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CANONICAL QUANTIZATION OF GAUGE THEORIES

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ABSTRACT

Canonical quantization scheme in a Hilbert-space with positive definite metric is proposed for local gauge theories. The procedure is based on Dirac's general theory of singular dynamical systems. A theorem on the subsidiary conditions, which permits perturbation treatment is stated and proved. Unitarity in the subspace of the allowed states is demonstrated. The method is applied to the cases of free electrodynamics in nonlinear gauges and Yang-Mills theory. The description does not require the introduction of ghost particles. The rules for calculating graphs are shown to be equivalent to those in Lagrangean approach with ghost.

РЕЗЮМЕ

Предложена схема канонического квантования локальных калибровочных полей в гильбертовом пространстве с положительно определенной метрикой, основанная на общей теории сингулярных динамических систем, разработанной Дираком. Формируется и доказывается теорема о дополнительных условиях, позволяющая применение теории возмущений. Доказана унитарность в подпространстве допустимых состояний. Рассматривается свободное электромагнитное поле в нелинейной калибровке и теория полей типа Йанг-Миллс. Фиктивных частиц в теории не возникает. Показано, что правила диаграммной техники эквивалентны правилам, полученным по лагранжевому методу с введением фиктивных частиц.

KIVONAT

Lokális gauge elméletek kanonikus kvantálását tárgyaljuk pozitiv metrikáju Hilbert térben. Az eljárás Dirac szinguláris rendszerekre kidolgozott elméletén alapul. Bebizonyitunk egy tételt a mellékfeltételekkel kapcsolatban, mely lehetővé teszi a perturbációs tárgyalást. Megmutatjuk az unitaritást a megengedett állapotok terén. Az eljárást nem-lineáris gauge-ban felirt elektrodinamikára és a Yang-Mills terek esetére alkalmazzuk. A módszer nem teszi szükségessé ghost részek bevezetését. Megmutatjuk, hogy a gráf-szabályok ekvivalensek azokkal a szabályokkal, melyeket ghost részek bevezetésével kapunk a Lagrange-eljárás esetén.

1. INTRODUCTION

Quantization of field theories, invariant with respect to a group of non-Abelian local transformations, encounters certain difficulties. If one calculates scattering amplitudes, using graphical rules, analogous to those of quantum electrodynamics in covariant gauge, than, as pointed out first by Feynman [1], one comes into conflict with unitarity. The origin of this difficulty lies in the fact, that in a covariant gauge the polarization states of the particles on the internal lines of a diagram are more numerous, than on the cut lines, appearing in the unitarity relations. Feynman suggested [1], that unitarity could be restored by inclusion of fictitious "ghost" particles into the theory. Following the suggestion of Feynman several authors [2-5] have succeeded in constructing manifestly covariant unitary quantum theory of fields, possessing local gauge invariance.

In the classical theories of this type constraints $\phi=0$ always occur. In order to guarantee the correspondence between the quantum and classical theories, the physical state vectors $|\psi\rangle$ must satisfy the subsidiary condition $\phi|\psi\rangle=0$.

It is well known, that covariant quantization requires introduction of indefinite metric and, in addition, such a quantization can be carried out only, if the condition $\phi|\psi\rangle=0$ is reducible to the form $\phi^{+}|\psi\rangle=$ $=0$, suggested by Gupta [6] and Bleuler [7]. This reduction is allowed only if

a./ the operator ϕ satisfies the equation $\Box\phi=0$, making the separation of the positive and negative frequency parts possible,

b./ the Heisenberg-equations of motion contain the field operators in such combination, that the terms, proportional to the constraints disappear, when expectation value is taken.

It seems rather difficult to satisfy both these requirements. As it is shown in [5], a nontrivial modification of the Lagrange multiplicator method, developed for the quantization of the electromagnetic field in different covariant gauges [8,9] is necessary to meet condition a/ in the case of non-Abelian gauge theories.

In the present work we propose a canonical quantization scheme, based on Dirac's general treatment [10] of singular dynamical systems. Following Dirac, we keep the subsidiary condition in the original form $\phi|\psi\rangle=0$. This requires relativistically noncovariant quantization procedure without indefinite metric, and, as a result, the propagator of the particles will contain noncovariant terms. It will be shown, that the S-matrix elements between the mathematically complicated physical states are equal to the matrix elements between simple states, not satisfying the subsidiary conditions, provided the propagator is supplemented by additional noncovariant terms. However, it can be proved, that all the noncovariant terms in the propagator can be omitted, if one takes into consideration ghost particles.

We will consider free electrodynamics in non-linear gauges and Yang-Mills theory. In the last case the expressions ϕ contain the coupling constant g explicitely. However, on the ingoing unperturbed states $|\varphi\rangle$ only the subsidiary condition $\phi_{g=0}|\varphi\rangle=0$, independent of g, has to be imposed, since - as it will be shown - the scattering states $|\psi^{\pm}\rangle$, calculated to any given order, up to this order authomatically satisfy the full subsidiary condition $\phi|\psi^{\pm}\rangle=0$. The unitarity of the S-matrix on the subset of asymptotic states, defined by $\phi_{g=0}|\varphi\rangle=0$, is shown to be a consequence of this theorem.

2. DIRAC'S THEORY OF SINGULAR DYNAMICAL SYSTEMS

A dynamical system is called singular, if the expressions of the momenta $p_i=\frac{\partial L(q,\dot{q})}{\partial \dot{q}_i}$ do not determine unambiguously the velocities $\dot{q}_i$, as the functions of momenta p_i. Dirac [10] worked out the general framework for the canonical quantization of singular systems. We give here a very brief summary of the method. For a detailed treatment we refer the reader to Dirac's book.

In a singular theory the expressions $\frac{\partial L(q,\dot{q})}{\partial \dot{q}_i}$ and the coordinates satisfy a number of identities and, therefore, the momenta are subjected to the so called primary constraints $\phi_1^{\ell}(p,q)=0$. The consistency of these primary constraints with the equations of motion usually leads to a number of secondary constraints $\phi_2^{k}(p,q)=0$.

