$

where β_{XY} is the inverse temperature of the X-Y model, and β_{pe} is the inverse temperature of the 2-d Coulomb gas. Using eqns. /11.8/ and /11.9/, the relationship between the X-Y model and the 1-d Fermi gas is expressed as

$$4\pi\beta_{XY} J = l_1 \left[\frac{v_\sigma + \frac{1}{2\pi} g_{1\parallel}}{v_\sigma - \frac{1}{2\pi} g_{1\parallel}} \right]^{1/2}, \quad e^{-\beta_{XY}\pi^2 J} = \frac{g_{1\perp}}{4\pi^2 u_\sigma}.$$

/11.19/

The properties of this model were studied by Kosterlitz /1974/ using a cutoff scaling on the partition function. These scaling equations agree with the corresponding scaling equations of the 1-d Fermi gas model in the weak coupling case. In accord with the two distinct regions $g_{1\parallel} \geq |g_{1\perp}|$ and $g_{1\parallel} < |g_{1\perp}|$, the behaviour of the X-Y model is very different whether $\pi\beta_{XY} J - 1 > 2\pi \exp(-\beta_{XY}\pi^2 J)$ or $\pi\beta_{XY} J - 1 < 2\pi \exp(-\beta_{XY}\pi^2 J)$. The critical temperature is obtained from

$$\frac{\pi J}{k_B T_c} - 1 = 2\pi \exp\left(-\frac{\pi^2 J}{k_B T_c}\right).$$

/11.20/

In the low temperature region, i.e. for $T < T_c$, the vortices are bound into pairs, similarly as in the Coulomb gas the particles with opposite charges are bound into pairs, and they can be scaled out when the lattice constant is scaled to larger values. The problem is then equivalent to a spin wave problem and the correlation functions decay according to a power law as in eq. /11.15/, the dependence of η on the physical /unrenormalized/ T and J can be obtained from the scaling curves shown in fig. 34.

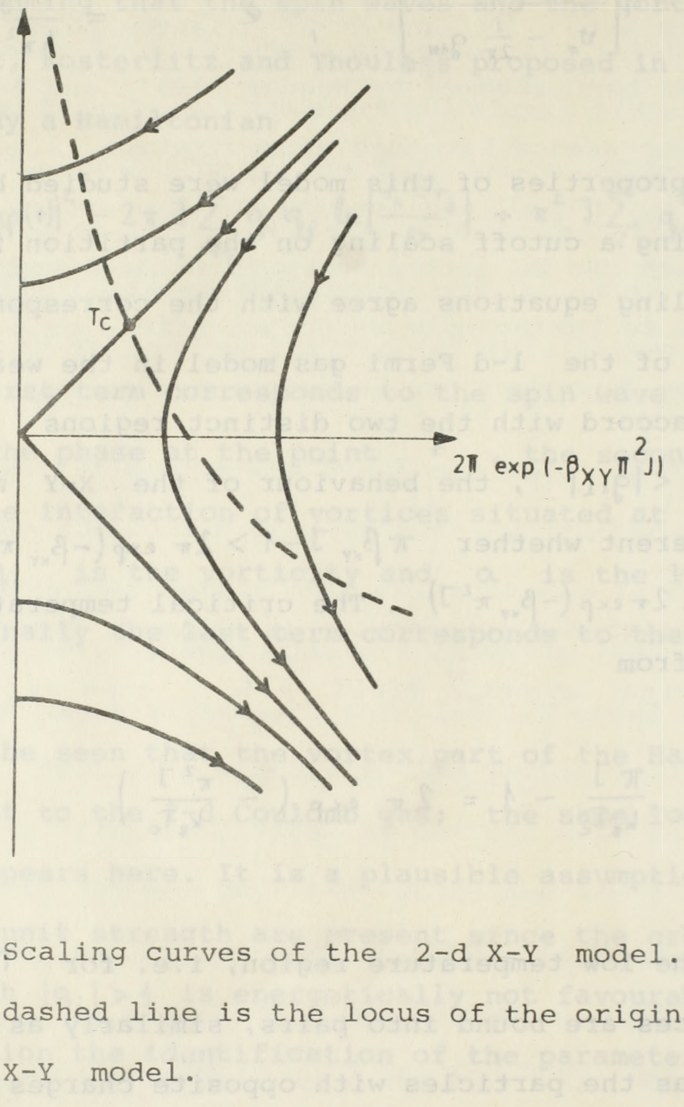

Fig. 34. Scaling curves of the 2-d X-Y model. The dashed line is the locus of the original X-Y model.

The X-Y model at T_c is equivalent to a model without vortices / $2\pi \exp(-\beta_{XY}\mu) = 0$ / and $\pi\beta_{XY}J - 1 = 0$. Putting in this value into the exponent in eq. /11.15/, $\eta = 1/4$ at T_c . For temperatures $T < T_c$ $\eta < 1/4$ and η vanishes at $T = 0$. Equivalence with the spin wave problem indicates that the susceptibility is divergent in the whole temperature range $T \leq T_c$.

The vortices have very drastic effect on the behaviour of the X-Y model at $T > T_c$. The vortex pairs start to break up resulting in an exponential decay of the correlation function. If the analogy with the 2-d Coulomb gas were complete, a further transition should occur at $\pi \beta_{XY} J - 1 = -\frac{1}{5}$ and in the high temperature region all the vortices should already be free. It is very probable that the analogy breaks down in the high temperature region and one cannot get a reliable description of the high temperature behaviour of the X-Y model in this way.

The analogy in the low temperature region has been further developed by Villain /1975/ and José et al. /1977/. While the introduction of vortices by Kosterlitz and Thouless was done intuitively, Villain has shown that a modified form of the X-Y model in which $\cos \varphi$ is replaced by a mathematically more tractable expression, which is, however, 2π periodic and thus allows the possibility of vortex states, can indeed be transformed into a decoupled system of spin waves and vortices. The difference between this model and the one treated before is that everywhere in eqns. /11.17/ - /11.20/ J should be replaced by $A(J)$,

$$A(J) = J\left(1 - \frac{k_B T}{4J} + \dots\right),$$

/11.21/

which in the low temperature region coincides with J. The physical picture and the values of the exponents at T_c are the same as in the model of Kosterlitz and Thouless /1973/.

José et al. /1977/ have pointed out that the Villain model or something very similar to it can be obtained from the original X-Y model by using the Migdal recursion relation /Migdal 1975/. This justifies the use of the Villain model to study the properties of the X-Y model. In this way it is possible to show that the above-mentioned treatment of the X-Y model gives a reasonable description at low temperatures but cannot be trusted above T_c.

Zittartz /1976a, 1976b/ approached the problem of the 2-d X-Y model in a different way. He relaxed the restriction $0 \le \varphi_i \le 2\pi$ by allowing the angles to vary in the interval $(-\infty, \infty)$. Furthermore he introduced a term $\frac{1}{2}\lambda(\varphi_i - \varphi_j)^2$ which favours small angular differences. The Hamiltonian

$$H_\lambda = \sum_{ij} J_{ij}\left[1 - \cos(\varphi_i - \varphi_j) + \frac{1}{2}\lambda(\varphi_i - \varphi_j)^2\right] - $$

$$- h\sum_i \cos\varphi_i ,$$

/11.22/

where h is the external magnetic field, is not 2π-periodic, but averages of 2π - periodic functions calculated with H_λ should lead to the correct expressions in the limit $\lambda \to 0$. In zero field Zittartz finds spin wave excitations only with the dispersion relation

$$\varepsilon_k = 2a^2 A(J,T) k^2 \qquad /11.23/$$

where a is the lattice constant and

$$A(J,T) = J\left(1 - \frac{T}{2J} + \dots\right) \qquad /11.24/$$

at low temperatures and it vanishes at high temperatures.

As a consequence of the k^2 dispersion, the correlation functions have power law behaviour for large distances, like in a spin wave theory, unless $A(J,T)$ vanishes at a finite temperature T' above which the decay is exponential. In the harmonic rotator /spin wave/ model where $\cos\varphi$ is approximated by $1 - \frac{1}{2}\varphi^2$, $A(J,T)$ is equal to J and is independent of T, thus the correlations are always power law like. Zittartz assumes that in the X-Y model a T'

$$A(J) = J\left(1 - \frac{k_B T}{4J} + \dots\right), \qquad /11.21/$$

which in the low temperature region coincides with J. The physical picture and the values of the exponents at T_c are the same as in the model of Kosterlitz and Thouless /1973/.

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exists and $A(J,T) = 0$ for any $T > T'$. In the low temperature region $T < T'$ he obtains for the two angle correlation function

$$\langle \varphi_i \varphi_j \rangle \sim |R_i - R_j|^{-\eta} \qquad /11.25/$$

where now

$$\eta = \frac{T}{4\pi A(J,T)} . \qquad /11.26/$$

The results obtained for the singular part of the free energy and for the magnetization are also similar to the results of the spin wave theory, namely

$$\langle m \rangle \sim h^{\frac{\eta}{4-\eta}} . \qquad /11.27/$$

At very low temperatures, where $A = J$, the results of the spin wave theory are reproduced. However, according to Zittartz /11.25/ and /11.27/ are valid in the whole temperature interval $0 < T < T'$. There are two characteristic temperatures in this temperature range. Below T_2 defined by

$\eta(T_2) = 2$ the susceptibility is divergent, while above it the susceptibility is finite. Above T_2 higher derivatives of the free energy with respect to the magnetic field can be divergent. When the temperature reaches T_∞ defined by $\eta(T_\infty) = 4$ the free energy is infinitely many times differentiable. The correlation length in zero magnetic field is divergent for any $T < T_\infty$ and behaves for finite field as

$$\xi \sim h^{-\frac{2}{4-\eta}} . \qquad /11.28/$$

Contrary to this behaviour, the phase transition found by Kosterlitz /1974/, Villain /1975/ and José et al. /1977/ occurs at $\eta(T_c) = 1/4$ already. Above this temperature the correlation length is finite, though it does not have the usual power law behaviour,

$$\xi \sim \exp\left[-b\,(T - T_c)^{1/2}\right]. \qquad /11.29/$$

The disagreement stems probably from the failure of Zittartz's method to account for the vortex excitations.

It is interesting to remark that Zittartz /1976a, b/ found an analogy between his model and the 2-d Coulomb gas.

This analogy is, however, very different from the one suggested by Kosterlitz and Thouless and described above. The partition function in a finite magnetic field h can be written as

$$Z = Z_\circ \cdot \Delta Z \tag{11.30}$$

where $Z_\circ$ is the partition function in the absence of field and ΔZ corresponds to the excess free energy due to the field. Expanding ΔZ in powers of h, Zittartz found an expression similar to /11.6/ and the following correspondence holds:

$$\frac{h}{2T} = e^{-\beta_{pl}\mu} = \frac{g_{1\perp}}{4\pi^2 u_\sigma} ,$$

$$\eta(T) = \beta_{pl} e^2 = 4 \left[\frac{v_\sigma + \frac{1}{2\pi} g_{1\parallel}}{v_\sigma - \frac{1}{2\pi} g_{1\parallel}} \right]^{1/2} \tag{11.31}$$

This analogy holds for ΔZ only. The scaling relations of the 2-d Coulomb gas and 1-d Fermi gas predict that h is a relevant perturbation if $\eta(T) < 4$ i.e. $T < T_\infty$ / $g_{1\perp}$ is relevant if $g_{1\parallel} < 0$ /, and it is irrelevant if $T > T_\infty$, in agreement with the result of José et al. /1977/. Although formally low temperatures in Zittartz's model correspond to high temperatures in the Coulomb gas and vice versa, this analogy does not help in the study of the X-Y model, because Z_0 - for which this analogy does not hold - will determine its behaviour.

A completely different approach to the study of the critical behaviour of the 2-d X-Y model has been proposed by Luther and Scalapino /1977/. They criticised the separation of spin wave and vortex excitations explicitly made by Kosterlitz and Thouless /1973/. This separation holds in the Villain model, too. Luther and Scalapino argue that this separation is not correct near the critical point and in contrast to Zittartz's treatment it is not sufficient to consider elementary excitations with quadratic dispersion. They transform the Hamiltonian of the 2-d X-Y model in a series of transformations to a 1-d fermion problem and then use the known properties of this problem to determine the correlation functions of the 2-d X-Y model.

