KFKI-76-30

M. HUSZÁR

IMPACT PARAMETER EXPANSION OF FIELDS

Hungarian Academy of Sciences

CENTRAL
RESEARCH
INSTITUTE FOR
PHYSICS

BUDAPEST

KFKI-76-30

IMPACT PARAMETER EXPANSION OF FIELDS

M. Huszár
High Energy Physics Department
Central Research Institute for Physics, Budapest, Hungary

ISBN 963 371 138 X

ABSTRACT

Impact parameter states are constructed by the aid of the generators of the Poincaré group and the expansion of scalar and Dirac field is given in terms of these states.

АННОТАЦИЯ

Построены состояния прицельного параметра с помощью генераторов группы Пуанкаре. Скалярное и спинорное поля разложены по этим состояниям.

KIVONAT

A Poincaré csoport generátorainak segitségével impakt paraméter állapotokat épitünk fel és megadjuk a skalár- és Dirac-tér ezen állapotok szerinti kifejtését.

1. INTRODUCTION

It has been shown in a preceeding paper [1] that the impact parameter can be endowed with a consequent group theoretical meaning within the Poincaré group. The aim of the present paper is to demonstrate that impact parameter states can be defined independently of the scattering amplitude and the fields can be expanded in terms of impact parameter states. In the expansion the emission and absorption operators describe emission and absorption of particles with given values of the impact parameter.

The usual interpretation of the impact parameter [2] is closely related to the eikonal approximation, however, within the framework of light front dynamics impact parameter states free of any approximation can be defined. Expansions of the scalar- and Dirac fields are given in terms of impact parameter states. The impact parameter picture is useful in high energy phenomenology since experimental data can be fitted by simple forms of the impact parameter amplitude. Therefore, it is possible that a simple form of interaction described by emission and absorption operators in impact parameter picture yield a good phenomenological field theory. In other words that means that it is probable that impact parameter states introduced in the present paper *incorporate a great deal of dynamics*. Although, the impact parameter emission and absorption operators exhibit a simple transformation property under the two-dimensional Galilean group, they transform in a rather complicated manner under the full Poincaré group. Therefore, it is hard to find a simple Poincaré invariant interaction in impact parameter picture.

The paper is organized as follows. In Sect. 2 the expansion of a complex scalar field in terms of impact parameter states is given. A characteristic feature of these states is a factor of $e^{i\frac{\mu}{2x}(\vec{x}-\vec{b})^2}$ oscillating rapidly in the transverse direction. Transformation properties of absorption operators under the Poincaré group are given. Sect. 3 contains the expansion of Dirac field. Dirac spinors are decomposed into "good" and "bad" components. Impact parameter states with up or down spin projection form a complete set in subspace of good components.

2. IMPACT PARAMETER STATES

Initial data for describing a dynamical system can be given on any spacelike plane or on the light front $x^+=x^o+x^3=0$. The generators of the Poincaré group can be divided into two sets with respect to initial conditions. The first set contains the generators of the subgroup which leave the plane of initial conditions invariant. The second set contains the generators of Poincaré transformations leading out of the surface. These later generators are called Hamiltonians by Dirac [3].

In the instant form of dynamics the invariance group is the three-dimensional Euclidean group E(3) and the Hamiltonians are the three boost generators N_1, N_2, N_3 and p^o. In front form the invariance group is the two-dimensional Galilean group enlarged with dilatations and the Hamiltonians are N_1+M_2, N_2-M_1 and $p^-=p^o-p^3$. These results are shown in the following table

	Invariance Group	Generators	Hamiltonians
Instant form $x^o=0$	$\begin{pmatrix}\alpha & \beta \\ -\beta^* & \alpha^*\end{pmatrix} = SU(2)$, $T_a = (a^1, a^2, a^3)$ } E(3) Three-dimensional Euclidean group	M_1, M_2, M_3 p^1, p^2, p^3	$M_{01}=N_1$, $M_{02}=N_2$, $M_{03}=N_3$, p^o
Front form $x^+=0$	$\begin{pmatrix}\alpha & 0 \\ \gamma & \alpha^{-1}\end{pmatrix} = D \otimes E(2)$, $T_a = (a^-, a^1, a^2)$ } $D \otimes G(2)$ Two-dimensional Galilean group with dilatation	M_{-1}, M_{-2} M_{12}, M_{+-} p^1, p^2, p^+	M_{+1}, M_{+2} p^-

