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PURELY ALGEBRAICAL CONSTRUCTION OF FIRST ORDER LOGICS

Hungarian Academy of Sciences

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ABSTRACT

The aim of this paper is to introduce classical first order logics as interrelated purely algebraic constructions. To this end a special class of universal algebras, namely the class of locally independently finite cylindric algebras, is defined. Some of its basic properties are investigated.

The constructions of first order logics are based on this class, and the investigations of the logical properties are purely algebraic arguments based on the properties of this class of algebras.

The paper contains the material of the lectures of the authors delivered on the "Logical semester - 1973" organized by the International S. Banach Center of Mathematics in Warsaw.

РЕЗЮМЕ

Целью данной работы является чисто алгебраическое построение логики первого порядка. Для этого определяется новый класс алгебр в рамках теории универсальных алгебр, а именно, локально-независимо конечный класс цилиндрических алгебр. Исследованы его основные характеристики. На базе этого локально-независимого конечного класса алгебр построена логика первого порядка. Ее основные логические свойства исследованы чисто алгебраическими методами, основанными на характеристиках вышеуказанного класса алгебр.

Данная работа содержит материал лекций, прочитанных авторами на "Семинаре по логике - 1973", организованном Международным Математическим Центром им. Ст. Банаха в Варшаве.

KIVONAT

A tanulmány célja az elsőrendü predikátumkalkulus tisztán algebrai felépitése. Ehhez az univerzális algebrák egy speciális algebraosztályát definiáljuk, nevezetesen a lokálisan függetlenül véges cilindrikus algebrák osztályát. Megadjuk ennek az algebraosztálynak néhány fontos tulajdonságát. Megkonstruáljuk az elsőrendü predikátumkalkulust ezen algebraosztály segitségével.

A predikátumkalkulus logikai tulajdonságait ezen algebraosztályra bizonyitott tételek segitségével tisztán algebrai uton vizsgáljuk.

Ez a tanulmány a szerzők az S. Banach Nemzetközi Matematikai Központ által szervezett "Logical semester - 1973"-on tartott előadásainak anyagát tartalmazza.

CONTENTS

LIST OF DEFINITIONS

0	the empty set
1	$\overset{d}{=} \{0\}$
2	$\overset{d}{=} \{0,1\}$
$\vdots$	
ω	$\overset{d}{=} \{0,1,2,\ldots\}$
Do f	domain of the function or relation f
Rg f	range of f
f_x, fx	x-th value of f: $f_x \overset{d}{=} fx \overset{d}{=} f(x)$
$\langle f(x)\rangle_{x\in A}$	way of defining functions: $\langle f(x)\rangle_{x\in A} \overset{d}{=} \{\langle x,f(x)\rangle : x\in A\}$
$\langle s_0, s_1, \ldots, s_n\rangle_{n<\alpha}$	is a function defined on the ordinal α that is: $\langle s_0,\ldots,s_n,\ldots\rangle_{n<\alpha} \overset{d}{=} \{\langle n, s_n\rangle : n<\alpha\}$
$X \upharpoonright f$	f domain-restricted to X: $X\upharpoonright f \overset{d}{=} \{\langle x,f(x)\rangle : x\in X\}$
B_A	power of A to B: ${}^BA \overset{d}{=} \{f : f:B\to A\}$
Sb A	class of subsets of A: $Sb\,A \overset{d}{=} \{Y : Y\subseteq A\}$
$r\circ q$	composition of r and q: $r\circ q \overset{d}{=} \{\langle b,a\rangle : (\exists c)(\langle c,a\rangle \in r \,\&\, \langle b,c\rangle\in q)\}$
$r\mid q$	relative product of r and q: $r\mid q \overset{d}{=} \{\langle a,b\rangle : (\exists c)(\langle a,c\rangle\in r \,\&\, \langle c,b\rangle\in q)\}$
f°	the equivalence-relation induced by the function f: $f^{\circ} \overset{d}{=} f\mid f^{-1}$
r^*	if $A\subseteq$ Do r, then r^*A is the r-image of A: $r^*A \overset{d}{=} \{y : (\exists x\in A)\ \langle x,y\rangle\in r\}$
$r^{\star}$	if $a\in$ Do r, then $r^{\star}a$ is the r-image of a: $r^{\star}a \overset{d}{=} \{y : \langle a,y\rangle\in r\} = r^*\{a\}$
pj_x	the x-th projection function: $pj_x(f) \overset{d}{=} f(x)$

$\mathfrak{Sg}^{(\mathfrak{A})}X$	subalgebra of $\mathfrak{A}$ generated by X, that is: $\mathfrak{Sg}^{(\mathfrak{A})}X$ is the least (by $\subseteq$) element of the set $\{\mathfrak{B} \subseteq \mathfrak{A} : X \subseteq B\}$
$\mathfrak{A} \subseteq \mathfrak{B}$	$\mathfrak{A}$ is subalgebra of $\mathfrak{B}$
$\mathfrak{A} \preccurlyeq \mathfrak{B}$	$\mathfrak{A}$ is homomorphic to $\mathfrak{B}$
$\mathfrak{A} \cong \mathfrak{B}$	$\mathfrak{A}$ is isomorphic to $\mathfrak{B}$
$\mathfrak{Ho}\,\mathfrak{A}$	class of homomorphisms on $\mathfrak{A}$
$\mathfrak{Ho}(\mathfrak{A},\mathfrak{B})$	set of homomorphisms from $\mathfrak{A}$ <u>on</u>to $\mathfrak{B}$
$\mathfrak{Hom}(\mathfrak{A},\mathfrak{B})$	set of homomorphisms from $\mathfrak{A}$ <u>in</u>to $\mathfrak{B}$
$\mathfrak{Co}\,\mathfrak{A}$	set of congruence-relations on $\mathfrak{A}$
$f^*\mathfrak{A}$	is defined only if f is a homomorphism on $\mathfrak{A}$, and then there is a unique $\mathfrak{B}$ such that $f \in \mathfrak{Ho}(\mathfrak{A},\mathfrak{B})$, now: $f^*\mathfrak{A} \overset{d}{=} \mathfrak{B}$
$\mathbb{P}_{i \in I} \mathfrak{A}_i$	direct product of the algebras $\mathfrak{A}_i$ according to the indexing I
$\mathbb{I}\,K$	class of algebras isomorphic to the elements of K: $\mathbb{I}K \overset{d}{=} \{\mathfrak{B} : \mathfrak{B} \cong \vert\in K\}$
$\mathbb{H}\,K$	class of algebras homomorphic to the elements of K: $\mathbb{H}K \overset{d}{=} \{\mathfrak{B} : \mathfrak{B} \preccurlyeq \vert\in K\}$
$\mathbb{S}\,K$	class of subalgebras of the elements of K: $\mathbb{S}K \overset{d}{=} \{\mathfrak{B} : \mathfrak{B} \subseteq \vert\in K\}$
$\mathfrak{Cr}_I^{(S)}K$	free congruence over K with I generators and with defining relation S: $\mathfrak{Cr}_I^{(S)}K \overset{d}{=} \bigcap\{R \in \mathfrak{Co}\,\mathfrak{Fr}_I : S \subseteq R,\ \mathfrak{Fr}_I/_R \in \mathbb{I}\mathbb{S}\,K\}$
$\mathfrak{Fr}_I^{(S)}K$	free algebra over K with I generators and with defining relation S: $\mathfrak{Fr}_I^{(S)}K \overset{d}{=} \mathfrak{Fr}_I/\mathfrak{Cr}_I^{(S)}K$

$\mathfrak{G}_I^{(t)}K \overset{d}{=} \mathfrak{G}_I^{(T)}K$
$\mathfrak{F}_I^{(t)}K \overset{d}{=} \mathfrak{F}_I^{(T)}K$ } where T is the defining relation: $\{\langle \varrho, c_i\varrho\rangle : \varrho \in I,\ i \notin t(\varrho)\}$

$s_j^{i(\mathfrak{A})}$ substitution operation in $\mathfrak{A}$, j for i:

$s_j^{i(\mathfrak{A})} x \overset{d}{=} c_i^{(\mathfrak{A})}(d_{ij}^{(\mathfrak{A})} \cdot^{(\mathfrak{A})} x)$

$s_\tau^{(\mathfrak{A})}$ is defined if $\mathfrak{A} \in Lf$ and τ is a finite transformation of ω , and then $s_\tau^{(\mathfrak{A})}$ is the unary operation defined as follows:

if $\tau = [\mu_0/\nu_0, \ldots, \mu_{k-1}/\nu_{k-1}]$ is the canonical representation of τ ($\mu, \nu \in {}^k\omega$, $\mu_0 < \ldots < \mu_{k-1}$) , if x is any element of A, and if $\pi_0, \ldots, \pi_{k-1}$ are in this order the first k ordinals in $\omega \setminus (\Delta^{(\mathfrak{A})}x \cup Rg\,\mu \cup Rg\,\nu)$, then

$$s_\tau^{(\mathfrak{A})} x \overset{d}{=} s_{\nu_0}^{\pi_0(\mathfrak{A})} \ldots s_{\nu_{k-1}}^{\pi_{k-1}(\mathfrak{A})} s_{\pi_0}^{\mu_0(\mathfrak{A})} \ldots s_{\pi_{k-1}}^{\mu_{k-1}(\mathfrak{A})} x$$

$[i_1/j_1, \ldots, i_n/j_n]_A$ a way of defining finite transformations:

$$[i_1/j_1, \ldots, i_n/j_n]_A \overset{d}{=} \{\langle i_1, j_1\rangle, \ldots, \langle i_n, j_n\rangle\} \cup \{\langle a, a\rangle : a \in A \setminus \{i_1, \ldots, i_n\}\}$$

$\mathbb{P}K$ class of direct products from K:

$$\mathbb{P}K \overset{d}{=} \mathbb{I}\{\mathop{\mathbb{P}}_{i\in I} \mathfrak{A}_i : I \text{ is arbitrary and } (\forall i \in I)\,\mathfrak{A}_i \in K\}$$

I. ALGEBRAIC NOTIONS

A function with range consisting of positive integers is called a type, that is t is a type if Rg t $\subseteq (\omega \setminus 1)$.

A structure of type $t \in {}^{I}\omega$ is a function $\mathfrak{A}$ having the following properties:

$$\mathrm{Do}\,\mathfrak{A} = I \cup \{I\},$$
$$(\forall \varrho \in I)\ \mathfrak{A}_\varrho \subseteq {}^{t(\varrho)}\mathfrak{A}_I, \quad \text{and}$$
$$\mathfrak{A}_I \neq 0$$

The last property serves only purposes of convenience and has nothing to do with the essence of the concept structure.

For example:
The triple $\langle +, \leq, \omega \rangle$ is a structure of type $\langle 3,2 \rangle$, where + and $\leq$ are the usual function and relation on ω. This follow-s from the fact that the series have been defined as ordinal functions:
$\langle +, \leq, \omega \rangle = \{\langle 0,+\rangle, \langle 1,\leq\rangle, \langle 2,\omega\rangle\}$. Since $2 = \{0,1\}$, in this case $I = \{0,1\}$ and $t = \{\langle 0,3\rangle, \langle 1,2\rangle\} = \langle 3,2\rangle$.