Universality of critical behaviour suggests that near T_c the 2-d X-Y model and a two-component 2-d Ginzburg-Landau model have the same properties, one can therefore study the model in which the free energy functional is

$$F\{\psi\} = \int dx\, dy \left[a |\psi(r)|^2 + b |\psi(r)|^4 + \right.$$

$$\left. + c_x \left|\frac{\partial \psi}{\partial x}\right|^2 + c_y \left|\frac{\partial \psi}{\partial y}\right|^2 \right] , \qquad /11.32/$$

where ψ is a complex two-dimensional field and the partition function can be expressed as a functional integral

$$Z = \int \mathcal{D}\psi \; \exp\left(-\beta F\{\psi\}\right). \tag{11.33}$$

The transfer-matrix method allows to reduce the d dimensional functional integral to a /d-1/ dimensional quantum mechanical problem as shown by Scalapino et al. /1972/ and Stoeckly and Scalapino /1975/. The free energy and the order parameter correlation function of the original problem can be obtained from the ground state energy and low lying excitations of the transfer Hamiltonian. Stoeckly and Scalapino /1975/ have shown that in the particular case of a complex /two-component/ 2-d Ginzburg-Landau model the transfer Hamiltonian corresponds to a 1-d set of linearly coupled anharmonic oscillators. This problem can be approximated if only low-lying states of the oscillators are considered in which case the equivalent Hamiltonian is a spin-one chain problem,

$$H = \Delta \sum_i \left(J_i^z\right)^2 - \sum_i \left(J_i^+ J_{i+1}^- + J_i^- J_{i+1}^+\right), \tag{11.34}$$

with J a spin-one operator and $\Delta = \left(b/|a|\right)^2 \beta^2 c_x c_y$.

The most serious approximation in the whole consideration of Luther and Scalapino is probably the one in which only the three lowest eigenstates of the individual oscillators are kept. A priori it is not clear that the vortices are correctly described in this approximation.

Luther and Scalapino /1977/ argue that in the low lying states of this system every site is in the triplet state, therefore J can be replaced by $J = L + S$ where both L and S are spin one-half operators. Then they show that using a continuum version of the Jordan-Wigner transformation, the transfer Hamiltonian can be expressed in terms of two spinless fermion fields corresponding to the two spin one-half operators. Finally the two spinless fermion fields are combined into one fermion field with spin. Introducing the charge and spin densities, the transfer Hamiltonian takes the form

$$H = H_0 + H_1 \tag{11.35}$$

with

$$H_0 = \frac{2\pi v_0}{L} \sum_{k>0} \left[\rho_1(k)\, \rho_1(-k) + \rho_2(-k)\, \rho_2(k) \right] + \\ + \frac{V_{\parallel}}{L} \sum_{k} \rho_1(k)\, \rho_2(-k) , \tag{11.36}$$

and

$$H_1 = \frac{2\pi v_1}{L} \sum_{k>0} \left[\sigma_1(k)\,\sigma_1(-k) + \sigma_2(-k)\,\sigma_2(k)\right] +$$

$$+ \frac{U_{\parallel}}{L} \sum_{k} \sigma_1(k)\,\sigma_2(-k) +$$

$$+ \frac{U_{\perp}}{(2\pi\alpha)^2} \int dx \left\{ \exp\left[\frac{2\pi}{L} \sum_{k} \frac{e^{-\frac{\alpha}{2}|k| - ikx}}{k} \left(\sigma_1(k) + \sigma_2(k)\right)\right] + h.c. \right\} . \quad /11.37/$$

Formally this is the same as the Hamiltonian of the backward scattering model in eqns. /6.6/ - /6.8/, the velocities and coupling constants are given by

$$2\pi v_0 = \frac{17\pi}{4} + \frac{\Delta}{2}\,, \qquad V_{\parallel} = -\frac{15\pi}{4} + \frac{\Delta}{2}$$

$$2\pi v_1 = \frac{17\pi}{4} - \frac{\Delta}{2}\,, \qquad U_{\parallel} = -\frac{15\pi}{4} - \frac{\Delta}{2}$$

/11.38/

and

$$U_{\perp} = -\frac{\pi}{2}\,.$$

Since the Hamiltonian is the sum of two commuting terms, the correlation functions factor into two pieces

$$\langle \psi(r)\,\psi^*(r')\rangle = C_0(r-r')\,C_1(r-r')\,,$$

/11.39/

where $C_0(r-r')$ and $C_1(r-r')$ are calculated with H_0 and

H_1 respectively. H_0 is a Tomonaga model Hamiltonian and the correlation functions have power law behaviour. Using the results of § 5 we obtain

$$C_0(\tau-\tau') \sim |\tau-\tau'|^{-\eta_0(T)} \qquad /11.40/$$

with

$$\eta_0(T) = \left[\frac{2\pi v_0 + V_\parallel}{2\pi v_0 - V_\parallel}\right]^{1/2} . \qquad /11.41/$$

The temperature dependence is incorporated into Δ , it varies smoothly with temperature and decreases with decreasing T . At high temperatures $\Delta > \pi/2$, $U_\parallel < -2\pi v_1$ and H_1 corresponds to a strongly attractive backward scattering model. This model cannot be studied by the conventional methods reviewed in this paper because for strong enough couplings the renormalized Fermi velocities become complex. Using the equivalence to the X-Y-Z model, the Fermi gas model with $U_\parallel < -2\pi v_1$ can be mapped to a problem with $U_\parallel > -2\pi v_1$ /Luther 1977/ and therefore in this case as well a gap exists in the excitation spectrum and there are also bound states. The finite excitation energy gives rise to an exponential decay in $C_1(\tau-\tau')$. With decreasing temperatures Δ approaches $\pi/2$, $U_{\parallel}$ goes to $-2\pi v_1$ and the energy of the bound states approaches zero and eventually at $\Delta=\pi/2$ the system becomes degenerate. This defines the critical temperature T_c. Above T_c $C_1(\tau-\tau')$ is expected to behave as

$$C_1(\tau-\tau') \sim |\tau-\tau'|^{-\eta_1} f\left((\tau-\tau')/\xi\right). \qquad /11.42/$$

At T_c $\eta_1=0$ and the correlation length ξ diverges, $C_1(\tau-\tau')$ becomes a constant. The critical behaviour is determined by η_0 only, which at T_c $(\Delta=\pi/2)$ takes the value $\eta_0(T_c)=1/\sqrt{8}$. This result differs from the one obtained by Kosterlitz /1974/. One way to understand this is to suppose that the critical behaviour is non-universal, it depends on the details of the particular model, the classical X-Y model and the two-component 2-d Ginzburg-Landau model do not belong to the same universality class. José et al. /1977/ suggested another explanation. It is possible that the 2-d planar model has two phase transitions, the lower one has been studied by Kosterlitz /1974/, Villain /1975/ and José et al. /1977/ and the transition studied by Luther and Scalapino /1977/ occurs at higher temperatures. The energy spectrum of the backward scattering model allows such an explanation because degeneracy occurs not only at $u_{\parallel} = -2\pi v_1$ but also at $u_{\parallel} = |u_{\perp}|$. The low temperature phase transition could be associated with this point in the fermion representation. Finally there is the possibility that the discrepancy arises from the approximations made in the transfer Hamiltonian.

11.3. Field theoretical models

A further analogy /Heidenreich et al. 1975/ can be obtained for the spin-density part of the Hamiltonian using eq. /6.10/. Introducing the field variable

$$\phi(x) = \frac{-i}{\sqrt{4\pi}} \frac{2\pi}{L} \sum_k \frac{e^{-\frac{\alpha}{2}|k| - ikx}}{k} \left(\sigma_1(k) + \sigma_2(k)\right)$$

/11.43/

and the canonical momentum

$$\pi(x) = \frac{1}{\sqrt{4\pi}} \frac{2\pi}{L} \sum_k e^{-\frac{\alpha}{2}|k| - ikx} \left(\sigma_1(k) - \sigma_2(k)\right)$$

/11.44/

which obey the canonical commutation relation

$$\left[\pi(x), \phi(x')\right] = -i\,\delta(x-x'),$$

/11.45/

the first term of eqn. /6.10/ can be written in terms of π and ϕ as a usual free field Hamiltonian; the second term with the Hermitian conjugate gives a cosine thus leading to

$$H = \int dx \, u_\sigma \left\{ \frac{1}{2}\left[\pi^2(x) + (\nabla\phi(x))^2\right] + 2\frac{g_{1\perp}}{(2\pi\alpha)^2 u_\sigma} \cos\left[\sqrt{4\pi}\sqrt{2}\, e^{\psi}\phi(x)\right]\right\}. \tag{/11.46/}$$

This is just the same as the Hamiltonian of the sine-Gordon theory /Coleman 1975/ measured in units of u_σ,

$$H = \int dx \left\{ \frac{1}{2}\left[\pi^2(x) + (\nabla\phi(x))^2\right] - \frac{\alpha_0}{\beta^2}\cos\beta\phi(x)\right\} \tag{/11.47/}$$

with

$$\frac{\alpha_0}{\beta^2} = -2\frac{g_{1\perp}}{(2\pi\alpha)^2 u_\sigma} \tag{/11.48/}$$

and

$$\beta = \sqrt{8\pi}\, e^{\psi} \quad \text{or} \quad \beta^2 = 8\pi\left[\frac{v_\sigma + \frac{1}{2\pi}g_{1\parallel}}{v_\sigma - \frac{1}{2\pi}g_{1\parallel}}\right]^{1/2}. \tag{/11.49/}$$

The cutoffs are defined in a different way in the two theories. Bergcan et al. /1978/ have shown that the cutoff α in the boson representation of the backward scattering model and the cutoff Λ in the sine-Gordon theory as defined by Coleman should satisfy the relation

$$\alpha \Lambda = c^{-1/2} , \qquad /11.50/$$

where $c^{1/2}$ =0.89 is related to Euler's constant. If this relationship is satisfied, both theories have similar behaviour for small space-like separations.

On the other hand Coleman /1975/ pointed out the equivalence of the sine-Gordon theory to the zero-charge sector of the massive Thirring model defined by the Lagrangian density

$$\mathcal{L} = \bar{\psi} i \gamma_\mu \partial^\mu \psi - \tfrac{1}{2} g j^\mu j_\mu - m' \bar{\psi} \psi , \qquad /11.51/$$

where ψ is a two-component spinor, $\bar{\psi} = \psi^+ \gamma_0$,

$$j^\mu = \bar{\psi} \gamma^\mu \psi \qquad /11.52/$$

and the γ matrices in one space and one time dimension can be expressed by the Pauli matrices

$$\gamma_0 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_1 = i\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

/11.53/

The equivalence holds in a term-by-term comparison of the perturbation series if the following identification is made between the coupling constants:

$$\frac{\beta^2}{4\pi} = \frac{1}{1 + \frac{1}{\pi} g}.$$

/11.54/

A recent work by Schroer and Truong /1977/ indicates that this equivalence holds for $0 \le \beta^2 \le 4\pi$ $(g>0)$ only. According to /11.49/ this region corresponds to $-2\pi v_\sigma \le g_{1\parallel} \le -\frac{6}{5}\pi v_\sigma$, i.e. to the region below the Luther-Emery line. We have seen that in this range of the couplings bound states appear within the gap. The same result has been obtained by Dashen et al. /1975/ who studied the sine-Gordon theory using a semiclassical WKB method. They have got soliton solutions with mass

$$M = \frac{8\sqrt{\alpha_0}}{\beta^2}\left(1 - \frac{\beta^2}{8\pi}\right)$$

/11.55/

and bound solution states at masses

$$M_n = 2M \sin\left[\frac{n\pi}{2}\,\frac{\frac{\beta^2}{8\pi}}{1-\frac{\beta^2}{8\pi}}\right], \qquad n = 1, 2, \ldots < \frac{1-\frac{\beta^2}{8\pi}}{\frac{\beta^2}{8\pi}} \qquad /11.56/$$

Comparing this result with eqn. /10.21/, the same structure appears, since from eqn. /10.16/ and /11.49/ the argument of the sine function is the same. The soliton mass is not correctly obtained in the semiclassical approach. It is interesting to remark that Bergcan et al. /1978/ obtained almost the same expression for the soliton mass using a variational approach as the one obtained from the correspondence with the $s = 1/2$ X-Y-Z chain.

