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ABSTRACT

The Karlsson-Zeiger three-body equations work exclusively with half-off-shell scattering amplitudes as two-body input. The half-off-shell phases, themselves, can be calculated by solving Sobel's non-linear differential equations. Present paper proposes an explicit form for the half-shell phase in terms of the on-shell phase functions, i.e. just the solution to the Sobel equation, in any partial wave.

АННОТАЦИЯ

Трехчастичные уравнения Карлсона-Зейгера работают исключительно с двухчастичными амплитудами рассеяния частично вне массовой поверхности. Решением нелинейного уравнения Собеля можно получить эти фазовые сдвиги. В настоящей работе выведено явное выражение для фазовых сдвигов частично вне массовой поверхности через фазовые функции на массовой поверхности.

KIVONAT

A háromtest-szórási probléma Karlsson és Zeiger által ujonnan felállitott integrálegyenletei a két.test inputot kizárólag félig-off-shell amplitudók formájában tartalmazzák. A félig off-shell fázistolások a Sobel által korábban felállitott fázisegyenletek megoldásából nyerhetők. Jelen cikk explicit és exakt megoldást ad Sobel nem-lineáris differenciálegyenletei számára.

1. TWO-BODY INPUT TO THREE-PARTICLE PROBLEMS

Barring three- and many-body forces, the N-particle dynamics is governed by the elementary interaction through the two-particle transition matrix $t(k_o^2+i\varepsilon)$. Concerning case N=3, as far as only the Faddeev-equations [1] were available, knowledge of all the off-the-energy shell matrix elements $t(k_2, k_1; k_o^2)$ was considered to be necessary to supply the two-body input for the three-particle theory. Calculation of the completely off-shell two-particle scattering amplitudes in terms of the potential $V(r)$ consists in solving integral equations of the Schwinger-Lippmann type. The partial wave t-matrix elements satisfy the equation [2]

$$t_\ell(k_2,k_1;k_o^2) = V_\ell(k_2,k_1) + \frac{2\mu}{\hbar^2}\int_o^\infty q^2 \, \frac{V_\ell(k_2,q)t_\ell(q,k_1;k_o^2)}{k_o^2 - q^2}\, dq \qquad /1.1/$$

with the notation

$$V_\ell(k,q) \equiv \frac{1}{(2\Pi)^3}\int_o^\infty r^2\hat{j}_\ell(kr)V(r)\hat{j}_\ell(qr)dr \ . \qquad /1.2/$$

When solving eq. /1.1/, one has to face, in fact, double integrations. The early modifications of the Faddeev-equations [3] require invariably the completely off-shell two-body input for the three-particle problem. Only recently, Karlsson and Zeiger [4] succeeded in transforming the three-body Faddeev equations in such a way that the new system of integral equations involves as two-body input exclusively the half-off-shell amplitudes $t(k_2,k_1;k_1^2)$ (for all momenta k_2 and k_1) and the bound state vertex functions.

Whether the Karlsson-Zeiger equations come up to essential simplifications in the three-body calculations depends critically on the easy availability of the half-shell scattering amplitudes. A simple recourse to eqs. /1.1/ - /1.2/ for case $k_o=k_1$ would not offer significant time saving on the computer in comparison to the conventional calculations. Nevertheless, there is also a more promising approach to the half-shell amplitudes. Within the framework of the variable phase approach [5,6] as developed by Calogero and coworkers, Sobel derived [7,8] non-linear differential equations for the two-particle half-off-shell phase functions in any partial wave working with the on-shell phase functions as input. Sobel's phase equation reads for the special case of S-waves as

$$\frac{d\Delta_o^{(211)}(a)}{da} = - V(a)\ \sin[k_1 a + \delta_o(k_1,a)]$$
$$\{k_2^{-1}\ \sin(k_2 a) + k_1^{-1}\ \cos[k_1 a + \delta_o(k_1,a)]\ \Delta_o^{(211)}(a). \qquad /1.3/$$

