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EQUATIONS OF STATE FOR RELATIVISTIC QUANTUM IDEAL GASES OF MASSIVE PARTICLES

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ABSTRACT

It is pointed out that the Newton-Wigner localization of relativistic particles yields equations of state for ideal gases different from those given by the usual "periodic boundary conditions." The differences appear only in the relativistic part of quantum corrections i.e. both procedures give identical results in the non-relativistic as well as in the Boltzmann limits.

АННОТАЦИЯ

Показывается что локализация Нютона-Вигнера релятивистских частиц дает уравнения состояния для идеальных газов не совпадающие с уровнениями выведенными из обичных "периодических граничных условий". Разница возникает только в релятивистской части квантовых поправок, то есть обе процедуры дают то же самые результаты так в нерелятивистком как и в больцмановском пределе.

KIVONAT

Rámutatunk arra, hogy a relativisztikus részecskék Newton-Wigner lokalizációja olyan ideális gáz állapotegyenleteket szolgáltat, amelyek különböznek a szokásos "periódikus határfeltételek" által adottaktól. A különbségek csak a kvantum korrekciók relativisztikus részében jelennek meg, tehát mindkét eljárás azonos eredményre vezet, mind a nem-relativisztikus, mind pedig a Boltzmann-határesetekben.

I. INTRODUCTION

Relativistic quantum gases of massive particles may occur in nature only under rather exceptional conditions. /"Relativistic" means that the average thermal motion is relativistic wheareas the attribute "quantum" refers to the importance of quantum corrections./ Examples of candidates are the electron gas at the initial period of a neutron star's life or the "pion gas" produced in energetic collisions of two heavy nuclei. The simplest approximation to such systems is to assume that the gases are ideal i.e. the particles are distributed uniformly among the possible free particle states.

The level density of a set of free particles in the thermodynamic limit is usually believed to be independent from the prescription of counting the number of states. Non-relativistically this question was investigated in detail and, indeed, it turned out that the equations of state are independent from the boundary conditions imposed on the wave function at the large volume's boundary. /For the ideal Bose-Einstein gas see e.g. [1]./ Therefore, the usual procedure is to make the simplest choice i.e. impose periodicity in space making momentum space discrete for finite volumes V /"box quantization"/ and go over to integration as $V \to \infty$ [2, 3].

In the relativistic case the level density and the partition function of the ideal quantum gas was investigated in detail by Chaichian, Hagedorn and Hayashi [4] using the method of box quantization. The grand canonical partition function in the case of the "invariant phase space measure" is [4] :

$$Z(\beta, V, A) = \exp\left\{V\left(\frac{m}{2\pi\beta}\right)^{3/2} G_2^{(2)}(A, m\beta)\right\} , \tag{1.1}$$

where $\beta = T^{-1}$ is the inverse temperature, A is the absolute activity /"fugacity"/ and m is the mass of the particle /in this paper we use the system of units where $\hbar = c = k_{\text{Boltzmann}} = 1$/. The function $G_\mu^{(i)}$ is defined by

$$G_\mu^{(i)}(x, y) = \sqrt{\frac{2y}{\pi}} \sum_{j=1}^{\infty} \frac{x^j}{j^\mu} K_i(jy) . \tag{1.2}$$

The equations of state can be obtained from the partition function (1.1) in the usual way:

$$\begin{aligned} P\beta &= V^{-1} \ln Z = \left(\frac{m}{2\pi\beta}\right)^{3/2} G_2^{(2)}(A, m\beta) , \\ \nu &= V^{-1} A \frac{\partial}{\partial A} \ln Z = \left(\frac{m}{2\pi\beta}\right)^{3/2} G_1^{(2)}(A, m\beta) , \\ \varepsilon &= -V^{-1} \frac{\partial}{\partial \beta} \ln Z = \left(\frac{m}{2\pi\beta}\right)^{3/2} \left[m G_1^{(1)}(A, m\beta) + \frac{3}{\beta} G_2^{(2)}(A, m\beta)\right] . \end{aligned} \tag{1.3}$$