The functions $F(p,q)$ of momenta and coordinates fall into two classes. The first class quantities are those, whose Poisson-bracket with any of the constraints is zero or equal to the linear combination of the constraints. In particular, the constraints themselves can be divided into first and second class constraints.

The Hamiltonian is determined up to a linear combination of first class constraints* with coefficients, which are arbitrary functions of the dynamical variables. Therefore, the time evolution of a dynamical quantity is determined only up to arbitrary functions. For those quantities, however, whose Poisson-bracket with the first class constraints is a linear combination of the constraints, these arbitrary functions are multiplied by constraints in the equation of motion and do not give contribution, when the constraints are satisfied. These quantities are, therefore, the physical quantities of the theory. In particular the Hamiltonian can be shown to be of first class,

$$[\phi^a, H] = r^{ab}\phi^b \qquad /1/$$

and it is a physical quantity.

The indeterminateness, connected with the arbitrary functions, reflects the gauge freedom of the theory. Different choices of these functions correspond to different gauge conditions, imposed on the generalized coordinates.

The change of the unphysical quantities under the influence of the first class constraints does not correspond to any change in the dynamical state of the system. Therefore, the first class constraints generate symmetry transformations of the system in the sense e.g. of the gauge transformations in electrodynamics.

In quantizing the theory the first class constraints are imposed on the state vectors as subsidiary conditions, while the second class constraints are satisfied as operator identities through a suitable redefinition of the Poisson-bracket. In the cases to be considered below, no second class constraints arise.

The method, outlined above, will be applied to special cases in the subsequent sections, and all the above statements will be illustrated explicitely.

3. THE SUBSIDIARY CONDITION THEOREM

When the first class constraints ϕ contain the coupling constant g, the subsidiary conditions $\phi|\psi\rangle=0$ depend explicitely on g. It can be shown, however, that in the perturbation expansion this dependence is taken into account authomatically. This is the containt of the following theorem:

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* The rigorous proof of this statement has recently been given by A. Frankel [11].

Let the first class constraints ϕ^a, the Hamiltonian H and the coefficients r^{ab} in /1/ be of the form

$$\phi^a(\underline{x})=\chi^a(\underline{x}) + g\xi^a(\underline{x}),\quad H=H_o+gH_1,\quad r^{ab}(\underline{x})=r_o^{ab}(\underline{x}) + gr_1^{ab}(\underline{x}) ,$$

where λ^a, ξ^a, H_o, H_1, r_o^{ab}, r_1^{ab} are independent of g*. Then the n-the order approximation $|\psi_n^+>$ to the solution $|\psi^+>$ of the Lippmann-Schwinger equation

$$|\psi^+> = |\varphi> + g \frac{1}{E-H_o+i\varepsilon} H_1|\psi^+> ,$$

satisfies the full subsidiary condition up to the n-th order:

$$\phi^a|\psi_n^+> = g^{n+1}\xi^a\left(\frac{1}{E-H_o+i\varepsilon} H_1\right)^n|\varphi> , \qquad /2/$$

provided the unperturbed state $|\varphi>$ satisfies the unperturbed subsidiary conditions

$$\chi^a|\varphi> = 0. \text{ **} \qquad /3/$$

Let us introduce the vector $\underline{\phi}$, whose components are the constraints ϕ^a, and the matrices $\hat{r}$ and $\hat{H}$ with the matrix elements r^{ab} and $H\delta^{ab}$ respectively. Then /1/ can be written in the form

$$[\underline{\phi}, \hat{H}] = \hat{r}\underline{\phi}$$

Comparing the coefficients of the different powers of g, we get

$$[\underline{\chi}, H_o] = \hat{r}_o\chi \qquad /4/$$

$$[\underline{\chi}, H_1] + [\underline{\xi}, \hat{H}_o] = \hat{r}_o\underline{\xi} + \hat{r}_1\chi \qquad /5/$$

$$[\underline{\xi}, H_1] = \hat{r}_1\underline{\xi} \qquad /6/$$

The theorem can be proved by induction. Consider first the case n=1. The state vector $|\psi_1^+>$ is given by the equation

$$|\psi_1^+> = |\varphi> + g \frac{1}{E-H_o+i\varepsilon} H_1|\varphi>$$

From this expression, using /3/, we get

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* This assumption of linearity in g, which is satisfied for instance in the case of fermion-photon interactions, simplifies the proof, but the theorem can be shown to hold also in the more general case.

** The common spectra of H_o and H is assumed.

$$\phi|\psi_1> = g[\underline{\xi} + \underline{\chi}\,\frac{1}{E-\hat{H}_o+i\epsilon}\,\hat{H}_1]\;|\varphi> + g^2\underline{\xi}\,\frac{1}{E-\hat{H}_o+i\epsilon}\,\hat{H}_1|\varphi>$$

We prove, that

$$\left(\underline{\xi} + \underline{\chi}\,\frac{1}{E-\hat{H}_o+i\epsilon}\,\hat{H}_1\right)|\varphi> = 0 \qquad /7/$$

as $i\epsilon \to 0$. To this end let us write /4/ in the form

$$\underline{\chi}\hat{H}_o = \left(\hat{H}_o + \hat{r}_o\right)\underline{\chi} \qquad /8/$$

Using the relations /3/, /5/ and /8/ we can write

$$\underline{\chi}\,\frac{1}{E-\hat{H}_o+i\epsilon}\,\hat{H}_1|\varphi> = \frac{1}{E-\hat{H}_o-\hat{r}_o+i\epsilon}\,\underline{\chi}\hat{H}_1|\varphi> = \frac{1}{E-\hat{H}_o-\hat{r}_o+i\epsilon}\,[\underline{\chi} + \hat{H}_1]|\varphi> =$$

$$= \frac{1}{E-\hat{H}_o-\hat{r}_o+i\epsilon}\left(\hat{r}_o\underline{\xi} + \hat{r}_1\underline{\chi} - [\underline{\xi}, \hat{H}_o]\right)|\varphi> = \frac{1}{E-\hat{H}_o-\hat{r}_o+i\epsilon}\left(\hat{r}_o+\hat{H}_o-E\right)\underline{\xi}|\varphi> = -\underline{\xi}|\varphi>+o(\varepsilon)$$

which proves /7/, and the theorem holds for n=1.