At $\beta^2 = 4\pi$ or $g = 0$ the sine-Gordon equation describes the zero-charge sector of a free massive Dirac field. This is again in agreement with the result of Luther and Emery who found that at $g_{1\parallel} = -\frac{6}{5}\pi v_F$ the backward scattering model can be transformed to a free spinless fermion problem.

The properties of the sine-Gordon theory for β^2 around 8π depend on the way the model is defined. Coleman /1975/ eliminates all ultraviolet divergences by an appropriate normal-ordering operation with an arbitrary mass m. This mass is then chosen so that the coefficient before the cosine should be finite. This leads to the result that β^2 is unrenormalized and the theory has no ground state for $\beta^2 > 8\pi$. This region corresponds to $g_{1\parallel} > 0$ irrespective of the value of $g_{1\perp}$, whereas on the basis of symmetry considerations the dividing line between the gapless region and the region with gap should be $g_{1\parallel} = |g_{1\perp}|$. Similar discrepancy is present in the relationship between the Fermi gas and the S=1/2 X-Y-Z chain.

Wiegmann /1978/ pointed out that this discrepancy can be eliminated if the sine-Gordon theory is defined on a lattice. In this case no instability arises, β^2 gets renormalized and the scaling trajectories around 8π agree, after the identification given in eqns. /11.48/ and /11.49/, with the scaling curves obtained earlier in the $(g_{1\parallel}, g_{1\perp})$ plane.

11.4. The Hubbard model

The discussion of the various properties of the 1-d Fermi gas was based in this paper until now on the model Hamiltonian given in eqns. /2.3/ and /2.4/. An alternative approach can be to start from the Hubbard model /Hubbard 1963/ which in its simplest form

$$H = t \sum_{i,\sigma} \left(c^+_{i\sigma} c_{i+1\,\sigma} + c^+_{i+1\,\sigma} c_{i\sigma} \right) + \\ + U \sum_i c^+_{i\uparrow} c_{i\uparrow} c^+_{i\downarrow} c_{i\downarrow} \qquad /11.57/$$

describes hopping between nearest neighbour sites labelled by i and Coulomb repulsion between electrons of opposite spin if they are on the same site. $c_{i\sigma}^{+}$ $(c_{i\sigma})$ is a creation /annihilation/ operator for an electron of spin σ in the Wannier state localized at the i-th lattice site. This model can be solved exactly in one dimension as was shown by Lieb and Wu /1968/. They find that in the case of half-filled band the ground state is insulating for any nonzero U and is antiferromagnetic. Each site is occupied by one electron and the spins are alternating.

Writing this Hamiltonian in Bloch state representation

$$c_k = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} e^{-ikR_i} c_i \qquad /11.58/$$

and approximating the spectrum $2t\cos(ka)$ by linear spectra around $\pm k_F$, a Hamiltonian of the form described in § 2 is obtained. All the couplings $g_{1\parallel}$, $g_{1\perp}$, g_2, g_3 and g_4 are equal since they are all related to the single coupling U of the Hubbard model. Considering the case when all the couplings are equal and positive we are in the situation $g_{1\parallel} - 2g_2 < g_3$ and as shown in § 6 and 7 a gap appears in the charge-density excitation spectrum but there is no gap in the spectrum of spin-density excitations. According to the discussion in § 8.1 a spin-density wave and a charge-density wave both with wave vector $2k_F$ are formed. This is in agreement with the result of Lieb and Wu that the ground state is antiferromagnetic and insulating.

Generalization of the Hubbard model taking into account the Coulomb interaction between electrons on neighbouring sites allows for nonequal values of the Fermi gas couplings. This model cannot be solved exactly but there is an extended literature on various approximate treatments of the extended Hubbard model. Here we mention only a few works which can be related to the models discussed in this review. In all our earlier considerations it was assumed that the interaction energies are small compared to the bandwidth, $g_i/2\pi v_F \ll 1$ or in the language of the Hubbard model $U \ll t$. Efetov and Larkin /1975/ and Emery /1976b/ studied the opposite limit, i.e. a 1-d electron gas with strong "on-site" interaction. They assume that the largest energy in the problem is U, the hopping term and the nearest neighbour Coulomb interaction V are small compared to it. The model they consider is defined by the Hamiltonian

$$H = U \sum_i c^+_{i\uparrow} c_{i\uparrow} c^+_{i\downarrow} c_{i\downarrow} + t \sum_{i,\sigma} \left(c^+_{i\sigma} c_{i+1\sigma} + c^+_{i+1\sigma} c_{i\sigma} \right) + $$
$$+ V \sum_{i,\sigma,\sigma'} c^+_{i\sigma} c_{i\sigma} c^+_{i+1\sigma'} c_{i+1\sigma'} . \qquad /11.59/$$

The "on-site" Coulomb interaction U is usually taken to be positive corresponding to repulsion between electrons on the same site, but in strongly polarizable molecules the direct Coulomb repulsion can be reduced and the indirect interaction can lead to $U<0$. In this case electrons of opposite spin form pairs and the sites are either doubly occupied or unoccupied. Making a second order perturbational calculation, an effective Hamiltonian can be introduced in the space of paired states which is

$$H' = -\frac{t^2}{|U|}\sum_{i,\sigma}\Big[c_{i\sigma}^{+} c_{i+1\sigma} c_{i+1\sigma}^{+} c_{i\sigma} + c_{i+1\sigma}^{+} c_{i\sigma} c_{i\sigma}^{+} c_{i+1\sigma} + c_{i\sigma}^{+} c_{i+1\sigma} c_{i,-\sigma}^{+} c_{i+1,-\sigma} + c_{i+1\sigma}^{+} c_{i\sigma} c_{i+1,-\sigma}^{+} c_{i,-\sigma}\Big] + V\sum_{i,\sigma,\sigma'} c_{i\sigma}^{+} c_{i\sigma} c_{i+1\sigma'}^{+} c_{i+1\sigma'} \,. \qquad /11.60/$$

Introducing the pair operators

$$b_i = c_{i\uparrow} c_{i\downarrow} \,, \qquad b_i^{+} = c_{i\downarrow}^{+} c_{i\uparrow}^{+} \,, \qquad /11.61/$$

and the operators

$$n_i = \frac{1}{2}\left(c_{i\uparrow}^{+} c_{i\uparrow} + c_{i\downarrow}^{+} c_{i\downarrow} - 1\right),$$

$$\sigma_i = \frac{1}{2}\left(c_{i\uparrow}^{+} c_{i\uparrow} - c_{i\downarrow}^{+} c_{i\downarrow}\right), \qquad /11.62/$$

the effective Hamiltonian H' has the form

$$H' = \frac{2t^2}{|U|} \sum_i \left[-b_i^+ b_{i+1} - b_{i+1}^+ b_i + 2n_i n_{i+1} + 2\sigma_i \sigma_{i+1} - \frac{1}{2} \right] +$$

$$+ 4V \sum_i \left(n_i n_{i+1} + n_i + \frac{1}{4} \right) .$$

/11.63/

The operators σ_i give zero acting on doubly occupied or unoccupied sites and can be ignored. The commutation relations of the b_i and n_i operators

$$\left[b_i , b_j^+ \right] = \left[b_i , b_j \right] = 0 \qquad \text{if} \quad i \neq j$$

$$b_i^2 = 0$$

$$\left[n_i , b_j^+ \right] = b_j^+ \delta_{ij} , \qquad \left[n_i , b_j \right] = - b_j \delta_{ij}$$

/11.64/

are the same as the commutation relations of spin-1/2 operators with the identification

$$b_i = S_i^- ,$$

$$b_i^+ = S_i^+ ,$$

$$n_i = S_i^z .$$

/11.65/

Thus the Hamiltonian /11.63/ is equivalent to a spin-1/2 anisotropic Heisenberg model Hamiltonian

$$H' = \frac{4t^2}{|U|} \sum_i \left[S_i^z S_{i+1}^z - \frac{1}{2}\left(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+\right)\right] +$$

$$+ 4V \sum_i S_i^z S_{i+1}^z + 4V \sum_i S_i^z \qquad /11.66/$$

We arrived at a model which has been shown already in § 10 to be equivalent to the 1-d fermion gas. Using the results of Johnson et al. /1973/, Fowler /1978/ studied the behaviour of this generalized Hubbard model. He concluded from this analogy that the ground state is a charge-density wave state for $V<0$, whereas for $V>0$ a superconducting state is stabilized at $T=0$.

Emery /1976b/ pointed out that in the strongly repulsive case $(U>0)$ with half-filled band when all sites are singly occupied, the generalized Hubbard model can again be mapped onto a spin-1/2 Heisenberg chain problem, now with the identification

$$S_i^+ = c_{i\downarrow} c_{i\uparrow}^+ ,$$

$$S_i^- = c_{i\uparrow} c_{i\downarrow}^+ ,$$

$$S_i^z = \frac{1}{2}\left(c_{i\uparrow}^+ c_{i\uparrow} - c_{i\downarrow}^+ c_{i\downarrow}\right) , \qquad /11.67/$$

and the ground state is a transverse or longitudinal spin-density wave state.

11.5. Summary of relationship of various models

The list of models which can be related to the 1-d Fermi gas model is not exhausted yet. The 2-d Coulomb gas and the 2-d X-Y model can be further related to roughening models. Knops /1977/ and independently José et al. /1977/ have shown that a general 2-d planar model

$$H = -\sum_{ij} V(\varphi_i - \varphi_j) \tag{11.68}$$

where $-\pi \leq \varphi_i \leq \pi$ is the polar angle at site R_i, with an arbitrary potential V can be transformed by a duality transformation into a general 2-d roughening model

$$\tilde{H} = -\sum_{ij} \tilde{V}(h_i - h_j), \tag{11.69}$$

where h_i takes on all integer values, provided

$$\exp\left(-\tilde{\beta}\,\tilde{V}(h)\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi\, \exp\left[-ih\varphi - \beta V(\varphi)\right], \tag{11.70}$$

$\tilde{\beta}$ is the inverse temperature in the dual model. The

Villain model /Villain 1975/ corresponds to a particular choice of $V(\varphi)$ for which $\tilde{V}(k)$ is quadratic, the dual model is called the discrete Gaussian model.

On the other hand the discrete Gaussian model can be mapped /Chui and Weeks 1976/ onto a neutral Coulomb gas with arbitrary integer charges. Villain obtained the relationship between his model and the Coulomb gas directly. When studying the properties of the Coulomb gas, particles with unit charge only are considered, which is probably a reasonable approximation at low temperatures only.

The various models mentioned in this paragraph and the transformations by which they can be related to each other are shown in Table 1. Table 2 summarizes the relationship between the parameters of the various models, and indicates the behaviour of the systems in the different regimes.

Table 1. Mappings between 1-d and 2-d models.
Dashed lines connect similar but inequivalent models.

