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SO/4/ SYMMETRIC SOLUTIONS
FOR CERTAIN TYPES OF ENERGY-MOMENTUM TENSOR

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SO/4/ SYMMETRIC SOLUTIONS FOR CERTAIN TYPES OF
ENERGY-MOMENTUM TENSOR

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ABSTRACT

If in space there is matter, the Einstein equations of general relativity can be solved for special cases only. However, if the gravitational field has several symmetries, the problem becomes simpler. Supposing that the Killing equation has six linearly independent solutions, that the operators $G_A \equiv K_A^{\rho} \partial_{\rho}$ formed of these solutions behave as generators of an $SO(4)$ group, and that the Killing vectors fill a three-dimensional space /"hyperspherical symmetry"/, the Robertson-Walker metric with $k = +1$ is obtained. Using this metric tensor the Einstein equations become ordinary differential equations, they can be integrated for viscous dust /supposing a certain run of viscous coefficient/, and, for Hookean elastic media, the solution can be traced back to an elliptic integral.

The viscous dust solutions may have cosmological importance and the elastic solutions may be used to write down Schwarzschild interiors.

АННОТАЦИЯ

Решение уравнений Эйнштейна общей теории относительности в присутствии материи возможно только для специальных случаев. Если же для пространства имеется симметрия определенного порядка, то проблема упрощается. Допуская, что для уравнения Киллинга имеется 6 линейных независимых решений, то составленные из них операторы $G_A \equiv K_A^{\rho} \partial_{\rho}$ ведут себя как генераторы группы $SO(4)$, и векторы Киллинга образуют 3-х мерное пространство /"гиперсферическая симметрия"/, получаем случай $k=+1$ пространства Робертсона-Валкера. Используя этот метрический тензор, уравнения Эйнштейна становятся обычными дифференциальными уравнениями, и могут быть решены для вязких жидкостей без давления /наряду с определенными условиями, принятыми для внутреннего коэффициента трения/, для случая же идеального упругого тела решением является эллиптический интеграл.

Решения для вязких жидкостей могут представлять космологический интерес, решения же, содержащие упругие материалы, возможно могут представлять собой внутренние решения Шварцшильда.

KIVONAT

Az általános relativitáselmélet Einstein-egyenleteinek megoldása anyag jelenlétében csak speciális esetekre lehetséges. Ha azonban a térnek bizonyos számú szimmetriája van, a probléma egyszerűbbé válik. Feltételezve, hogy a Killing-egyenletnek 6 lineárisan független megoldása van, az ezekből képzett $G_A \equiv K_A^{\rho} \partial_{\rho}$ operátorok $SO(4)$ csoport generátoraiként viselkednek, és a Killing-vektorok 3 dimenziós teret feszítenek ki /"hipergömbszimmetria"/, a Robertson-Walker terek $k = +1$ esetét kapjuk. E metrikus tenzort használva az Einstein-egyenletek közösleges differenciálegyenletek lesznek, és nyomásmentes viszkózus folyadékra /a belső surlódási együttthatóra tett bizonyos feltételezés mellett/ megoldhatók, ideális rugalmas test esetén pedig a megoldás egy elliptikus integrál.

A viszkózus folyadékmegoldásoknak kozmológiai érdekessége lehet, a rugalmas anyagot tartalmazó megoldások pedig esetleg belső Schwarzschild-megoldások lehetnek.

1. INTRODUCTION

In the presence of matter the Einstein-equations of general relativity can be solved for special cases only. However, if the gravitational field has sufficiently high symmetry, the equations can often be integrated in analytical form.

If the gravitational field possesses six Killing vectors K_A^μ , the operators

$$G_A \equiv K_A^\rho \partial_\rho \quad /1.1/$$

constitute the SO(4) group, and the quantity K_A^μ is a matrix of rank 3 with the indices A, μ , the metric tensor has the well known Robertson-Walker form with $k = +1$. This symmetry may be termed "hyperspherical".

The "SO(4) symmetric" solutions may be useful as models of the Universe or homogeneous interior solutions of spherically symmetric exterior fields. /E.g. the elliptic Friedmann solution is:

- a./ Universe solution with $p = 0$;
- b./ one of the interior Schwarzschild solutions, corresponding to homogeneous incoherent fluid./

This paper contains some SO(4) symmetric solutions for several types of energy-momentum tensor. In Sect. 2 we investigate the meaning of the notion of "SO(4) symmetry with three-dimensional transitivity". The well-known cases of the vacuum and electrovac problems are contained in Sects 3 and 4. Section 5 deals with the case of massless non-charged scalar meson field; Sect. 6 contains some new solutions for viscous fluid. Finally, in Sects. 7 and 8 we deal with elastic and visco-elastic media and give some solutions for these cases.