Here P is the pressure, ν is the particle number density and ε is the relativistic energy density /including the energy corresponding to the rest mass of particles/. Introducing the corresponding "reduced" /dimensionless/ quantities P_r, ν_r and ε_r Eq. (1.3) goes over into

$$P_r \equiv m^{-4} P = (2\pi m\beta)^{-3/2} \frac{G_2^{(2)}(A, m\beta)}{m\beta},$$

$$\nu_r \equiv m^{-3}\nu = (2\pi m\beta)^{-3/2} G_1^{(2)}(A, m\beta), \tag{1.4}$$

$$\varepsilon_r \equiv m^{-4}\varepsilon = (2\pi m\beta)^{-3/2} \left[G_1^{(1)}(A, m\beta) + \frac{3}{m\beta} G_2^{(2)}(A, m\beta)\right].$$

The periodicity of the wave function is clearly an unphysical assumption which is legitimate to use only if it gives the same result as some physically motivated prescription for counting level densities. For ideal gases consisting of free, non-interacting particles the simplest physical condition is to require localization to some large volume V i.e. to allow arbitrary wave functions inside V and require the vanishing of the wave functions outside V. Relativistic free particles are described by some covariant wave equation /Klein-Gordon, Dirac, .../. The solutions of the equations are physically interpreted in momentum space /spanning out an irreducible representation of the Poincaré group/. The construction of the relativistic position operator /and its eigenstates: the states localized to some space point/ in the momentum representation was given by Newton and Wigner [5] . In the present paper the relativistic quantum equations of states for ideal gases following from the Newton-Wigner /NW/ localization will be derived.

In Section II. the Bose-Einstein case will be investigated based on a previous publication where the partition function following from NW localization was given for arbitrary volumes [6] . /The main emphasis in Ref. [6] was put, in fact, on small volumes./ In Section III. the case of Fermi-Dirac particles will be considered along the same lines. The last Section IV. deals with the non-relativistic and Boltzmann limits.

II. IDEAL BOSON GAS

The physical states of an arbitrary number of non-interacting bosons are elements of a Fock-space $\mathcal{H}_F$ /here only the case of spin zero neutral bosons will be considered/. The Fock-space $\mathcal{H}_F$ is spanned out by the creation operator $\hat{a}^+(\underset{\sim}{p})$ of bosons with three-momentum $\underset{\sim}{p}$ over the vacuum state $|0\rangle$. The creation and annihilation operators satisfy the covariantly normalized commutation relation

$$\left[\hat{a}(\underset{\sim}{p}), \hat{a}^+(\underset{\sim}{p}')\right] = 2p_0 \mathcal{N}\, \delta^3(\underset{\sim}{p} - \underset{\sim}{p}') \quad , \qquad p_0 = \sqrt{m^2 + \underset{\sim}{p}^2} \qquad (2.1)$$

where $\mathcal{N}$ is an arbitrary normalization constant, say $\mathcal{N}=(2\pi)^3$. /Note that in what follows operators will be distinguished from ordinary numbers by a "hat"./

The localized single particle states are the eigensates of the NW position operator [5]

$$\hat{\underset{\sim}{x}} = i\nabla_{\underset{\sim}{p}} - \frac{i}{2p_0^2}\underset{\sim}{p} . \tag{2.2}$$

The momentum space wave function of the single particle state $|X(\underset{\sim}{x})\rangle$ localized at the point $\underset{\sim}{x}$ is:

$$\langle 0|\hat{a}(\underset{\sim}{p})|X(\underset{\sim}{x})\rangle = \sqrt{\frac{2p_0\mathcal{N}}{(2\pi)^3}}\, e^{-i\underset{\sim}{p}\underset{\sim}{x}} . \tag{2.3}$$

