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ABSTRACT

The motion of a spinless particle with electric charge e around a magnetic charge g is considered in the quasiclassical approximation. It is shown that the requirement of spherical symmetry leads to the Dirac quantization condition $\frac{1}{c}$ eg = $\frac{1}{2}$ $\hbar$n with an arbitrary integer n. For odd values of n the total angular momentum is half integer multiple of $\hbar$.

АННОТАЦИЯ

Рассматривается движение бесспиновой частицы заряда е вокруг магнитного заряда д в квазиклассическом приближении. Показано, что требование сферической симметрии приводит к условию квантования Дирака $\frac{1}{c}$ eg = $\frac{1}{2}$ $\hbar$n где n произвольное целое число. Когда n нечетно полный момент импульса системы равен полуцелому значению $\hbar$.

KIVONAT

Spinnélküli e töltésü tömegpont mozgását vizsgáljuk g nagyságu mágneses töltés terében félklasszikus közelitésben. Megmutatjuk, hogy a gömbszimmetriát megkövetelve az $\frac{1}{c}$ eg = $\frac{1}{2}$ $\hbar$n Dirac-féle kvantumfeltételre jutunk, ahol n tetszőleges egész szám. Amikor n páratlan, a rendszer teljes impulzusmomentuma a $\hbar$ félegészszerese.

Dirac[1] showed that in quantum mechanics the peaceful coexistence of a magnetic charge g and an electric charge e is possible only if the product of the charges is quantized according to the rule $\frac{1}{c}\,eg = \frac{1}{2}\,\hbar n$. Recently it has been widely recognized that when n is odd the total angular momentum is equal to a half integer value of $\hbar$ in spite of the fact that the system is described by some one component wave function, i.e. does not have spin degrees of freedom [2-4]. On the other hand, doubts have been expressed[5-6] as to whether odd values of n are actually allowed or not. The decision depends on a rather involved integrability condition of the Lie-algebra of three complicated operators with angular momentum commutation rules[7]. The answer is affirmative[4,7], i.e. the original quantization condition of Dirac with an arbitrary integer n has been justified. Nevertheless, the study of the problem in the quasiclassical limit seems instructive. On this level no mathematical subtilities arise but, instead, physical assumptions become crucial.

In order to concentrate on the angular momentum quantization a kind of spherical pendulum will be considered instead of a free electron, moving in the field of a magnetic monopole. It differs from the ordinary spherical pendulum in that the moving point mass m possesses an electric charge e while the center of the pendulum has a magnetic charge g. As it is known[8], in this case a constant angular momentum of the magnitude $\frac{1}{2}\,\sigma = \frac{1}{c}\,eg$ acts along the radius. We assume that it points toward the center.

The classical free motion of this "magnetic spherical pendulum" is completely determined by the conservation of the total angular momentum $\underline{J}$ /Fig.1/ which must satisfy the condition

$$J \geq \frac{\sigma}{2}, \tag{1}$$

Since $\frac{1}{2}\sigma = J\cos\alpha$, the angle α is constant during the motion. The mass point therefore, moves along the circle of radius $CP = r\sin\alpha = r\sqrt{1-\frac{\sigma^2}{4J^2}}$. The magnitude of the orbital angular momentum is $L = mrv$ and the relation $L\sin\alpha + \frac{1}{2}\,\sigma\cos\alpha = J$ gives $E = \frac{1}{2}\,mv^2 = \frac{1}{2mr^2}\,(J^2 - \frac{1}{4}\,\sigma^2)$.

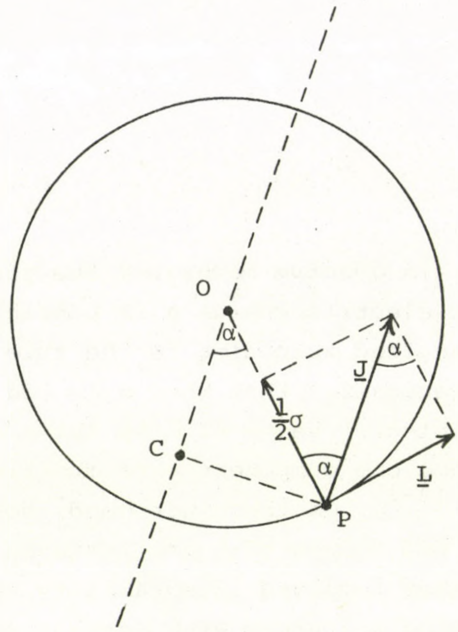

Fig. 1.