Here, function $\Delta_o^{(211)}(a)$ is defined by the identity

$$e^{i\delta_o(k_1,a)}\ \Delta_o^{(211)}(a) \equiv t_o^a(k_2,\ k_1;\ k_1^2) \qquad /1.4/$$

with t^a denoting the transition matrix due to the potential $V^a(r)$ which is obtained by cutting off $V(r)$ at $r=a$. The input to eq. /1.3/ is the conventional S-wave phase function $\delta_o(k,a)$ that satisfies the on-shell phase equation [6]

$$\frac{d\delta_o(k,a)}{da} = -k^{-1}\ V(r)\ \sin^2[ka + \delta_o(k,a)] \qquad /1.5/$$

The pair of equations /1.3/ and /1.5/ should be compared to eqs. /1.1/ - /1.2/ applied to the case $k_o=k_1$. In order to solve integral equation /1.1/ at given values of k_1 and k_2 , one has first to integrate eq. /1.2/ once through for each value of parameter q in the range $q=(o,\infty)$ while to provide, at fixed k_1 and k_2 , complete input to Sobel's equation, a single integration of eq. /1.5/ over the range $a=(o,\infty)$ will do.

Focussing attention to the phase approach, present paper proposes an explicit solution to the Sobel equation for any partial wave.

2. THE HALF-SHELL VS. ON-SHELL RELATIONSHIP

An integral representation of the S-wave half-shell phase function follows from relationship /1.4/ as [7]

$$\Delta_o^{(211)}(a) \equiv \sin\delta_o^{(211)}(a) =$$
$$= -\frac{1}{k_2}\int_o^a \sin(k_2 r)\ V(r)\ u_o^a(k,r)dr. \qquad /2.1/$$

Here, wave function $u_o^a(r)$ denotes the particular physical solution to the cut-off problem $V^a(r)$, introduced as

$$V^a(r) = V(r)\ ,\quad r \leq a,$$
$$= 0\ ,\quad r > a, \qquad /2.2/$$

and is specified by the boundary condition

$$u_o^a(k,r) \underset{r\to\infty}{\to} \sin\left[kr + \delta_o^a(k)\right]. \qquad /2.3/$$

The quantity $\delta_o^a(k) \equiv \delta_o(k,a)$ is the phase function for potential $V(r)$, i.e. the phase shift accumulated by potential $V^a(r)$. Owing to the cut-off, asymptotic condition /2.3/ can be recast, so as to refer to finite distances, as

$$u_o^a(k,r) = \sin\left[kr + \delta_o(k,a)\right]\ , \qquad r \geq a. \qquad /2.4/$$

Starting from the cut-off point r=a, wave function $u_o^a(k,r)$ can be continued to the inner region r<a where it coincides but normalization with the physical wave function $u_o(k,r)$ of problem V(r) which solution, in turn, is uniquely fixed by phase function $\delta_o(k,r)$. That wave function vs. phase function relationship is obtained by a standard procedure of the phase approach [6] through parametrizing the wave function by some functions $c_o(r)$ and $\omega_o(r)$ as follows

$$u_o(k,r) = c_o(r)\,\sin\,\omega_o(r)\ , \qquad /2.5/$$

$$\frac{du_o(k,r)}{dr} = c_o(r)k\,\cos\,\omega_o(r)\ . \qquad /2.6/$$

A straightforward elimination of c_o establishes the relationship between u_o and ω_o which can be written in the particular way

$$\tan\left[\omega_o(r) - kr\right] = \frac{u_o'(r)\,\sin(kr) - k\,u_o(r)\cos(kr)}{u_o'(r)\,\cos(kr) + k\,u_o(r)\sin(kr)} \qquad /2.7/$$

The r.h.s. of this equation is easily recognized as the well known expression of the tangent of the phase function $\delta_o(k,r)$. Hence, meaning of parameter $\omega_o(r)$ is recovered as

$$\omega_o(k,r) = kr + \delta_o(k,r)\ , \qquad \mathrm{mod}(\Pi)\ . \qquad /2.8/$$