The momentum space representation of the projection operator $\hat{K}_V^{(1)}$ of single particle states localized in some volume V is

$$\langle 0|\hat{a}(\underset{\sim}{p})\,\hat{K}_V^{(1)}\,\hat{a}^+(\underset{\sim}{p}')|0\rangle = \int_V d^3x\, \langle 0|\hat{a}(\underset{\sim}{p})|X(\underset{\sim}{x})\rangle\langle X(\underset{\sim}{x})|\hat{a}^+(\underset{\sim}{p}')|0\rangle = \tag{2.4}$$

$$= 2\mathcal{N}\sqrt{p_0p_0'}\; k_V(\underset{\sim}{p}-\underset{\sim}{p}') \equiv K_V(\underset{\sim}{p},\underset{\sim}{p}') ,$$

where the function k_V is equal to

$$k_V(\underset{\sim}{q}) = \frac{1}{(2\pi)^3}\int_V d^3x\, e^{-i\underset{\sim}{q}\underset{\sim}{x}} . \tag{2.5}$$

A non-trivial problem arising at the determination of the level density of states with total energy E is due to the fact that the projection operator $K_V^{(1)}$ does not commute with energy /it is non-diagonal in the momentum representation (2.4) although the off diagonal elements vanish in the limit $V \to \infty$ /. The problem manifests itself in the non-uniqueness of the definition of the density operator $\hat{R}_V[E;A]$ of the statistical ensemble of states belonging to total energy E, fugacity A and localized in the volume V.

Consider first the density operator $\hat{R}_V^{(1)}[E;A]$ in the single particle subspace. This is the quantum mechanical operator corresponding to the classical function

$$A\,\delta(p_0 - E)\,\Theta_V(\underset{\sim}{x}) \quad . \tag{2.6}$$

Here $\Theta_V(\underset{\sim}{x})$ denotes the characteristic function of the volume V /equal to zero outside V and to unity inside V/. It is well known that the definition of operators belonging to classical functions of quantum mechanically non-commuting observables is to a certain extent arbitrary. This gives the non-uniqueness of the density operator.

One way to define the operators is the Weyl-prescription [7] . According to it the operator belonging to the function

(2.6) is

$$\hat{R}_V^{(1)}[E;A] = \frac{1}{(2\pi)^6} \int d^3x\, d^3p\, A\, \delta(p_0 - E)\, \Theta_V(\underset{\sim}{x}) \cdot \int d^3\xi\, d^3\eta\, \exp\{i\underset{\sim}{\xi}(\hat{\underset{\sim}{p}} - \underset{\sim}{p}) + i\underset{\sim}{\eta}(\hat{\underset{\sim}{x}} - \underset{\sim}{x})\} \quad . \tag{2.7}$$

Here $\hat{\underset{\sim}{p}}$ is the three-momentum operator and $\hat{\underset{\sim}{x}}$ is the NW position operator (2.2) . Using Eqs. (2.1-5) it is straightforward to calculate the momentum space representation of $\hat{R}_V^{(1)}$. The result is:

$$\langle 0|\, \hat{a}(\underset{\sim}{p})\, \hat{R}_V^{(1)}[E;A]\, \hat{a}^+(\underset{\sim}{p}')\, |0\rangle = A\; \delta\left[E - \sqrt{m^2 + \tfrac{1}{4}(\underset{\sim}{p} + \underset{\sim}{p}')^2}\,\right] K_V(\underset{\sim}{p}, \underset{\sim}{p}') \quad . \tag{2.8}$$

In another way this means:

$$\hat{R}_V^{(1)}[E;A] = A \int \frac{d^3p\, d^3p'}{2p_0\mathcal{N}\, 2p_0'\mathcal{N}}\, K_V(\underset{\sim}{p}, \underset{\sim}{p}')\; \delta\left[E - \sqrt{m^2 + \tfrac{1}{4}(\underset{\sim}{p} + \underset{\sim}{p}')^2}\,\right] \cdot \hat{a}^+(\underset{\sim}{p})\, |0\rangle \langle 0|\, \hat{a}(\underset{\sim}{p}') \quad . \tag{2.9}$$

As a "compensation" for the arbitrariness in the definition of the density operator it is true that from the point of view of statistics the exact form of it is to a certain extent irrelevant. The statistical distribution of particles