German letters stand for structures and the corresponding capital latin letter stands for the universe of the structure, that is $\mathfrak{A}_I \overset{d}{=} A$.

A structure is called <u>algebra</u>, if all its relations are functions defined on its universe.

Now we fix a type which shall be used thoroughout the paper:

$$\ell \overset{d}{=} \{ \langle \wedge, 3\rangle, \langle \neg, 2\rangle, \langle \exists_i, 2\rangle, \langle =_{ij}, 1\rangle \; : \; i,j\in\omega \} .$$

We shall discuss only algebras of type ℓ.

Any ℓ-type algebra $\mathfrak{A}$ can be defined in the following manner:

$$\mathfrak{A} \overset{d}{=} \langle \mathfrak{A}_\wedge , \mathfrak{A}_\neg , \mathfrak{A}_{\exists_i} , \mathfrak{A}_{=_{ij}} \rangle_{i,j\in\omega}$$

We shall use the next symbols for operators of ℓ-type algebras:

$$\mathfrak{A}_\wedge \overset{d}{=} \cdot^{(\mathfrak{A})}$$

$$\mathfrak{A}_\neg \overset{d}{=} -^{(\mathfrak{A})}$$

$$\mathfrak{A}_{\exists_i} \overset{d}{=} c_i^{(\mathfrak{A})}$$

$$\mathfrak{A}_{=_{ij}} \overset{d}{=} d_{ij}^{(\mathfrak{A})}$$

We usually omit the index $(\mathfrak{A})$.

Let us introduce the <u>dimension-sensitivity</u> function $\Delta^{(\mathfrak{A})}$:

$$\Delta^{(\mathfrak{A})} x \overset{d}{=} \{ i : c_i^{(\mathfrak{A})} x \neq x \}$$

One of our basic tool will be the well-known universal algebraic concept <u>"word-algebra"</u> or "absolutely free algebra".

The definition of the word-algebra is:

We define $\mathfrak{M}^t$ as the t-type algebra for which

a/ the universe W is the set of all n-tuples of the elements of $X \cup Do\,t$, that is:

$$W \stackrel{d}{=} (X \cup Do\,t) \cup (X \cup Do\,t) \times (X \cup Do\,t) \cup (X \cup Do\,t) \times ((X \cup Do\,t) \times (X \cup Do\,t)) \cup \ldots$$

b/ for all $\varrho \in Do\,t$

$$\mathfrak{M}^t_\varrho (x_1, \ldots, x_{t(\varrho)-1}) \stackrel{d}{=} \langle \varrho, \langle x_1, \ldots, x_{t(\varrho)-1} \rangle \rangle$$

in the case $t(\varrho)-1=0$

$$\mathfrak{M}^t_\varrho \stackrel{d}{=} \varrho \in W \quad .$$

$$\mathfrak{F}_{X,t} \stackrel{d}{=} \mathfrak{G}_g^{(\mathfrak{M}^t)} X$$

$\mathfrak{F}_{X,t}$ is the absolutely free algebra or word algebra of type t generated by X.

Since we devote ourselves to algebras of type ℓ, we set

$$\mathfrak{F}_X \stackrel{d}{=} \mathfrak{F}_{X,\ell} \quad ,$$

and we call $\mathfrak{F}_X$ the word algebra generated by X.

II. SOME IMPORTANT CLASSES OF ℓ-TYPE ALGEBRAS

1/ The veriety of <u>cylindric algebras (CA)</u>, ([1], 1.1.1)

Let us introduce the following shorthands:

$$x+y \stackrel{d}{=} -(-x\cdot -y)$$
$$0 \stackrel{d}{=} x\cdot -x$$
$$1 \stackrel{d}{=} -0$$

Now we can define CA the class of cylindric algebras:

For any ℓ-type algebra $\mathfrak{A}$,

$\mathfrak{A}\in$ CA if for all $x,y,z\in A$ and $i,j,n\in\omega$ the following equations hold:

(C0) $\langle \mathfrak{A}_1, \mathfrak{A}_1, A\rangle$ is a Boolean algebra, that is

a./ $x\cdot y = y\cdot x$

b./ $x\cdot(y+z) = x\cdot y + x\cdot z$

c./ $x\cdot 1 = x$

(C1) $c_i 0 = 0$

(C2) $c_i x\cdot x = x$

(C3) $c_i(x\cdot c_i y) = c_i x\cdot c_i y$

(C4) $c_i c_j x = c_j c_i x$

(C5) $d_{ii} = 1$

(C6) $i\neq j,n \Rightarrow d_{jn} = c_i(d_{ji}\cdot d_{in})$

(C7) $i\neq j \Rightarrow c_i(d_{ij}\cdot x)\cdot c_i(d_{ij}\cdot -x) = 0$

2/ The class of __locally finite cylindric algebras__ (Lf), ([1], 1.11.1)

$$Lf \stackrel{d}{=} \{ \mathfrak{A} \in CA : (\forall x \in A) |\Delta^{(\mathfrak{A})} x| < \omega \}$$

3/ The class of __full cylindric set algebras__ (Th), ([1], 1.1.5)

The full cylindric set algebra induced by the set A is:

$$\mathcal{L}_A \stackrel{d}{=} \langle \cap, \setminus^{(A)}, C_i^{(A)}, D_{ij}^{(A)}, Sb\,{}^{\omega}A \rangle_{i,j \in \omega}$$

where

$$\setminus^{(A)} X \stackrel{d}{=} A \setminus X$$

$$C_i^{(A)} X \stackrel{d}{=} \{ s \in {}^{\omega}A : (\exists x \in X)(\forall j \in \omega \setminus \{i\})\, s_j = x_j \} \quad \text{(See fig. 1.)}$$

$$D_{ij}^{(A)} \stackrel{d}{=} \{ s \in {}^{\omega}A : s_i = s_j \} \quad \text{(See fig. 2.)}$$

We often omit the superscript (A) that is for example we write D_{ij} instead of $D_{ij}^{(A)}$.

$$Th \stackrel{d}{=} \{ \mathcal{L}_A : A \neq 0 \}$$

4/ The class of __cylindric set algebras__ ($\mathcal{H}a$), ([1], 1.1.5)

$$\mathcal{H}a \stackrel{d}{=} \mathsf{S}\, Th$$

II4.1 Lemma: $A = B$ iff $\mathsf{S}\mathcal{L}_A = \mathsf{S}\mathcal{L}_B$

Proof: $\mathfrak{A} \in \mathsf{S}\mathcal{L}_A \cap \mathsf{S}\mathcal{L}_B$ implies that $1^{(\mathfrak{A})} = 1^{(\mathcal{L}_A)} = {}^{\omega}A = 1^{(\mathcal{L}_B)} = {}^{\omega}B$, and this implies that $A = B$ ▲

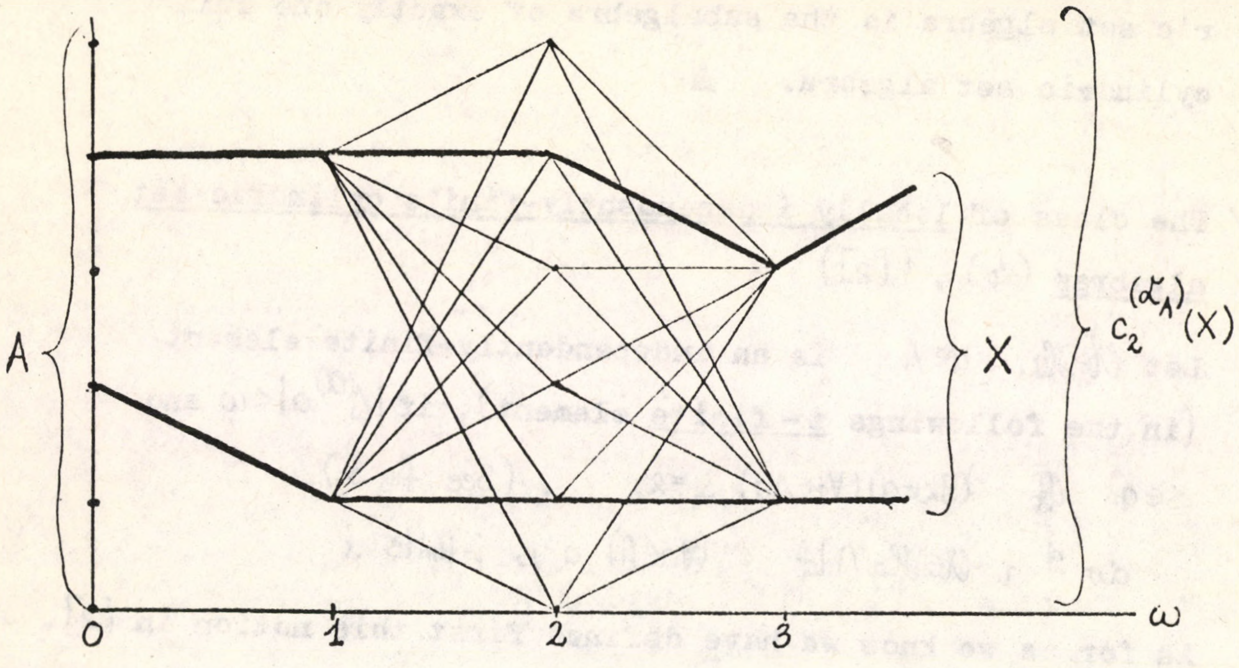


Fig. 1.

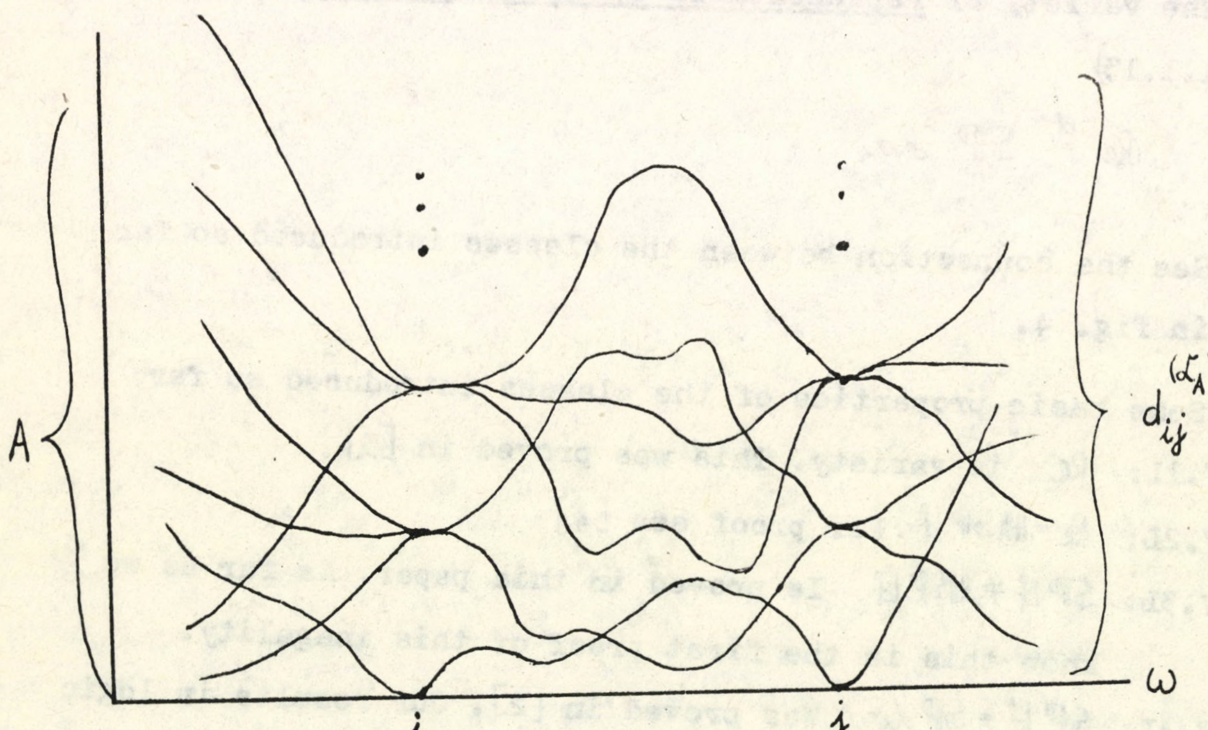


Fig. 2.