Now assume, that the theorem is valid up to $|\psi_n^+>$. Let us write $|\psi_{n+1}^+>$ in the form

$$|\psi_{n+1}^+> = |\psi_n^+> + g^{n+1}\left(\frac{1}{E-\hat{H}_o+i\epsilon}\,\hat{H}_1\right)^{n+1}|\varphi>$$

Using /2/ we have

$$\underline{\Phi}|\psi_{n+1}^+>=g^{n+1}\left[\underline{\xi}\left(\frac{1}{E-\hat{H}_o+i\epsilon}\,\hat{H}_1\right)^n+\underline{\chi}\left(\frac{1}{E-\hat{H}_o+i\epsilon}\,\hat{H}_1\right)^{n+1}\right]|\varphi>+g^{n+2}\underline{\xi}\left(\frac{1}{E-\hat{H}_o+i\epsilon}\,\hat{H}_1\right)^{n+1}|\varphi> \qquad /9/$$

The first term can be written as

$$\left[\underline{\xi} + \underline{\chi}\,\frac{1}{E-\hat{H}_o+i\epsilon}\,\hat{H}_1\right]\left(\frac{1}{E-\hat{H}_o+i\epsilon}\,\hat{H}_1\right)^n|\varphi> \qquad /10/$$

The following relation can now be proved:

$$\left[\underline{\xi}+\underline{\chi}\,\frac{1}{E-\hat{H}_o+i\epsilon}\,\hat{H}_1\right]\frac{1}{E-\hat{H}_o+i\epsilon}\,\hat{H}_1=\frac{1}{E-\hat{H}_o-\hat{r}_o+i\epsilon}\left(\hat{H}_1+\hat{r}_1\right)\left[\underline{\xi}+\underline{\chi}\,\frac{1}{E-\hat{H}_o+i\epsilon}\,\hat{H}_1\right] \qquad /11/$$

Really, using the relations /8/, /5/ and /6/ we can write

$$\chi \frac{1}{E-\hat{H}_o+i\epsilon} \hat{H}_1 \frac{1}{E-\hat{H}_o+i\epsilon} \hat{H}_1 = \frac{1}{E-\hat{H}_o-\hat{r}_o+i\epsilon} \chi\hat{H}_1 \frac{1}{E-\hat{H}_o+i\epsilon} \hat{H}_1 =$$

$$= \frac{1}{E-\hat{H}_o-\hat{r}_o+i\epsilon} \left([\chi, \hat{H}_1] + \hat{H}_1\chi\right) \frac{1}{E-\hat{H}_o+i\epsilon} \hat{H}_1 =$$

$$= \frac{1}{E-\hat{H}_o-\hat{r}_o+i\epsilon}\left(\hat{r}_o\xi + \hat{r}_1\chi - [\xi, \hat{H}_o] + \hat{H}_1\chi\right) \frac{1}{E-\hat{H}_o+i\epsilon} \hat{H}_1 =$$

$$= \frac{1}{E-\hat{H}_o-\hat{r}_o+i\epsilon} \left[\left(\hat{H}_o + \hat{r}_o - E\right)\xi - \xi\left(\hat{H}_o - E\right) + \left(\hat{r}_1 + \hat{H}_1\right)\chi \right] \frac{1}{E-\hat{H}_o+i\epsilon} H_1 =$$

$$= - \xi \frac{1}{E-\hat{H}_o+i\epsilon} \hat{H}_1 + \frac{1}{E-\hat{H}_o-\hat{r}_o+i\epsilon} \xi\hat{H}_1 + \frac{1}{E-\hat{H}_o-\hat{r}_o+i\epsilon} \left(\hat{r}_1 + \hat{H}_1\right)\chi \frac{1}{E-\hat{H}_o+i\epsilon} \hat{H}_1 =$$

$$= - \xi \frac{1}{E-\hat{H}_o+i\epsilon} \hat{H}_1 + \frac{1}{E-\hat{H}_o-\hat{r}_o+i\epsilon}\left(\hat{r}_1 + \hat{H}_1\right)\xi + \frac{1}{E-\hat{H}_o-\hat{r}_o+i\epsilon}\left(\hat{r}_1 + \hat{H}_1\right)\chi \frac{1}{E-\hat{H}_o+i\epsilon} \hat{H}_1 =$$

$$= - \xi \frac{1}{E-\hat{H}_o+i\epsilon} \hat{H}_1 + \frac{1}{E-\hat{H}_o-\hat{r}_o+i\epsilon}\left(\hat{H}_1 + \hat{r}_1\right) \left[\xi + \chi \frac{1}{E-\hat{H}_o+i\epsilon} \hat{H}_1\right] ,$$

which proves /11/.

By means of /11/ the first factor in /10/ can be brought in front of $|\varphi\rangle$, and the expression /10/ vanishes as a result of /7/. Therefore, the first term in /9/ is zero, and /9/ is identical to /2/ when n is replaced by (n+1).

We have, therefore, shown, that in order to have $\phi|\psi^+\rangle=0$ it is sufficient to satisfy the condition $\chi|\varphi\rangle=0$. The latter is also a necessary condition, i.e. if $|\psi^+\rangle$ is the scattering state, developed from $|\varphi\rangle$, which is an eigenstate of H_o, then from $\phi|\psi^+\rangle=0$ it follows the relation $\chi|\varphi\rangle=0$. To see this it is sufficient to expand $\phi|\psi^+\rangle$ in powers of g. When $\phi|\psi^+\rangle=0$, the coefficients of this power series vanish, and we get $\chi|\varphi\rangle=0$.

It can be mentioned, that the theorem holds for the scattering states $|\psi^-\rangle$ also.

In the following the asymptotic states $|\varphi\rangle$, satisfying the condition $\chi|\varphi\rangle=0$, will be called allowed states.

