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GELL-MANN AND LOW TYPE RENORMALIZATION GROUP AND THE EQUATION OF STATE OF THE HEISENBERG FERROMAGNET

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ABSTRACT

It is shown that the Lie differential equation of the modified Gell-Mann and Low renormalization group is a "natural tool" for obtaining the scaled equation of state of the Heisenberg ferromagnet.

АННОТАЦИЯ

В статье показано, что дифференциальное уравнение Ли модифицированной ренормализационной группы Гэл-Мана и Ло является "натуральным средством" для получения уравнения состояния Геизенбергского ферромагнита.

KIVONAT

Megmutatjuk, hogy a Gell-Mann és Low féle renormalizációs csoport módositott változatának Lie differenciál egyenlete "természetes eszköz" a Heisenberg ferromágnes állapotegyenletének felirásához.

As it can be seen from the title we are not going to discuss anything new in this short note, since using renormalization group technique, scaled equations of state have already been obtained /Brezin et al, 1974/. The purpose of this work is to demonstrate that using the Lie equations of the modified version of the Gell-Mann and Low renormalization group /MGLRG/, rather than the Callan-Symanzik equations, the equation of state can be obtained in an extremely simple way. By MGLRG we mean a method worked out by Sólyom, in which the intuitive picture of the Kadanoff cut-off scaling is combined with the Gell-Mann and Low renormalization group /Gell-Mann and Low, 1954/. The method is based on an assumption the validity of which must be checked order by order in perturbation theory. MGLRG physically is much nearer to Wilson's ideas than the original Gell-Mann and Low method, on the other hand from mathematical point of view it uses the same simple Lie differential equations as the traditional formulation. A thorough review of the above method and also applications can be found in the work of Forgács et. al./1976/.

To get an equation of state for the Heisenberg ferromagnet /near the critical point/ we use the φ^4 theory. The assumptions of MGLRG in this case are the following/see Forgács et al. 1976/:

$$G(q^2, t, \Lambda, u) = z_1\left(\frac{\tilde{\Lambda}}{\Lambda}, u\right) G(q^2, \tilde{t}, \tilde{\Lambda}, \tilde{u}) , \qquad /1/$$

$$\bar{\Gamma}_4(q^2, t, \Lambda, u) = z_2\left(\frac{\tilde{\Lambda}}{\Lambda}, u\right) \bar{\Gamma}_4(q^2, \tilde{t}, \tilde{\Lambda}, \tilde{u}) , \qquad /2/$$

$$\tilde{u} = u\left(\frac{\tilde{\Lambda}}{\Lambda}\right)^{-\varepsilon} z_1^2\left(\frac{\tilde{\Lambda}}{\Lambda}, u\right) z_2\left(\frac{\tilde{\Lambda}}{\Lambda}, u\right) , \qquad /3/$$

$$\tilde{t} = t\, z_3\left(\frac{\tilde{\Lambda}}{\Lambda}, u\right) . \qquad /4/$$

Here q is the momentum variable, G is the full propagator, $\bar{\Gamma}_4$ is the dimensionless four-point function, with $\bar{\Gamma}_4^{BARE} = 1$ /the momenta of the external lines of $\bar{\Gamma}_4$ are chosen in such a way that $\bar{\Gamma}_4$ depends only on one external momenta variable/. t is proportional to $T - T_c$, u is the dimensionless coupling constant, $\varepsilon = 4 - d$ and d is the dimension of space. Λ and $\tilde{\Lambda}$ are the original and the "new" cut-offs in momentum space. The main assumption is that the Z factors depend only on $\frac{\tilde{\Lambda}}{\Lambda}$ and u . Equations /1-4/ determine the Z factors and it can be shown that for higher order vertex functions similar equations are valid with Z factors not independent of Z_1, Z_2 and Z_3 . The Z factors as power series in u and ε are given in the Appendix. From the above equations all the critical indeces and corrections to scaling have been obtained /Forgács et al, 1976/.

In order to get the equation of state we start with /Jona-Lasinio, 1964/

$$F(t, u, \Lambda) = \sum_{n=2}^{\infty} \frac{M^n}{n!} \Gamma_n(t, u, \Lambda) \qquad /5/$$

for the free energy. Here Γ_n are the proper n-point functions /not dimensionless/. $\Gamma_2 = G^{-1}$ Using dimensional analysis, dimensionless quantities and the transformation properties /1-4/ /also for higher order vertex functions/, it is easy to show that

$$\bar{F}\,(x, y, u) = \left(\frac{\tilde{\Lambda}}{\Lambda}\right)^{d} \bar{F}(\tilde{x}, \tilde{y}, \tilde{u}) \qquad /6/$$

where $\bar{F}$ is the dimensionless free energy, and

$$x = \frac{M}{\Lambda^{\frac{d-2}{2}}}\,, \quad y = \frac{t}{\Lambda^{2}}\,; \quad \tilde{x} = \frac{M z_1^{-1/2}}{\tilde{\Lambda}^{\frac{d-2}{2}}}\,, \quad \tilde{y} = \frac{t z_3}{\tilde{\Lambda}^{2}} \qquad /7/$$

It has to be stressed that /5/ is only the magnetic part of the free energy. It is only this magnetic part which is multiplicatively renormalizable according to /6/. We know that the specific heat is not multiplicatively renormalizable, and since the second derivative of the nonmagnetic free energy is just the specific heat, therefore the nonmagnetic free energy is not multiplicatively renormalizable either.