II4.1C Corollary: $(\forall \mathfrak{A} \in \mathcal{H}a)(\exists! B)\ \mathfrak{A} \subseteq \mathcal{L}_B$, that is any cylindric set algebra is the subalgebra of exactly one full cylindric set algebra. ▲

5/ The class of <u>locally independently-finite cylindric set algebras</u> $(\mathcal{L}v)$, ([2])

Let $\mathfrak{A} \in \mathcal{H}a$. $a \in A$ is an independently-finite element (in the followings <u>i- finite</u> element), if $|\Delta^{(\mathfrak{A})} a| < \omega$ and $s \in a$ iff $(\exists x \in a)(\forall i \in \Delta a)\ s_i = x_i$. (See fig.3.)

$$\mathcal{L}v \overset{d}{=} \{ \mathfrak{A} \in \mathcal{H}a \cap Lf \ : \ (\forall a \in A)\ a \text{ is i-finite} \}$$

As for as we know we have defined first this notion in [2].

6/ The variety of <u>representable cylindric algebras</u> (Re), ([1], 1.1.13)

$$Re \overset{d}{=} \mathbf{S}\mathbf{P}\ \mathcal{H}a$$

See the connection between the classes introduced so far in fig. 4.

7/ Some basic properties of the classes introduced so far

II7.1L: Re is variety. This was proved in [3].

II7.2L: $Re = \mathbf{H}\mathbf{S}\mathbf{P}\ Lf$. For proof see [4].

II7.3L: $\mathbf{S}\mathbf{P}\ Lf \neq \mathbf{H}\mathbf{S}\mathbf{P}\ Lf$. Is proved in this paper. As far as we know this is the first proof of this inegality.

II7.4L: $\mathbf{S}\mathbf{P}\ Lf = \mathbf{S}\mathbf{P}\ \mathcal{L}v$. Was proved in [2]. Our results in logic are based on this equality.

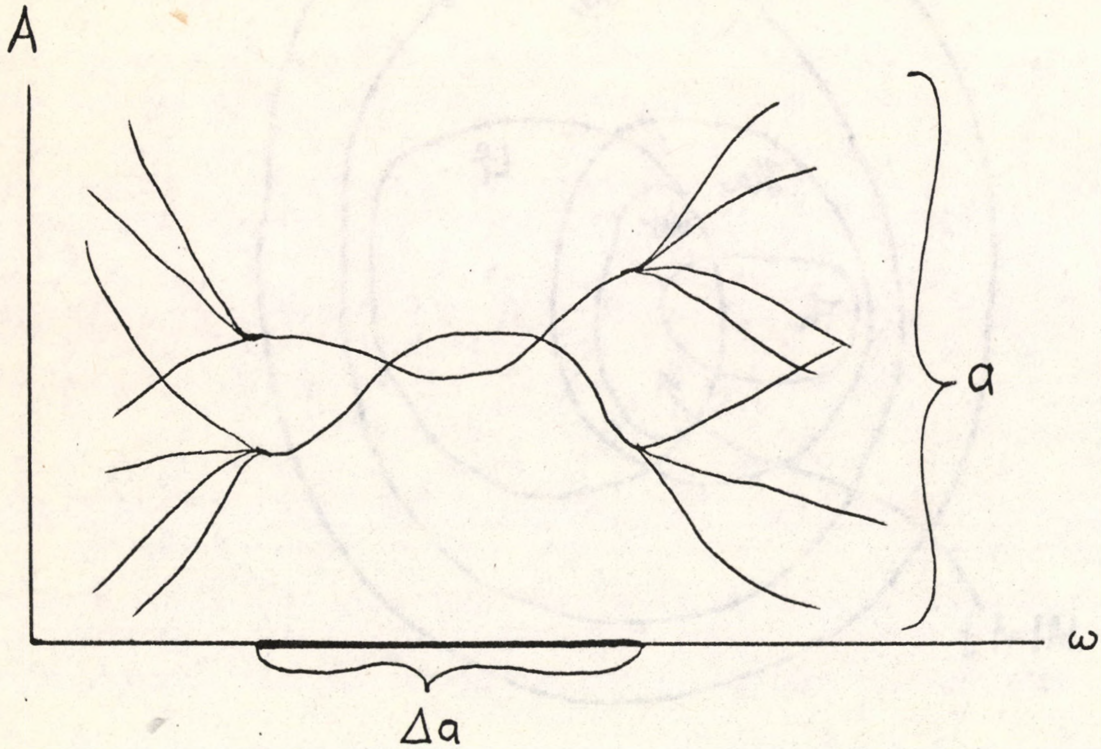


Fig. 3.

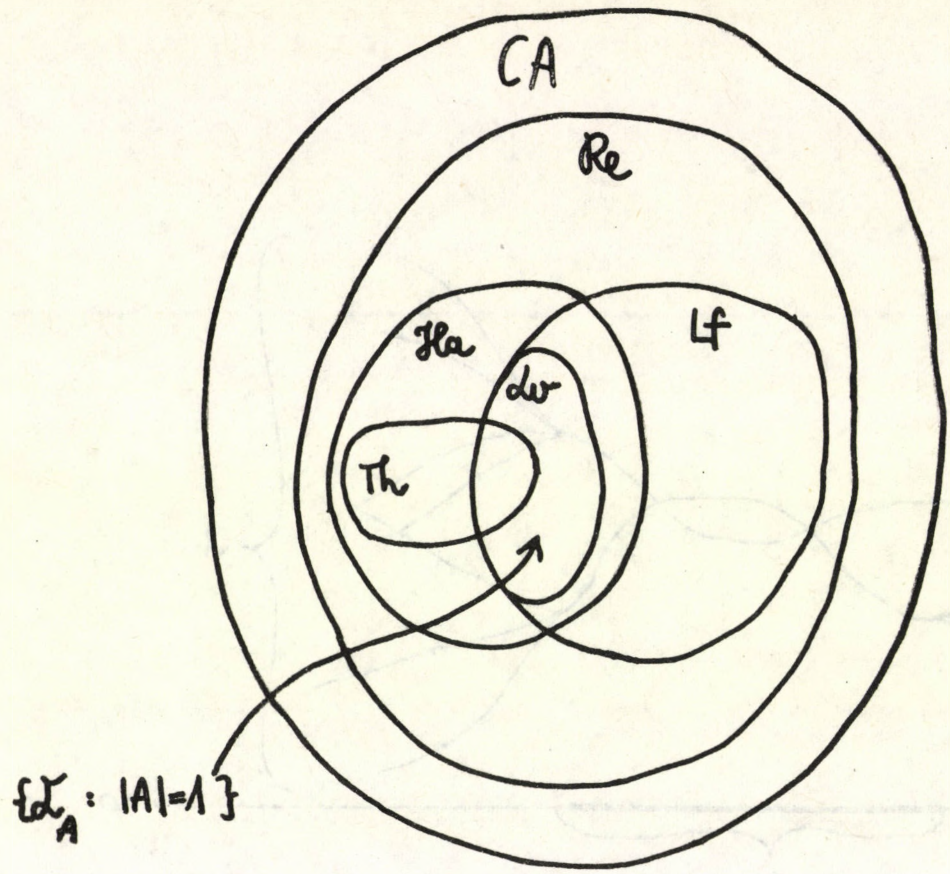
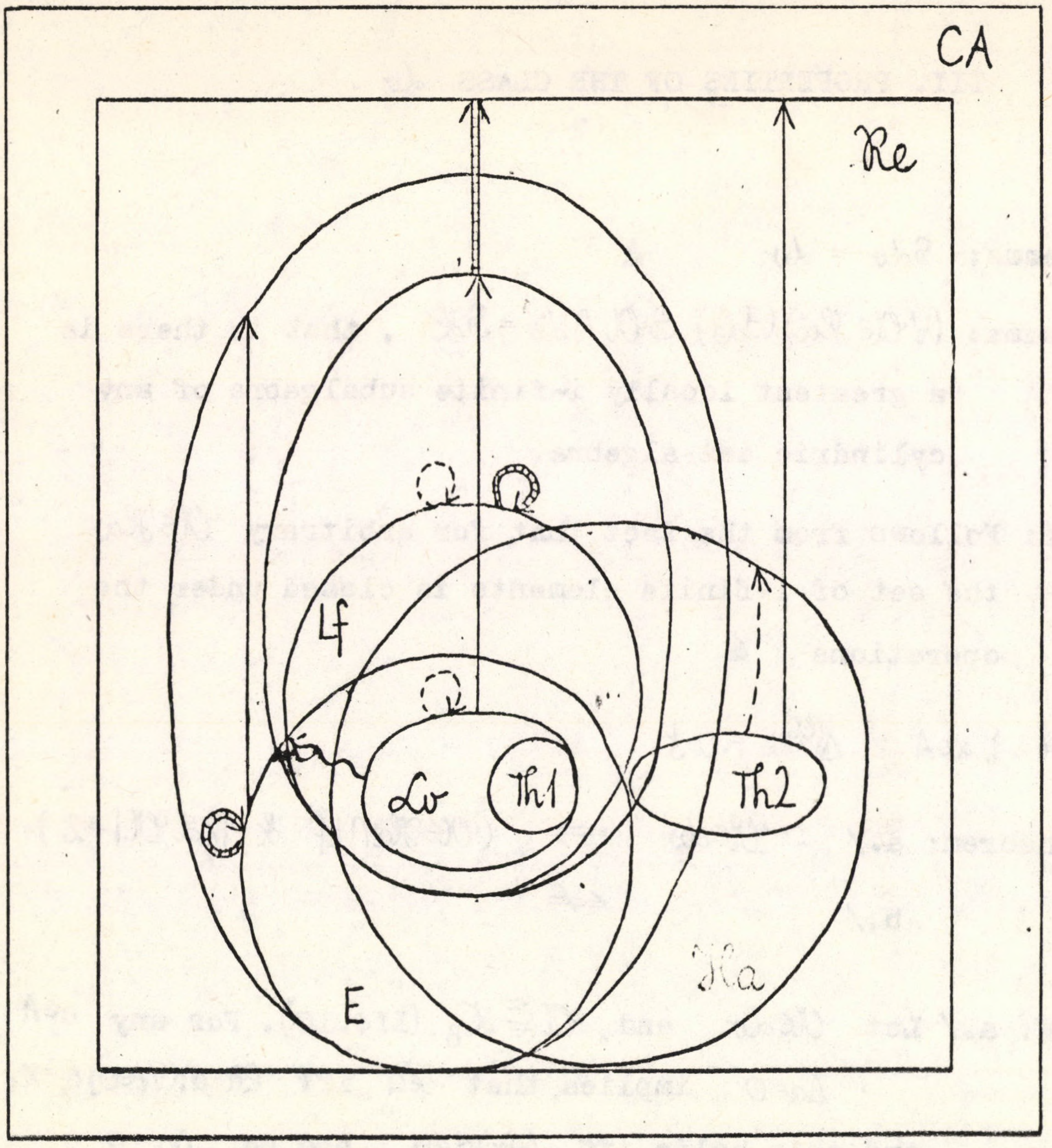