The above theorem is basically important for the perturbation treatment of the Yang-Mills theory, and justifies the usual calculational procedure in QED, where noninteracting subsidiary conditions are always used. In the latter case one of the constraint equations is $(\mathrm{div}\,\underline{E}(\underline{x})-e\rho(\underline{x}))|\psi\rangle=0$, but generally the solutions of the equation $\mathrm{div}\,\underline{E}(\underline{x})\;|\psi'\rangle=0$ are used as asymptotic states. However, in this case one could build up the true physical states from the solutions $|\psi'\rangle$ by an appropriate similarity transformation [12]. But in the Yang-Mills theory such a transformation does not exist, because the commutation relations for ϕ and $\phi_{g=0}$ are different. On the other hand, the smallest part of H, which has common eigenvector with the constraints, is the total H itself, which would prevent us from using perturbation theory effectively.

An important straightforward consequence of the theorem discussed above is, that when $|\varphi\rangle$ is an allowed state, than $S|\varphi\rangle$ is also allowed. In order to see this we show, that the two sets of states $\Omega_+|\varphi\rangle\equiv H_+$ and $\Omega_-|\varphi\rangle\equiv H_-$, which can be obtained from the allowed states $|\varphi\rangle$, coincide. Suppose, that this is not true. Than one can find a vector $|\bar{\psi}\rangle$ in H_+, which is outside H_-. But according to the theorem proved above for the states $|\psi\rangle$ in H_+ or H_- we have $\underline{\phi}|\psi\rangle=0$, while for the states outside H_+ or H_-, $\underline{\phi}|\psi\rangle\neq 0$. Hence the vector $|\bar{\psi}\rangle$ cannot exist, and $H_+ = H_- = H$.

Let $H'_\pm$ be the sets $\Omega_\pm|\varphi'\rangle$, where $|\varphi'\rangle$ are orthogonal to all the allowed states. If there are no bound states, than $\Omega_\pm$ are unitary operators, and the sets $H'_\pm$ are orthogonal to H. Hence $H'_+=H'_-=H'$, and we have $(H',H)=0$. Consider the S-matrix element $S_{\beta\alpha} = \langle\psi^-_\beta|\psi^+_\alpha\rangle$, where $|\psi^+_\alpha\rangle = \Omega_+|\varphi\rangle$ and $|\psi^-_\beta\rangle = \Omega_-|\varphi'\rangle$. Since $|\psi^+_\alpha\rangle\in H$ and $|\psi_\beta\rangle \in H'$ we have $S_{\beta\alpha}=0$, which is the statement, we wanted to prove.

From the Hermiticity of the Hamiltonian it follows, that S is unitary in the Hilbert-space of all of the asymptotic states. As a consequence of the above considerations the S-matrix is unitary also in the Hilbert-space of the allowed states alone.

4. CANONICAL QUANTIZATION OF THE FREE ELECTROMAGNETIC FIELD IN NONLINEAR GAUGES

We start from the gauge invariant Lagrangean

$$L = -\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}d^3x$$

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$$

of the free electromagnetic field. The canonical momenta are

$$B^{\mu} = \frac{\partial L}{\partial A_{\mu,o}} = F^{\mu o} \qquad /12/$$

The nonzero Poisson-bracket is defined by the relation

$$\{A_{\mu}(\underline{x}),\ B^{\nu}(\underline{x}')\} = \delta^{\nu}_{\mu}\ \delta^{3}(\underline{x}-\underline{x}')$$

Since $\frac{\partial L}{\partial A_{o,o}}$ is identically zero, the primary constraints of the theory are

$$\phi_1(\underline{x}) \equiv B^{o}(\underline{x}) = 0 \qquad /13/$$

i.e. one primary constraint at each point of space. The quantity $\int B^{\mu}A_{\mu,o}d^3x -$ $- L$ can be expressed through the momenta, potentials and their space derivatives without solving /12/ for the velocities $A_{\mu,o}$, and in this way one can obtain one of the possible Hamiltonians

$$H = \int\left(\frac{1}{4}\ F_{rs}F^{rs} + \frac{1}{2}\ B^{r}B^{r} - A_{o}B^{r}_{\ ,r}\right)d^3x$$

The indices r, s, t take on the values 1,2,3. Computing the time derivative of the primary constraints $B^{o}(\underline{x})$ with this Hamiltonian one finds, that it is equal to $B^{r}_{\ ,r}(\underline{x})$. Therefore, the consistency of the constraint /13/ with the equation of motion leads to the secondary constaints

$$\phi_2(\underline{x}) \equiv B^{r}_{\ ,r}(\underline{x}) = 0 \qquad /14/$$

The consistency requirement of /14/ leads to no further secondary constraints. The Poisson-bracket of the constraints is zero, so they are of first class.

In order to obtain the most general Hamiltonian H_E one has to add to H the constraints ϕ_1, ϕ_2, multiplied by arbitrary functions of the dynamical variables:

$$H_E = \int\left(\frac{1}{4}\ F_{rs}F^{rs} + \frac{1}{2}\ B^{r}B^{r} - A_{o}B^{r}_{\ ,r} + c_1\phi_1 + c_2\phi_2\right)d^3x \qquad /15/$$

This Hamiltonian leads to the equation of motion

$$A_{o,o} = C_1 + \frac{\partial C_1}{\partial B^{o}}\ \phi_1 + \frac{\partial C_2}{\partial B^{o}}\ \phi_2$$

$$A_{s,o} = B^{s} + A_{o,s} - C_{2,s} + \frac{\partial C_1}{\partial B^{s}}\ \phi_1 + \frac{\partial C_2}{\partial B^{s}}\ \phi_2 \qquad /16/$$

$$B^{o}{}_{,o} = B^{r}{}_{,r} - \frac{\partial C_1}{\partial A_o}\phi_1 - \frac{\partial C_2}{\partial A_o}\phi_2 \qquad /16/$$

$$B^{s}{}_{,o} = A_{s,rr} - A_{r,rs} - \frac{\partial C_1}{\partial A_s}\phi_1 - \frac{\partial C_2}{\partial A_s}\phi_2$$

In the classical theory one can set here $\phi_1 = \phi_2 = 0$. The arbitrary functions C_i need not disappear from the equations, since the potentials are second class quantities, and cannot be defined unambiguously. If, however, one identifies the first class quantities $\underline{B}$ and rot $\underline{A}$ with the electric and magnetic fields respectively, then for these quantities /14/ and /16/ lead to the Maxwell-equations, independent of the functions C_i.