Let us introduce a Z axis through the center. The relation

$$J \geq |J_Z| \tag{2}$$

is obviously satisfied. When $|J_Z| = \frac{1}{2}\sigma$ the orbit crosses the Z axis. When $|J_Z| > \frac{1}{2}\sigma$ the Z axis intersects the plane of the orbit inside the circle, while for $|J_Z| < \frac{1}{2}\sigma$ the intersection occurs outside. In this latter case the azimuthal angle φ does not perform complete revolution but rather oscillates within an interval less than π. The ordinary spherical pendulum does not possess this type of free motion, since it moves always on main circles.

All these features of the motion are incorporated into the angular part of the Hamiltonian given in[1] which up to a constant term can be written as

$$J^2(\vartheta, P_\vartheta, P_\varphi) = P^2_\vartheta + \frac{P^2_\varphi}{\sin^2\vartheta} - \frac{\sigma}{2}\sec^2\frac{\vartheta}{\alpha}, P_\varphi + \frac{\sigma^2}{4}\,tg^2\frac{\vartheta}{2} + \frac{\sigma^2}{4}\,. \tag{3}$$

Here φ, P_φ and ϑ, P_ϑ are canonically conjugate pairs. We take this expression for the Hamiltonian, assuming that the moment of inertia $mr^2 = \frac{1}{2}$. Computing ϑ from (3) as a Hamiltonian through the canonical equations one can explicitely verify that the motion it describes is indeed identical to that discussed above. Therefore, (3) can be taken for the Hamiltonian of the magnetic spherical pendulum. For future use we notice that $J_Z = L_Z - \frac{\sigma}{2} \cdot \frac{Z}{r} = [\underline{r}xm\dot{\underline{r}}]_Z - \frac{\sigma}{2}\frac{Z}{r}$ can be expressed through the canonical variables as

$$J_Z = P_\varphi - \frac{1}{2}\sigma.$$

Quantization requires knowledge of a solution of the Hamilton-Jacobi equation. Introducing new canonical variables and P_x by $x = \cos\vartheta$, $P_\vartheta = -P_x\sqrt{1-x^2}$ we can rewrite (3) as[4]

$$J^2(x,P_x,P_\varphi) = \frac{P_\varphi^2}{1-x^2} + (1-x^2)P_x^2 - \frac{\sigma}{1+x}P_\varphi + \frac{\sigma^2}{4}\frac{1-x}{1+x} + \frac{\sigma^2}{4}. \tag{4}$$

We seek the solution $S(x,\varphi)$ of the Hamilton-Jacobi equation

$$J^2(x, \frac{\partial S}{\partial x}, \frac{\partial S}{\partial \varphi}) = J^2 = \text{const.}$$

in the form $S = P_\varphi \varphi + K(x)$ where P_φ = const. Substituting this into (4), solving for $\frac{dK}{dx}$ and integrating we get

$$K(x) = \frac{1}{2}\int^x \frac{dy\sqrt{R(y)}}{1-y^2} \tag{5}$$

where

$$R(x) = a + bx + cx^2 \tag{6}$$

with

$$a = 4(J^2 - P_\varphi^2 + P_\varphi\sigma - \frac{1}{2}\sigma^2)$$

$$b = 2\sigma(\sigma - 2P_\varphi)$$

$$c = -4J^2$$

The motion in x is an oscillation between the limits x_{min} and x_{max} which are the roots of the equation $R(x) = 0$. The inequalities (1) and (2) are just all conditions under which $R(x) = 0$ has solutions in the interval $-1 \leq x \leq 1$.