Fig. 4.a.



~~~> $\mathbb{I}$ - closure

- - -> $\mathbb{S}$ - closure

$\Rightarrow$ $\mathbb{H}$ - closure

$\longrightarrow$ $\mathbb{SP}$ - closure

□ variety

○ non-variety

$E \overset{d}{=} \{$simple cylindric algebras with the trivial cylindric algebra$\}$

$\mathrm{Th}1 \overset{d}{=} \{\mathcal{L}_A : |A| = 1\}$

$\mathrm{Th}2 \overset{d}{=} \mathrm{Th} \setminus \mathrm{Th}1 = \{\mathcal{L}_A : |A| > 1\}$

Fig. 4.b.
~~~

III. PROPERTIES OF THE CLASS Lo .

III.1 Lemma: $\mathfrak{S}\mathit{Lo} = \mathit{Lo}$ ▲

III.2 Lemma: $(\forall\mathfrak{A}\in \mathit{Ha})(\exists\mathfrak{B})\ \mathfrak{S}\mathfrak{A}\cap \mathit{Lo}=\mathfrak{S}\mathfrak{B}$, that is there is a greatest locally i-finite subalgebra of any cylindric set algebra.

Proof: Follows from the fact that for arbitrary $\mathfrak{A}\in \mathit{Ha}$ the set of i-finite elements is closed under the operations ▲

$Zd\,\mathfrak{A} \overset{d}{=} \{x\in A : \Delta^{(\mathfrak{A})}x=0\}$

III.3 Theorem: a./ $\mathfrak{A}\in \mathit{Lo} \Longrightarrow (\mathfrak{A}\in \mathit{Ha}\cap \mathit{Lf}\ \&\ |Zd\,\mathfrak{A}|=2)$

b./ $\not\Longleftarrow$

Proof: a./ Let $\mathfrak{A}\in \mathit{Lo}$ and $\mathfrak{A}\subseteq\mathfrak{L}_B$. (II4.1C). For any $a\in A$ $\Delta a=0$ implies that $s\in a$ iff $(\exists z\in a)(\forall i\in 0)\, s_i=z_i$, and this holds iff $(\exists z)\, z\in a$, and so $\Delta a=0$ implies that $a=0$ or $a={}^{\omega}B$

b./ We define an algebra $\mathfrak{A}$ such that $\mathfrak{A}\in \mathit{Ha}\cap \mathit{Lf}$ and $|Zd\,\mathfrak{A}|=2$ and $\mathfrak{A}\notin \mathit{Lo}$. Let $x \overset{d}{=} \{s\in {}^{\omega}2 : |s^{-1*}0|=\omega$ iff $s_0=0\}$, and $\mathfrak{A}\overset{d}{=}\mathfrak{S}\mathfrak{g}^{(\mathfrak{L}_2)}\{x\}$. It follow-s from this definition that $\mathfrak{A}\in \mathit{Ha}$, and $\Delta^{(\mathfrak{A})}x=1$ implies by [1].2.1.5 that $\mathfrak{A}\in \mathit{Lf}$.

Since x is not i-finite, $\mathfrak{A}\notin \mathit{Lo}$. Now we show that $|Zd\,\mathfrak{A}|=2$.

We define a property on the elements of A. It can be proved by induction, that all elements of A have this property. However since this proof is long but mechanical, we omit it. To complete the proof we show that this implies that $|Zd\,\mathfrak{A}|=2$.

$$y \text{ is good} \overset{d}{\Longleftrightarrow} (\exists n\in\omega)(\exists a\subseteq {}^{n}2)\; y = \{s\in{}^{\omega}2 : n|s\in a \;\&\; |s^{-1*}0|=\omega\} \cup$$
$$\cup \{s\in{}^{\omega}2 : n|s\in d(a) \;\&\; |s^{-1*}0|<\omega\}$$

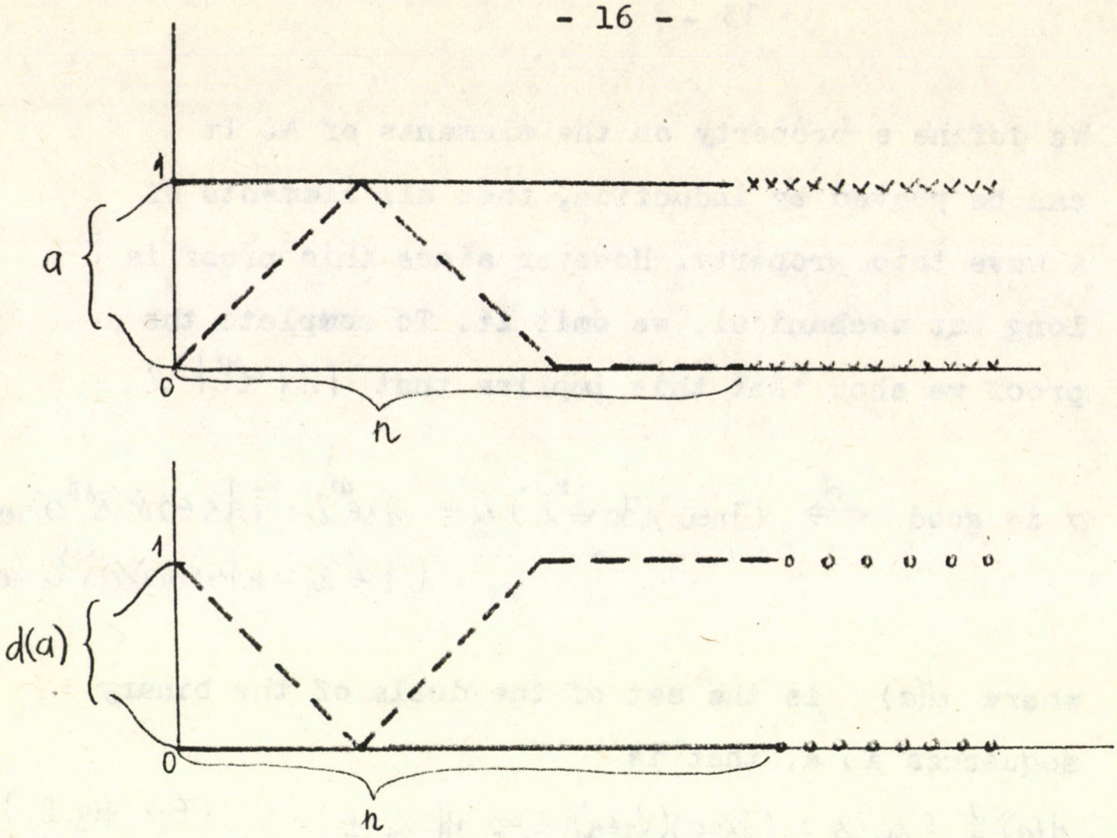
where $d(a)$ is the set of the duals of the binary sequences in a, that is

$d(a) \overset{d}{=} \{s\in{}^{n}2 : (\exists x\in a)(\forall i\in n)(s_i=0 \text{ iff } x_i=1)\}$. (See fig. 5.)

It can be proved by induction that all the elements of A are good. Now let $y\in Zd\,\mathfrak{A}$ good. Since y is good, there exists an "a" and "n". From $\Delta y=0$ follow-s that $c_0c_1\ldots c_{n-1}\,y=y$, and so either $a=0$ or $a={}^{n}2$. This implies that $y=0$ or $y=1^{(\mathfrak{A})}$ ▲

III.3. Corollary: $\mathsf{Lw} \neq \mathsf{Ha} \cap \mathsf{Lf}$

Proof: The algebra constructed in the proof of the second part of the above theorem is an element of $(\mathsf{Ha}\cap\mathsf{Lf}) \sim \mathsf{Lw}$. We note that this can be proved without the second part of the above theorem because it is easy to construct an algebra $\mathfrak{A}$ such that $\mathfrak{A}\in \mathsf{Ha}\cap\mathsf{Lf}$ and $|Zd\,\mathfrak{A}|>2$. ▲



A line ending with xxxx stands for all the sequences starting with that line and having infinitely many zeros. The ending ooooo has the same meaning but with finitely many zeros.

Fig. 5.

III.4. Corollary: $Lf \neq \mathbb{I}Lo$

Proof: There exists an $\mathfrak{A} \in Lf$ such that $|Zd\,\mathfrak{A}| > 2$ ▲

$\mathfrak{A}$ is <u>simple</u> $\overset{d}{\Leftrightarrow}$ $|Co\,\mathfrak{A}| = 2$

III.5. Corollary: For all $\mathfrak{A} \in \mathcal{H}a \cap Lf$

a./ $\mathfrak{A} \in Lo \rightleftarrows\!\!\!\!/\; \mathfrak{A}$ is simple

b./ $\mathfrak{A} \in Lo \rightleftarrows\!\!\!\!/\; \mathfrak{A}$ is directly indecomposable*/

c./ $\mathfrak{A} \in Lo \rightleftarrows\!\!\!\!/\; \mathfrak{A}$ is subdirectly indecomposable*/

d./ $\mathfrak{A} \in Lo \rightleftarrows\!\!\!\!/$ is weekly subdirectly indecomposable*/

Proof: b./ followes from [1] 2.4.14

a./ c./ d./ follows from [1] 2.4.43

Now we define a natural correspondance between the locally i-finite algebras (Lo) and the structures. By this correspondance Lo can serve as a basic tool for investigating structures and their interrelationships.

Let "a" be an i-finite element of the cylindric set algebra $\mathfrak{A}$. The relation belonging to "a" is:

$$r(a) \overset{d}{=} \{ (\cup \Delta^{(\mathfrak{A})} a) \,|\, s \;:\; s \in a \}$$

---

* These well known universal algebraic notions can be found in [1].

The importance of r(a) follows from the fact that

$s \in a$ iff $(\bigcup \Delta^{(\alpha)} a) | s \in r(a)$

This fact is illustrated in fig. 6.

We define p as a mapping of Lr into the class of structures such as: if $\mathcal{L}_B \supseteq \mathfrak{A} \in \mathsf{Lr}$ (II4.1C), then

$$p(\mathfrak{A}) \stackrel{d}{=} \{\langle A, B\rangle, \langle a, r(a)\rangle : a \in A\}$$

The correctness of this definition follows from the II4.1C corollary.