/containing much more information than the equation of states only/ can be obtained from the density operator $\hat{R}_V[E;A]$ via the generating functional $V[E;\varphi(\cdot)]$ /see Ref. [6] / defined like

$$V[E;\varphi(\cdot)] = \frac{\mathrm{Tr}\{\hat{R}_V[E;A]\,\hat{I}[\varphi(\cdot)]\}}{\mathrm{Tr}\{\hat{R}_V[E;A]\}} \;. \tag{2.10}$$

Here the operator $\hat{I}[\varphi(\cdot)]$ is given by

$$\hat{I}[\varphi(\cdot)] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} \left\{ \frac{d^3 p_i}{2p_{i0}\mathcal{N}}\, \varphi(p_i) \right\} \cdot \hat{a}^+(\underset{\sim}{p}_n) \dots \hat{a}^+(\underset{\sim}{p}_1)|0\rangle\langle 0|\,\hat{a}(\underset{\sim}{p}_1) \dots \hat{a}(\underset{\sim}{p}_n) \tag{2.11}$$

From Eq. (2.9) it follows that

$$\mathrm{Tr}\{\hat{R}_V^{(1)}[E;A]\,\hat{I}[\varphi(\cdot)]\} = \int \frac{d^3 p}{2p_0\mathcal{N}}\, A\varphi(p)\, \delta(E-p_0)\, K_V(\underset{\sim}{p},\underset{\sim}{p}) \;. \tag{2.12}$$

As long as this relation is maintained any change in $\hat{R}_V^{(1)}[E;A]$ is irrelevant for statistical quantities.

In Ref. [6] this freedom in the definition of the density operator was exploited to obtain a form better suited for calculations. Instead of the density operator given by the Weyl-prescription (2.9) another form consistent with Eq. (2.12) was used, namely:

$$\hat{R}_V^{(1)}[E;A] = A \int \frac{d^3p\, d^3p'}{2p_0\mathcal{N}\, 2p_0'\mathcal{N}} K_V(\underset{\sim}{p}, \underset{\sim}{p}')\, \delta\left[E - \tfrac{1}{2}(p_0 + p_0')\right]$$

$$\cdot\, \hat{a}^+(\underset{\sim}{p})|0\rangle\langle 0|\, \hat{a}(\underset{\sim}{p}') \tag{2.13}$$

A suitable generalization of this density operator to multi-particle states is [6] :

$$\hat{R}_V[E;A] = \sum_{n=0}^{\infty} \frac{A^n}{n!} \int \prod_{i=1}^{n} \left[\frac{d^3p_i}{2p_{i0}\mathcal{N}} \frac{d^3p_i'}{2p_{i0}'\mathcal{N}} K_V(\underset{\sim}{p}_i, \underset{\sim}{p}_i') \right] \cdot$$

$$\cdot\, \delta\left[E - \frac{1}{2}\sum_{i=1}^{n}(p_{i0} + p_{i0}')\right] \hat{a}^+(\underset{\sim}{p}_n)\ldots \hat{a}^+(\underset{\sim}{p}_1)|0\rangle\langle 0|\hat{a}(\underset{\sim}{p}_1')\ldots \hat{a}(\underset{\sim}{p}_n') \quad . \tag{2.14}$$

In Ref. [6], in fact, besides energy conservation also total momentum conservation was imposed. This is irrelevant for the derivation of the equations of state, therefore is omitted here, for simplicity.

The grand canonical partition function Z_V belonging to the density operator (2.14) was determined in Ref. [6] for an arbitrary volume V /small or large/:

$$Z_V[\beta;A] = \int dE\, e^{-\beta E}\, \mathrm{Tr}\{\hat{R}_V[E;A]\} =$$

$$= \exp\left\{ \sum_{j=1}^{\infty} j^{-1} z_V^{(j)}[\beta;A] \right\} \quad . \tag{2.15}$$