The formula (5) can be transformed into

$$K(x) = 2J^2\int^x \frac{dy}{\sqrt{R(y)}} + (J_Z + \frac{\sigma}{2})^2 \int^{\frac{1}{x-1}} \frac{dy}{\sqrt{R'(y)}} - (J_Z - \frac{\sigma}{2})^2 \int^{\frac{1}{x+1}} \frac{dy}{\sqrt{R''(y)}} \tag{7}$$

where R'and R" are also expressions of the form[x/] (6) with

$$a' = c \qquad a'' = c$$
$$b' = b + 2c \qquad b'' = b - 2c$$
$$c' = a + b + c \qquad c'' = a - b + c.$$

The integration of (7) gives

$$K(x) = -J \arcsin \frac{2cx+b}{\sqrt{-\Delta}} + \frac{1}{2}|J_z + \frac{1}{2}\sigma| \arcsin \frac{2c'\frac{1}{x-1}+b'}{\sqrt{-\Delta}} - \frac{1}{2}|J_z - \frac{1}{2}\sigma| \arcsin \frac{2c''\frac{1}{x+1}+b''}{\sqrt{-\Delta}}$$

where $\Delta = 4ac-b^2$.

Let us denote the argument of the arcsin functions by $\psi(x), \psi'(x), \psi''(x)$ respectively. It is straightforward to verify that

$$\psi(x_{max}) = \psi'(x_{min}) = \psi''(x_{min}) = -1$$

$$\psi(x_{min}) = \psi'(x_{max}) = \psi''(x_{max}) = +1.$$

This means that the change of all three arcsin functions during the full period of the **x** motion is equal to 2π in magnitude. Taking into account the sign, we have for the full change of $K(x)$ the expression

$$\Delta K = 2\pi\left[J + \frac{1}{2}\left(|J_z + \frac{1}{2}\sigma| - |J_z - \frac{1}{2}\sigma|\right)\right]. \tag{8}$$

Let us now turn to the quantization. The method of Einstein, Brillouin and Keller will be adopted [(9-11)]. According to this procedure[xx/] the expression $\sum_r \oint P_r d_{qr}$ is to be quantized where the summation is over the degrees of freedom. The contours are taken on an extended configuration space /"invariant torus"/, consisting in general of several sheets chosen in such a way as to make the momentum a single valued vector space. All the possible contours are divided into classes so that contours within a class can be continuously deformed into each other. Then the quantization condition is applied to an arbitrarily selected contour from each class. A characteristic feature of this procedure is that in general quantum states

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x/ The prime does not imply differentiation.
xx/ The E.B.K. method has recently been employed in [(12)].

cannot be identified with appropriately selected classical motions. Rather, to each quantum state there corresponds a whole family of classical orbits, and the magnetic spherical pendulum is a simple but striking example of this.

First we review briefly the familiar case of the ordinary spherical pendulum /σ=0/. The contour integrals which are to be quantized are $\oint P_\varphi d\varphi$ and $\oint P_x dx$. For the first we have

$$\oint P_\varphi d\varphi = hm. \tag{9}$$

Since P_φ is a constant of motion this leads to

$$P_\varphi = \hbar m. \tag{10}$$

For the second integral we have to take into account the oscillatory behaviour of the coordinate x. According to(11), the quantum condition has to be assumed in the form

$$\oint P_x dx = h(\ell + \frac{\alpha}{4}),$$

where α is the so called "caustic index". In the present case it must be taken equal to 2, corresponding to the fact, that the velocity becomes zero twice within a period. Therefore

$$\oint P_x dx = h(\ell + \frac{1}{2}).$$

Now, since $P_x = \frac{\partial S}{\partial x} = \frac{\partial K}{\partial x}$, the contour integral is equal to ΔK. Putting in (8) σ= 0, we have

$$J = \hbar(\ell + \frac{1}{2}), \tag{11}$$

The kinematical conditions (1) and (2) imply $\ell \geq 0$ and $|m| \leq \ell$. $\ell = \frac{1}{\hbar}J - \frac{1}{2}$ is the total angular momentum quantum number in the sense, that the correct /non-semiclassical/ value of the angular momentum is $\sqrt{\ell(\ell+1)}$. Since $\sqrt{\ell(\ell+1)} = \ell + \frac{1}{2} + O(\frac{1}{\ell})$ we see that the quasiclassical method gives correct results in the limit of large quantum numbers.