We define q as a mapping of the class of structures into Lr such as, if $\mathfrak{B}$ is an arbitrary structure and t stands for the type of $\mathfrak{B}$ then

$$q(\mathfrak{B}) \stackrel{d}{=} \mathfrak{S}_g^{(\mathcal{L}_B)} \{\{s : t_g | s \in \mathfrak{B}_g\} : g \in \mathrm{Dot}\}$$

III.6 Remark: a./ $q \supseteq p^{-1}$, that is $q \circ p$ is the identity transformation on Lr

b./ $p \circ q$ correlates with any structure $\mathfrak{B}$ a structure with the same universe and all the relations, elementarily definable in $\mathfrak{B}$. ▲

Summing up the relations between Lr and the other classes of cylindric algebras:

$$\mathsf{Lr} \subsetneqq \mathsf{Lf} \cap \mathsf{Ha}$$

$$\mathbf{SP}\,\mathsf{Lr} = \mathbf{SP}\,\mathsf{Lf} \neq \mathbf{SP}\,\mathsf{Ha} = \mathbf{SP}\,\mathsf{Th} = \mathsf{Re}$$

$$\mathbf{HSP}\,\mathsf{Lr} = \mathbf{HSP}\,\mathsf{Lf} = \mathbf{HSP}\,\mathsf{Ha} = \mathbf{HSP}\,\mathsf{Th} = \mathsf{Re}$$

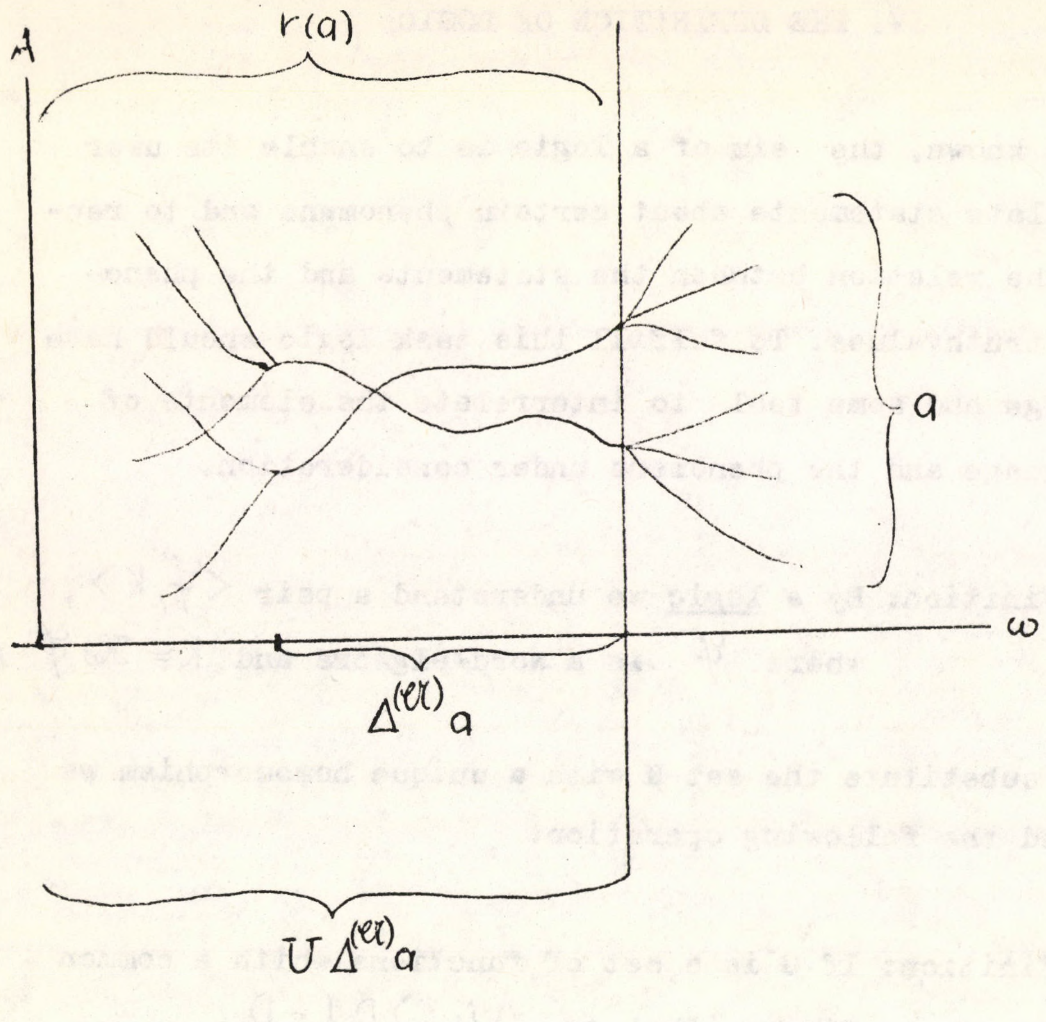


Fig. 6.

IV. THE DEFINITION OF LOGIC

As it is known, the aim of a logic is to enable its user to formulate statements about certain phenomena and to represent the relation between the statements and the phenomena by truthvalues. To fulfill this task logic should have a language and some tool to interrelate the elements of the language and the phenomena under consideration.

IV.1 Definition: By a <u>logic</u> we understand a pair $\langle \mathcal{F}, K \rangle$, where $\mathcal{F}$ is a word-algebra and $K \subseteq \mathcal{H}o\,\mathcal{F}$ ▲

To substitute the set K with a unique homomorphism we need the following operation:

IV.2 Definition: If G is a set of functions whith a common domain, that is $(\forall f \in G)\, Do\, f = D$

, then $\Pi G \overset{d}{=} \langle \langle f_x \rangle_{f \in G} \rangle_{x \in D}$

/see fig. 7./

We now introduce some concepts related to the concept of logic: $k \overset{d}{=} \Pi K$.

The set F is called <u>language</u>, its elements are called <u>formulas.</u> The elements of K are called <u>interpreting functions</u>,

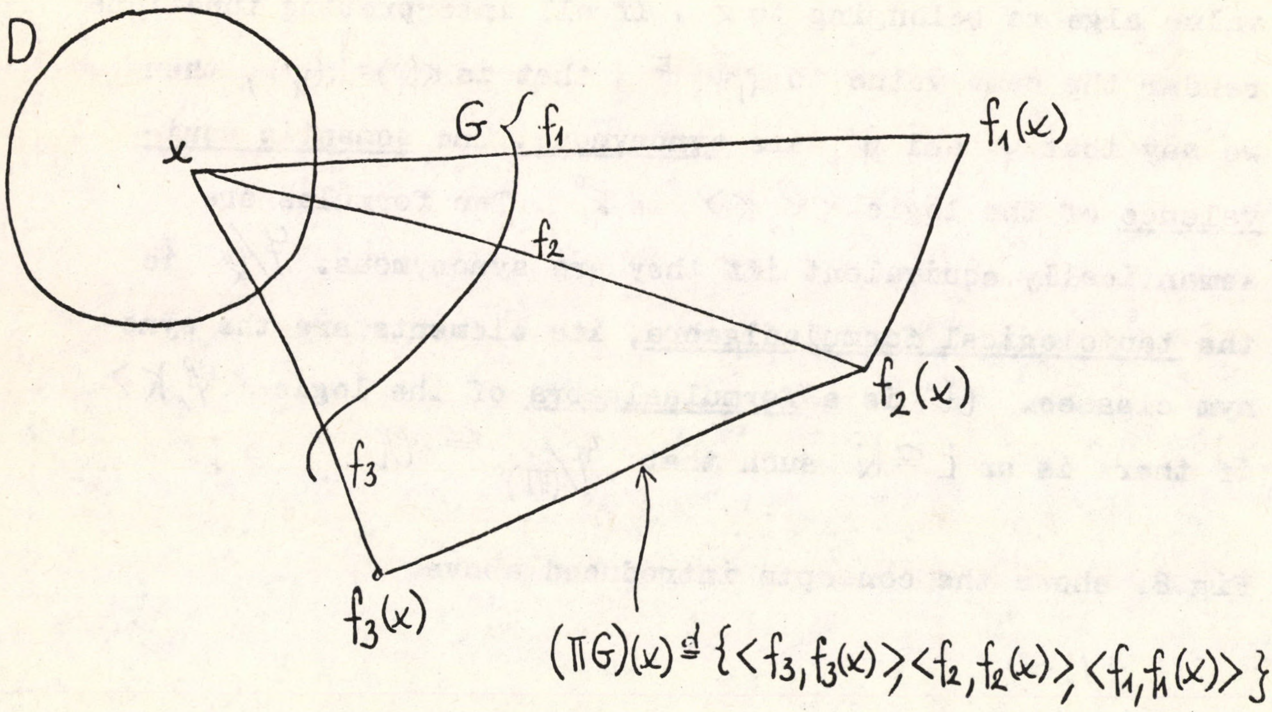

D
x
G
f_1
$f_1(x)$
f_2
$f_2(x)$
f_3
$f_3(x)$
$(\Pi G)(x) \stackrel{d}{=} \{\langle f_3, f_3(x)\rangle, \langle f_2, f_2(x)\rangle, \langle f_1, f_1(x)\rangle\}$

Fig. 7.

these render meanings to the elements of the language. For all $x \in K$ and $\varphi \in F$, $k(\varphi)_x$ is the <u>truthvalue</u> of the formula φ according to the interpreting function x. For all $x \in K$, ${}_x\mathfrak{Q} \overset{d}{=} x^*\mathcal{F}$. The algebra ${}_x\mathfrak{Q}$ is called the truthvalue algebra belonging to x. If all interpreting functions render the same value to $\varphi, \psi \in F$, that is $k(\varphi) = k(\psi)$, then we say that φ and ψ are <u>synonymous.</u> The <u>semantic equivalence</u> of the logic $\langle \mathcal{F}, K \rangle$ is k^0. Two formulas are semantically equivalent iff they are synonymous. $\mathcal{F}/_{k^0}$ is the <u>tautological formulaalgebra</u>, its elements are the synonym classes. $\mathfrak{A}$ is a <u>formulaalgebra</u> of the logic $\langle \mathcal{F}, K \rangle$ if there is an $L \subseteq K$ such that $\mathcal{F}/_{(\Pi L)^0} \cong \mathfrak{A}$.

Fig.8. shows the concepts introduced above.

<u>Interpretations</u>

To make more convenient the use of logic, we can render "labels" to the interpreting functions, which serve to identify the interpreting functions. These labels are called interpretations. That is, we can pick any class M with a function $h \in {}^M K$, and consider the elements of M as interpretations, which label the interpreting functions through h. Let $m \in M$, now $k(\varphi)_{h(m)}$ is the truthvalue of the formula φ in the interpretation m. ${}_m\mathfrak{Q} \overset{d}{=} h_m^* \mathcal{F}$. The algebra ${}_m\mathfrak{Q}$ is the truthvalue algebra of the interpretation m.