Here the "j-particle cluster partition function" $z_V^{(j)}$ is given by /with $\underset{\sim}{x}_0 \equiv \underset{\sim}{x}_j$ /:

$$z_V^{(j)}[\beta;A] = A^j \prod_{i=1}^{j} \left\{ \int_V \frac{d^3x_i}{(2\pi)^3} \beta z\left(\sqrt{\beta^2 + (\underset{\sim}{x}_i - \underset{\sim}{x}_{i-1})^2}\right) \right\},$$

$$z(\beta) \equiv 4\pi m^2 \beta^{-2} K_2(m\beta) \tag{2.16}$$

In the thermodynamic limit the expression for the partition function is [6] :

$$Z_V[\beta;A] = \exp\left\{ \frac{Vm^3}{v_Q(m\beta)} G_{5/2}\left(\frac{A}{D(m\beta)}\right) \right\}. \tag{2.17}$$

Here the function $G_\mu(x)$ defined as

$$G_\mu(x) = \sum_{j=1}^{\infty} j^{-\mu} x^j = \frac{1}{\Gamma(\mu)} \int_0^\infty dt \, \frac{t^{\mu-1} x e^{-t}}{1 - xe^{-t}} \tag{2.18}$$

is well known from the theory of the non-relativistic ideal quantum gases /see e.g. [2]/. The functions D and v_Q /this latter denoted in Ref. [6] by v_{BE} / are given like

$$D(x) = \sqrt{\frac{\pi}{2}} \, \frac{\{4K_0(x) + K_1(x)[x + \frac{8}{x}]\}^{3/2}}{x^2 \{K_0(x) + \frac{2}{x} K_1(x)\}^{5/2}},$$

$$v_Q(x) = \left\{ 2\pi x^2 \frac{[K_0(x) + \frac{2}{x} K_1(x)]}{[4K_0(x) + K_1(x)(x + \frac{8}{x})]} \right\}^{3/2}. \tag{2.19}$$

As it was discussed in Ref. [6] $v_Q(m\beta)$ is the characteristic volume of correlations due to quantum effects /i.e. the symmetrization of wave functions/.

From the partition function (2.17) one can obtain the following "reduced" equations of state:

$$P_r = \frac{G_{5/2}\left(\frac{A}{D(m\beta)}\right)}{m\beta\, v_Q(m\beta)} ,$$

$$\nu_r = \frac{G_{3/2}\left(\frac{A}{D(m\beta)}\right)}{v_Q(m\beta)} , \tag{2.20}$$

$$\varepsilon_r = \nu_r \frac{D'(m\beta)}{D(m\beta)} + P_r m\beta \frac{v'_Q(m\beta)}{v_Q(m\beta)} .$$

The functions D' and v'_Q in the last equation are the derivatives of D and v_Q , respectively. These equations are, in general, different from the above equations in (1.4).

III. IDEAL FERMION GAS

The results of the previous Section can be easily carried over to the case of fermions. Here, for definiteness and simplicity a spin $1/2$ neutral fermion will be considered. The creation and annihilation operators of fermions

/ $\hat{a}^+[p]_\sigma$ resp. $\hat{a}[p]_\sigma$ / satisfy the anticommutation relation

$$\{ \hat{a}[p]_\sigma , \hat{a}^+[p']_{\sigma'} \} = 2p_0 \mathcal{N} \delta^3(\underset{\sim}{p} - \underset{\sim}{p}') \delta_{\sigma\sigma'} . \qquad (3.1)$$

The state created by $\hat{a}^+[p]_\sigma$ has momentum p and spin index σ . It gives the irreducible Poincaré-group representation with Wigner rotations [8] .

In order to define the NW localized states one has to perform a Foldy-Wouthuysen transformation [9, 10] on the Dirac spinors. The Foldy-Wouthuysen spinor $u_{FW}(p,\sigma)$ satisfies the relations

$$\begin{aligned} \tilde{u}_{FW}(p,\sigma)\, u_{FW}(p,\sigma') &= \frac{p_0}{m}\, \delta_{\sigma\sigma'} , \\ \sum_\sigma u_{FW}(p,\sigma)\, \tilde{u}_{FW}(p,\sigma) &= \frac{p_0}{m}\, \frac{1+\gamma_0}{2} . \end{aligned} \qquad (3.2)$$