The arbitrary functions serve to fix the gauge. Let us choose them, for example, in the following way:

$$C_1 = A_{r,r} - \lambda A_\nu A^\nu - \frac{1}{2} B^o$$

$$C_2 = 0 \qquad /17/$$

Then the first of the equations /16/ leads to the expression

$$B^o = -(\partial_\nu A^\nu + \lambda A_\nu A^\nu)$$

and /13/ is equivalent to the nonlinear gauge condition

$$\partial_\nu A^\nu + \lambda A_\nu A^\nu = 0$$

From H_E one can go back to the Lagrangean

$$L = \int\left[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\nu A^\nu + \lambda A_\nu A^\nu)^2\right] d^3x \qquad /18/$$

The theory, corresponding to this Lagrangean, has been discussed in great detail in [13] on the basis of graph combinatorics, since it serves as a good introduction to the non-Abelian gauge theories. For the same reason we also discuss the canonical quantization first in this case.

The nonzero commutation relations are

$$\left[A_\nu(x), B^\mu(x')\right]_{x^o = x^{o\prime}} = i\delta_\nu{}^\mu\, \delta^3(\underline{x}-\underline{x}')$$

and we get the following Heisenberg equations of motion in the gauge chosen

$$A_{o,o} = A_{r,r} - \lambda A_\nu A^\nu - B^o$$

$$A_{r,o} = B^r + A_{o,r} \qquad /19/$$

$$B^o{}_{,o} = B^r{}_{,r} + 2\lambda A_o B^o$$

$$B^s{}_{,o} = A_{s,rr} - A_{r,rs} + B^o{}_{,s} - 2\lambda A_s B^o$$

Using these equations, the commutation rule can be written also as

$$\left[A_\nu(x), A_{\mu,o}(x')\right]_{x^o = x^{o'}} = -\, i g_{\mu\nu}\delta^3(x-x')$$

i.e. in the same form, as in the Feynman-gauge $/\lambda=0/$.
The subsidiary conditions, corresponding to the constraints are

$$B^o|\Omega\rangle = 0; \qquad B^r{}_{,r}|\Omega\rangle = 0$$

These conditions cannot be weakened to the form $B^{o(+)}|\Omega\rangle = B^{r(+)}{}_{,r}|\Omega\rangle = 0$, since for $\lambda \neq 0$ B^o does not satisfy the equation $\Box B^o = 0$, and hence it is not possible to carry through covariant quantization with indefinite metric.

The operators a_μ, b^μ in the interaction picture satisfy the equations /19/ with $\lambda=0$, from which one gets

$$\Box a_\mu(x) = 0; \qquad \Box b^\mu(x) = 0$$

The subsidiary conditions in the interaction picture are

$$b^o(x)|\omega(t)\rangle = 0; \qquad b^r{}_{,r}(x)|\omega(t)\rangle = 0 \qquad /20/$$

Here the operators could be replaced by their positive frequency part, but then the correspondence of the Heisenberg-equations of motion /19/ to the corresponding classical equations would be destroyed.

The equations of motion permit us to write /20/ in the form

$$\partial^\mu a_\mu(x)|\omega(t)\rangle = 0; \qquad (\partial_\mu a^\mu(x))_{,o}|\omega(t)\rangle = 0 \qquad /21/$$

The interaction Hamiltonian is the following:

$$H_1(x^o) = -\lambda \int a_\nu(x) a^\nu(x) b^o(x) d^3x \qquad /22/$$

Using the Fourier-expansions

$$a_o(x) = \frac{1}{\sqrt{V}} \sum_{\underline{k}} \frac{1}{\sqrt{2\omega}} \left(e^{-ikx} a_o^+(\underline{k}) + e^{ikx} a_o(\underline{k}) \right)$$

$$a_r(x) = \frac{1}{\sqrt{V}} \sum_{\underline{k}} \frac{1}{\sqrt{2\omega}} \left(e^{-ikx} a_r(\underline{k}) + e^{ikx} a_r^+(\underline{k}) \right)$$

$/k^o=\omega=|\underline{k}|/$ one gets for the commutation relations of the Fourier-amplitudes the oscillator-type rules

$$\left[a_\mu(\underline{k}),\ a_\nu^+(\underline{k}') \right] = \delta_{\mu\nu} \delta_{\underline{k}\underline{k}'}$$

which lead to positive metric for all four components.

In momentum space the subsidiary conditions /21/ can be written as

$$\left(a_o^+(\underline{k}) + C(\underline{k}) \right) |\omega(t)\rangle = 0$$
$$\qquad /23/$$
$$\left(a_o(\underline{k}) + C^+(\underline{k}) \right) |\omega(t)\rangle = 0$$

where $C(\underline{k}) = \frac{1}{\omega} k^r a_r(\underline{k})$.

Let $|0)$ be the mathematical vacuum, i.e. the Fock-state, defined by the equations $a_\mu(k)|0) = 0$. It is not the physical vacuum, since it does not satisfy /23/. The physical vacuum is given by the expression [14]

$$\prod_{\underline{k}} e^{-a_o^+(\underline{k}) C^+(\underline{k})} |0) \equiv |0\rangle$$

The unperturbed Hamiltonian H_o in the interaction picture can be obtained from /15/ and /17/, replacing A_ν, B^μ by a_ν, b^μ and taking the limit $\lambda=0$:

$$H_o = \int d^3x \left[\frac{1}{4} f_{rs} f^{rs} + \frac{1}{2} b^r b^r - a_o b^r{}_{,r} + (a_{r,r} - \tfrac{1}{2} b^o) b^o \right] =$$

$$= \sum_{\underline{k}} \omega \left[a_r^+(\underline{k}) a_r(\underline{k}) - a_o^+(\underline{k}) a_o(\underline{k}) \right] + c - \text{number}$$

$$= \sum_{\underline{k}} \omega \left[a_{T_1}^+(\underline{k}) a_{T_1}(\underline{k}) + a_{T_2}^+(\underline{k}) a_{T_2}(\underline{k}) + \left(C^+(\underline{k}) C(\underline{k}) - a_o^+(\underline{k}) a_o(\underline{k}) \right) \right] + C - \text{number}$$