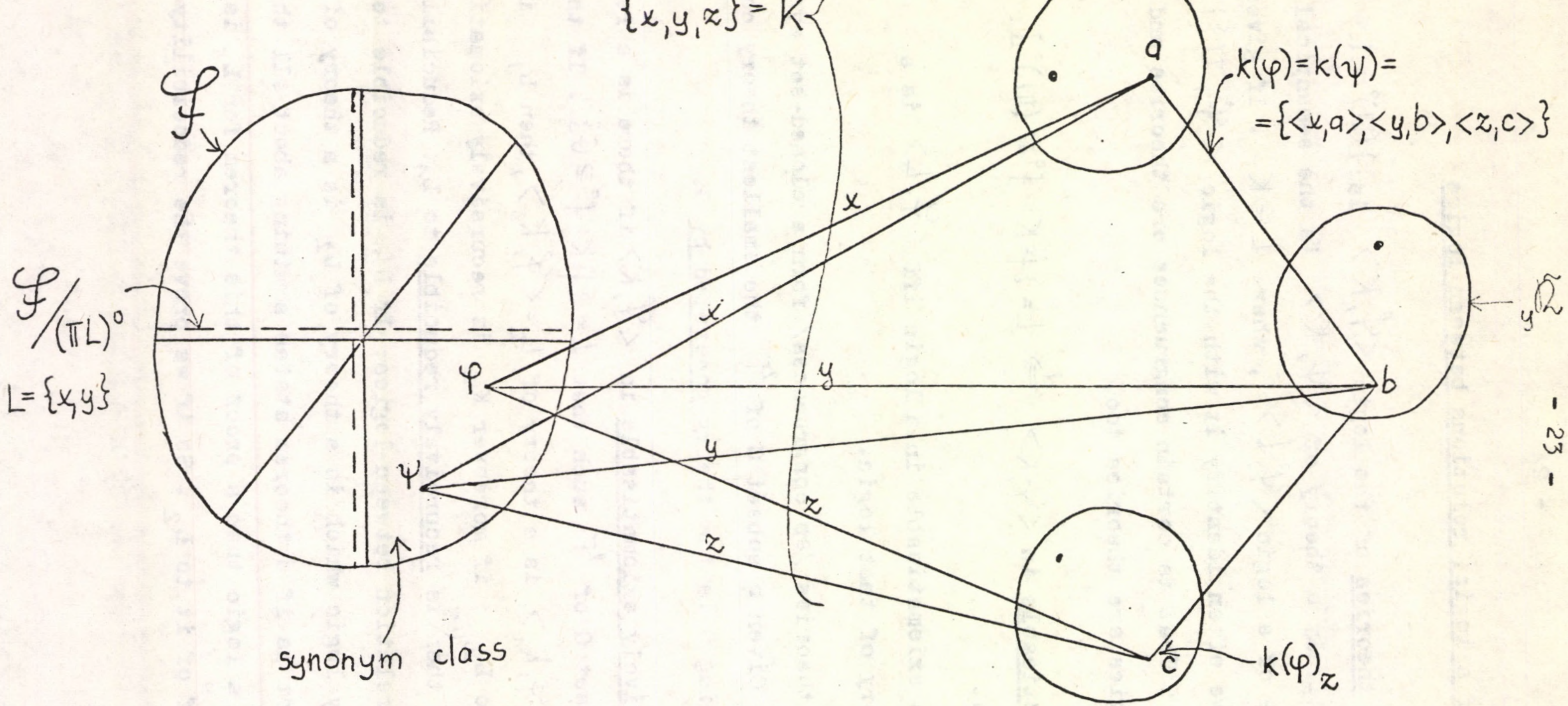

$\{x,y,z\}=K$
$\mathcal{F}$
$\mathcal{F}/(\Pi L)^0$
$L=\{x,y\}$
φ
ψ
synonym class
x
x
y
y
z
z
a
b
c
$k(\varphi)=k(\psi)=$
$=\{\langle x,a\rangle,\langle y,b\rangle,\langle z,c\rangle\}$
$^y\Omega$
$k(\varphi)_z$

Fig.8.

Theories of a logic, relations between logics

The set of theories of the logic $\langle \mathcal{F}, K\rangle$ is: $\{(\Pi L)^\circ : L \subseteq K\}$. More intuitively a theory of $\langle \mathcal{F}, K\rangle$ is the semantical equivalence of a logic $\langle \mathcal{F}, L\rangle$, where $L \subseteq K$. If given theory R we often identify it with the logic $\langle \mathcal{F}, \bigcup\{L \subseteq K : R = (\Pi L)^\circ\}\rangle$. That is certain congruences are theories and certain logics are theories too.

L is axiomatisable in $\langle \mathcal{F}, K\rangle \overset{d}{\Longleftrightarrow} L = \{f \in K : f^\circ \supseteq (\Pi L)^\circ\}$.
We note that

a./ L is axiomatisable in a logic iff $\langle \mathcal{F}, L\rangle$ is a theory of that logic.

b./ The theories /as congruences/ form a closed-set system. Given a subset G of ${}^2F$ the smallest theory containing G is the theory generated by G.

L is recursively axiomatisable in $\langle \mathcal{F}, K\rangle$ if there is a recursive subset G of ${}^2F$ such that $L = \{f \in K : f^\circ \supseteq G\}$. If the logic $L_1 = \langle \mathcal{F}, K_1\rangle$ is a theory of $L_2 = \langle \mathcal{F}, K_2\rangle$, then L_1 is reducible to L_2; if moreover K_2 is recursively axiomatisable in L_1 then L_2 is recursively reducible to L_1. Reducibility is a close relation between logics: If L_2 is reducible to L_1 then any logic which is a theory of L_2 is a theory of L_1 too. That is if a theorem states something about all the theories of a logic than a proof of this theorem for L_1 is also a proof of it to L_2. So if we prove the reducibility of L_1 to L_2, then all such proof to L_1 become superfluous. Theorems of this kind are the compactness theorem, the Löveinheim-Skolem th., the ultraproduct-th., and also the completeness theorem can be reformulated in such a form.

Shorthands

There is another means to make the use of a logic $\langle \mathcal{F}, K\rangle$ more convenient /the other one was the use of interpretations/. We can introduce shorthands for the formulas, that is instead of the elements of F we can use their names. Of course, just as it was the case with the interpretations, different purposes may require different kinds of shorthands for the same logic. The definition of shorthands goes as follows: We define a relation $\vdash$ on finite sequences. We do this by listing elements of $\vdash$ in the form $\alpha \vdash \beta$, where α and β are given sequences /or shemes of sequences/, and then saying that $\vdash$ is the smallest relation for which if $\alpha, \beta, \gamma, \delta$ are arbitrary sequences /the empty sequence included/ and $\delta \vdash \gamma$, then $\alpha\delta\beta \vdash \alpha\gamma\beta$ */. Now we define $\Vdash$ as the smallest transitive relation containing $\vdash$. If $\alpha \Vdash \beta$ and $\beta \in F$ we say that α is the name of the formula β. The set of names is $N \overset{d}{=} \Vdash^{-1*}F$. and we say that the definition of $\Vdash$ is correct if $N \upharpoonright \Vdash$ is a function. This can be

---

* More precisely, for any sequence $\langle s_0, \dots, s_n\rangle$ we state:
$\langle s_0, \dots, \alpha, \dots s_n\rangle \vdash \langle s_0, \dots, \beta, \dots s_n\rangle$

also a tool of forming new logics:

We pick a $B \subseteq N$ such that $\Vdash^* B = F$ and define an algebra $\mathcal{B}$ on B such that $\Vdash^* \mathcal{B} = \mathcal{F}$. Now the new logic is $\langle \mathcal{B}, \{B \upharpoonright (f \circ \Vdash) : f \in K\} \rangle$. /Of course care should be taken for $\mathcal{B}$ to be a word-algebra./ We call this a logic with built in shorthands. Of course this logic is recursively reducible to the original one since $B \upharpoonright \Vdash$ is a recursive function. For example well known shorthands are: $(\varphi \vee \psi) \vdash \neg(\neg\varphi \wedge \neg\psi)$; $(\varphi \wedge \psi) \vdash \langle \wedge, \langle \varphi, \psi \rangle \rangle$, and $\forall_i \vdash \neg \exists_i \neg$ for any $\varphi, \psi \in F$.

V. TYPELESS LOGIC

Before introducing typeless logic we have to introduce the following auxiliary concepts:

In the followings I is an arbitrary but fixed set, and $\mathcal{A}$ is an arbitrary class of similar algebras.

V.1 Definition: A set $S \subseteq {}^2\mathfrak{F}r_I$ is called a defining relation.* ▲

V.2 Definition: $\Gamma_I^{(S)}\mathcal{A}$ is the class of homomorphisms over $\mathcal{A}$ with I generators and with def. relation S: $\Gamma_I^{(S)}\mathcal{A} \overset{d}{=} \{g \in Ho\,\mathfrak{F}r_I : g^*\mathfrak{F}r_I \in \mathbf{S}\mathcal{A},\ g^\circ \supseteq S\}$

in case S = 0 we omit the superscript:

$$\Gamma_I^{(0)}\mathcal{A} \overset{d}{=} \Gamma_I\mathcal{A} \quad ▲$$

V.3 Definition: The <u>typeless logic</u> of index set I is the pair $\langle \mathfrak{F}r_I, \Gamma_I \mathcal{L}o \rangle$. ▲

We introduce the shorthand for the class of interpreting functions of this logic: $K^I \overset{d}{=} \Gamma_I \mathcal{L}o$.

Now we fix the class of interpretations with which we use this logic:

$$M^I \overset{d}{=} \{\mathfrak{A} : Do\,\mathfrak{A} = I \cup \{I\}\}$$

---

* We know that by calling S defining relation instead of a set of defining relations we have broken the tradition.

This is the class of structures the relation symbols of which are from the set I. We note that for other purposes other classes of interpretations might said better, e.g. the class of structures with proper relations. /A relation "r" over A is proper if there is no "q" such that r = q x A./

The labelling function $h \in {}^{M^I}K^I$ is as follows:

V.4 Definition: For any $\mathfrak{A} \in M^I$, $h_{\mathfrak{A}} \in Hom(\mathfrak{F}_I, \mathcal{L}_A)$ such that for all $g \in I$

$$h_{\mathfrak{A}}(g) \stackrel{d}{=} \{ s \in {}^{\omega}A : (\exists n \in \omega) \langle s_0, \ldots, s_n \rangle \in \mathfrak{A}_g \}. \blacktriangle$$

V.5 Theorem: M^I together with h form a class of interpretations to the typeless logic of i.s. I, that is $h^* M^I = K^I$

Proof: 1./ $h^* M^I \subseteq K^I$. We know that $h_{\mathfrak{A}}^* \mathfrak{F}_I \in \mathcal{H}a$ and it is easy to see that for any $g \in I$ the element $h_{\mathfrak{A}}(g) = \{ s : (\exists n) n \mid s \in \mathfrak{A}_g \}$ is i-finite. Since $h_{\mathfrak{A}}^* I$ generates $h_{\mathfrak{A}}^* \mathfrak{F}_I$ we have $h_{\mathfrak{A}}^* \mathfrak{F}_I \in \mathcal{L}o$.