Differentiating /6/ with respect to x and then putting $\varrho = \frac{\tilde{\Lambda}^2}{\Lambda^2} = y z_3 \left(\frac{\tilde{\Lambda}^2}{\Lambda^2} = \varrho\right)$, one gets the Lie equation for the free energy.

$$\frac{\partial \bar{F}(x,y,u)}{\partial x} = \frac{\varrho^{d/2}}{z_1^{1/2}\,\varrho^{\frac{d-2}{4}}}\,\frac{\partial}{\partial s}\,\bar{F}\left(s, 1, u\varrho^{-\varepsilon} z_1^{2}(\varrho) z_2(\varrho)\right)\Bigg|_{\substack{s = x z_1^{-1/2} \varrho^{\frac{2-d}{4}} \\ \varrho = y z_3}} \qquad /8/$$

Near to the critical point $\tilde{u} = u\varrho^{-\varepsilon} z_1^{2} z_2$ can be replaced by its fix point value u^* and from the definition of the z factors and their value given in the Appendix one can see that

$$z_1(\varrho, u^*) = \varrho^{\sigma(u^*)}\,, \qquad /9/$$

$$z_3(\varrho, u^*) = \varrho^{\varphi(u^*)} \qquad /10/$$

where

$$\sigma(u^*) = \frac{\partial z_1(\varrho, u^*)}{\partial \varrho}\Big|_{\varrho=1}, \qquad /11/$$

$$\varphi(u^*) = \frac{\partial z_3(\varrho, u^*)}{\partial \varrho}\Big|_{\varrho=1} \qquad /12/$$

Using /9/ and /lo/ from $\varrho = y z_3(\varrho)$ it follows that

$$\varrho^{1-\varphi(u^*)} = y \qquad /13/$$

Putting this value of ϱ into /8/, we finally get

$$\frac{\partial \bar{F}(x,y,u)}{\partial x} = y^{\frac{1}{4}\frac{d+2-2\sigma(u^*)}{1-\varphi(u^*)}}\,\phi\left(x\,y^{-\frac{1}{4}\frac{d-2+2\sigma(u^*)}{1-\varphi(u^*)}}\right) \qquad /14/$$

Here

$$\phi(x) = \frac{\partial}{\partial s}\bar{F}(s, 1, u^*)\Big|_{s=x} \qquad /15/$$

is the generator of the corresponding Lie equation. Calculating ϕ from perturbation theory $\bar{F}$ can be determined from /14/. However, since we are interested in the equation of state, we do not have to calculate $\bar{F}$. As

$$\bar{H} \text{ /dimensionless magnetic field/} = \frac{\partial \bar{F}}{\partial x} \qquad /16/$$

we see that the Lie equation /14/ is just the equation of state. Comparing /14/ with

$$H = M^{\delta} f\left(\frac{t}{M^{1/\beta}}\right) \qquad /17/$$

we get

$$\beta = \frac{1}{4}\,\frac{d-2+2\sigma(u^*)}{1-\varphi(u^*)} \qquad /18/$$

$$\delta = \frac{d+2-2\sigma(u^*)}{d-2+2\sigma(u^*)} \qquad /19/$$

We would like to stress that in this formalism one does not have to solve any differential equation /unless the aim is to get the scaled equation of state/, because the Lie equation for $\bar{F}$ coincides with the equation of state. Comparing this method with others it seems to us that this is the simplest way to get the scaled equation of state. Nothing sophisticated, such as renormalizability of the theory has been used and therefore all the above is easily digestible for a statistical mechanician. The only thing one has to do is to calculate the Z factors. If there are no such Z factors which depend only on the ratio of the cut-offs and the dimensionless coupling constants, the method can not be used. As it has been shown in the work of Forgács et al /1976/ when there is scaling in the theory equations similar to /1-4/ always can be satisfied.

The author is indebted to J. Sólyom for stimulating remarks.

Appendix

$$Z_1 = 1 + \frac{s^2}{48}\left(1-\frac{\varepsilon}{4}\right)\ln\varrho + \frac{s^3}{64}\ln^2\varrho \qquad /A.1/$$

$$Z_2 = 1 + s\frac{3}{4}\left(\ln\varrho - \frac{\varepsilon}{4}\ln^2\varrho\right) + s^2\left(-\frac{3}{4}\ln\varrho + \frac{9}{16}\ln^2\varrho\right) \qquad /A.2/$$

$$Z_3 = 1 + \frac{s}{4}\left(\ln\varrho - \frac{\varepsilon}{4}\ln^2\varrho\right) - \frac{s^2}{48}\left(5\ln\varrho - 6\ln^2\varrho\right) \qquad /A.3/$$

Here $s = uK_d$, $K_d = \left[2^{d-1}\pi^{d/2}\Gamma(d/2)\right]^{-1}$, $\varrho = \frac{\tilde{\Lambda}^2}{\Lambda^2}$.

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