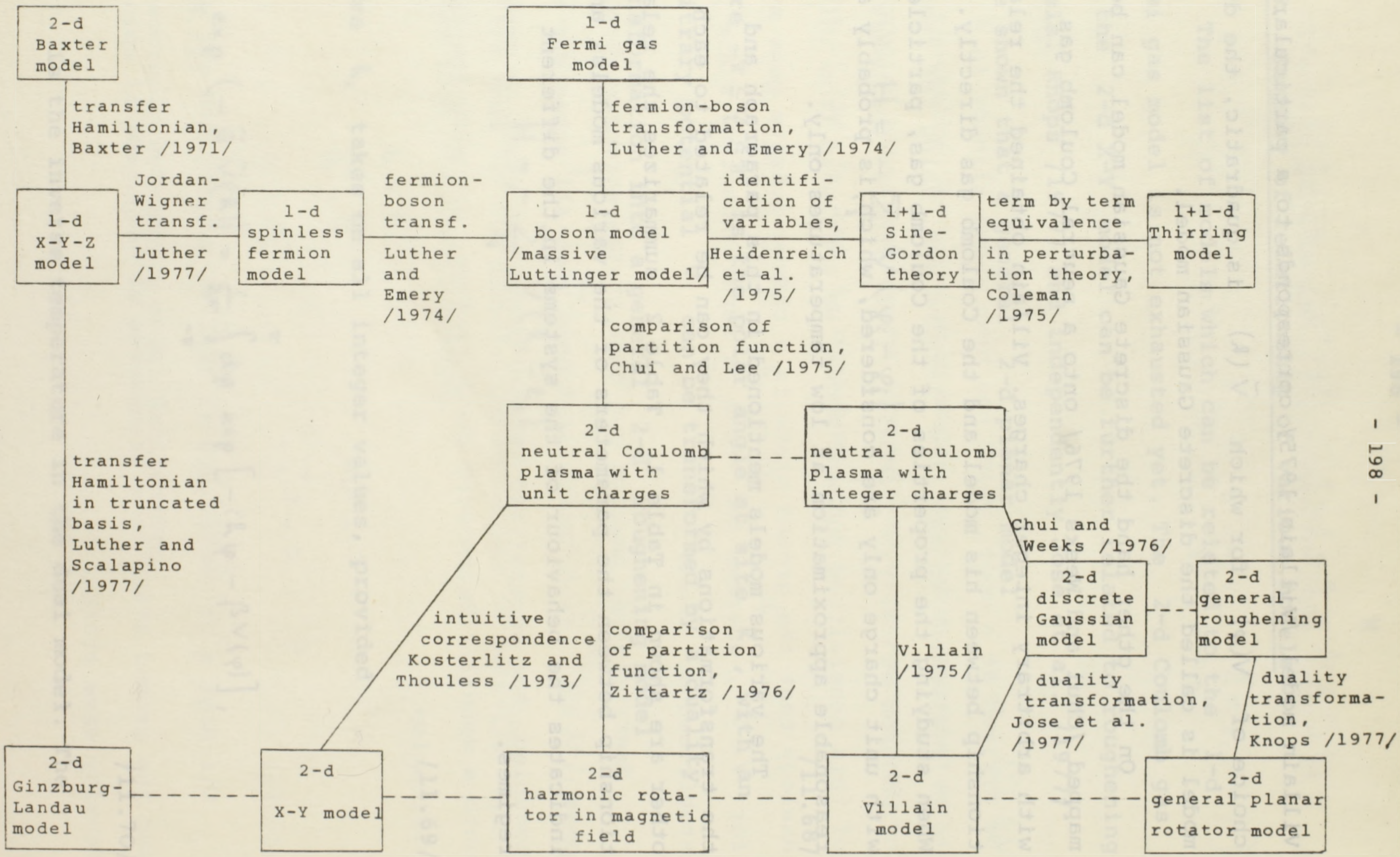

Table 2. Relationship in the behaviour and between the parameters of the various models.

model	parameters and their relationships	region I.	region II.	region III.
spin-density part of the 1-d Fermi gas model	$g_{1\parallel}$, $g_{1\perp}$, $v_\sigma = v_F + \frac{1}{2\pi}(g_{1\parallel} - g_{1\perp})$ or $u_\sigma = \left[v_\sigma^2 - \left(\frac{1}{2\pi} g_{1\parallel}\right)^2\right]^{1/2}$	$g_{1\parallel} \ge \lvert g_{1\perp}\rvert$ This region scales to the Tomonaga model with $g_{1\perp} = 0$. There is no gap in the excitation spectrum.	$-\frac{6}{5}\pi v_\sigma \le g_{1\parallel} < \lvert g_{1\perp}\rvert$ This region scales to a strong coupling regime. There is a gap in the excitation spectrum.	$-2\pi v_\sigma \le g_{1\parallel} < -\frac{6}{5}\pi v_\sigma$ There are bound states in the excitation spectrum.
charge-density part of the 1-d Fermi gas model	$g_{1\parallel} - 2g_2$, g_3, $v_\rho = v_F + \frac{1}{2\pi}(g_{1\parallel} + g_{1\perp})$ or $u_\rho = \left\{v_\rho^2 - \left[\frac{1}{2\pi}(g_{1\parallel} - 2g_2)\right]^2\right\}^{1/2}$	$g_{1\parallel} - 2g_2 \ge \lvert g_3\rvert$ This region scales to the Tomonaga model with $g_3 = 0$. There is no gap in the excitation spectrum.	$-\frac{6}{5}\pi v_\rho \le g_{1\parallel} - 2g_2 < \lvert g_3\rvert$ This region scales to a strong coupling regime. There is a gap in the excitation spectrum.	$-2\pi v_\rho \le g_{1\parallel} - 2g_2 < -\frac{6}{5}\pi v_\rho$ There are bound states in the excitation spectrum.
2-d Coulomb gas	$\exp(-\beta_{pl}\mu) = \frac{g_{1\perp}}{4\pi^2 u_\sigma}$, $\beta_{pl} e^2 = 4\left[\frac{v_\sigma + \frac{1}{2\pi} g_{1\parallel}}{v_\sigma - \frac{1}{2\pi} g_{1\parallel}}\right]^{1/2}$	$\beta_{pl} e^2 \ge 4$ Insulating phase in the low temperature region.	$2 \le \beta_{pl} e^2 < 4$ Metallic behaviour only for large distances in an intermediate temperature region.	$0 \le \beta_{pl} e^2 < 2$ Metallic phase in the high temperature region.
1-d X-Y-Z model	$\frac{1}{2}(J_x - J_y) = \frac{g_{1\perp}}{2\pi\alpha}$, $4 J_z a = -\frac{5}{4} g_{1\parallel} - \frac{3}{2}\pi v_\sigma$, $a\left[\frac{1}{2}(J_x + J_y) - \frac{1}{\pi} J_z\right] = \frac{5}{4} v_\sigma + \frac{3}{8\pi} g_{1\parallel}$	Extension of the analogy to this region is questionable.	$J_z \le 0$ There is a gap in the spin wave spectrum.	$J_z > 0$ There exist bound spin wave states.
2-d X-Y model	$\exp(-\beta_{xy}\pi^2 J) = \frac{g_{1\perp}}{4\pi^2 u_\sigma}$, $4\pi\beta_{xy} J = 4\left[\frac{v_\sigma + \frac{1}{2\pi} g_{1\parallel}}{v_\sigma - \frac{1}{2\pi} g_{1\parallel}}\right]^{1/2}$	$\pi\beta_{xy} J - 1 \ge 2\pi \exp(-\beta_{xy}\pi^2 J)$ The vortices are bound into pairs in the low temperature region.	$\pi\beta_{xy} J - 1 < 2\pi \exp(-\beta_{xy}\pi^2 J)$ The vortices become free in the high temperature region.	Extension of the analogy to this region is questionable.
2-d harmonic rotator model	$\frac{h}{2\pi} = \frac{g_{1\perp}}{4\pi^2 u_\sigma}$, $\eta(T) = \frac{T}{4\pi A(J,T)} = 4\left[\frac{v_\sigma + \frac{1}{2\pi} g_{1\parallel}}{v_\sigma - \frac{1}{2\pi} g_{1\parallel}}\right]^{-1/2}$	$\eta(T) \ge 4$ The magnetic field is an irrelevant perturbation in the high temperature region.	$2 \le \eta(T) < 4$ The susceptibility is finite, the magnetic field is a relevant perturbation in an intermediate temperature region	$0 \le \eta(T) < 2$ The susceptibility is divergent in the low temperature region.
Sine-Gordon theory	$\frac{\alpha_0}{\beta^2} = -2\frac{g_{1\perp}}{(2\pi\alpha)^2 u_\sigma}$, $\beta^2 = 8\pi\left[\frac{v_\sigma + \frac{1}{2\pi} g_{1\parallel}}{v_\sigma - \frac{1}{2\pi} g_{1\parallel}}\right]^{1/2}$	$\beta^2 \ge 8\pi$ The theory is meaningful on a lattice only.	$4\pi \le \beta^2 < 8\pi$ The solitons have finite mass.	$0 \le \beta^2 < 4\pi$ There exist bound soliton states.
Thirring model	$g = \frac{4\pi^2}{\beta^2} - \pi$	$g \le -\frac{\pi}{2}$	$-\frac{\pi}{2} < g \le 0$	$g > 0$

§ 12. SYSTEM OF WEAKLY COUPLED CHAINS

So far a strictly one-dimensional model has been treated in this paper. The so-called "one-dimensional conductors" which are studied very extensively nowadays experimentally are in fact real three-dimensional systems. Their one-dimensional or quasi-one-dimensional character arises from their particular structure, namely they consist of weakly coupled linear chains. It would be natural to assume that this weak coupling does not modify strongly the behaviour of the system. It is known, however, that there are very important dimensionality effects in one-dimensional systems and a weak interchain coupling, however weak it is, can have drastic effects. It is therefore of great interest how a system of weakly coupled chains behaves and a comparison of existing experimental data should be done with the results obtained for coupled chains. In this respect the single chain problem serves as an example where the different approaches have been elaborated and then these methods are easily applied to the coupled chain problem. Analogously to the strictly one-dimensional case there are exactly soluble models for the coupled chain problem, but unfortunately they do not correspond to physically relevant situation. Therefore usually a leading logarithmic or next to leading logarithmic approximation is made in both the intrachain and interchain couplings or the single chain problem is considered as a zeroth order approximation and the interchain couplings are taken into account in a mean field approximation. In this paragraph the main results obtained for a system of weakly coupled chains are presented.

There are various interaction mechanisms that can couple the one-dimensional chains. The phonon mediated coupling has been treated by several authors /Rice and Strässler 1973b, Bjelis et al. 1974, Dieterich 1974, Leung 1975, Brazovsky and Dzyaloshinsky 1976, Brazovsky et al. 1977, Barisic 1978/. This problem will not be delt with in this paper and we will restrict ourselves to electron-electron interactions. The retardation effects which can be important in the case of a phonon mediated effective electron-electron interaction will not be considered here.

If the effect of phonons is not taken into account, two types of electronic interactions are usually considered to couple the chains. One is a Coulomb type two-body interaction between electrons on different chains, the other is a direct hopping of electrons between the chains. The two mechanisms will be considered successively, starting first with the case when there is no hopping between the chains.

Similarly to the earlier treatment of the single chain problem, in a first step the backward scattering will be neglected and an exact solution is given for the generalized Tomonaga model. Then the various approaches to treat the backward scattering will be presented. Finally the effect of interchain hopping will also be considered. The crossover from 1-d to 3-d behaviour will be briefly reviewed.

12.1. Chains coupled by interchain forward scattering

The Tomonaga model for a single chain can be solved exactly as was shown in § 5. This is due to the fact that the Hamiltonian of the Tomonaga model can be written as a bilinear expression of the charge- and spin-density, respectively, and an exact diagonalization is possible. The Ward identities used in § 5 are the consequences of this simple form of the Hamiltonian. For a system of coupled chains the same situation occurs if the interchain interaction has forward scattering components only. Using the same procedure as in § 5 the Green's function and response functions can be calculated exactly.

The Hamiltonian is best written in a mixed representation with an index λ denoting the chain on which the electron moves and an index k denoting the momentum along the chain. In this notation the Hamiltonian can be written as

$$H = H_0 + H_{int}\,, \tag{12.1}$$

where

$$H_0 = \sum_{\lambda, k, \alpha} \varepsilon_k\, a^+_{\lambda k \alpha}\, a_{\lambda k \alpha} + \sum_{\lambda, k, \alpha} \varepsilon_k\, b^+_{\lambda k \alpha}\, b_{\lambda k \alpha}\,, \tag{12.2}$$

and

$$H_{int} = \frac{1}{L} \sum_{\substack{\lambda\mu \\ k_1, k_2, p \\ \alpha\beta}} \left(g_{2\parallel\lambda\mu}\, \delta_{\alpha\beta} + g_{2\perp\lambda\mu}\, \delta_{\alpha,-\beta} \right) a^+_{\lambda k_1 \alpha}\, b^+_{\mu k_2 \beta}\, b_{\mu k_2 + p \beta}\, a_{\lambda k_1 - p \alpha} +$$

$$+ \frac{1}{2L} \sum_{\substack{\lambda\mu \\ k_1, k_2, p \\ \alpha\beta}} \left(g_{4\parallel\lambda\mu}\, \delta_{\alpha\beta} + g_{4\perp\lambda\mu}\, \delta_{\alpha,-\beta} \right) \Big(a^+_{\lambda k_1 \alpha}\, a^+_{\mu k_2 \beta}\, a_{\mu k_2 + p \beta}\, a_{\lambda k_1 - p \alpha} +$$

$$+ b^+_{\lambda k_1 \alpha}\, b^+_{\mu k_2 \beta}\, b_{\mu k_2 + p \beta}\, b_{\lambda k_1 - p \alpha} \Big). \tag{12.3}$$

$g_{\lambda\mu}$ is the coupling strength for the interaction of electrons moving on chains λ and μ and we assumed that the chains are equivalent, i.e. the dispersion relation is the same on all the chains. The cutoff will be on the momentum transfer p and therefore $g_{4\parallel}$ is also kept.