This complets the proof of 1./[*]/

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[*]/Beyond completing the proof we got that the labelling h coincids with our natural correspondence q between the structures and the $\mathcal{L}o$ algebras that is: $h_{\mathfrak{A}}^* \mathfrak{F}_I = q(\mathfrak{A})$

2./ $h^* M^I \supseteq K^I$. Let $g \in \Gamma_I \mathcal{L}\omega = K^I$. To g we construct an $\mathfrak{A} \in M^I$ such that $h_{\mathfrak{A}} = g$.

$\mathcal{B} \overset{d}{=} g^* \mathfrak{F}_I \in \mathcal{L}\omega$. According to II4.1C there is a unique A such that $\mathcal{B} \in \mathcal{L}_A$. Now we define a structure $\mathfrak{A}$ on A: for all $\varrho \in I$ we pick an $n \in \omega$ such that $\Delta^{(\mathcal{B})} g(\varrho) \subseteq n$ and then we fix $\mathfrak{A}_\varrho \overset{d}{=} \{\langle s_0, \ldots, s_{n-1}\rangle : s \in g(\varrho)\}$. Now we show that $h_{\mathfrak{A}} = g$. For all $\varrho \in I$:

$h_{\mathfrak{A}}(\varrho) =$

$$= \{s \in {}^{\omega}A : (\exists m \in \omega) m | s \in \mathfrak{A}_\varrho\} \overset{\text{because the def. of } \mathfrak{A}}{=} \{s \in {}^{\omega}A : (\exists z \in g(\varrho)) \langle z_0, \ldots, z_{n-1}\rangle = \langle s_0, \ldots, s_{n-1}\rangle\} \overset{\text{because } \mathcal{B} \in \mathcal{L}\omega \text{ and } \Delta^{(\mathcal{B})} g(\varrho) \subseteq n}{=} g(\varrho)$$

And since $\mathfrak{A}_I = A$ and $\mathcal{B} \in \mathcal{L}_A$ implies that $h_{\mathfrak{A}}, g \in \mathit{Hom}(\mathfrak{F}_I, \mathcal{L}_A)$, we have: $h_{\mathfrak{A}} = g$.

That is for any $g \in K^I$ there is an $\mathfrak{A} \in M^I$ such that $h_{\mathfrak{A}} = g$ and so $h^* M^I \supseteq K^I$ ▲

Now we start to investigate the algebraic properties of typeless logic. We prove that the semantical equivalence coincides with the free congruence over $\mathcal{L}\omega$ and so the tautological formulaalgebra is the free algebra over $\mathcal{L}\omega$. To this we need the following five purely algebraic lemmas. From now on $\mathcal{A}$ is an arbitrary class of similar algebras and S is an arbitrary defining relation.

V.6 Lemma: $(\Pi\Gamma_I^{(S)}\mathcal{A})^\circ = Cr_I^{(S)}\mathcal{A}$

Proof: $\langle x,y\rangle \in (\Pi\Gamma_I^{(S)}\mathcal{A})^\circ$ iff $(\forall g\in\Gamma_I^{(S)}\mathcal{A})\, g(x)=g(y)$ **iff**

iff $(\forall g\in \mathcal{H}o\,\mathcal{F}_I)[(g^*\mathcal{F}_I\in \mathbb{S}\mathcal{A}\ \&\ g^\circ \supseteq S)\Rightarrow \langle x,y\rangle \in g]$ **iff**

iff $(\forall r\in Co\,\mathcal{F}_I)[(\mathcal{F}_I/_r\in \mathbb{I}\mathbb{S}\mathcal{A}\ \&\ r\supseteq S)\Rightarrow \langle x,y\rangle\in r]$ iff

iff $\langle x,y\rangle\in Cr_I^{(S)}\mathcal{A}$. ▲

V.7 Lemma: For all $g\in\Gamma_I^{(S)}\mathbb{S}\mathbb{P}\mathcal{A}$ **there is a** $G\subseteq\Gamma_I^{(S)}\mathcal{A}$ **such that** $g^\circ = (\Pi G)^\circ$

Proof: Since $g\in\Gamma_I^{(S)}\mathbb{S}\mathbb{P}\mathcal{A}$ **, there exists an J index set and** $f\in {}^J\!\mathcal{A}$ **for which** $g^*\mathcal{F}_I \cong|\in \underset{j\in J}{\mathbb{P}} f_j$ **If i stands for the isomorphism, for all** $j\in J$ **we have** $\varepsilon_j\circ i\circ g\in\Gamma_I^{(S)}\mathcal{A}$ **, and since** $(\forall x\in\mathcal{F}_I)\,Do\, g(x)=J$ **, we have:** $g^\circ = (\Pi\{\varepsilon_j\circ i\circ g:\ j\in J\})^\circ$ ▲

V.8 Lemma: For all set I and congruence R

$\mathcal{F}_I/_R\in\mathbb{S}\mathbb{P}\mathcal{A}$ iff $(\exists L\subseteq\Gamma_I\mathcal{A})\ (\Pi L)^\circ = R$

Proof: 1./ $\mathcal{F}_I/_R\in\mathbb{S}\mathbb{P}\mathcal{A} \Rightarrow (\exists L\subseteq\Gamma_I\mathcal{A})(\Pi L)^\circ = R$

$R^*\in\mathcal{H}o(\mathcal{F}_I,\mathcal{F}_I/_R)\subseteq\Gamma_I\mathbb{S}\mathbb{P}\mathcal{A}$ **since** $\mathcal{F}_I/_R\in\mathbb{S}\mathbb{P}\mathcal{A}$.

By this V.7 gives $(\exists L\subseteq\Gamma_I\mathcal{A})\ (\Pi L)^\circ = (R^*)^\circ$.

This with the fact that for all equivalence-relation r, $(r^*)^\circ = r^\circ$ **completes the proof of 1./.**

2./ $(\exists L\subseteq\Gamma_I\mathcal{A})\ (\Pi L)^\circ = R \Rightarrow \mathcal{F}_I/_R\in\mathbb{S}\mathbb{P}\mathcal{A}$

$L\subseteq\Gamma_I\mathcal{A}\Rightarrow \Pi L\in\mathcal{H}om(\mathcal{F}_I, \underset{l\in L}{\mathbb{P}}\ l^*\mathcal{F}_I)$, **since for all** $l\in L$, $l^*\mathcal{F}_I\in\mathcal{A}$ **we have** $\mathcal{F}_I/_{(\Pi L)^\circ}\in\mathbb{S}\mathbb{P}\mathcal{A}$

Now $(\prod L)^{\circ} = R$ completes the proof of 2./ ▲

V.9 Lemma: $\mathfrak{G}_I^{(S)} \mathcal{A} = \mathfrak{G}_I^{(S)} \mathbf{SP} \mathcal{A}$

Proof:

$$\langle x,y\rangle \in \mathfrak{G}_I^{(S)} \mathbf{SP}\mathcal{A} \underset{\text{because V.6}}{\text{ iff }} (\forall g \in \Gamma_I^{(S)} \mathbf{SP}\mathcal{A}) \langle x,y\rangle \in g^{\circ} \underset{\text{because V.7}}{\text{ iff }} (\forall G \subseteq \Gamma_I^{(S)} \mathcal{A}) \langle x,y\rangle \in (\prod G)^{\circ} \underset{\text{because V.6}}{\text{ iff }} \langle x,y\rangle \in \mathfrak{G}_I^{(S)} \mathcal{A}$$

▲

V.10 Lemma: $\mathfrak{G}_I \mathcal{A} = \mathfrak{G}_I \mathbf{HSP} \mathcal{A}$

Proof: Can be found in [1] ▲

V.11 Definition: The semantical equivalence of the typeless logic of index set I: $\equiv^{I} \overset{d}{=} (\prod K^{I})^{\circ}$ ▲

V.12 Theorem: $\equiv^{I} = \mathfrak{G}_I \mathcal{L}v$

Proof: $\mathfrak{G}_I \mathcal{L}v = (\prod \Gamma_I \mathcal{L}v)^{\circ}$ by V.6. ▲

V.13 Corollary: $\mathfrak{F}_I/_{\equiv^I} = \mathfrak{F}_I \mathcal{L}v$

V.14 Definition: The class of typeless formulaalgebras:

$$\mathfrak{S}^{(F)} \overset{d}{=} \mathbf{I}\{ \mathfrak{F}_I/_{(\prod L)^{\circ}} : \text{I is an arbitrary set}, L \subseteq K^{I} \}$$

▲

V.15 Theorem: $\mathfrak{S}^{(F)} = \mathbf{SP}\, \mathcal{L}v$

Proof: 1./ $\mathfrak{S}^{(F)} \subseteq \mathbb{S}\mathbb{P}\,\mathcal{L}v$. According to the definition of $\mathfrak{S}^{(F)}$ for any $\mathfrak{A} \in \mathfrak{S}^{(F)}$ there is an I and $L \subseteq \Gamma_I \mathcal{L}v$ such that $\mathfrak{A} \cong \mathfrak{F}_I/(\pi L)^\circ$. From this and V.8 follows that $\mathfrak{A} \in \mathbb{S}\mathbb{P}\,\mathcal{L}v$.

2./ $\mathfrak{S}^{(F)} \supseteq \mathbb{S}\mathbb{P}\,\mathcal{L}v$. For any $\mathfrak{A} \in \mathbb{S}\mathbb{P}\,\mathcal{L}v$ there is a set I such that $\mathfrak{F}_I \succcurlyeq \mathfrak{A}$. /e.g. $\mathfrak{F}_A \succcurlyeq \mathfrak{A}$ ./ By V.8. this implies that $(\exists L \subseteq \Gamma_I \mathcal{L}v)\, \mathfrak{A} \cong \mathfrak{F}_I/(\pi L)^\circ$ that is $\mathfrak{A} \in \mathfrak{S}^{(F)}$ ▲

We have proved so far that the semantic equivalence of the typeless logic is the free congruence over $\mathcal{L}v$, the tautological formulaalgebra is the free algebra over $\mathcal{L}v$ and the class of formulaalgebras is $\mathbb{S}\mathbb{P}\,\mathcal{L}v$.

We note that the same is true for the propositional logic if we replace $\mathcal{L}v$ by $\{\langle \wedge, \neg, 2 \rangle\}$. For the algebraic purposes the definition of $\mathcal{L}v$ is not algebraic enough. So we try to replace it with more algebraic classes. E.g. the fact that the tautological formulaalgebra of the propositional logic is the free Boolean algebra is more algebraic as our V.13, since the class of Boole algebras is a variety. In the followings we succeed in replacing $\mathcal{L}v$ by Lf as well as $\mathcal{R}e$, both having purely algebraic definitions. /The presently known algebraic definition of $\mathcal{R}e$ is more complicated than that of $\mathcal{L}v$, however it has the advantage that $\mathcal{R}e$ is a variety and a set of equations is known for it./

V.16 Theorem: a./ $\equiv^{I} = \mathfrak{Cr}_I\ Lf$

b./ $\mathfrak{Fr}_I/_{\equiv^I} = \mathfrak{Fr}_I\ Lf$

c./ $\mathfrak{S}^{(F)} = \mathbb{S}\mathbb{P}\ Lf$

Proof: by II7.4 and V.9 ▲

V.17 Corollary: a./ $\equiv^{I} = \mathfrak{Cr}_I\ \mathcal{R}e$

b./ $\mathfrak{Fr}_I/_{\equiv^I} = \mathfrak{Fr}_I\ \mathcal{R}e$

Proof: by II7.2 and V.10 and V.16 ▲

Remark: The corollary V.17 does not generalize part c./ of V.16. This generalization ($\mathfrak{S}^{(F)} = \mathbb{S}\mathbb{P}\ \mathcal{R}e = \mathcal{R}e$) is easily seen to be equivalent with the equality $\mathbb{S}\mathbb{P}\ Lf = \mathbb{H}\mathbb{S}\mathbb{P}\ Lf$, which however fails, as will be seen later.