Here $p^+=p^o+p^3$, $p^-=p^o-p^3$ for which the notation $p^-=\mu$ will be used too since it plays the role of mass in the two-dimensional Galilean group. Furthermore, $M_{\alpha\beta}$ denotes the (+, -, 1,2) light front components of the Lorentz group. These are related to the familiar angular momentum $\underset{\sim}{M}$ and boost momentum $\underset{\sim}{N}$ by

$$M_{+1} = \frac{1}{2}(N_1+M_2) \qquad M_{+2} = \frac{1}{2}(N_2-M_1)$$

$$M_{-1} = \frac{1}{2}(N_1-M_2) \qquad M_{-2} = \frac{1}{2}(N_2+M_1)$$

$$M_{12} = M_3, \qquad M_{+-} = -\frac{1}{2}N_3 .$$

It is a remarkable fact that from the Hamiltonians N_1, N_2, N_3, p^o of instant dynamics no reasonable basis can be constructed, whereas, three Hamiltonian p^-, M_{+1}, M_{+2} of light front dynamics defines a basis corresponding to reduction of the Poincaré group according to the subgroup $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$ of SL(2,C) and translations in the direction x^+. Introducing the notations

$$B_1 = \frac{2M_{+1}}{p^-} = \frac{N_1+M_2}{p^o-p^3}\ , \quad B_2 = \frac{2M_{+2}}{p^-} = \frac{N_2-M_1}{p^o-p^3} \qquad /2.1/$$

the impact parameter states are characterized by the following eigenvalue equations

$$B_1|\vec{b},\mu,\sigma\rangle = b_1|\vec{b},\mu,\sigma\rangle \qquad /2.2/$$

$$B_2|\vec{b},\mu,\sigma\rangle = b_2|\vec{b},\mu,\sigma\rangle \qquad /2.3/$$

$$p^-|\vec{b},\mu,\sigma\rangle = \mu|\vec{b},\mu,\sigma\rangle \qquad /2.4/$$

where $\vec{b} = (b_1,b_2)$. It has been shown in [1] that the above states can really be interpreted as impact parameter states. One more label is required for the spin projection denoted above by σ . This is the eigenvalue of the operator

$$\hat{\sigma} = \frac{w^o - w^3}{p^o - p^3} = \frac{w^-}{p^-}$$

where $w^\mu = -\frac{1}{2}\,\epsilon^{\mu\alpha\beta\nu}\,M_{\alpha\beta}p_\nu$ is the Pauli-Lubanski vector (/cf. [4]) i.e.

$$\hat{\sigma}|\vec{b},\mu,\sigma\rangle = \sigma|\vec{b},\mu,\sigma\rangle \qquad /2.5/$$

It can be easily seen that in the rest frame of the particle the above operator coincides with the third component of spin. States /2.2-2.5/ form a basis within an irreducible representation of the Poincaré group characterized by some fixed value of mass and spin.

3. IMPACT PARAMETER EXPANSION OF SCALAR FIELD

For expansion of the scalar field the generators of the Poincaré group are represented in coordinate space in the form

$$M_{\mu\nu} = i(x_\mu\partial_\mu - x_\nu\partial_\mu),\quad p_\mu = i\partial_\mu$$

and eigenvalue equations /2.2/, /2.3/, /2.4/ are imposed:

$$2i\frac{\partial}{\partial x^+}\varphi_{\mu,b}(x) = \mu\,\varphi_{\mu,b}(x) \qquad /3.1/$$

$$\left(i\frac{x^-}{\mu}\frac{\partial}{\partial x^k} + x^k\right)\varphi_{\mu,b}(x) = b_k\varphi_{\mu,b}(x) \quad (k=1,2) \qquad /3.2/$$

$$(\Box+m^2)\varphi_{\mu,b}(x) = 0 \qquad /3.3/$$

where m is the Poincaré mass, $\mu=p^0-p^3$ is the Galilean mass and x^+, x^- are the usual light front coordinates, $x^+=x^0+x^3$, $x^-=x^0-x^3$. The solution of these equations can be written in the form