The calculational procedure of § 5 can be repeated with a few trivial notational changes. The Ward identities in eqns. /5.16/ - /5.17/ and the generalized relations shown in figs. 22 and 23 are again valid for vertices which have the same chain index on the two fermion legs. The effective interactions can be introduced by the same diagrammatic equations /fig. 17/. In these equations a summation over intermediate chain indices has to be performed. In the Fourier transformed form a momentum component q perpendicular to the chain direction appears and in this momentum representation the effective couplings have the same form as before /see eqns./5.9/ - /5.15// but everywhere g should be replaced by the q dependent g

$$g_j(q) = \sum_{\lambda} g_{j\lambda\mu} e^{-iq(R_\lambda - R_\mu)} \, , \qquad j = 2\parallel,\ 2\perp,\ 4\parallel,\ 4\perp \qquad /12.4/$$

where R_λ and R_μ give the positions of the chains. The renormalized velocities u_σ and u_ρ are also q dependent.

$$u_\sigma^2(q) = \left[v_F + \frac{1}{2\pi}\left(g_{4\parallel}(q) - g_{4\perp}(q)\right)\right]^2 - \left[\frac{1}{2\pi}\left(g_{2\parallel}(q) - g_{2\perp}(q)\right)\right]^2 , \qquad /12.5/$$

$$u_\varrho^2(q) = \left[v_F + \frac{1}{2\pi}\left(g_{4\parallel}(q) + g_{4\perp}(q)\right)\right]^2 - \left[\frac{1}{2\pi}\left(g_{2\parallel}(q) + g_{2\perp}(q)\right)\right]^2 .$$

/12.6/

Since the Green's function is diagonal in the chain index, the solution of the Dyson equation has a product form

$$G_{+\lambda}(x,t) = \frac{1}{2\pi} \; \frac{x - v_F t + i/\Lambda(t)}{x - v_F t + i\,\delta(t)} \times$$

$$\times \prod_q \left\{ \frac{1}{\left(x - u_\sigma(q)t + i/\Lambda(t)\right)^{1/2} \left(x - u_\varrho(q)t + i/\Lambda(t)\right)^{1/2}} \times \right.$$

$$\times \left[\Lambda^2(t)\left(x - u_\sigma(q)t + i/\Lambda(t)\right)\left(x + u_\sigma(q)t - i/\Lambda(t)\right)\right]^{-\alpha_\sigma(q)} \times$$

$$\left. \times \left[\Lambda^2(t)\left(x - u_\varrho(q)t + i/\Lambda(t)\right)\left(x + u_\varrho(q)t - i/\Lambda(t)\right)\right]^{-\alpha_\varrho(q)} \right\}^{1/N} ,$$

/12.7/

with

$$\alpha_\sigma(q) = \frac{1}{4u_\sigma(q)}\left[v_F + \frac{1}{2\pi}\left(g_{4\parallel}(q) - g_{4\perp}(q)\right) - u_\sigma(q)\right] ,$$

/12.8/

$$\alpha_\varrho(q) = \frac{1}{4u_\varrho(q)}\left[v_F + \frac{1}{2\pi}\left(g_{4\parallel}(q) + g_{4\perp}(q)\right) - u_\varrho(q)\right] .$$

/12.9/

The product has to be taken for all the perpendicular momenta in the Brillouin zone and N is the number of chains i.e. the number of possible q values. If the q dependence of the velocities can be neglected, the Green's function has the same form as for a single chain /see eqn. /5.19//, but α_σ and α_ρ are replaced by α'_σ and α'_ρ, respectively, with

$$\alpha'_\sigma = \frac{1}{N}\sum_q \frac{1}{4u_\sigma}\left[v_F + \frac{1}{2\pi}\left(g_{4\parallel}(q) - g_{4\perp}(q)\right) - u_\sigma\right], \qquad /12.10/$$

and

$$\alpha'_\rho = \frac{1}{N}\sum_q \frac{1}{4u_\rho}\left[v_F + \frac{1}{2\pi}\left(g_{4\parallel}(q) + g_{4\perp}(q)\right) - u_\rho\right]. \qquad /12.11/$$

A similar result is valid for the response functions. Considering again the same response functions as for the single chain problem, i.e. charge-density wave, spin-density wave, singlet and triplet Cooper pair response functions, they are all diagonal in the chain index and the response functions are obtained in product form. Neglecting again the q dependence in u_σ and u_ρ, the response functions have the same form as in eqns. /5.28/ and /5.30/, but the exponents β_σ and β_ρ are replaced by β'_σ and β'_ρ where

$$\beta'_\sigma = \frac{1}{N}\sum_q \frac{1}{4\pi u_\sigma}\left(g_{2\parallel}(q) - g_{2\perp}(q)\right), \qquad /12.12/$$

$$\beta'_\rho = \frac{1}{N}\sum_q \frac{1}{4\pi u_\rho}\left(g_{2\parallel}(q) + g_{2\perp}(q)\right) \qquad /12.13/$$

This result shows that at least for the weak coupling case when the q dependence of u_σ and u_ρ can be neglected, the interchain forward scattering alone does not change the structure of the response functions compared to the single chain problem, it only renormalizes the exponents. An excitation of a charge- or spin-density wave or a Cooper pair has no response on other chains, the chains act as if they were decoupled. Therefore this system may have no phase transition at any finite temperature.

This result has been first obtained by Klemm and Gutfreund /1976/. They used the bosonization transformation, which in this model leads to the same result as the diagrammatic approach. Furthermore the bosonization transformation allows to solve the model in which the intrachain backward scattering is included. The Luther-Emery solution can be used for a particular value of the strength of this process. The interchain forward scattering gives rise only to a renormalization of the exponents, but the singularities are not shifted to finite temperature, no real phase transition can occur with interchain forward scattering.

12.2. Chains coupled by interchain backward scattering

Since the interchain forward scattering does not lead to correlation between different chains, the interchain backward scattering or hopping have to be considered as possible mechanisms giving rise to three-dimensional ordering. The treatment of backward scattering is rather difficult for the single chain problem already. A similar treatment of the system of chains is even more complicated. There are two usual approaches, extensions of two approaches used in the single chain problem.

The Fermi gas model of a system of coupled chains can be obtained by a simple extension of the model described in § 2. If there is no hopping between the chains, the dispersion relation in the quasi-1-d case is like the one in the 1-d problem. The logarithmic contributions appearing in the perturbational calculation are the consequences of the linear dispersion and therefore this problem is again a logarithmic one. We have seen that the parquet approximation is a usual approximate treatment of logarithmic problems. Gorkov and Dzyaloshinsky /1974/ have studied the properties of the system of coupled chains in this approximation. In this way only the leading logarithmic corrections are properly collected, but next to leading and subsequent corrections may be equally important. These corrections can be accounted for in a straightforward manner by using the renormalization group approach as in § 4. This method has been applied to the present problem by Mihály and Sólyom /1976/. A detailed discussion of the effect of various couplings and of the possible phase transitions has been presented by Lee et al. /1977/.

In an alternative treatment of the system of coupled chains the linear dispersion relation allows to use the bosonization transformation, in the same way as for a single chain. Klemm and Gutfreund /1976/ applied this technique using the Luther-Emery solution of the single chain problem /see § 6 / and treating the interchain backward scattering in mean field approximation, though the effect of fluctuations has also been estimated.

The Hamiltonian of even the single chain problem contains so many coupling constants in the most general spin-

-dependent case, that it is difficult to handle it. A generalization of the model to a system of coupled chains will increase the number of coupling constants enormously. In all the treatments so far a much simpler Hamiltonian is used in which spin dependence, umklapp processes and g_4 type processes are neglected. We are then left with a problem with intrachain and interchain backward and forward scatterings, respectively. The Hamiltonian is written in the form

$$H = H_0 + H_{int} \,, \qquad /12.14/$$

where

$$H_0 = \sum_{\lambda, k, \alpha} v_F \left(|k| - k_F \right) \left(a^+_{\lambda k \alpha} a_{\lambda k \alpha} + b^+_{\lambda k \alpha} b_{\lambda k \alpha} \right) , \qquad /12.15/$$

and

$$H_{int} = \frac{1}{L} \sum_{\substack{\lambda \mu \\ k_1, k_2, p \\ \alpha \beta}} \Big[g_{1 \lambda \mu} a^+_{\lambda k_1 \alpha} b^+_{\mu k_2 \beta} a_{\mu k_2 + 2k_F + p \beta} b_{\lambda k_1 - 2k_F - p \alpha} + g_{2 \lambda \mu} a^+_{\lambda k_1 \alpha} b^+_{\mu k_2 \beta} b_{\mu k_2 + p \beta} a_{\lambda k_1 - p \alpha} \Big] . \qquad /12.16/$$

It is assumed again that the chains are equivalent, the Fermi velocities are equal on all the chains. In the case of TTF-TCNQ the two types of chains have inverted bands with $v_{TTF} \approx - v_{TCNQ}$. Making an electron-hole transformation on one type of chains, the Fermi velocity changes sign and the sign of the coupling constants of the interactions coupling two different types of chains is also changed. Therefore without losing generality we will assume that the velocities are equal and the couplings can have any sign.

Since the problem defined by this Hamiltonian is a logarithmic one due to the linear dispersion, the renormalization group treatment allows a consistent calculation of leading, next to leading etc. logarithmic terms. In the same way as for the single-chain problem, the cutoff scaling can be formulated as a multiplicative renormalization /see Mihály and Sólyom /1976// and the Lie equations for the invariant couplings can be obtained in a straightforward way, generalizing eqns. /4.31/ - /4.33/:

$$\frac{d g_{1\lambda\mu}}{dx} = \frac{1}{x}\left\{ \frac{1}{\pi v_F}\sum_{\nu} g_{1\lambda\nu}\, g_{1\nu\mu} + \frac{1}{\pi v_F} g_{1\lambda\mu}\left(g_{2\lambda\mu} - g_{2\lambda\lambda}\right) + \right.$$

$$+ \frac{1}{2\pi^2 v_F^2} g_{1\lambda\mu}\Big[g_{1\lambda\lambda}\left(g_{2\lambda\mu} - g_{2\lambda\lambda}\right) +$$

$$\left. + \sum_{\nu}\left(g_{1\lambda\nu}^2 + g_{2\lambda\nu}^2 - g_{2\lambda\nu}\, g_{2\nu\mu}\right)\Big] + \dots \right\}, \qquad /12.17/$$

$$\frac{d g_{2\lambda\mu}}{dx} = \frac{1}{x}\left\{ \frac{1}{2\pi v_F} g_{1\lambda\mu}^2 + \frac{1}{4\pi^2 v_F^2} g_{1\lambda\mu}^2 g_{1\lambda\lambda} + \right.$$

$$\left. + \frac{1}{2\pi^2 v_F^2} \sum_\nu g_{1\lambda\nu}^2 \left(g_{2\lambda\mu} - g_{2\nu\mu} \right) + \ldots \right\} .$$

/12.18/

In the Lie equations for the invariant couplings x is the ratio of the new and old cutoffs. When the invariant couplings are used to calculate the temperature dependent response functions, the argument x of the invariant couplings has to be identified with the argument T/E_0 of the response functions. Keeping this identification in mind we will speak about the temperature dependence of the invariant couplings.

Keeping only the second order terms, the parquet equations of Gorkov and Dzyaloshinsky /1974/ are recovered. In this approximation as well as in the better one, in which the third order terms are also taken into account, we find two different regimes leading to two very different solutions.