Let us turn now to the case of $\sigma > 0$. Equation (10) remains true but instead of (11) we get

$$J + \frac{1}{2}\left(|J_z + \frac{1}{2}\sigma| - |J_z - \frac{1}{2}\sigma|\right) = \hbar(\ell + \frac{1}{2}) \tag{12}$$

Again, the total angular momentum quantum number j must be identified with $\frac{1}{\hbar}J - \frac{1}{2}$. Using this notation, (12) can be written as

$$J = \begin{cases} \ell + \frac{1}{2}\frac{\sigma}{\hbar} & \text{if} \quad J_z \le -\frac{\sigma}{2} \\ \ell - \frac{1}{\hbar}J_z & \text{if} \quad -\frac{\sigma}{2} \le J_z \le \frac{\sigma}{2} \\ \ell - \frac{1}{2}\frac{\sigma}{\hbar} & \text{if} \quad \frac{\sigma}{2} \le J_z. \end{cases}$$

It may be seen that for a general value of $\sigma/\hbar$ the quantization has violated rotation symmetry. Indeed, let us select a particular value A for J_z such that $A > \frac{\sigma}{2}$. For some arbitrary ℓ we have a quantum state in which the total angular momentum quantum number is $j=\ell - \frac{1}{2}\frac{\sigma}{\hbar}$. If the system is rotationally invariant we must have for $J_z = -A$ also a state with the same total angular momentum j as before. But this is possible only if for the arbitrary ℓ an integer ℓ' can always be found such that the relation $\ell - \frac{1}{2}\frac{\sigma}{\hbar} = \ell' + \frac{1}{2}\frac{\sigma}{\hbar}$ holds. This requires $\sigma/\hbar$ to be equal to an integer n which is just the Dirac quantization condition. If this condition is fulfilled the quantum states can be arranged in multiplets with a given j and $\frac{1}{\hbar} J_z = m - \frac{n}{2}$ running from -j to +j. Within such a multiplet ℓ does not remain constant. This reflects the fact that the canonical angular momentum $|[\underline{r} \times \underline{p}]|$ is not an integral of motion.

For odd /even/ values of n j is half integer /integer/. The kinematical conditions can be written as $j \ge \frac{n}{2}$, $j \ge |m - \frac{n}{2}|$. States, violating these inequalities, must not be taken into account.

We see that it is the spherical symmetry which plays the role of the quantization condition for σ. Without this requirement not included authomatically into the E.B.K. quantization conditions the total angular momentum j would have remained unquantized.

Let us now remember that on many of the classical orbits φ oscillates within a range smaller than π. One might suppose that for these orbits modified quantization condition with the caustic index $\alpha=2$ should be used. However, the configuration space must be cut into sheets and, accordingly, a nonzero caustic index has to be chosen only if otherwise the momentum would not be a single valued function of the coordinates. This is certainly not the case for a conserved momentum like P_φ. We, therefore, have to retain the quantization condition of P_φ in the form of (9) but stress on the difference between the contour of integration in (9) and the orbits in the actual motion.

We note finally that the possibility for j to take half integer values can also be expected on purely physical grounds on the basis of the physical completeness of quantum mechanics. The point is that the magnetic spherical pendulum can be imagined without involving magnetic monopoles. To this end we assume that the rigid rod, keeping the point mass at the constant distance from the center, is made from a ferromagnetic material which is magnetized. We retain the idealizations /the proper degrees of freedom of the rod can be ignored/ usually made. Now, since the electrons are half spin particles, $\frac{1}{2}\sigma$ can take the value $\frac{1}{2}\hbar n$ with an arbitrary integer and the quantum number j can be both integer and half-integer. If quantum mechanics is physically complete it must cover all these cases. But the Hamiltonian (3) is independent of the way the pendulum is realized. Therefore, the real existence of half spin electrons ensures that in the Dirac quantization condition n may be an arbitrary integer.

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