Furthermore, by comparing eq. /2.6/ with the first derivative of eq. /2.5/, a differential equation is obtained for the parameter $c_o(r)$ as

$$\frac{d\,\ell n\,c_o(r)}{dr} = \frac{d\,\delta_o(r)}{dr}\,\mathrm{cotg}\,\omega_o(r)\ . \qquad /2.9/$$

Solution of this equation

$$c_o(k,r) = e^{\int_{r_o}^{r} \delta_o^{(\prime)}(\rho)\,\text{cotg}\,\omega_o(\rho)\,d\rho} \qquad /2.10/$$

involves the free parameter r_o that determines normalization of the wave function. Equations /2.5/ through /2.10/, valid for wave function $u_o(r)$ throughout the range $(0,\infty)$, also hold for wave function $u_o^a(r)$ in the restricted range $r=(0,a)$. As regards $u_o^a(r)$, integration constant r_o^a is fixed by boundary condition /2.4/ which combines with eqs. /2.5/, /2.8/ and /2.10/ to yield

$$r_o^a = a \ . \qquad /2.11/$$

According to the above argument, the properly normalized solution to the cut-off problem, due for insertion in eq. /2.1/, is given for the inner region by

$$u_o^a(k,r) = e^{-\int_{r}^{a} \delta_o^{(\prime)}(\rho)\,\text{cotg}\,\omega_o(\rho)\,d\rho} \sin\omega_o(r) \quad , \quad r<a. \qquad /2.12/$$

On account of eq. /2.1/, the integral representation of the half-off-shell phase function for partial wave $\ell=0$ in terms of the on-shell phase function rather than the wave function is given by

$$\sin\delta_o^{(211)}(a) =$$

$$= -\frac{1}{k_2}\int_0^a \{\sin(k_2 r)\,V(r)\,e^{-\int_{r}^{a} \delta_o^{(\prime)}(\rho)\,\text{cotg}\,\omega_o(\rho)\,d\rho} \sin\omega_o(r)\}dr \ , \qquad /2.13/$$

where abbreviation $\omega_o(r)$ is to be understood in terms of eq. /2.8/. At the same time, eq. /2.13/ furnishes the sought-for explicit solution for Sobel's phase equation /1.3/. Differentiation of e.q. /2.13/ with respect to the variable a, with due regard of its double appearence on the r.h.s., reproduces eq. /1.3/ as it, indeed, should do.

3. GENERALIZATION TO HIGHER PARTIAL WAVES

Extension of the method just developed to cases $\ell>0$ is straightforward although not quite trivial. An integral representation of the half-off--shell phase function is given [5] by the equation

$$\sin\delta_\ell^{(211)}(a) = -\frac{1}{k_2}\int_0^a \hat{j}_\ell(k_2 r)\ V(r)\ \tilde{u}_\ell^a(k_1,r)dr \qquad /3.1/$$

where the wave function is subject to the boundary condition

$$\tilde{u}_\ell^a(k,r) \underset{r\to\infty}{\to} \sin\left[kr - \ell\frac{\Pi}{2} + \delta_\ell^a(k)\right] . \qquad /3.2/$$

Here again, superscript a refers to the cut-off problem $V^a(r)$. Consequently, the asymptotic boundary condition /3.2/ can be recast so as to refer to any finite value of the variable r beyond the cut-off point r=a. To do so, one has to consider that wave function $\tilde{u}_\ell^a(r)$: (i) should satisfy for $r\geq a$ the force free partial wave equation; (ii) should imply δ_ℓ^a as the phase shift in the ℓth partial wave; (iii) should reproduce the asymptotical behaviour fixed by eq. /3.2/. The expression

$$\tilde{u}_\ell^a(k,r) = \cos\delta_\ell^a(k)\ \hat{j}_\ell(kr) - \sin\delta_\ell^a(k)\ \hat{n}_\ell(kr)\ , \quad r \geq a\ , \qquad /3.3/$$

is easily seen to fulfil requirements (i) through (iii), by considering the properties of the Riccati-Bessel functions involved.