2. SO(4) SYMMETRY WITH THREE-DIMENSIONAL TRANSITIVITY

If space-time has a symmetry, the Killing equation

$$K_\mu{}_{;\nu} + K_\nu{}_{;\mu} = 0 \quad /2.1/$$

has a K^μ solution. In the case of more than one symmetry there exists a set of the linearly independent Killing vectors, namely, if the number of linearly independent vectors is N , then

$$\varphi^{\dagger R \mu}_{K_R} \neq 0; \quad R \leq N \quad /2.2/$$

in any point of space-time if and only if the scalars φ^{R1} are zeros. We may define the generators as in eq. /1.1/ and their commutators

$$[G_A, G_B] \psi = (K_A^\rho K_B^\sigma{}_{;\rho} - K_B^\sigma K_A^\rho{}_{;\sigma}) \psi_\sigma \equiv Q_{AB}^\rho \psi_{;\rho} \quad /2.3/$$

Here ψ is scalar. But it is well known that Q_{AB}^α is again a Killing vector, thus it can be written in the following form:

$$Q_{AB}^\alpha = C_{AB}^R K_{R1}^\alpha \quad /2.4/$$

C_{AB}^R -s are constants with antisymmetry in the indices A, B . They are structure constants of a Lie-algebra [1].

In the case of $SO/4/$ symmetry there are six linearly independent Killing vectors, and, introducing new notation in the indices:

$$G_{ab} \equiv \frac{1}{2} \epsilon_{abr} G_r; \quad G_{a0} \equiv G_{a+3}; \quad i = 1, 2, 3 \quad /2.5/$$

the commutators are the following:

$$[G_{\alpha\beta}, G_{\gamma\delta}] = \delta_{\beta\gamma} G_{\alpha\delta} + \delta_{\alpha\delta} G_{\beta\gamma} - \delta_{\alpha\gamma} G_{\beta\delta} - \delta_{\beta\delta} G_{\alpha\gamma} \\ \mu = 0, a. \quad /2.6/$$

The Killing vectors stretch a subspace in which the symmetry transformations can mix the coordinates. The number of this subspace /hypersurface of transitivity/ is equal to the rank of the matrix:

$$M_{A\alpha} \equiv K_{A1}^\alpha; \quad A \leq N. \quad /2.7/$$

For $SO/4/$ symmetry the rank is 3 or 4. In the following we will deal with the case of three-dimensional transitivity only.

The eqs. /1.1/, /2.3/, /2.6/ are partial differential equations for the K_A^α -s, and these equations can be solved. The coordinate system can be chosen such that:

$$K_1^\alpha = (0, \sin x^2 \sin x^3, \operatorname{ctgx}^1 \cos x^2 \sin x^3, \operatorname{ctgx}^1 \sin^{-1} x^2 \cos x^3)$$

$$K_2^\alpha = (0, \sin x^2 \cos x^3, \operatorname{ctgx}^1 \cos x^2 \cos x^3, \operatorname{ctgx}^1 \sin^{-1} x^2 \sin x^3)$$

$$K_3^\alpha = (0, \cos x^2, -\operatorname{ctgx}^1 \sin x^2, 0)$$

$$K_4^\alpha = (0, 0, \sin x^3, \operatorname{ctgx}^2 \cos x^3)$$

$$K_5^\alpha = (0, 0, \cos x^3, -\operatorname{ctgx}^2 \sin x^3)$$

$$K_6^\alpha = (0, 0, 0, 1)$$

$$0 \leq x^1 < \pi, \quad 0 \leq x^2 < \pi, \quad 0 \leq x^3 < 2\pi \quad /2.8/$$

The forms /2.8/ are preserved by the following transformations:

$$x^{i'} = x^i, \quad x^{0'} = \vartheta(x^0); \quad i = 1, 2, 3. \quad /2.9/$$

Now the eq. /2.1/ for $g_{\mu\nu}$ can be solved. We obtain /after a special choice of x^0 /:

$$ds^2 = a^2(x^0) \left[-dx^{02} + dx^{12} + \sin^2 x^1 (dx^{22} + \sin^2 x^2 dx^{32}) \right], \quad /2.10/$$

This is a Robertson-Walker line element /with $k=+1$ /.