/Here γ_0 denotes, of course, the Dirac-matrix and $\tilde{u} = u^+\gamma_0$ /. The spin of the localized states can also be described by a Foldy-Wouthuysen spinor $\chi(\sigma)$ satisfying

$$\begin{aligned} \gamma_0 \chi(\sigma) = \chi(\sigma) , \quad & \Sigma_3 \chi(\sigma) = \sigma \chi(\sigma) , \\ \chi^+(\sigma)\, \chi(\sigma') &= \delta_{\sigma\sigma'} , \\ \sum_\sigma \chi(\sigma) \chi^+(\sigma) &= \frac{1+\gamma_0}{2} . \end{aligned} \qquad (3.3)$$

where Σ_3 is the matrix corresponding to the third component of the spin.

The momentum space wave function of the single particle state $|X(\underset{\sim}{x})_\sigma\rangle$ with spin index σ localized at the point $\underset{\sim}{x}$ is

$$\langle 0|\,\hat{a}[p]_\tau\,|X(\underset{\sim}{x})_\sigma\rangle = \sqrt{\frac{2m\mathcal{N}}{(2\pi)^3}}\; e^{-i\underset{\sim}{p}\underset{\sim}{x}}\; \tilde{u}_{FW}(p,\tau)\,\chi(\sigma) \tag{3.4}$$

Correspondingly, the momentum space representation of the projection operator $\hat{K}_V^{(1)}$ of single particle states localized in volume V is:

$$\langle 0|\,\hat{a}[p]_\sigma\,\hat{K}_V^{(1)}\,\hat{a}^+[p']_{\sigma'}|0\rangle =$$

$$= \sum_\tau \int d^3x\, \langle 0|\,\hat{a}[p]_\sigma\,|X(\underset{\sim}{x})_\tau\rangle\langle X(\underset{\sim}{x})_\tau|\,\hat{a}^+[p']_{\sigma'}|0\rangle = \tag{3.5}$$

$$= 2m\mathcal{N}\,\tilde{u}_{FW}(p,\sigma)\,u_{FW}(p',\sigma')\,k_V(\underset{\sim}{p}-\underset{\sim}{p}') \equiv K_V\{[p]_\sigma\,;[p']_{\sigma'}\} .$$

This replaces the expression for spin zero bosons in Eq. (2.4).

The density operator of the ensemble with total energy E, fugacity A is defined in full analogy with Eq. (2.14) :

$$\hat{R}_V[E;A] = \sum_{n=0}^{\infty} \frac{A^n}{n!} \int \prod_{i=1}^{n} \left[\frac{d^3p_i \, d^3p'_i}{2p_{io}\mathcal{N} \, 2p'_{io}\mathcal{N}} \sum_{\sigma_i \sigma'_i} K_V\{[p_i]_{\sigma_i}; [p'_i]_{\sigma'_i}\} \right]$$

$$\cdot \, \delta\left[E - \frac{1}{2}\sum_{i=1}^{n} (p_{io} + p'_{io})\right] \hat{a}^+[p_n]_{\sigma_n} \cdots \hat{a}^+[p_1]_{\sigma_1} |0\rangle\langle 0| \hat{a}[p'_1]_{\sigma'_1} \cdots \hat{a}[p'_n]_{\sigma'_n} \quad (3.6)$$

A calculation completely analogous to the one in Ref. [6] yields the grand canonical partition function:

$$Z_V[\beta;A] = \exp\left\{2 \sum_{j=1}^{\infty} (-1)^{j+1} z_V^{(j)}[\beta;A]\right\} . \quad (3.7)$$

Appart from the alternating sign in the cluster expansion, this is rather similar to the result for spin zero in Eq. (2.15) . The factor 2 in the exponent is due to the spin degeneracy of single particle states.