The next step of the argument is continuation of wave function $\tilde{u}_\ell^a(r)$ into the inner region r<a. To this end, $\tilde{u}_\ell^a(r)$ will be separeted into two r--dependent factors such as an a-independent phase factor and an amplitude function that carries the a-dependence implied in boundary condition /3.2/. This factorization is conveniently done by the familiar parametrization procedure of Calogero [6] as

$$\tilde{u}_\ell^a(k,r) = c_\ell^a(r)\{\cos\Theta_\ell^a(r)\ \hat{j}_\ell(kr) - \sin\Theta_\ell^a(r)\ \hat{n}_\ell(kr)\} \qquad /3.4/$$

and

$$\frac{d\,\tilde{u}_\ell^a(k,r)}{dr} = k\ c_\ell^a(r)\{\cos\Theta_\ell^a(r)\ \hat{j}_\ell^{(\prime)}(kr) - \sin\Theta_\ell^a(r)\ \hat{n}_\ell^{(\prime)}(kr)\}\ . \qquad /3.5/$$

In order to extract physical meaning for parameters c_ℓ^a and Θ_ℓ^a , one has to resolve this system of equations for Θ_ℓ^a . Then, one has the expression

$$\tan\Theta_\ell^a(r) = \frac{\tilde{u}_\ell^{a(\prime)}(r)\,\hat{j}_\ell(kr) - k\,\tilde{u}_\ell^a(r)\,\hat{j}_\ell(kr)}{\tilde{u}_\ell^{a(\prime)}(r)\,\hat{n}_\ell(kr) - k\,\tilde{u}_\ell^a(r)\,\hat{n}_\ell(kr)}\,, \qquad /3.6/$$

which is seen to be independent of the particular normalization of the wave function involved. In view of the proportionality, for r<a, of the physical solutions $\tilde{u}_\ell^a(r)$ and $\tilde{u}_\ell(r)$, eq. /3.6/ proves at once the equivalence

$$\Theta_\ell^a(r) = \delta_\ell(r), \qquad \mathrm{mod}(\Pi), \qquad r \leqq a\,. \qquad /3.7/$$

Parameter $\Theta_\ell^a(r)$ coincides thus inside the cut-off point with the phase function of the problem V(r). As regards the other parameter, a differential equation can be extracted by combining eqs. /3.4/ and /3.5/ as

$$\frac{d\,\ell n\,c_\ell^a(r)}{dr} = \frac{d\,\delta_\ell(r)}{dr}\,\tau_\ell^{-1}(r), \qquad r < a, \qquad /3.8/$$

where the notation

$$\tau_\ell^{-1}(r) \equiv \frac{\sin\delta_\ell(r)\,\hat{j}_\ell(kr) + \cos\delta_\ell(r)\,\hat{n}_\ell(kr)}{\cos\delta_\ell(r)\,\hat{j}_\ell(kr) - \sin\delta_\ell(r)\,\hat{n}_\ell(kr)} \qquad /3.9/$$

was introduced. The solution of eq. /3.8/ is given by

$$c_\ell^a(r) = e^{\int_{r_\ell(a)}^{r} \delta_\ell^{(\prime)}(\rho)\,\tau_\ell^{-1}(\rho)\,d\rho}\,, \qquad /3.10/$$

involving the constant of integration $r_\ell(a)$ that is responsible for the normalization of wave function $\tilde{u}_\ell^a(r)$. Note that boundary condition /3.3/ holds at and beyond the cut-off point r=a while the parametrization procedure implied by eq. /3.4/ through /3.10/ is valid at and inside the cut-off. Therefore, by putting just r=a in both equations /3.3/ and /3.4/, condition /3.3/ is reworded for incorporation into eq. /3.4/ as

$$c_\ell^a(a) = 1\,. \qquad /3.11/$$

Hence, $r_\ell(a)$=a and thus parameter c_ℓ^a is finally fixed for the inner region as

$$c_\ell^a(r) = e^{-\int_r^a \delta_\ell^{(\prime)}(\rho)\,\tau_\ell^{-1}(\rho)\,d\rho}\,, \qquad r \leqq a\,. \qquad /3.12/$$