The Ricci tensor has four nontrivial components, for example R_{00} , R_{01} , R_{11} , R_{12} . The Einstein equations have the following form:

$$\begin{aligned} K T_{11} &= 1 + 2 \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + \lambda a^2 \\ K T_{00} &= -3 \left(1 + \frac{\dot{a}^2}{a^2} \right) - \lambda a^2 \quad \dot{\vartheta} \equiv \frac{\partial \vartheta}{\partial x^0} \\ T_{01} &= T_{12} = 0. \end{aligned} \quad /2.11/$$

We shall now investigate the cases of vacuum, electrovac, massless noncharged scalar meson field, viscous fluid, Hookean elastic medium and Kelvin-Voigt viscoelastic systems. In the following we require that the characteristic components of $T_{\mu\nu}$ /for example pressure, velocity, vector potential, etc./ also have vanishing Lie derivatives along the K_A^α fields.

3. VACUUM CASE

This well known case is dealt with for the sake of completeness only. $T_{\mu\nu} = 0$, and the solution of eq. /2.11/ is

$$a(x^0) = \sqrt{\frac{3}{-\lambda}} \cos^{-1} x^0 \quad /3.1/$$

If $\lambda \geq 0$, solution with SO/4/ symmetry does not exist.

An observer, lying at $x^1=0$, feels time pass according to

$$dt = a(x^0) dx^0. \quad /3.2/$$

This time is a physical quantity, the coordinate x^0 has no physical meaning. In the following Sections "time" means this "t" quantity. From the eqs. /3.1 and 3.2/:

$$a(t) = \sqrt{\frac{3}{-\lambda}} \cosh t. \quad /3.3/$$

4. ELECTROVAC CASE

The electrovac solution with SO/4/ symmetry does not exist. Namely, for the electrovac case

$$\begin{aligned} T_{\mu\nu} &= F_{\mu\rho} F_{\nu}^{\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \\ F_{\mu\nu} &= A_{\nu;\mu} - A_{\mu;\nu}. \end{aligned} \quad /4.1/$$

But the Lie derivative of A^{μ} must vanish along K_A^{α} , thus

$$A^I = 0; A^0 = A^0(x^0); I = 1, 2, 3; \Rightarrow F_{\mu\nu} = 0. \quad /4.2/$$

using the K^{μ} vectors of eq. /2.8/. This is the vacuum case.

5. NONCHARGED SCALAR MESON FIELD

In this case [3]

$$T_{\mu\nu} = \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} (\phi_{,\rho} \phi^{,\rho} + m^2 \phi^2) g_{\mu\nu} \quad /5.1/$$

and ϕ is real. The symmetry condition is the following:

$$\phi, {}_{\rho}K_A^{\rho}| = 0 \Rightarrow \phi = \phi(x^0) \quad /5.2/$$

Let us confine ourselves to the case $m = \lambda = 0$, where the Einstein equations have the following forms:

$$\begin{aligned} 2 + \frac{a^2}{a^2} + \frac{\ddot{a}}{a} &= 0 \\ 2(-1-2 \frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}) &= \kappa \dot{\phi}^2 \end{aligned} \quad /5.3/$$

Their solution is the following:

$$\begin{aligned} a &= a_0 \sqrt{\cos 2x^0} \\ \phi &= \phi_0 + \sqrt{\frac{3}{-2\kappa}} \ln \frac{1+\sin 2x}{1-\sin 2x} \end{aligned} \quad /5.4/$$

The a/t , ϕ/t functions cannot be constructed in analytical form. But, when already $a/a_0 \ll 1$, integration is possible:

$$\begin{aligned} a &\approx \left[a_1^3 - 3a_0^2(t-t_1) \right]^{1/3} \\ \phi &\approx \phi_0 + \sqrt{\frac{3}{-2\kappa}} \ln \frac{a_0^2 + \sqrt{a_0^4 - a^4}}{a_0^2 - \sqrt{a_0^4 - a^4}} \quad a, \equiv a(t_1) < a_0 \end{aligned} \quad /5.5/$$

This result shows that there is collapse with singularity at a finite t .

6. THE FLUID PROBLEM

The fluid energy-momentum tensor has a more complicated structure than that of the meson or electromagnetic field. The general form for the Newtonian viscous fluid is the following [4], [5], [6]:

$$\begin{aligned} T_{\mu\nu} &= \rho U_{\mu} U_{\nu} + (p - \eta' U^{\rho}; \rho) h_{\mu\nu} - \eta (U_{\mu};_{\nu} + U_{\nu};_{\mu} + U_{\mu} U^{\rho};_{\nu} + U_{\nu} U^{\rho};_{\mu}) + q_{\mu} U_{\nu} + q_{\nu} U_{\mu}; \\ h_{\mu\nu} &\equiv g_{\mu\nu} + U_{\mu} U_{\nu}; \quad U_{\rho} U^{\rho} = -1 \\ &\quad q_{\rho} U^{\rho} = 0. \end{aligned} \quad /6.1/$$