The renormalization group equations have such fixed points for which $g_{1\lambda\mu}^* = 0$ and $g_{2\lambda\mu}^*$ have non-universal values. This corresponds to a system of Tomonaga chains coupled by interchain forward scattering only. This model was treated in § 12.1 and as it was shown, it has no real phase transition.

The physically more interesting solution is a fixed point with strong attractive interchain backward scattering. In order to find this fixed point we will neglect the forward scattering terms in eqn. /12.17/. The reason for this is that, as we have seen, forward scattering alone does not lead to real phase transition, and it can also be seen from eqns. /12.17/ - /12.18/ that in the case of a large number of interacting chains the enhancement of the backward scattering is much stronger during the renormalization than that of the forward scattering. Near the fixed point eqn. /12.17/ can be approximated as

$$\frac{d g_{1\lambda\mu}}{dx} = \frac{1}{x}\left\{\frac{1}{\pi v_F}\sum_{\nu} g_{1\lambda\nu}\, g_{1\nu\mu} + \frac{1}{2\pi^2 v_F^2}\, g_{1\lambda\mu}\sum_{\nu} g_{1\lambda\nu}^2 + \ldots\right\}. \qquad /12.19/$$

Starting with intrachain and nearest neighbour interchain bare interactions, the renormalization will generate interactions between next nearest neighbours and further neighbours. Near the fixed point the renormalized interaction can be a long range one and therefore it is not sufficient to restrict the summation in eqn. /12.19/ to a few neighbours only. It is useful to transform eqn. /12.19/ by Fourier transformation into

$$\frac{d g_1(q)}{dx} = \frac{1}{x}\left\{\frac{1}{\pi v_F} g_1^2(q) + \frac{1}{2\pi^2 v_F^2}\, g_1(q)\,\frac{1}{N}\sum_{q'} g_1^2(q') + \ldots\right\},$$

/12.20/

where q is the wave vector perpendicular to the chain directions and N is the number of chains.

For particular values of the bare coupling constants this equation can have a solution where the q and x dependences are separable, i.e. all the interchain couplings are enhanced in the same way as the intrachain coupling. This can happen for long range bare interchain interaction only /Gorkov and Dzyaloshinsky 1974/ and this situation is not very interesting physically.

In the more physical situation when the unrenormalized interchain interaction is of short range, the q and x dependences are not separable. The shape of $g_1(q)$ changes in the course of renormalization when the cutoff is scaled down /the temperature is decreased/. In that region where $g_1(q)$ is negative and renormalization leads to a strengthening of the attractive interaction, a dip develops in $g_1(q)$ at some value q_0 of the perpendicular wave vector. The position of q_0 is approximately determined by the minimum of the Fourier transform of the bare coupling. The depth of this dip decreases with decreasing temperature and diverges at a temperature T_c, which is the transition temperature. Near q_0 and T_c the solution of eqn. /12.20/ can be written in the form /Gorkov and Dzyaloshinsky 1974, Mihály and Sólyom 1976/

$$g_1(q, x) \sim \frac{1}{\ln \frac{T_c}{T}} \frac{\kappa^2(x)}{(q-q_0)^2 + \kappa^2(x)} ,$$

/12.21/

where $\kappa^2(x)$ decreases with decreasing temperature and vanishes at T_c . In the parquet approximation $\kappa^2(x = T/E_0) = \ln T/T_c$ while if next corrections are also considered, this form is somewhat modified.

Assuming that the bare interchain coupling is attractive and acts between nearest neighbour chains only, $g_1(q)$ for the bare coupling $(x = 1)$ is a combination of cosine curves with its minimum at $q_0 = 0$. The singularity in the invariant coupling will appear at $q_0 = 0$. This means that at the phase transition point all the far distant neighbours are equally strongly coupled attractively. If, however, the coupling between the nearest neighbour chains is repulsive, the cosine functions in $g_1(q)$ have their minima at $q_0 = \left(\pm\frac{\pi}{a}, \pm\frac{\pi}{a}\right)$ i.e. at the zone boundary of the Brillouin zone. At the fixed point the couplings between the chains are alternatively repulsive or attractive.

In order to determine what kind of phase transitions can take place in this system the temperature dependence of the response functions has to be studied. Mihály and

Sólyom /1976/ investigated the charge-density wave, spin--density wave and singlet-superconductor responses. Lee et al. /1977/ extended these investigations to the triplet-superconductor and excitonic responses and interchain Cooper pairing. These temperature dependent response functions can be obtained by the analytic continuation to the upper half-plane of the correlation functions

$$R(k,q,\omega_\nu) = -\int_0^{1/T} d\tau\, e^{i\omega_\nu \tau} \sum_{\lambda\mu} e^{iq(R_\lambda - R_\mu)} \langle T_\tau \{ O_\lambda(k,\tau)\, O_\mu(k,0)\} \rangle , \tag{12.22}$$

where for the charge-density wave response $N(k,q,\omega_\nu)$

$$O_\lambda(k,\tau) = \sum_\alpha \frac{1}{L^{1/2}} \sum_p c^+_{\lambda p \alpha}(\tau)\, c_{\lambda\, p+k\, \alpha}(\tau) , \tag{12.23}$$

for the spin-density wave response $\chi(k,q,\omega_\nu)$

$$O_\lambda(k,\tau) = \frac{1}{L^{1/2}} \sum_p c^+_{\lambda p \uparrow}(\tau)\, c_{\lambda\, p+k\, \downarrow}(\tau) , \tag{12.24}$$

for the singlet-superconductor response $\Delta_s(k,q,\omega_\nu)$

$$O_\lambda(k,\tau) = \frac{1}{L^{1/2}} \sum_p c_{\lambda p \uparrow}(\tau)\, c_{\lambda\, k-p\, \downarrow}(\tau) , \tag{12.25}$$

for the triplet-superconductor response $\Delta_t(k, q, \omega_\nu)$

$$O_\lambda(k, \tau) = \frac{1}{L^{1/2}} \sum_p c_{\lambda p \uparrow}(\tau)\, c_{\lambda\, k-p \uparrow}(\tau),$$

/12.26/

for the excitonic response $X_{\alpha\alpha'}(k, q, \omega_\nu)$

$$O_{\lambda\, \alpha\alpha'}(k, \tau) = \frac{1}{L^{1/2}} \sum_p c^+_{\lambda p \alpha}(\tau)\, c_{\lambda+1\, p+k\, \alpha'}(\tau),$$

/12.27/

for the interchain Cooper pairing $A_{\alpha\alpha'}(k, q, \omega_\nu)$

$$O_{\lambda\, \alpha\alpha'}(k, \tau) = \frac{1}{L^{1/2}} \sum_p c_{\lambda p \alpha}(\tau)\, c_{\lambda+1\, k-p\, \alpha'}(\tau).$$

/12.28/

p and k in these expressions are momenta parallel to the chain direction and q is the perpendicular momentum. The density responses are most singular for $k = 2k_F$, whereas the pairing responses for $k = 0$. In what follows k will be fixed at these values, q and ω will be kept as variables.

Similarly as in the 1-d model, the response functions do not obey Lie equations if the cutoff is used as a scaling parameter. It can be assumed, however, using the perturbational expression for the response functions, that auxiliary quantities $\bar{R} \sim x \cdot dR/dx$ with $x = \omega/E_o$ can be defined that satisfy the scaling equations. The Lie equations for the appropriately normalized auxiliary quantities are obtained from the perturbational expressions in the following form:

$$\frac{d \ln \bar{N}(q,x)}{dx} = \frac{1}{x}\left\{ \frac{2}{\pi v_F} g_1(q) - \frac{1}{\pi v_F} \frac{1}{N} \sum_{q'} g_2(q') + \dots \right\}, \qquad /12.29/$$

$$\frac{d \ln \bar{\chi}(q,x)}{dx} = \frac{1}{x}\left\{ - \frac{1}{\pi v_F} \frac{1}{N} \sum_{q'} g_2(q') + \dots \right\}; \qquad /12.30/$$

$$\frac{d \ln \bar{\Delta}_s(q,x)}{dx} = \frac{1}{x}\left\{ \frac{1}{\pi v_F} \frac{1}{N} \sum_{q'} \left[g_1(q') + g_2(q') \right] + \dots \right\}, \qquad /12.31/$$

$$\frac{d \ln \bar{\Delta}_t(q,x)}{dx} = \frac{1}{x}\left\{ - \frac{1}{\pi v_F} \frac{1}{N} \sum_{q'} \left[g_1(q') - g_2(q') \right] + \dots \right\}, \qquad /12.32/$$

$$\frac{d \ln X_{\alpha\alpha'}(q,x)}{dx} = \frac{1}{x}\left\{\frac{1}{\pi v_F}\frac{1}{N}\sum_{q'}\left[g_1(q') + g_2(q')\right]f(q') + \dots\right\}$$

/12.33/

$$\frac{d \ln \bar{A}_{\alpha\alpha'}(q,x)}{dx} = \frac{1}{x}\left\{-\frac{1}{\pi v_F}\frac{1}{N}\sum_{q'} g_2(q')\, f(q') + \dots\right\},$$

/12.34/

where $f(q')$ depends on the arrangements of the chains. For a two-dimensional square lattice of the chains $f(q') = \cos(q_x a) + \cos(q_y a)$, where a is the distance between the chains. It is easily seen in the perturbational series that except for $N_{\lambda\mu}(k, \omega_\nu)$, all other response functions involve a δ function in the chain index, i.e. except for $N(k, q, \omega_\nu)$ all other response functions are independent of q . The inter-chain correlation exists for charge-density fluctuations only, for all other perturbations the chains are effectively decoupled. Thus it is expected that if a phase transition occurs in this system, it will be to a charge-density wave state. The analysis of the Lie equations for the response functions confirms that in fact only the charge-density wave response function develops a true singularity, i.e. only the charge-density wave state can be stabilized by the interchain backward scattering. This can easily be understood since we have neglected the spin-exchange between the chains which would favour a three-dimensional spin ordering and we have also neglected the tunnelling of Cooper pairs from one chain to the other, which would stabilize a three-dimensional superconducting state.

Lee et al. /1977/ studied the behaviour of the $4k_F$ as well as the uniform response functions. A strong inter-chain backward scattering can couple the $4k_F$ charge-density waves on different chains and could explain the observation of strong $4k_F$ correlations /Pouget et al. 1976, Kagoshima et al. 1976/. The interchain interactions are, however, weak and become stronger near the three-dimensional ordering only. In this situation the $4k_F$ correlation should be a harmonic of the $2k_F$ charge-density correlation. This is not in agreement with the experimental facts according to which the $4k_F$ correlations are observed at high temperatures already, where the $2k_F$ correlations are still very weak. No generally accepted theory exists as yet to explain the $4k_F$ correlations.

Using different methods, Klemm and Gutfreund /1976/ studied the same model and arrived at essentially the same conclusions. The Hamiltonian was transformed to a bosonized form using the fermion-boson transformation /see eqns.

/5.54/ - /5.55//. The intrachain interactions are taken into account exactly by using the Luther-Emery solution and its extension to general values of the intrachain backward scattering. The effect of interchain backward scattering is considered in mean field approximation. They found that a nearest neighbour interchain backward scattering gives rise to a phase transition at finite temperatures into a charge--density wave state. This occurs for those values of the intrachain interactions for which the charge-density wave response function of a single chain is divergent at T=0.

12.3. Simultaneous treatment of backward and forward scattering

As we have seen, a real three-dimensional ordering can take place only if the interchain backward scattering is present in the system. More precisely the phase transition to the charge-density wave state occurs if the interchain backward scattering is renormalized to a strong coupling fixed point and no transition occurs if the backward scattering goes to zero in the renormalization procedure. The domains of attraction of the two fixed points are very simply obtained if the forward scattering is neglected. The two fixed points are $g^*_{1\lambda\mu} = 0$ i.e. $g^*_1(q) = 0$ and $|g^*_{1\lambda\mu}| = 2\pi v_F$ what means that $g^*_1(q)$ is singular for some value of q.

The first fixed point is reached if the unrenormalized $g_1(q)$ is positive for all q and the strong coupling fixed point is reached if the unrenormalized $g_1(q)$ is negative for some values of q. Assuming strong intrachain and weak nearest neighbour interchain interactions, the condition for the occurrence of a phase transition is an attractive intrachain backward scattering. Here we will consider the effect of interchain forward scattering on the domain of attraction of the fixed points.