The term

$$C^+(\underline{k}) C(\underline{k}) - a_o^+(\underline{k}) a_o(\underline{k}) = \left[\left(C^+(\underline{k}) - a_o(\underline{k}) \right) \left(C(\underline{k}) + a_o^+(k) \right) + \left(C(\underline{k}) - a_o^+(\underline{k}) \right) \left(C^+(\underline{k}) + a_o(k) \right) \right] \frac{1}{2}$$

acting on physical states gives zero, therefore the physical states obtained from $|0\rangle$ by transverse creation operators are those, which contain definite number of photons. However, the state $|0\rangle$ is not normalizable, since the operators $a_o^+ + C$ and $a_o + C^+$ possess continuous spectrum. Let us confine ourselves to a single momentum component, and following [15], define the state

$$|0_\gamma\rangle = \hat{0}_\gamma |0) \equiv e^{-\gamma a_o^+(\underline{k}) C^+(\underline{k})} |0) \qquad /24/$$

which for $|\gamma|<1$ has the norm

$$\langle 0_\gamma | 0_\gamma \rangle = \frac{1}{1-\gamma^2}$$

and for $\gamma \to 1$ satisfies /23/. It is well known, that matrix elements between states, belonging to the same eigenvalue in a continuous spectrum are not meaningful quantities, and using them, one could come to contradictions. For example, the matrix element of the commutator $[\partial^\mu a_\mu(x), a_\nu(y)]$ between physical states, satisfying /23/, can be proved to be zero in spite of the fact, that the commutator itself is a nonzero c-number. Contradictions of this type can be circumvented, if the physical bra-states are generated from the vacuum bra $\langle 0_\gamma|$ with $\gamma \neq 1$, and the limit $\gamma = 1$ is taken only in the final step. This limit, when exists, can be considered as the correct matrix element between physical states.

The S-matrix is given by the formal expression

$$S = T_D \, e^{-i \int dx_o H_1(x_o)} \qquad /25/$$

where T_D is the Dyson chronological ordering operator.* It is straightforward to calculate amplitudes, corresponding to any diagram, between states $|0,tr)$, obtained from the mathematical /unphysical/ vacuum $|0)$ with the aid of the transverse creation operators. These amplitudes correspond to the Lagrangean /18/. The H_1 of /22/ appearently does not contain the characteristic interaction term $-\frac{1}{2}\gamma^2 a_\nu a^\nu a_\mu a^\mu$. Contributions from such term authomatically arise, when the Dyson chronological product is systematically replaced by the Wick chronological product /for a similar procedure see [16]/. However we do not get ghost contribution, without which the theory is known [13] to contradict unitarity.

This is because the calculation has been based incorrectly on unphysical states. The ghost contribution arises, when the matrix elements of /25/ are taken between the states, satisfying the subsidiary condition. These states are complicated, since they contain longitudinal and time-like photons. They can be written as

$$|0_\gamma tr\rangle = \hat{0}_\gamma |0,tr)$$

and the S-matrix elements are

$$\frac{1}{\langle 0_\gamma | 0_1\rangle} \langle 0_\gamma tr'|S|0_1,tr\rangle = \frac{1}{\langle 0_\gamma | 0_1\rangle} (0,tr'|(\hat{0}_\gamma^+ \hat{0}_1)\bar{S}|0,tr) \qquad /26/$$

where $\bar{S}=\hat{0}_1^{-1} S\hat{0}_1$. Since $\hat{0}_1$ is independent of the space-time coordinates, this transformation can be accounted for by the replacement

$$a_\mu(x) \to \bar{a}_\mu(x) = \hat{0}_1^{-1} a_\mu(x)\hat{0}_1 \qquad /27/$$

In Section 5 it will be shown, that the operator $\frac{1}{\langle 0_\gamma | 0_1\rangle} \hat{0}_\gamma^+ \hat{0}_1$ can in fact be omitted. Therefore, it is allowed to use transverse Fock-states, built up on the mathematical vacuum $|0)$, provided the propagator is identified with the expectation value of the chronological product of the operators $\bar{a}_\mu(x)$. Using /24/ and /27/ it is straightforward to verify, that

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* The Dyson chronological operator T_D is defined by the relation $T_D(A(t)B(t'))=\Theta(t-t')A(t)B(t')+\Theta(t'-t)B(t')A(t)$, and does not commute with differentiation with respect to time. In a Lagrangean theory the Wick chronological product is used. Acting on a product of field operators and their derivatives, the operator T_W is defined to commute with differentiation.

$$\bar{a}_o(x) = a_o(x) - \frac{1}{\sqrt{V}} \sum_{\underline{k}} \frac{1}{\sqrt{2\omega}} e^{ikx} c^+(\underline{k})$$

$$\bar{a}_r(x) = a_r(x) - \frac{1}{\sqrt{V}} \sum_{\underline{k}} \frac{1}{\sqrt{2\omega}} \frac{k^r}{\omega} a_o^+(\underline{k}) e^{-ikx}$$

From these expressions one obtains the expressions /in continuous Fourier-representation/

$$(0|T_D\left(\bar{a}_\mu(x)\bar{a}_\nu(y)\right)|0) = \int d^4k e^{ik(x-y)} \frac{1}{(2\pi)^4 i} \left[\frac{g_{\mu\nu}}{k^2+ie} + \frac{2\pi i\delta(k^2)\Theta(k^o)\delta_{\mu o}k_\nu}{k^o} + \right.$$

$$\left. + \frac{2\pi i\delta(k^2)\Theta(-k^o)\delta_{\nu o}k_\mu}{k^o} \right] \qquad /28/$$

$$(0|T_D\left(\partial^\mu \bar{a}_\mu(x)\bar{a}_\nu(y)\right)|0) = \int d^4k e^{ik(k-y)} ik_\nu \frac{1}{(2\pi)^4 i} \frac{1}{k^2-iek^o}$$

The integrand in the last line contains the Fourier-component of the retarded propagator, and the integral vanishes, unless $x^o > y^o$. This is the consequence of the fact, that from the subsidiary condition $\partial^\mu a_\mu|0\rangle = 0$ it follows the equation $\partial^\mu \bar{a}_\mu|0\rangle = 0$.