/The epithet "Newtonian" does not mean the character of gravitational field but the linearity of the energy-momentum tensor in the derivatives of velocity. U^{μ} is a vector field, representing the average velocity of matter; $h_{\mu\nu}$ is the projection tensor of the observer, comoving with matter /it projects into the observer's three-dimensional space/; ρ is the density of the rest energy; p is the pressure; q_{μ} is the energy flux /e.g. thermal/ observed by the

comoving observer; η and η' are the usual coefficients of viscosity./

Of course, there are further equations, namely the equations of thermal conduction and entropy production, and the equations of state. However, first it is convenient to deal with the restrictions for the energy-momentum tensor which are the consequences of the symmetries. All characteristic quantities must have vanishing Lie derivatives along the Killing vector fields.

Thus we obtain:

$$\begin{aligned} \rho &= \rho(x^0), \quad p = p(x^0), \quad \eta = \eta(x^0), \quad \eta' = \eta'(x^0); \\ u^I &= 0, \quad u^0 = u^0(x^0) = \frac{1}{a}; \\ q^I &= 0, \quad q^0 = q^0(x^0) \end{aligned} \quad /6.2/$$

But, because of the orthogonality of u^μ and q^μ , $q^\mu = 0$: this means, that the temperature T is also the function of the x^0 coordinate only [4], [7]. Thus the equation of thermal conduction is unnecessary.

If matter contains conserved particles, the equation of particle number conservation is valid:

$$(nu^0)_{;0} = 0 \quad /6.3/$$

where n is the particle number density. This equation can be integrated:

$$n = \frac{N}{2\pi^2} \frac{1}{a^3}; \quad /6.4/$$

N being the total number of particles.

If matter consists of one component, the specific entropy s is a function of n and T only. The equation of entropy production can be obtained from the contracted Bianchi identity by contraction with u^μ :

$$T^{\rho\sigma};_{\sigma} u_\rho = 0. \quad /6.5/$$

/The procedure is the same as that described in Ref. 5 but the influence of the bulk viscosity has been calculated too./ The obtained equation is the following:

$$nT\dot{s} = 9\left(\eta' + \frac{2}{3}\eta\right)\frac{\dot{a}^2}{a} \quad /6.6/$$

It is necessary to know even the form of the following equations of state:

$$\rho = \rho(n, T), \quad (\eta' + \frac{2}{3} \eta) = f(n, T), \quad s = s(n, T). \quad /6.7/$$

The equation $p = p(n, T)$ can be obtained from the First Law of Thermodynamics./

First we deal with the perfect fluid. The viscosity coefficients are zeros, thus the specific entropy is constant. It is sufficient to give only one equation of state, e.g. in the form

$$p = p(n, s_0). \quad /6.8/$$

This is a well known case. If $p=0$, the Friedmannian dust solution is obtained.

$$\begin{aligned} a &= \frac{a_0}{2} (1 + \cos x^0) \\ t &= \frac{a_0}{2} (x^0 + \sin x^0) \end{aligned} \quad /6.9/$$

It can be seen that there are alternating expansions and collapses with constant maximal radius and constant finite time of period.

We now come to the viscous fluid. However, the general case is a hopeless matter. Since the interesting effect is the viscosity, it is supposed that

$$p = \lambda = 0. \quad /6.10/$$

In the Einstein equations one combination of the viscosity coefficients appears only: $3\eta' + 2\eta = d > 0$. If we suppose that its dependence from n and T is the following:

$$d(n, T) = \text{const.} \cdot n^{-1/3} = D/a \quad /6.11/$$

then the Einstein equations can be integrated in analytical form. /Eq. /6.11/ means that the viscosity coefficient is inversely proportional to the average distance between two neighbouring particles. Evidently the real situation is far more complicated, but it is reasonable to assume that the viscosity increases with increasing density./

The Einstein- and entropy-production equations are:

$$\begin{aligned} 1 + 2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - E \frac{\dot{a}}{a} &= 0 \\ -3\left(1 + \frac{\dot{a}^2}{a^2}\right) &= \kappa \rho a^2 \quad E \equiv -\kappa D \\ T\dot{s} &= \frac{6\pi^2}{N} D \frac{\dot{a}^2}{a}. \end{aligned} \quad /6.12/$$

The solutions have six different forms corresponding to the cases $E = 2$; $E \neq 2$; $E < 2$. The functions a/x^0 , ρ/x^0 , t/x^0 can be seen in Table 1. If $E \neq 2$, the "universe" expands to infinity. If $E < 2$, collapses and expansions alternate /but in the final stages of collapses, pressure increases rapidly, thus the condition $p = 0$ becomes wrong for the real Universe/. The maximal radius and the time of cycle increase monotonously. This behaviour agrees with Tolman's result, deduced from the entropy increase only [13]. Going back in time it can be seen that the "Universe" has existed for finite time.