$$\varphi_{\mu,b}(x) = i(2\pi)^{-3/2}\frac{\mu}{x^-}\ \exp\frac{i}{2}\left\{-\mu x^+-\frac{m^2}{\mu}x^-+\frac{\mu}{x^-}(\vec{x}-\vec{b})^2\right\} \qquad /3.4/$$

with $\vec{x}=(x^1,x^2)$, $b=(\vec{b}_1,b_2)$. These states are normalized on the light front x^+=const. according to

$$i\int dx^-d^2x\varphi_{\mu',b'}(x)^*\frac{\overleftrightarrow{\partial}}{\partial x^-}\varphi_{\mu,b}(x) = 2\mu\delta(\mu'-\mu)\delta^2(b'-b)\,. \qquad /3.5/$$

In terms of states /3.4/ the impact parameter expansion of a complex scalar field with mass m assumes the form

$$\varphi(x) = \int_0^\infty\frac{d\mu}{2\mu}\int d^2b(\varphi_{\mu,b}(x)a(\mu,b)+\varphi_{\mu,b}(x)^*d^+(\mu,b)) \qquad /3.6/$$

Here $a(\mu,b)$ absorbs a particle and $d^+(\mu,b)$ emits an antiparticle with a given value of impact parameter and Galilean mass μ.

By making use of Dirac's method for quantization of constrained systems it can be shown [5] that the field satisfies the equal x^+ commutation relations

$$\left[\varphi^+(x),\frac{\partial}{\partial y^-}\varphi(y)\right]_{x^+=y^+} = \frac{i}{2}\delta(x^--y^-)\delta^2(\vec{x}-\vec{y})$$

From this the following nonvanishing commutation relations between emission and absorption operators can be derived

$$\left[a(\mu',b'),\ a^+(\mu,b)\right]=\left[d(\mu',b'),\ d^+(\mu,b)\right]= 2\mu\delta(\mu'-\mu)\delta^2(b'-b) \qquad /3.7/$$

Transformation properties of absorption operators under $D\otimes E(2)=\begin{pmatrix}\alpha & \beta\\ 0 & \alpha^{-1}\end{pmatrix}\in SL(2,C)$ group are very simple,

$$U_{\alpha,\beta}a(\mu,b)U^{-1}_{\alpha,\beta} = e^{-\frac{\beta}{\alpha}\mu b+\frac{\beta^*}{\alpha^*}\mu b^*} a(|\alpha|^{-2}\mu, \frac{\alpha^*}{\alpha}b). \qquad /3.8/$$

Here and hereafter the notation $b=\frac{1}{2}(b_1+ib_2)$ is used. The simplest way for obtaining /3.8/ and subsequent transformation formulas is to use the overlap coefficients between the usual momentum eigenstates and impact parameter states [1]

$$\langle p^-, \vec{p} \mid \vec{b},\mu\rangle = 2\mu\delta(p^- -\mu)\ (2\pi)^{-1}\ e^{-i\vec{p}\vec{b}}$$

$$(\vec{p}\vec{b} = p^1b_1+p^2b_2)$$

Action of the subgroup $\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$ is much more complicated. By making use of the overlap coefficients again one gets

$$U_\gamma a(\mu,b)U_\gamma^{-1} = \int \frac{d\mu'}{2\mu'}\ d^2b'\Gamma(\mu',b';\mu,b)a(\mu';b')$$

where

$$\Gamma(\mu',b';\mu,b) = \frac{1}{(2\pi)}\frac{\mu'\mu}{|\gamma|^2}\exp\{\frac{b^*\mu+b'^*\mu'}{\gamma^*} - \frac{b\mu+b'\mu'}{\gamma}\}\ J_o(2\left|\frac{b}{\gamma} - \frac{b'^*}{\gamma^*}\right|\sqrt{\mu'\mu-m^2|\gamma|^2})$$

and J_o is the Bessel-function of zero order.

Transformation properties of $a(\mu,b)$ under the translation group $a^\mu=(a^+,a^-,a^1,a^2)$ are somewhat simpler

$$U_a\ a(\mu,b)\ U_a^{-1} = \int d^2b'A(b',b;\mu)\quad a(\mu,b')$$

with

$$A(b',b;\mu) = \frac{1}{2\pi i}\frac{\mu}{a^=}\exp\frac{1}{i}\{\frac{a^-}{\mu}m^2+\frac{\mu}{a^=}a^\mu a_\mu + \frac{\mu}{a^=}\vec{a}(\vec{b}'-\vec{b}) - \frac{2\mu}{a^=}(\vec{b}'-\vec{b})^2\}\ .$$

Here $a^\mu a_\mu$ denotes the invariant a squared, $a^\mu a_\mu = a^+a^- -\vec{a}^2$.