As a first step Lee et al. /1977/ studied the problem of two coupled chains in the lowest order of the renormalization group approach. Denoting by $g_{1\lambda}$ and $g_{2\lambda}$ $(\lambda = 1, 2)$ the coupling constants of the intrachain backward and forward scattering for the two chains / λ is the chain index/ and by w_1 and w_2 the coupling constants of the interchain backward and forward scattering, respectively, the Lie equations /12.17/ - /12.18/ for the renormalized couplings have the form

$$\frac{d g_{1\lambda}}{dx} = \frac{1}{x} \left\{ \frac{1}{\pi v_F} \left(g_{1\lambda}^2 + w_1^2 \right) + \dots \right\}, \qquad /12.35/$$

$$\frac{d g_{2\lambda}}{dx} = \frac{1}{x} \left\{ \frac{1}{2\pi v_F} g_{1\lambda}^2 + \dots \right\}, \qquad /12.36/$$

$$\frac{dw_1}{dx} = \frac{1}{x}\left\{ \frac{1}{\pi v_F}\left(g_{1\lambda} w_1 + g_{1\mu} w_1 + w_1 w_2 - \frac{1}{2} w_1 g_{2\lambda} - \frac{1}{2} w_1 g_{2\mu} \right) + \dots \right\},$$

/12.37/

$$\frac{dw_2}{dx} = \frac{1}{x}\left\{ \frac{1}{2\pi v_F} w_1^2 + \dots \right\}.$$

/12.38/

For two equivalent chains / $g_{1\lambda}$ and $g_{2\lambda}$ are independent of the chain index λ / the renormalization group equations have two fixed points:

i/ $g_1^* = 0$, $w_1^* = 0$, g_2^* and w_2^* have non-universal values at the fixed point,

ii/ $g_1^* \to -\infty$, $|w_1^*| \to +\infty$, $g_1^* - 2g_2^* + 2w_2^* \to -\infty$

with $g_1^*/(g_1^* - 2g_2^* + 2w_2^*) = 1$, $|w_1^*|/(g_1^* - 2g_2^* + 2w_2^*) = -1$.

A better theory, taking into account higher order corrections in the Lie equations, may modify the value of the strong-coupling fixed point. The important feature is that as in the single chain problem, or in the problem with interchain backward scattering alone, there are two different regimes, one which is scaled onto a fixed point with forward scattering only /Tomonaga chains coupled by interchain forward scattering/ and the other which is scaled onto the strong-coupling fixed point.

The domain of attraction of the strong-coupling fixed point in the single chain problem is $g_1 < 0$. In the case of two coupled chains this domain is larger. In addition to the region $g_1 < 0$, the regions

a/ $g_1 > 0$ and $g_1 - 2g_2 + 2w_2 < 0$

and

b/ $g_1 > 0$, $g_1 - 2g_2 + 2w_2 > 0$ and $g_1(g_1 - 2g_2 + 2w_2) < w_1^2$

scale also onto the strong-coupling fixed point. The effect of forward scattering is that it can renormalize the originally repulsive intrachain backward scattering to attractive couplings. Similar conclusions can be drawn for two coupled inequivalent chains. The response functions of the two-chain problem have been studied by Klemm et al. /1977/.

The analytic treatment of the N-chain problem is not possible. It can be expected, however, that the forward scattering has similar effect and the strong-coupling attractive fixed point can be reached even from repulsive initial couplings. This was verified by Lee et al. /1977/ by solving the scaling equations /12.17/ -/12.18/ numerically. They found that starting even from a repulsive δ-

-function interaction, the coupling constants scale onto the strong-coupling attractive fixed point. This behaviour was also noticed by Menyhárd /1977a/ and Obukhov /1977/. Lee et al. calculated the response functions by a numerical integration of the Lie equation. It may happen that the spin-density wave response function is more strongly enhanced than the charge-density wave response function over a large temperature range, but the true singularity develops in the charge-density wave response function only. This confirms that near the fixed point the forward scattering can be neglected compared to the backward scattering and our earlier analysis of the possible phase transitions is true, but the approach to the fixed point is strongly influenced by the forward scattering.

12.4. Crossover from 1-d to 3-d behaviour

The treatments mentioned until now have concentrated on what states are stabilized for a given set of coupling constants. Menyhárd /1977b/ looked at the problem of quasi-one-dimensional systems from a different point of view. She assumed a weak interchain coupling - weak compared with the intrachain couplings, $|\sum_{\lambda\neq\mu} g_{1\lambda\mu}| \ll |g_{1\lambda\lambda}|$ and $|\sum_{\lambda\neq\mu} g_{2\lambda\mu}| \ll |g_{2\lambda\lambda}|$ - and studied how the behaviour of the system changes from a 1-d like behaviour to a 3-d behaviour as the temperature is decreased. In the approximation mentioned above and assuming further that $g_{2\lambda\mu}$ $(\lambda \neq \mu)$ does not contribute essentially to the renormalization, the Lie equations /12.17/ - /12.18/ for the invariant couplings have the form

$$\frac{d g_{1\lambda\mu}}{dx} = \frac{1}{x}\left\{ \frac{1}{\pi v_F} \sum_{\substack{\lambda\neq\nu \\ \mu\neq\nu}} g_{1\lambda\nu}\, g_{1\nu\mu} + \frac{1}{\pi v_F} g_{1\lambda\mu}\left(2 g_{1\lambda\lambda} - g_{2\lambda\lambda}\right) + \right.$$

$$\left. + \frac{1}{2\pi^2 v_F^2} g_{1\lambda\mu}\left(g_{1\lambda\lambda}^2 + g_{2\lambda\lambda}^2 - g_{1\lambda\lambda}\, g_{2\lambda\lambda}\right) + \dots \right\}, \quad \lambda \neq \mu \qquad /12.39/$$

$$\frac{d g_{2\lambda\mu}}{dx} = 0 \quad , \quad \lambda \neq \mu \qquad /12.40/$$

$$\frac{d g_{1\lambda\lambda}}{dx} = \frac{1}{x}\left\{ \frac{1}{\pi v_F} g_{1\lambda\lambda}^2 + \frac{1}{2\pi^2 v_F^2} g_{1\lambda\lambda}^3 + \dots \right\}, \qquad /12.41/$$

$$\frac{d g_{2\lambda\lambda}}{dx} = \frac{1}{x}\left\{ \frac{1}{2\pi v_F} g_{1\lambda\lambda}^2 + \frac{1}{4\pi^2 v_F^2} g_{1\lambda\lambda}^3 + \dots \right\}. \qquad /12.42/$$

In the second and third terms of the right hand side of eqn. /12.39/ we can recognize the coupling constant combination which appears in the Lie equation determining the the charge-density response function of a single chain /see eqn./4.52//. Denoting by N_{1d} this response function and by $\overline{N}_{1d}$ the auxiliary function

$$\overline{N}_{1d} = \pi v_F \, x \frac{d N_{1d}}{dx} , \qquad x = T/E_0 ,$$

/12.43/

/12.39/ can be solved in the form

$$g_{1t}(q,x) = \frac{g_{1t}^{(0)}(q)\,\overline{N}_{1d}(x)}{1 - g_{1t}^{(0)}(q)\,N_{1d}(x)} ,$$

/12.44/

where $g_{1t}^{(0)}(q)$ is the bare coupling and the Fourier transform is defined by

$$g_{1t}(q,x) = \sum_{\lambda \neq \mu} g_{1\lambda\mu}(x)\, e^{iq(R_\lambda - R_\mu)} .$$

/12.45/

Thus g_{1t} characterizes the interchain /transverse/ coupling. The mean field transition temperature is obtained from the pole of eqn. /12.44/. It gives a crude approximation for the ordering temperature, it indicates, however, that

T_C depends strongly on the interchain couplings as well as on the fluctuations within the chains. Above T_C, but close to it, the system behaves like a real three-dimensional system. Far above T_C the denominator in /12.44/ is near to unity, $g_{1t}(q,x) \approx g_{1t}^{(0)}(q)\,\overline{N}_{1\alpha}(x)$, i.e. the singularity of $g_{1t}(q,x)$ seems to be at x=0, the system behaves effectively as a 1-d system. Inbetween a crossover temperature can be defined, where the renormalized interchain and intrachain coupling constants become of the same order of magnitude. Above this T_{cross} the system shows 1-d fluctuations without correlation between the chains, below T_{cross} the 3-d nature of the system becomes perceptible.

12.5. The effect of interchain hopping

The interchain scattering of electrons is only one mechanism which can couple the chains in quasi-1-d systems. The other mechanism is a direct hopping /tunnelling/ of electrons from one chain to the other. The Hamiltonian describing the hopping is

$$H_h = \sum_{\substack{\lambda\mu \\ k\alpha}} t_{\lambda\mu}\, a^+_{\lambda k\alpha}\, a_{\mu k\alpha} . \tag{12.46}$$

It gives rise to a change in the energy spectrum, which in the case of $t_{\lambda\mu} \ll \varepsilon_F$ can be written as

$$\varepsilon(k,q) - \varepsilon_F = v_F\left(|k| - k_F\right) + w(q),$$

/12.47/

where k and q are the momentum components parallel and perpendicular, respectively, to the chain direction. In the nearest neighbour tight binding approximation for a square lattice with lattice constant a

$$w(q) = 2t\left[\cos(q_x a) + \cos(q_y a)\right].$$

/12.48/

In the absence of hopping /t=0/ the Fermi surface is flat. The hopping will distort this flat surface and the curvature is proportional to the rate of hopping. At a temperature T the electrons which are in a range of width $k_B T$ around the Fermi surface will contribute essentially to the scattering. If the distortion of the Fermi surface is within this range, in a first approximation the hopping will play no role. It becomes important only if the distortion of the Fermi surface is larger than the thermal smearing, in which case the 1-d character of the system is lost and a 3-d behaviour is obtained.

The renormalization group treatment is again very suitable for the study of this crossover. Looking first at the elementary bubbles of the parquet diagrams, the Cooper--pair diagram is not sensitive to the hopping as it was noticed by Dzyaloshinsky and Kac /1968/, and in the logarithmic approximation the same result is obtained as for the 1-d case /see eqn. /3.1//. The zero-sound channel with external momentum $k = 2k_F$ is, however, modified and instead of eqn. /3.2/ the integral can be approximated by

$$\frac{1}{2\pi v_F} \ln \frac{(\omega^2 + t^2)^{1/2}}{E_0} \approx \frac{1}{2\pi v_F} \ln \frac{\max\{\omega, t\}}{E_0} . \qquad /12.49/$$

In the finite temperature formalism ω is replaced by T .

In the renormalization group treatment the cutoff is scaled down until a fixed point or singularity is reached. In the case of a 3-d system of coupled chains this singularity occurs at a finite value of the scaled cutoff and the corresponding temperature determines the critical temperature T_c. Considering now a system in which interchain scattering and interchain hopping are present simultaneously, the contribution of the diagrams in the high temperature region $T > t$ is the same as for $t = 0$ and the scaling equations /12.17/ - /12.18/ can be used. If the critical temperature obtained from these equations satisfies $T_c > t$, the hopping does not contribute to the 3-d ordering and our earlier analysis in § 12.2 and 12.3 is valid. Only charge-density wave state can arise in this system. If, however, no singularity occurs until the scaled cutoff reaches a value of the order of t , the behaviour changes dramatically. Once the cutoff is smaller than t , the zero--sound channel is not logarithmic any more and only the Cooper-pair diagrams give contribution to the Lie equations, as it is the case in real three-dimensional systems. The scaling equations are easily obtained and they have the form

$$\frac{d g_{1\lambda\mu}}{dx} = \frac{1}{x}\left\{ \frac{1}{\pi v_F} g_{1\lambda\mu} g_{2\lambda\mu} + \dots \right\}, \qquad /12.50/$$

$$\frac{d g_{2\lambda\mu}}{dx} = \frac{1}{x}\left\{ \frac{1}{2\pi v_F} \left(g_{1\lambda\mu}^2 + g_{2\lambda\mu}^2 \right) + \dots \right\}. \qquad /12.51/$$

These equations are equivalent to the parquet equations of Gorkov and Dzyaloshinsky /1974/. They were also obtained by Prigodin and Firsov /1977/. When solving the problem with $T_c < t$, the scaling has to be performed in two steps. First we have to use eqns. /12.17/ - /12.18/ until the new cutoff is scaled to t, the couplings take on the value $g_{1\lambda\mu}(t/E_0)$ and $g_{2\lambda\mu}(t/E_0)$. Then, starting with these values, eqns. /12.50/ - /12.51/ have to be used in the further scaling. The procedure is somewhat similar to the two-cutoff problem discussed in § 11.2.