The calculation of temperature is a difficult problem. For entropy we are not able to use the ideal gas formula because the viscous interaction shows that the gas is not ideal. We must also take the radiation entropy into account. Nevertheless, if we use the ideal gas or radiation approach for entropy, it can be seen that temperature remains finite, except for the singular states, and its average value increases with time. /A further discussion can be found in the Appendix./

The case $[\eta' + 2/3\eta] = \text{const}$, $p=0$ has been investigated by Heller, Klimek and Suszycki [9] for the Robertson-Walker metric with $k=0$ and arbitrary λ . Of course, these solutions show different characters from ours because of the condition $k = 0$ rather than the different density-dependence of the viscosity coefficient. Their radiation-filled solutions are also different from our solutions because of the presence of the radiation pressure $p = \rho/3$.

These models are not suitable to accurately describe the details of the real Universe because of the arbitrary choice of viscous coefficient and the condition $p = 0$. But, if it is necessary, the Einstein equations and the entropy equation can be integrated numerically.

7. THE CASE OF ELASTICITY

Rayner constructed the field equations for Hookean elastic medium; the energy-momentum tensor being the following [10] :

$$T_{\mu\nu} = \rho u_\mu u_\nu - \frac{1}{2} C_{\mu\nu}^{\rho\sigma} (h_{\rho\sigma} - h_{\rho\sigma}^0)$$

$$h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu. \quad /7.1/$$

Table 1

$$E < 2$$

$$a = \frac{a_0}{2} (1 + \cos Qx^0) e^{\frac{E}{2} x^0} \quad Q \equiv \sqrt{1 - \frac{E^2}{4}}$$

$$\rho = -\frac{12}{\kappa a_0^2} e^{-Ex^0} \left\{ \left(1 + \frac{E^2}{4}\right) (1 + \cos Qx^0)^{-2} - EQ \sin Qx^0 (1 + \cos Qx^0)^{-3} + \right. \\ \left. + Q^2 \sin^2 Qx^0 (1 + \cos Qx^0)^{-4} \right\}$$

$$t = \frac{a_0}{2} \left\{ e^{\frac{E}{2} x^0} \left(\frac{2}{E} + \frac{E}{2} \cos Qx^0 + Q \sin Qx^0 \right) - \frac{2}{E} - \frac{E}{2} \right\}.$$

$$E = 2$$

$$a = K^2 x^{0^2} e^{x^0}$$

$$\rho = -\frac{6}{\kappa K^4} \frac{x^{0^2} + 2x^0 + 2}{x^{0^6}} e^{-2x^0}$$

$$t = K^2 \{ e^{x^0} (x^{0^2} - 2x^0 + 2) - 2 \}.$$

$$E > 2$$

$$a = K^2 (\operatorname{ch} Qx^0 - 1) e^{\frac{E}{2} x^0} \quad Q \equiv \sqrt{\frac{E^2}{4} - 1}$$

$$\rho = \frac{3}{\kappa K^4} e^{-Ex^0} \left\{ \left(1 + \frac{E^2}{4}\right) (\operatorname{ch} Qx^0 - 1)^{-2} + EQ \operatorname{sh} Qx^0 (\operatorname{ch} Qx^0 - 1)^{-3} + \right. \\ \left. + Q^2 \operatorname{sh}^2 Qx^0 (\operatorname{ch} Qx^0 - 1)^{-4} \right\}$$

$$t = K^2 \left\{ e^{\frac{E}{2} x^0} \left(\frac{E}{2} \operatorname{ch} Qx^0 - Q \operatorname{sh} Qx^0 - \frac{2}{E} \right) - \left(\frac{E}{2} - \frac{2}{E} \right) \right\}.$$

and

$$a = K^2 (\operatorname{ch} Qx^0 + 1) e^{\frac{E}{2} x^0}$$

$$t = K^2 \left\{ e^{\frac{E}{2} x^0} \left(\frac{2}{E} + \frac{E}{2} \operatorname{ch} Qx^0 - Q \operatorname{sh} Qx^0 \right) - \left(\frac{2}{E} + \frac{E}{2} \right) \right\}$$

$$\rho = -\frac{3}{\kappa K^4 (1 + \operatorname{ch} Qx^0)^4} \left[2 + \frac{E^2}{4} + 2 \left(1 + \frac{E^2}{4}\right) \operatorname{ch} Qx^0 + EQ \operatorname{sh} Qx^0 + \right. \\ \left. + \frac{E^2}{4} \operatorname{ch} 2Qx^0 + \frac{EQ}{2} \operatorname{sh} 2Qx^0 \right] e^{-Ex^0}$$

$$E \geq 2$$

$$a = a_0 e^{\left(\frac{E}{2} \pm Q\right) x^0}$$

$$t = a_0 \left(\frac{E}{2} \pm Q\right)^{-1} \left[e^{\left(\frac{E}{2} \pm Q\right) x^0} - 1 \right]$$

$$\rho = -\frac{3}{\kappa a_0^2} E \left(\frac{E}{2} \pm Q\right) e^{-(E \pm 2Q)x^0}$$