Equations /3.1/, /3.4/ and /3.12/ combine now into the explicit expression of the half-off-shell phase function in terms of the on-shell phase function in the ℓth partial wave as

$$\sin\delta_\ell(k_2,\, k_1;\, k_1^2;\, a) = \qquad /3.13/$$

$$= -\frac{1}{k_2}\int_0^a \hat{j}_\ell(k_2 r)\, V(r)\, c_\ell^a(k_1,r)\, \delta_\ell(k_1,r)\, dr,$$

where the notation

$$\sigma_\ell(k,r) \equiv \cos\delta_\ell(k,r)\, \hat{j}_\ell(kr) - \sin\delta_\ell(k,r)\, \hat{n}_\ell(kr) \qquad /3.14/$$

for the phase factor was introduced. If one prefers, also potential $V(r)$ can be eliminated from formula /3.13/ by means of the phase equation [7]

$$\frac{d\delta_\ell(k,r)}{dr} = -\frac{1}{k}\, V(r)\, \sigma_\ell^2(k,r). \qquad /3.15/$$

In doing so, the half-shell versus on-shell relationship /3.12/ - /3.14/ for the phase functions can be reworded as

$$\sin\delta_\ell(k_2,\, k_1;\, k_1^2;\, a) =$$

$$= \frac{k_1}{k_2}\int_0^a \hat{j}_\ell(k_2 r)\, \frac{d\delta_\ell(k_1,r)}{dr}\, \frac{c_\ell^a(k_1,r)}{\sigma_\ell(k_1,r)}\, dr, \qquad /3.16/$$

with the notations of eqs. /3.12/ and /3.14/.

Just as in the S-wave case, integral representation /3.13/ of the half-shell phase can be converted into a differential equation. Differentiating eq. /3.13/ with respect to the cut-off distance a, one has first to calculate the derivative of parameter c_ℓ^a which is worth writing down here /and comparing to eq, /3.8//:

$$\frac{d\, c_\ell^a(r)}{da} = \frac{d\, \delta_\ell(a)}{da}\quad \tau_\ell^{-1}(a)\, c_\ell^a(r)\, . \qquad /3.17/$$

With this equation in mind, one obtains in a straightforward way the non-linear differential equation for the half-shell phase function:

$$\frac{d\, \sin\delta_\ell^{(211)}(a)}{da} =$$

$$= \hat{j}_\ell(k_2 a)\, V(a)\, \sigma_\ell(k_1,a) - \frac{d\, \delta_\ell(k_1,a)}{da}\, \tau_\ell^{-1}(a)\, \sin\delta_\ell^{(211)}(a)\, . \qquad /3.18/$$

This equation can be cast in new forms in different ways. One can eliminate the potential by means of phase equation /3.15/ or, conversely, replace the derivative $\delta_\ell^{(,)}(a)$ by means of the potential. Proceeding with the latter choice, eq. /3.18/ becomes

$$\frac{d \sin\delta_\ell^{(211)}(a)}{da} = - V(a)\ \sigma_\ell(k_1,a)$$

$$\{k_2^{-1}\ \hat{j}_\ell(k_2 a) - k_1^{-1}\ \kappa_\ell(k_1,a)\ \sin\delta_\ell^{(211)}(a)\}\ , \qquad /3.19/$$

with the notation

$$\kappa_\ell(k,r) \equiv \sin\delta_\ell(k,r)\ \hat{j}_\ell(kr) + \cos\delta_\ell(k,r)\ \hat{n}_\ell(k,r)\ , \qquad /3.20/$$

This equation is, however, identical with Sobel's general half-off-shell phase equation [8], as it, indeed, should be so.

Summarizing the above considerations, one sees that explicit formulae have been derived for the half-off-shell phase functions in terms of the on--shell phase functions. Simultaneously, the general Sobel equation has been rederived. For this first order differential equation one has thus found the exact solution. As regards the two-body input to the Karlsson-Zeiger three--body equations, the half-shell phases are obtained from the half-shell phase functions $\delta^{(211)}(a)$ by simply going to the limit $a\to\infty$.

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