It is seen that transformation properties under the Galilean group are very simple but it is hard to satisfy covariance under the full Poincaré group. This is why a simple Poincaré covariant interaction cannot be given easily in terms of $a(\mu,b)$. Although, it is possible to transcribe a Poincaré covariant interaction, say $\varphi(x)^3$, in terms of $a(\mu,b)$, however it would be interesting to have a simple interaction just in b-space.

4. EXPANSION OF THE DIRAC FIELD

The Dirac field can be decomposed into "good" and "bad" components according to $\psi(x)=\psi^+(x)+\psi^-(x)$ with

$$\psi^+(x) = \frac{1+\alpha_3}{2}\psi(x) , \qquad \psi^-(x) = \frac{1-\alpha_3}{2}\psi(x) \qquad /4.1/$$

In terms of these components Dirac equation takes the form

$$2i \frac{\partial}{\partial x^+}\psi^+(x) = (\frac{1}{i}\alpha^k \frac{\partial}{\partial x^k} + m\beta)\psi(x) \qquad /4.2/$$

$$2i \frac{\partial}{\partial x^-}\psi^-(x) = (\frac{1}{i}\alpha^k \frac{\partial}{\partial x^k} + m\beta)\,\psi^+(x) \qquad (k = 1,2) \qquad /4.3/$$

Only the first of these equations is an equation of motion because this contains a derivative with respect to time, x^+, and correspondingly only ψ^+ is considered as a dynamical variable. Equation /4.3/ determines the subsidiary quantity ψ^- provided some boundary condition, e.g. $\psi^- \to 0$ as $x^- \to -\infty$, is imposed. Then an equation of motion directly for $\psi^+(x)$ can be obtained by eliminating $\psi^-(x)$,

$$i \frac{\partial}{\partial x^+}\psi^+(x) = \frac{i}{4}\left(\frac{\partial^2}{\partial \vec{x}^2} - m^2\right)\int_{-\infty}^{x^-} dy^- \;\psi^+(x^+,y^-,\vec{x}) \qquad /4.4/$$

In order to find impact parameter states the generators of the Lorentz group are represented in the form of a sum of orbital and spin parts,

$$M_{\mu\nu} = M^{orb}_{\mu\nu} + M^{spin}_{\mu\nu} \qquad /4.5/$$

where $M^{orb}_{\mu\nu}$ is the generator used earlier for the scalar field, and $M^{spin}_{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu]$. In what follows the spinor representation of γ-matrices is used i.e. $\gamma^0=\beta=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\gamma^\lambda=\gamma^0\alpha^\lambda=\begin{pmatrix}0&-\sigma^\lambda\\\sigma^\lambda&0\end{pmatrix}(\lambda = 1,2,3)$.

According to /4.5/ the operator of the impact parameter is also composed of orbital and spin parts

$$B_k = \frac{2M^{orb}_{+k}}{\mu} + \frac{2M^{spin}_{+k}}{\mu} = B^{orb}_k + B^{spin}_k \qquad (k = 1,2)$$

where

$$B^{orb}_k = i\,\frac{x^-}{\mu}\frac{\partial}{\partial x^k} + x^k$$

and

$$B_1^{spin} = \frac{2M_{+1}^{spin}}{\mu} = -\frac{i}{\mu}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$B_2^{spin} = \frac{2M_{+2}^{spin}}{\mu} = -\frac{i}{\mu}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The eigenvalue equation /3.2/ for the impact parameter now can be written as

$$(i\frac{x^-}{\mu}\frac{\partial}{\partial x^k} + x^k + B_k^{spin})\ \psi_{\mu,b,\sigma}(x) = b_k\psi_{\mu,b,\sigma}(x) \qquad /4.6/$$