Introducing the quantity:

$$C_{\mu\nu\kappa\lambda} \equiv h_{\kappa\rho}^0 h_{\lambda\sigma}^0 C_{\mu\nu}^{\rho\sigma} \quad /7.2/$$

the quantities $C_{\mu\nu\kappa\lambda}$ and $h_{\mu\nu}^0$ fulfil four conditions:

- a./ Their Lie derivatives vanish along the vector field u^μ ,
- b./ They are orthogonal to u^μ ,
- c./ They are symmetric in all indices,
- d./ $C_{AB}^{\mu\nu} = C_{\mu\nu}^{AB}$, $\kappa\lambda = B$ is a matrix of rank 6 and $h_{\mu\nu}^0$ is a positive semi-definite matrix of rank 3.

If the elastic medium is isotropic /has no macroscopic crystallic structure/, the tensor $C_{\mu\nu\kappa\lambda}$ has the simple form:

$$C_{\mu\nu\kappa\lambda} = \nu h_{\mu\nu}^0 h_{\kappa\lambda}^0 + \mu (h_{\mu\kappa}^0 h_{\nu\lambda}^0 + h_{\mu\lambda}^0 h_{\nu\kappa}^0) \quad /7.3/$$

where, because of condition a./

$$\nu_{,\rho} u^\rho = \mu_{,\rho} u^\rho = 0. \quad /7.4/$$

Dealing with isotropic matter and stipulating the symmetry conditions for the quantities $\rho, \mu, \nu, u^\mu, h_{\mu\nu}^0$ the following results are obtained:

$$\rho = \rho(x^0); \quad \mu, \nu \text{ are constants};$$

$$u^I = 0, \quad u^0 = \frac{1}{a};$$

$$h_{\mu\nu}^0 = A^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin^2 x^1 & 0 & 0 \\ 0 & 0 & \sin^2 x^1 \sin^2 x^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A \text{ is constant.} \quad /7.5/$$

The non-trivial components of the Einstein equations have the following forms:

$$(1 + A^2 Q^2) + (\lambda - Q^2) a^2 + 2 \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = 0 \quad /7.6.a/$$

$$-3(1 + \frac{\dot{a}^2}{a^2}) - \lambda a^2 = \kappa \rho a^2 \quad /7.6.b/$$

$$Q^2 \equiv -\kappa \frac{3\nu + 2\mu}{2} \quad /7.6.c/$$

The eq. /7.6.a/ cannot be solved in analytical form. Its solution is the following:

$$x^0 = \int_a^{\infty} (\sqrt{Ca - (1+A^2Q^2)a^2 - \frac{1}{3}(\lambda-Q^2)a^4})^{-1} da \quad /7.7.a/$$

or

$$t = \int_a^{\infty} a (\sqrt{Ca - (1+A^2Q^2)a^2 - \frac{1}{3}(\lambda-Q^2)a^4})^{-1} da \quad /7.7.b/$$

and the right-hand sides cannot be expressed using elementary functions.

There are two simpler special cases. If $\lambda < Q^2$, there is a static solution:

$$a = \sqrt{\frac{1+A^2Q^2}{Q^2-\lambda}}$$

$$\rho = -\frac{1}{\kappa} \left[\lambda + 3 \frac{Q^2 - \lambda}{1+A^2Q^2} \right] \quad /7.8/$$

This solution is unstable against small perturbations in "a".

If $\lambda = Q^2$, the solution can be obtained in analytical form:

$$a = \frac{a_0}{2} (1 + \cos Bx^0)$$

$$t = \frac{a_0}{2} (x^0 + \frac{1}{B} \sin Bx^0) \quad B \equiv \sqrt{1+A^2Q^2} \quad /7.9/$$

There is collapse, similarly to the Friedmannian dust solution, but the time-scale is divided by $B = \sqrt{1+A^2Q^2}$.

We do not investigate the general formulae /7.7/ in detail because the "ideal elastic Universe" cannot describe the properties of the real Universe. One rough approach for the motion $a = a(t)$ can be seen in Appendix B.