One can look for the equivalent Lagrangean description also in this case. The difference between the Dyson and Wick chronological product can be expressed through the value of the commutators. Since the latters are invariant under similarity trasformation, the vertices of the Lagrangean equivalent to $H_1(\bar{a}_\mu, \bar{b}^\nu)$ are also given by /18/ without ghost contribution. However, in contrast to the previous case, when the subsidiary conditions were neglected, the propagators to be used in the diagrams are those of /28/. This difference is sufficient to make the theory unitary.

5./ CANONICAL QUANTIZATION OF THE YANG-MILLS FIELD

The Lagrangean of the Yang-Mills theory [17]

$$L = -\frac{1}{4} \int d^3x F^a_{\mu\nu} F^{a\,\mu\nu}$$

$$F^a_{\mu\nu} = A^a_{\nu,\mu} - A^a_{\mu,\nu} + gf^{abc} A^b_\mu A^c_\nu$$

leads to the particular Hamiltonian

$$H = \int \left(\frac{1}{4} F^a_{rs} F^{a\,rs} + \frac{1}{2} B^{a\,r} B^{a\,r} - A^a_o \nabla^{ab}_r B^{b\,r} \right) d^3x$$

where

$$\nabla^{ab}_r = \delta^{ab} \partial_r + g f^{acb} A^c_r$$

means invariant differentiation, and the indices a, b, c label the generators of the underlying compact group.

The constraints of the theory are

$$\phi^a_1 \equiv B^{a\,o} = 0 \qquad /29/$$

$$\phi^a_2 \equiv \nabla^{ab}_r B^{b\,r} = 0 \qquad /30/$$

The consistency requirement $\dot{\phi}^a_2 = 0$ fulfills as a consequence of /30/ since

$$\dot{\phi}^a_2 = i\left[H, \phi^a_2\right] = - g f^{abc} A^b_o \phi^c_2$$

and no further constraints arise. The constraints are of first class:

$$\left[\phi^a_1(x), \phi^b_1(y)\right]_{x^o=y^o} = \left[\phi^a_1(x), \phi^b_2(y)\right]_{x^o=y^o} = 0$$

$$\left[\phi^a_2(x), \phi^b_2(y)\right]_{x^o=y^o} = g f^{abc} \phi^c_2(x) \delta^3(\underline{x}-\underline{y}) \qquad /31/$$

The most general Hamiltonian is therefore

$$H_E = \int d^3\!x \left[\frac{1}{4} F^a_{rs} F^{a\,rs} + \frac{1}{2} B^{a\,r} B^{a\,r} - A^a_o \nabla^{ab}_r B^{b\,r} + C^a_1 \phi^a_1 + C^a_2 \phi^a_2 \right]$$

The subsidiary conditions in Heisenberg- and interaction-picture are

$$B^{a\,o} | \Omega > = 0 \qquad /32/$$

$$\nabla^{ab}_r B^{b\,r} | \Omega > \doteq 0 \qquad /33/$$

$$b^{a}{}_{o}(x)|\omega(t)\rangle = 0 \qquad /34/$$

$$\left(b^{a}{}_{r}{}_{,r}(x) + gf^{acb}\, a^{c}{}_{r}(x) b^{b\,r}(x)\right)|\omega(t)\rangle = 0 \qquad /35/$$

The subsidiary condition /33/ in the Heisenbert-picture is the time derivative of the subsidiary condition /32/. As a consequence, /35/ is the time derivative of /34/. However, the operator in the condition /35/ is not equal to $\dot{b}^{a}{}_{o}$ since in the Yang-Mills theory the term $b^{a}{}_{o}\,\frac{d}{dt}|\omega(t)\rangle$ does not vanish /in the free electrodynamics, discussed above, this term gives zero as a consequence of the primary constraint/. In spite of the fact, that the subsidiary conditions contain the coupling constant explicitely, it is sufficient to impose the noninteracting subsidiary conditions $b^{a}{}_{o}|\varphi\rangle = b^{a}{}_{r}{}_{,r}|\varphi\rangle = 0$ on the asymptotic states $|\varphi\rangle$, since according to the theorem, proved in Section 3, the scattering states authomatically satisfy /35/.

The procedure of the previous section can now be immediately applied to the Yang-Mills theory. The only step, we have to add, is to prove, that the operator $\frac{1}{\langle 0_\gamma | 0_1\rangle}\ \hat{0}^{+}_{\gamma}\ \hat{0}_1$ can indeed be omitted from the right hand side of /26/.

Using /24/ and computing the norm $\langle 0_\gamma|0_1\rangle$ we have

$$\frac{1}{\langle 0_\gamma|0_1\rangle}\ (0,tr'|\hat{0}^{+}_{\gamma}\ \hat{0}_1 = (1-8)(0,tr'|\ e^{-\gamma a_o(\underline{k})c(\underline{k})}\ e^{-a^{+}_{o}(\underline{k})c^{+}(\underline{k})}$$

After a simple but lengthy algebra this expression can be brought into the form

$$(0,tr'|\sum_{k=0}^{\infty} a_o^{k}\ C^{k}\ .\ \alpha_k(\gamma) \qquad /36/$$

where

$$\alpha_o(\gamma) = 1$$

$$\alpha_k(\gamma) = \frac{(-\gamma)^k}{(k!)^2}\ (1-\gamma)\frac{d^k}{d\gamma}\ \left(\frac{\gamma^k}{1-\gamma}\right) \qquad k \neq 0$$

In the limit $\gamma=1$ all the coefficients α_k tend to infinity except α_o, which goes to unity. However, the matrix element /26/ has to be calculated with the aid of /36/ before taking the limit $\gamma=1$, and it turns out, that the terms with $k \neq 0$ give no contribution.