The eigenvalue equation /2.5/ characterizing spin projection is left which reads now

$$(\frac{w^-}{p^-} - \sigma)\ \psi_{\mu,b,\sigma}(x) = \{\frac{1}{2}\Sigma_3 + \frac{1}{2\mu i}\vec{\Sigma}\frac{\partial}{\partial\vec{x}} + \frac{1}{2\mu}(\alpha_1\frac{\partial}{\partial x^2} - \alpha_2\frac{\partial}{\partial x^1}) - \sigma\}\psi_{\mu,b,\sigma}(x) = 0 \qquad /4.7/$$

Here Σ^α-s are the familiar spin matrices $\Sigma^\alpha = \frac{i}{2}\ e^{\alpha\beta\gamma}\ \gamma^\beta\gamma^\gamma$ $(\alpha,\beta,\gamma=1,2,3)$.

The eigenvalue equation /3.1/ remains unchanged

$$2i\frac{\partial}{\partial x^+}\ \psi_{\mu,b,\sigma}(x) = \mu\psi_{\mu,b,\sigma}(x) \qquad /4.8/$$

The solution of the eigenvalue equations /4.6/, /4.7/ /4.8/ can be found in form

$$\psi_{\mu,b,\sigma}(x) = \varphi_{\mu,b}(x)\ \chi_{\mu,b,\sigma}(x)$$

where $\varphi_{\mu,b}(x)$ is the impact parameter state /3.4/ obtained for the scalar field and $\chi_{\mu,b,\sigma}(x)$ is some four-component spinor. The solution is simply

$$\psi_{\mu,b,\frac{1}{2}} = \varphi_{\mu,b}(x)(1,0,0,0)\ ;\quad \psi_{\mu,b,-\frac{1}{2}} = \varphi_{\mu,b}(x)(0,0,0,1)$$

This satisfies also the equation of motion /4.4/

It has been mentioned earlier that only the good spinors $\psi^+(x)$ are dynamical quantities, whereas, ψ^- is determined by /4.3/. The impact parameter states $\psi_{\mu,b,\sigma}(x)$ thus obtained provide a basis in the space of good components.

Expansion of $\psi^{+}(x)$ now reads

$$\psi^{+}(x) = \int_{0}^{\infty} \frac{d\mu}{2\mu} \int d^{2}b \sum_{\sigma=\pm\frac{1}{2}} \left(a(\mu,b,\sigma)\psi_{\mu,b,\sigma}(x) + d^{+}(\mu,b,\sigma)\psi_{\mu,b,\sigma}(x)^{*}\right)$$

where $a(\mu,b,\sigma)$ absorbs a particle and $d^{+}(\mu,b,\sigma)$ emits an antiparticle with Galilean mass μ and impact parameter $\vec{b}$. The "good" component $\psi^{+}(x)$ satisfies the following equal x^{+} anticommutator relation

$$\{\psi^{+}(x), \left[\psi^{+}(y)\right]^{\dagger}\}_{x^{+}=y^{+}} = \frac{1+\alpha_{3}}{2}\, \delta(x^{-}-y^{-})\, \delta^{2}(\vec{x}-\vec{y})$$

where $\dagger$ denotes the adjoint spinor.

Impact parameter expansion of the electromagnetic field can be given in an analogous manner. We come back to this question as well as to the problem of interactions in a subsequent paper.

REFERENCES

[1] M. Huszár, Nuovo Cimento 31A, 237 /1976/

[2] M.L. Goldberger, K.M. Watson, Collision Theory, John Wiley and Sons, Inc. New York, London, Sydney 1964, Chapter 6 §7

[3] P.A.M. Dirac Rev. Mod. Phys 21, 392 /1949

[4] Y.B. Novozhilov, E.V. Prokvatilov, Teor,i Mat. Fiz. 1, 101 /1969/

[5] A.J. Hanson, T. Regge, C. Teitelboim, Constrained Hamiltonian Systems Accademia Naz. dei Lincei Rome, Italy,
M. Huszár, Preprint KFKI-75-49 /Journal of Physics in print/

Kiadja a Központi Fizikai Kutató Intézet
Felelős kiadó: Pintér György, a KFKI
Részecske- és Magfizikai Tudományos Tanácsának szekcióelnöke
Szakmai lektor: Tóth Kálmán
Nyelvi lektor : Hraskó Péter
Példányszám: 355 Törzsszám: 76-602
Készült a KFKI sokszorosító üzemében
Budapest, 1976. junius hó