We remark that Rayner's first condition for $C_{\mu\nu\kappa\lambda}$ seems to us an excessively strong one. It can be seen that in the SO(4) symmetric cases, if $C_{\mu\nu\kappa\lambda}$ has vanishing Lie derivatives along u^μ , the elastic coefficients must be constants. However, in the classical theory of elasticity, these coefficients may be chosen as /almost/ optional functions of ρ , T , etc. / ρ is time-dependent in our case/. Moreover, even the relativistic causality condition says the following only [10]:

$$\mu/\rho \leq 1, \quad (2\mu+\nu)/\rho \leq 1. \quad /7.10/$$

It seems that one could drop the condition that the Lie derivative of $C_{\mu\nu\alpha\kappa}$ vanishes along u^μ .

8. ONE VISCO-ELASTIC CASE

In classical continuum mechanics one class of visco-elastic systems /the Kelvin-Voigt system/ can be written down by addition of the viscous and elastic stress tensors [11]. The relativistic counterpart of this case /if we neglect the influence of the thermal flux q^μ on matter/ has the following energy-momentum tensor [12]:

$$T_{\mu\nu} = \rho U_\mu U_\nu - \frac{1}{2} C_{\mu\nu}^{\rho\sigma} (h_{\rho\sigma} - h_{\rho\sigma}^0) - \eta' U^\rho{}_{;\rho} h_{\mu\nu} - \eta (U_\mu{}_{;\nu} + U_\nu{}_{;\mu} + U_\mu U^\rho{}_{;\rho} U_\nu + U_\nu U^\rho{}_{;\rho} U_\mu) + q_\mu U_\nu + q_\nu U_\mu \quad /8.1/$$

If we assume that the viscous coefficients are the same functions as in Sect. 6, for the SO(4) symmetric case we obtain the following equations:

$$(1+A^2 Q^2) + 2 \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + (\lambda - Q^2) a^2 - E \frac{\dot{a}}{a} = 0. \quad /8.2.a/$$

$$-3(1+\frac{\dot{a}^2}{a^2}) - \lambda a^2 = \kappa \rho a^2. \quad /8.2.b/$$

$$nT\dot{s} = -3\frac{E}{\kappa} \frac{\dot{a}^2}{a} \quad /8.2.c/$$

/Here we assumed that the elastic compression is adiabatic./ The third equation can be integrated separately again. Eq. /8.2.a/ can be integrated in analytical form if $\lambda = Q^2$; the solutions then are obtained by performing substitutions

$$\begin{aligned} x^0 &\equiv Bx_{\text{new}}^0 \\ E &\equiv B^{-1} E_{\text{new}} \\ t &\equiv Bt_{\text{new}} \\ B &\equiv \sqrt{1+A^2 Q^2}. \end{aligned} \quad /8.3/$$

as the solutions of Table 1, and eq. /8.2.b/ gives ρ writing $a(x^0)$ into it. These solutions do not differ substantially from the solutions of Table 1. The more general cases $\lambda \neq Q^2$ can be solved only numerically. Since the visco-elastic model does not describe the behaviour of Universe, we do not intend to investigate this case further.

The elastic and visco-elastic solutions might have some importance only if we regard them as internal solutions, in which case we ought to fit them with external spherical solutions at certain boundaries. Further investigations are necessary with regard to this problem.

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APPENDIX A

THE VISCOSITY COEFFICIENTS

The run of function $\eta = \eta/n, T/$ is different in each case for fluids and gases. Fluid viscosity decreases with increasing temperature, its density-dependence can only be measured with great difficulty because of the practical incompressibility of fluids.

The viscosity coefficient of gases can be calculated theoretically when the mean free path is much greater than the molecular diameter. In this case

$$\eta = \frac{2}{3\pi^{3/2}} \frac{\sqrt{mk}}{d_0^2} \frac{T^{1/2}}{1+CT^{-1}} \quad /A.1/$$

Here d_0 is the molecular diameter and C is Sutherland's correction parameter [16]. η does not depend explicitly on particle density [15]. But now the gas is approximately an ideal one because the free path is great by comparison with the molecular size. Thus the specific entropy of monatomic gas is approximately equal to the following:

$$s = s_0 + \frac{3}{2}k \ln\left(\frac{T^{3/2}}{n}\right) \quad /A.2/$$

where s_0 depends neither on density nor on temperature. For adiabatic processes /e.g. for the whole Universe, if the viscous interaction is weak/

$$T_{ad} = \frac{K}{a^2} \quad /A.3/$$

Writing this formula into eq. /A.1/ and dealing with a sufficiently high temperature $/T \gg C$; e.g. $C = 123,6$ °K for air/ we get:

$$\eta = \frac{\text{const}}{a} \quad /A.4/$$

If the radiative entropy dominates the gas entropy, $T \sim 1/a$ [5] therefore

$$\eta = \frac{\text{const}}{\sqrt{a}} \quad /A.5/$$

Thus the supposition $\eta \sim 1/a$ is an approach which is valid for monatomic gas, at high temperature and moderate density, if the gas entropy dominates the radiative entropy, and it is valid only for such time interval during which the entropy increase is relatively small /e.g. for a few cycles in the case $E \ll 1$ /. Of course, the exact function $\eta = \eta/n, T/$ for all n and T is unknown even approximately.