Depending on the initial conditions, $g_{1\lambda\mu} + g_{2\lambda\mu}$ or $-g_{1\lambda\mu} + g_{2\lambda\mu}$ will diverge first. These are just the combinations which appear in the Lie equations /12.29/ - /12.34/ for the singlet- and triplet-superconductor responses, respectively. Accordingly the system will have a superconducting phase transition with singlet or triplet Cooper pairs.

Horovitz /1976/ and later Prigodin and Firsov /1977/ pointed out that in contrast to the situation discussed in connection with eqn. /12.49/, the zero-sound channel is not sensitive to the hopping if the dispersion relation has the symmetry property

$$\varepsilon(k, q) - \varepsilon_F = -\left[\varepsilon(k-k_0, q-q_0) - \varepsilon_F\right] \qquad /12.52/$$

for a particular value of (k_0, q_0) and the zero sound bubble is calculated with these external momenta. The nearest neighbour tight binding approximation /12.48/ satisfies this relation with $k_0 = 2k_F$ and $q_0 = \left(\frac{\pi}{a}, \frac{\pi}{a}\right)$. Since the zero-sound channel is again logarithmic for all temperatures, the scaling equations are the same as for $t = 0$ and the charge-density wave response function with wave vector (k_0, q_0) is singular. In the ordered phase the charge-density waves on the neighbouring chains are in opposite phase.

Klemm and Gutfreund /1976/ approached the problem of 3-d ordering due to interchain hopping in a different way. As a first step they studied the effect of interchain hopping in a perturbational calculation, i.e. the response functions were calculated with strictly one-dimensional energy spectrum to fourth order in the hopping Hamiltonian. The interchain couplings $g_{1\lambda\mu}$ and $g_{2\lambda\mu}$ were neglected. They concluded from this calculation that the most important contribution to the response functions comes from the "pair self-avoiding random walk" processes. The hopping of an electron creates a disturbance which propagates along the chain with the velocity characteristic for the spin-density modes of the bosonized Hamiltonian. This disturbance causes another particle - electron or hole, depending on whether the transition is to a superconducting or density wave state - to hop to the same adjacent chain. This second hopping takes place before the first particle could propagate to another chain. The important processes are those in which the pairs never return to the same chain.

The response functions were then calculated in a pair approximation, considering the propagation of pairs between the chains, by the use of the bosonization transformation. After a rather lengthy calculation they found that a superconducting or density-wave state appears, depending on whether $g_1 - 2g_2 > 0$ or $g_1 - 2g_2 < 0$. This demarcation between the superconducting and density-wave state is the same as for the single chain problem. Thus one can say that the effect of interchain hopping is to stabilize at a finite temperature that state which would be obtained as a ground state of the single chain problem.

Further clarification is necessary for $g_1 > 0$ where both the singlet- and triplet-superconductor responses and the charge-density and spin-density wave responses are singular for $g_1 - 2g_2 > 0$ and $g_1 - 2g_2 < 0$, respectively. This region was studied by Suzumura and Fukuyama /1977/ in greater detail. In a mean field theory of the pair hopping they found that the critical temperature of triplet-superconductivity is higher than that of singlet-superconductivity, if $g_1 - 2g_2 > 0$. Similarly, for $g_1 - 2g_2 < 0$ the critical temperature of the spin-density wave state lies higher than that of the charge-density wave state. The results are summarized in fig. 35, where we indicated the type of ordering when the coupling between the chains is due to interchain hopping only. This phase diagram is very similar to the one shown in fig. 13, obtained for a strictly one-dimensional system.

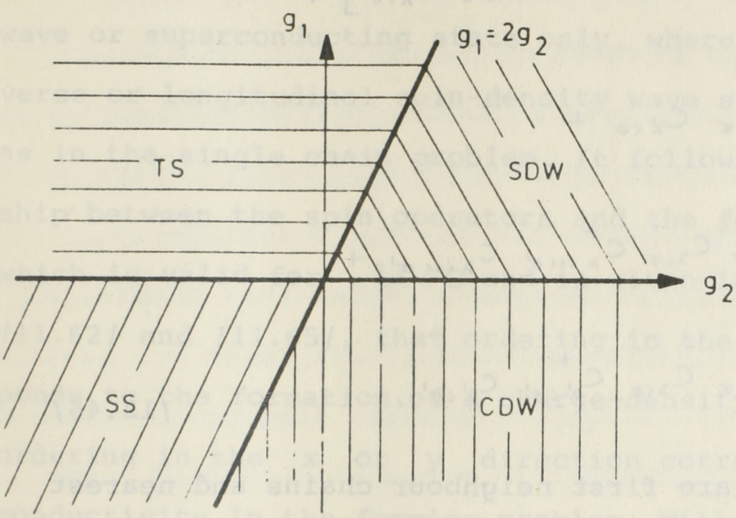

Fig. 35. Type of ordered state for a system of chains, with intrachain backward and forward scattering, coupled by interchain hopping.

12.6. Quasi-1-d fermion model with strong on-site interaction

The 1-d fermion model with strong on-site interaction has been considered in § 11.4. This model can also be extended to a set of coupled chains by assigning a chain index λ to the creation and annihilation operators:

$$H = U \sum_{\lambda i} c^{+}_{\lambda i \uparrow} c_{\lambda i \uparrow} c^{+}_{\lambda i \downarrow} c_{\lambda i \downarrow} +$$

$$+ t_{\parallel} \sum_{\lambda i \sigma} \left[c^{+}_{\lambda i \sigma} c_{\lambda i+1 \sigma} + c^{+}_{\lambda i+1 \sigma} c_{\lambda i \sigma} \right] +$$

$$+ t_{\perp} \sum_{\lambda \lambda' i \sigma} c^{+}_{\lambda i \sigma} c_{\lambda' i \sigma} +$$

$$+ V_{\parallel} \sum_{\lambda i \sigma \sigma'} c^{+}_{\lambda i \sigma} c_{\lambda i \sigma} c^{+}_{\lambda i+1 \sigma'} c_{\lambda i+1 \sigma'} +$$

$$+ V_{\perp} \sum_{\lambda \lambda' i \sigma \sigma'} c^{+}_{\lambda i \sigma} c_{\lambda i \sigma} c^{+}_{\lambda' i \sigma'} c_{\lambda' i \sigma'} , \qquad /12.46/$$

where λ and λ' are first neighbour chains and nearest neighbour interaction has been assumed. If the on-site interaction U is the largest energy in the problem, this Hamiltonian has been shown by Emery /1976b/ to be equivalent to the problem of interacting spin-1/2 chains in the same way as eqn. /11.59/ is equivalent to eqn. /11.66/. In this way we get

$$H = \frac{4 t_{\parallel}^{2}}{|U|} \sum_{\lambda i} \left[S^{z}_{\lambda i} S^{z}_{\lambda i+1} - \frac{1}{2} \left(S^{+}_{\lambda i} S^{-}_{\lambda i+1} + S^{-}_{\lambda i} S^{+}_{\lambda i+1} \right) \right] +$$

$$+ \frac{4 t_{\perp}^{2}}{|U|} \sum_{\lambda \lambda' i} \left[S^{z}_{\lambda i} S^{z}_{\lambda' i} - \frac{1}{2} \left(S^{+}_{\lambda i} S^{-}_{\lambda' i} + S^{-}_{\lambda i} S^{+}_{\lambda' i} \right) \right] +$$

$$+ 4 V_{\parallel} \sum_{\lambda i} S^{z}_{\lambda i} S^{z}_{\lambda i+1} + 4 V_{\perp} \sum_{\lambda \lambda' i} S^{z}_{\lambda i} S^{z}_{\lambda' i} +$$

$$+ 4 \left(V_{\parallel} + z V_{\perp} \right) \sum_{\lambda i} S^{z}_{\lambda i} , \qquad /12.47/$$

where z is the number of nearest neighbour chains.

Emery /1976b/ and Fowler /1978/ studied this model by taking over results from the literature on the Heisenberg model. They conclude that $U<0$ allows charge-density wave or superconducting state only, whereas for $U>0$ transverse or longitudinal spin-density wave state can be formed, as in the single chain problem. It follows from the relationship between the spin operators and the fermion operators which is valid for $U<0$ and is given in eqns. /11.61/, /11.62/ and /11.65/, that ordering in the z-direction corresponds to the formation of a charge-density wave, whereas ordering in the x or y direction corresponds to superconductivity in the fermion problem. Without interchain hopping $(t_{\perp}=0)$ the chains are coupled by the z components of the spin, only charge-density wave state can be stabilized, in agreement with the discussion in § 12.2. Taking into account the interchain hopping superconductivity could be favoured.

§ 13. CONCLUDING REMARKS

In this paper we have reviewed the properties of the 1-d Fermi gas model. We have seen that depending on the values of the coupling constants, the behaviour of the system can be quite different. Coulomb coupling between the chains favours the appearance of charge-density wave state. If the coupling between the chains is dominantly due to interchain hopping, the type of order that occurs is the same as the type of order that would set in at in a single chain. The corresponding phase diagram is given in § 8.1. The question concerning the application of these results to real systems is what the values of the couplings could be. Unfortunately the answer to this question is rather controversial. As an example we will consider TTF - TCNQ /tetrathiafulvalene - tetracyanoquinodimethane/ which is a typical representative of quasi - 1-d conductors and on which extensive study has been made.

There are two different schools advocating quite different descriptions. The Pennsylvania group /Heeger and Garito 1975/ believe that the on-site Coulomb interaction is reduced by screening and due to the strong polarizability of molecules to such a small value that its effect is minor compared to that of the electron-phonon interaction. They try to understand the properties of TTF - TCNQ on the basis of a Fröhlich Hamiltonian neglecting completely the electron-electron interaction.

Contrary to this view the IBM group /Torrance, Tomkiewicz and Silverman 1977/ claim that the effective Coulomb interaction is comparable or larger than the bandwidth and is more important than the electron-phonon coupling. This assertion is based on the interpretation of the optical absorption spectrum /Torrance, Scott and Kaufman 1975/, of the enhancement of the magnetic susceptibility, of the enhancement of the NMR relaxation rate T_1^{-1} /Jerome and Weger 1977/ and of the $4k_F$ scattering /Pouget et al. 1976 and Kagoshima et al. 1976/ discussed in § 4.3. The latter can be observed only if the Coulomb interaction is strongly repulsive.

Other charge transfer salts have different ratio of the effective Coulomb interaction to the bandwidth. In NMP - TCNQ /N-methyl-phenazinium-tetracyanoquinodimethane/ Epstein et al. /1972/ found $t_{\parallel} = 2.1 \cdot 10^{-2}$ eV and $U = 0.17$ eV, thus $U/4t_{\parallel} \approx 2$, whereas in HMTSF - TCNQ /hexamethylene - tetraselenofulvalene-tetracyanoquinodimethane/ $U/4t_{\parallel}$ is considerably smaller than unity

/Jerome and Weger 1977/. It is interesting to attempt to situate the various systems in the space of couplings as Jerome and Weger /1977/ did and to interpret all the properties in an interacting Fermi gas model. It seems that if we do so, many of the physically most interesting systems fall into the strong coupling regime where the results described in this paper are not easily applicable.

The Fermi gas model as presented here is an interesting one in its own right apart from its possible relevance to real systems. As a model system is shows particular dimensionality effects. It is an interesting theoretical problem how to get a good description of its behaviour. Furthermore, as it has been emphasized in this review, this model is related to a large number of other models. Though in most of the cases the relationship is approximate only and it is hard to estimate to what extent results obtained for one model can be extended to the others, it is hoped that further study of the related models will contribute to a better understanding of the behaviour of the Fermi gas model.

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