In the thermodynamic limit the partition function is

$$Z_V[\beta;A] = \exp\left\{\frac{2Vm^3}{v_Q(m\beta)} F_{5/2}\left(\frac{A}{D(m\beta)}\right)\right\} . \quad (3.8)$$

Here the function $F_\mu(x)$ can be expressed by the corresponding boson function (2.18) like

$$F_\mu(x) = -G_\mu(-x) . \quad (3.9)$$

The "reduced" equations of state have a rather similar form to Eq. (2.20), namely:

$$P_r = \frac{2F_{5/2}\left(\frac{A}{D(m\beta)}\right)}{m\beta\, v_Q(m\beta)}$$

$$\nu_r = \frac{2F_{3/2}\left(\frac{A}{D(m\beta)}\right)}{v_Q(m\beta)} \qquad (3.10)$$

$$\varepsilon_r = \nu_r \frac{D'(m\beta)}{D(m\beta)} + P_r\, m\beta\, \frac{v_Q'(m\beta)}{v_Q(m\beta)}$$

The last equation expressing the energy density in terms of number density and pressure is, in fact, identical for bosons and fermions.

IV. THE NON-RELATIVISTIC AND BOLTZMANN LIMITS

In the non-relativistic limit the thermic kinetic energy is small compared to the rest energy of the particles, that is $m\beta \to \infty$. The equations of state in Eqs. (1.4), (2.20) and (3.10) can be obtained in this limit from the asymptotic expansion of the modified Bessel functions [11] :

$$K_\mu(x) \to \sqrt{\frac{\pi}{2x}}\, e^{-x}\left(1 + \frac{4\mu^2-1}{8x} + \cdots\right) ; \qquad (4.1)$$

$$(x \to \infty) .$$

From Eqs. (1.2), (2.18) it follows that in this case

$$G^{(i)}_{\mu}(x,y) \to G_{\mu+\frac{1}{2}}(xe^{-y}) \quad , \tag{4.2}$$

therefore Eq. (1.4) in the non-relativistic limit is

$$\begin{aligned} P_{r} &= (2\pi m\beta)^{-3/2} \frac{G_{5/2}(Ae^{-m\beta})}{m\beta} \quad , \\ \nu_{r} &= (2\pi m\beta)^{-3/2} \; G_{3/2}(Ae^{-m\beta}) \quad , \\ \varepsilon_{r} &= \nu_{r} + \frac{3}{2} P_{r} \quad . \end{aligned} \tag{4.3}$$

/In deriving the last relation the second term in the asymptotic series (4.1) has to be considered, too./ For fermions, forgetting about the factor 2 of spin degeneracy, the function G_{μ} has to be replaced by F_{μ} .

Considering now the equations of state (2.20) following from the NW localization, Eqs. (4.1) and (2.19) give the asymptotic behaviour in the non-relativistic limit $m\beta \to \infty$:

$$\begin{aligned} D(m\beta) &\to e^{m\beta} \quad , \\ v_{Q}(m\beta) &\to (2\pi m\beta)^{3/2} \quad , \\ \frac{D'(m\beta)}{D(m\beta)} \to 1 \quad &, \qquad \frac{v'_{Q}(m\beta)}{v_{Q}(m\beta)} \to \frac{3}{2m\beta} \quad . \end{aligned} \tag{4.4}$$

Substituting these expressions into Eq. (2.20) gives again Eq. (4.3).

The Boltzmann limit corresponds to the situation when due to the small average occupation number of states the multi-particle clusters are negligible, i.e. the terms with j = 1 dominate in the sums (1.2) and (2.18) . Introducing the single particle partition function

$$Q_m(\beta) \equiv \frac{m^2}{2\pi^2\beta} K_2(m\beta) = \frac{m^3}{v_Q(m\beta)\, D(m\beta)} \tag{4.5}$$

the equations of state in Eq. (1.4) go over into:

$$\begin{aligned} P_r &= \frac{\nu_r}{m\beta} , \\ \nu_r &= \frac{A}{m^3} Q_m(\beta) = \frac{A}{v_Q(m\beta)\, D(m\beta)} , \\ \varepsilon_r &= \nu_r \left[\frac{3}{m\beta} + \frac{K_1(m\beta)}{K_2(m\beta)} \right] . \end{aligned} \tag{4.6}$$

This is identical to the Boltzmann limit of the equations in (2.20) /or of the corresponding fermion equations/. Summarizing: the NW localization gives the same equations of state in the non-relativistic as well as in the Boltzmann-limits as the usual "box quantization".

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