APPENDIX B

AN APPROACH FOR THE $a/t/$ FUNCTIONS OF ELASTIC UNIVERSES

The integral /7.7a/ cannot be expressed by elementary functions but we can carry out a rough approach, preserving the mean features of function $a/t/$. Let us write the integral in the following form:

$$t = \int \frac{\sqrt{a} da}{\sqrt{k f(a)}} \quad a \geq 0$$

$$f \equiv a^3 + \alpha a + \beta; \quad \alpha \equiv -\frac{B^2}{K}; \quad \beta \equiv \frac{C}{K}$$

$$K \equiv \frac{1}{3}(Q^2 - \lambda); \quad B^2 = 1 + A^2 Q^2.$$

The equation $f/a=0$ has zero, one, two or three non-negative real solutions a_1 according to the values of α and β . The condition that Kf must be non-negative, means that generally "a" moves between two extremal values only /of course these may be zero and infinite too/. Thus function $f/a/$ can be reduced to components, some of which cannot vanish at the possible values of "a".

It is easy to see that the eq. $f/a=0$ cannot have two positive solutions between which Kf is positive. Thus there is no oscillation between two finite radii. The remaining cases are:

- a./ $f/a=P_2/a/ \cdot /a-a_1/$, where $P_2/a/$ is a quadratic function of a , which does not change sign at the possible values of a .
- b./ $f/a=P_3/a/$, $P_3/a/$ behaves similarly.

The approach will be the following: If the maximal possible value of a is finite, we substitute a constant "average" value for $P/a/$ in the integral. If the "maximal" possible value is infinite, we keep the term a^2 in P_2 and a^3 in P_3 . Thus we can integrate and obtain some different cases. /Before listing cases it is necessary to note that if an $a/t/$ function is a solution of eq. /7.6.a/, then $a^*/t/ = a/-t/$ is a solution too, because the equation contains second time-derivative and square of first derivative only. Thus the motion is reversible./

Case I. $K \neq 0$. Motion exists for $C > 0$ only. There is one and only one a_0 , at which

$$f(a_0) = 0; \quad a \geq 0$$

/B.2/

Thus

$$t \approx \frac{2}{\sqrt{-K}} (\arcsin \vartheta - \vartheta \sqrt{1-\vartheta^2})$$

$$\vartheta \equiv \sqrt{\frac{a}{a_0}}.$$

/B.3/

Case II. $K = 0$. This case can be integrated in analytical form, and this solution was written down in Sect. 7.

Case III. $K > 0$. We introduce the following new quantities:

$$L \equiv \sqrt{K}; \quad \xi_+ \equiv \frac{B}{\sqrt{3}L} \quad /B.4/$$

Case III.a. When $C < \frac{2}{3\sqrt{3}} \frac{B^3}{L}$, it is possible motion between a minimal $\xi > \xi_+$ and infinity:

$$a \approx \xi \operatorname{ch}^2 \frac{L}{2} t. \quad /B.5/$$

Case III.b. If $0 < C < \frac{2}{3\sqrt{3}} \frac{B^3}{L}$, there is another possible motion too, motion between 0 and $\xi < \xi_+$:

$$t \approx \frac{2}{L} (\arcsin \vartheta - \vartheta \sqrt{1-\vartheta^2}).$$

$$\vartheta \equiv \sqrt{\frac{a}{\xi}}$$

/B.6/

Case III.c. If $C = \frac{2B^3}{3\sqrt{3}L}$, there are both of these motions. If at an arbitrary moment $a > \xi_+$, the motion happens according to Case III.a., if $a < \xi_+$, it happens according to Case III.b. In both cases $\xi = \xi_+$. If $a = \xi_+$, we get the static solution, mentioned in Sect. 7.

Case III.d. If $C > \frac{2}{3\sqrt{3}} \frac{B^3}{L}$, radius monotonously grows with time:

$$a \sim e^{Lt}$$

/B.7/

It can be seen that there are four different types of motion: Expansion from singular state followed by collapse back; expansion from singular state to infinity /or inversely/; contraction from infinity and expansion to infinity; static /unstable/ state.

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