Really, we have shown in Section 3, that $\chi S|0,tr) = 0$, so that $\bar{S}|0,tr)$ contains only transversal particles. That means, $\frac{1}{\langle O_\gamma | O_1 \rangle} \hat{O}_\gamma^+ \hat{O}_1$ can indeed be omitted.

The rules for calculating graphs are, therefore, similar to those of the previous section. Let us choose, for example,

$$C_1^a = A_{r,r}^a - \frac{1}{2} B^a{}_o - \lambda f^a (A) \qquad /37/$$

$$C_2^a = O$$

where $f^q(A)$ is a function of the potentials /and at least quadratic in A/. Then we get the Lagrangean

$$L = \int \left[-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2} (\partial_\mu A^{a\mu} + \lambda f^a (A) \right]^2 d\overset{3}{x} \qquad /38/$$

the interaction part of which defines the vertices of the diagrams. For the propagators one has to use the expression, given in /28/.

Amplitudes, corresponding to these prescriptions, satisfy unitarity and are equal to the S-matrix elements, calculated by using the usual Feynman-propagator and ghost Lagrangean. In order to prove this, it is sufficient to establish the validity of the Slavnov-Taylor identities [18].

In the usual description one has to add to L a suitable ghost contribution L_g in order to get a Lagrangean, which satisfies the Slavnov-Taylor identities. The form of L_g depends on the second term of L, breaking local gauge invariance, and can be determined by the rules, given in [18]. According to these rules, the propagator of the ghost particle is that of a zero mass scalar particle up to the $i\varepsilon$ prescription. This prescription cannot be chosen arbitrarily, and can be determined for instance from the Slavnov-Taylor identities in the lowest order [13,18]. In our gauges they give $k^\mu D_{\mu\nu}(k) = (-\hat{m}^{-1}) k_\mu$, where $D_{\mu\nu}$ and $(-\hat{m}^{-1})$ are the propagators of the vector particle and the ghost respectively. Taking into account the form /28/ of our vector meson propagator, we see, that the Lagrangean /38/ satisfies the Slavnov-Taylor identities, if we add to it the usual L_g and for the ghost use the retarded propagator

$$\frac{1}{(2\pi)^4 i} \qquad \frac{1}{k^2 - i\varepsilon k^o} \qquad /39/$$

One now sees, that the addition of L_g to L is completely irrelevant, when S-matrix elements are calculated. In this case the ghost lines occur always in closed loops, which with retarded propagators give no contribution. Therefore, from the point of view of the observable quantities the theory, based on $L+L_g$ is equivalent to the canonical quantization scheme, described above.

The Slavnov-Taylor identities serve as the basis for the proof of the invariance with respect to the $i\varepsilon$ prescription of longitudinal and time-like vector particles, from which it follows, that S-matrix elements, corresponding to the canonical quantization scheme are equal to those, calculated by using Feynman propagators and ghost Lagrangean.

Actually we will prove more: the S_α-matrix elements, calculated with the aid of propagators

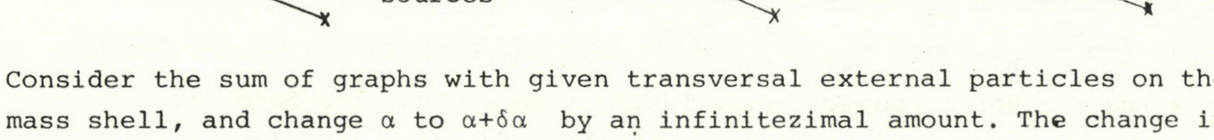
$$\underset{\rightarrow k}{\mu \;\text{———}\; \alpha \;\text{———}\; \nu} \qquad \frac{1}{(2\pi)^4 i}\left[\frac{g_{\mu\nu}}{k^2+i\varepsilon} + \alpha\left(\frac{\delta_{\mu 0}}{k^0}\, 2\pi i\delta(k^2)\theta(k^0)k_\nu + \frac{\delta_{\nu 0}}{k^0}\, 2\pi i\delta(k^2)\theta(-k^0)k_\mu\right)\right]$$

$$\underset{\rightarrow k}{\text{---}\;\alpha\;\text{-->--}} \qquad \frac{1}{(2\pi)^4 i}\left[\frac{1}{k^2+i\varepsilon} + \alpha\; 2\pi i\;\delta(k^2)\theta(k^0)\right] \equiv \underset{\rightarrow k}{\text{---}\;\text{>---}} + \alpha \underset{\rightarrow k}{|||||>|||||}$$

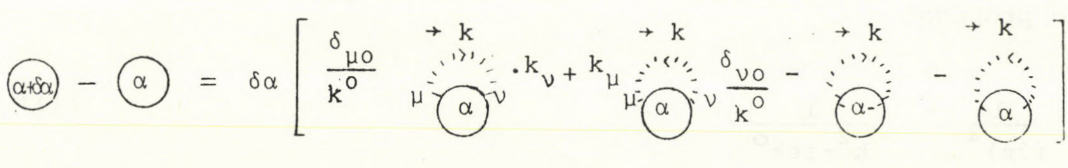
are independent of the arbitrary parameter α. The values $\alpha=1$ and $\alpha=0$ correspond to the cases, mentioned above.

Let us start from the Slavnov-Taylor identity

$$C^a = \partial_\mu A^{a\mu}$$



Consider the sum of graphs with given transversal external particles on the mass shell, and change α to $\alpha+\delta\alpha$ by an infinitezimal amount. The change in the S_α-matrix element can be represented graphically as



Using the Slavnon-Taylor identities to transform the first and second term on the right hand side /k is on the mass shell in these terms!/ we see immediately, that all the terms on the right hand side cancel, showing, that S_α is indeed independent of the parameter α. Choosing $\alpha=0$ one can prove the unitarity again [19].

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