V.18 Theorem: Let $\langle \mathfrak{Fr}_I, \Gamma_I \mathcal{A} \rangle$ be an arbitrary logic, that is $\mathcal{A}$ is an arbitrary class of algebras /not necessarily of type ℓ, howevere we do not take care about this in the notation/. If $\mathbb{S}\mathbb{P}\ \mathcal{A}$ is a variety then the <u>compactness</u> theorem is valid for $\langle \mathfrak{Fr}_I, \Gamma_I \mathcal{A} \rangle$

Proof: Let us suppose that $\mathbb{S}\mathbb{P}\mathcal{A} = \mathbb{H}\mathbb{S}\mathbb{P}\mathcal{A}$, and $K \stackrel{d}{=} \Gamma_I \mathcal{A}$. Let I be an arbitrary set and $\equiv\ \subseteq R \in \mathfrak{Co}\ \mathfrak{Fr}_I$.

Since $\mathfrak{F}_I/_\equiv = \mathfrak{F}_I\mathcal{A} \in \mathbb{SP}\mathcal{A}$, we have that $\mathfrak{F}_I/_R \in \mathbb{HSP}\mathcal{A}$. This, by the hypothesis, gives that $\mathfrak{F}_I/_R \in \mathbb{SP}\mathcal{A}$, and so by V.8 $(\exists L \subseteq K)(\Pi L)^\circ = R$, that is R is a theory of $\langle \mathfrak{F}_I, K\rangle$. This means that the set $\{R \in C_o\mathfrak{F}_I : R \supseteq \equiv\}$ coincides with the set of theories (on $\langle \mathfrak{F}_I, K\rangle$). Since it is well known [1] that the set of congriences containing a fixed congruence is an inductive closed-set system, we have proved the compactness th. for this logic. ▲

We note that the above theorem states e.g. the compactness of the propositional logic since the latter has the form: $\langle \mathfrak{F}_I, \Gamma_I\{\langle \cap, \smallsetminus, 2\rangle\}\rangle$.

Remark: It follows from the above theorem that the hypthesis that $\mathbb{SP}\, Lf$ is a variety implies the compactness of the typeless logic. However the compactness theorem holds for the typeless logic $\langle \mathfrak{F}_I, \Gamma_I \mathcal{L}\sigma\rangle$ iff I=0, for, as it easily seen, the set of formulas $\{\exists_0\neg\varrho, \exists_1\varrho, \exists_2\varrho, \dots \exists_i\varrho \dots\}$ has no model, while every finite subset of it has. As a corollary we get, that $\mathbb{SP}\, Lf$ is not a variety.

<u>Calculuses for typeless logic</u>

By a complete calculus we understand an algorithm which lists the semantical equivalence of the logic in consideration. That is a calculus of $\langle \mathfrak{F}_I, K^\Gamma\rangle$ lists the set $\equiv^\Gamma$. It is easy to find such a calculus by using that $\equiv^\Gamma = C_I\, \mathcal{R}e$, and a system of equations defining $\mathcal{R}e$ is known [1]. Thus starting from the

equations defining $\mathcal{Re}$ and by using the usual transformations on equations an algorithm can deduce any element of $\equiv^{I}$. This calculus can also lists the consequences of any finite set of formulas, however we know that there exist a recursive infinite set of formulas, the consequences of which cannot be listed by this calculus.
/For example: $\{\exists_0 \neg \varrho, \exists_1 \varrho, \exists_2 \varrho, \ldots\}$./
The correspondence $\equiv^{I} = Cr_I \mathcal{Re}$ can be a tool not only to construct new calculuses but also to check calculuses to be complete.* /We note that the completeness of the propositional calculus for instance can be proved in this manner in very few steps [2] /since the variety of Boolean algebras can be defined by three equations/.

Shorthands for typeless logic

We remind the reader that at the end of the chapter "def. of logic" we discussed the use of shorthands and fixed some definitions. For the typeless logic of index set I we can introduce the usual shorthands, e.g. $\vee, \rightarrow, \forall_i$, etc. However we cannot introduce shorthands for substitutions that is variables. We would like to have:

$$h_{\mathfrak{A}}(\Vdash(\varrho v_{i_0} \ldots v_{i_n})) = \{s \in {}^{\omega}A : (\exists m \in \omega)\, m \{ \langle s_{i_0}, \ldots, s_{i_n}, s_{n+1}, \ldots \rangle \in \mathfrak{A}_{\varrho} \}$$

---

* We have to check that the relation listed by the calculus is a congruence and contains the equations defining $\mathcal{Re}$.

We can not define this because $\Delta^{(\mathfrak{F}_I/\equiv)}\ g/_{\equiv} = \omega$. However if we substitute, in the definition of typeless logic, K^I with $K^* M^I$ where $k_{\mathfrak{A}}(g) = \{ s \in {}^{\omega}A : (\exists m \in \omega) \langle s_0, s_2, \ldots, s_{2m} \rangle \in \mathfrak{A}_g \}$, for all $g \in I$, than we get a modified version of typeless logic in which shorthands for substitution can be introduced, but unfortunately we lose with this the nice algebraic properties of the typeless logic.

Examples

1./ Let $I \overset{d}{=} \{g\}$, and for each $n \in \omega$ the structure ${}_n\mathfrak{A}$:

$${}_n\mathfrak{A} \overset{d}{=} \{ \langle \{g\}, \omega \rangle, \langle g, \{ \langle a_1, \ldots, a_n \rangle \in {}^n\omega : a_1 < a_2 < \ldots < a_n \} \rangle \}$$

It is easy to see that for any $k, n \in \omega$; $h_{{}_n\mathfrak{A}}(\exists_0 \ldots \exists_k g) = {}^{\omega}\omega$ iff $k \geq n$.

From this example it follows that $\Delta^{(\mathfrak{F}_I/\equiv)}(g/_{\equiv}) = \omega$.

2./ We would like to produce a formula φ such that $h_{\mathfrak{A}}(\varphi) = \{ s \in {}^{\omega}\omega : s_1 < s_0 \}$, where $\mathfrak{A} \overset{d}{=} {}_2\mathfrak{A}$ of the example 1./. We shall see that $\varphi = \exists_2(\exists_1(\exists_0(g \wedge =_{02}) \wedge =_{10}) \wedge =_{12})$ has just the required truthvalue in $\mathfrak{A}$ /see fig.9./

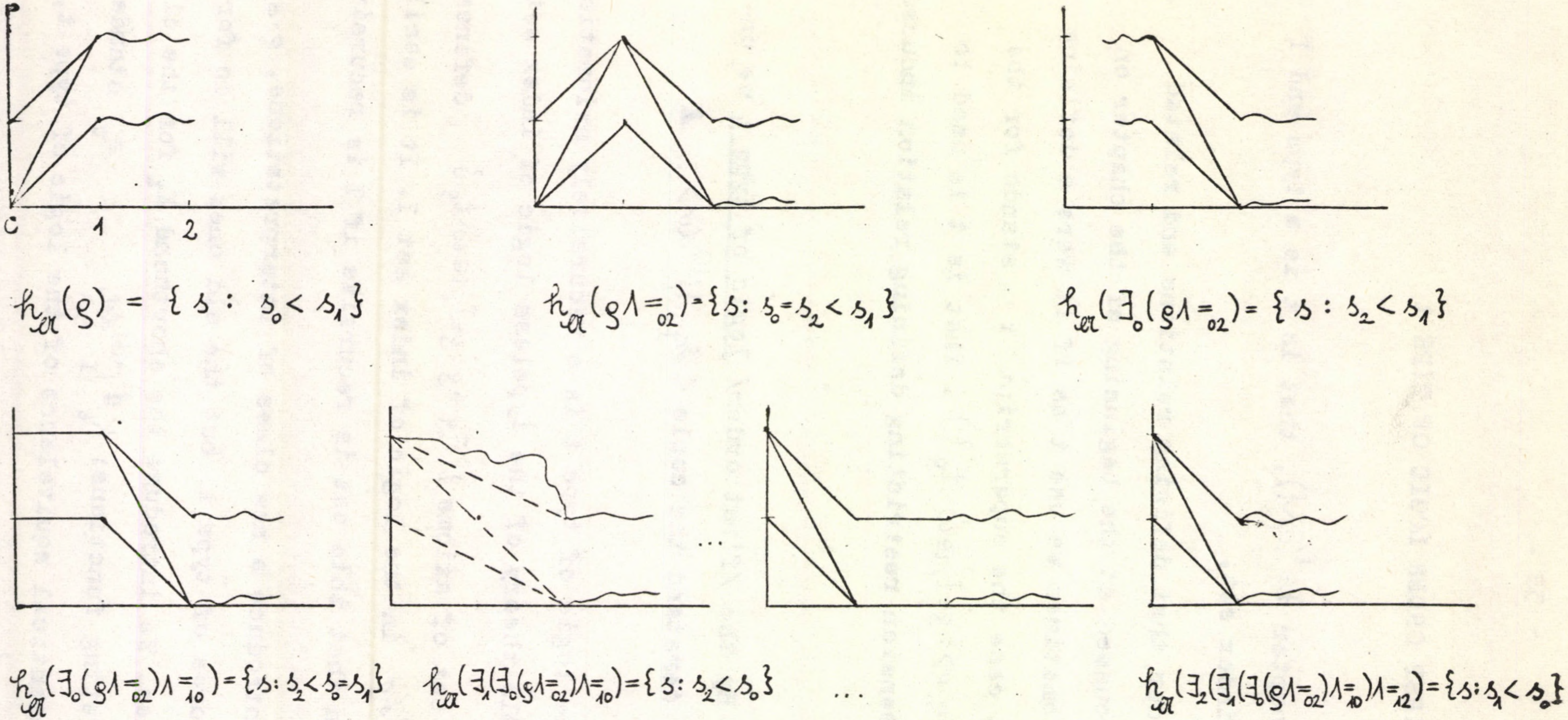
P
C
1
2
$h_{er}(g) = \{s : s_0 < s_1\}$
$h_{er}(g \wedge =_{02}) = \{s : s_0 = s_2 < s_1\}$
$h_{er}(\exists_0(g \wedge =_{02})) = \{s : s_2 < s_1\}$
$h_{er}(\exists_0(g \wedge =_{02}) \wedge =_{10}) = \{s : s_2 < s_0 = s_1\}$
$h_{er}(\exists_1(\exists_0(g \wedge =_{02}) \wedge =_{10})) = \{s : s_2 < s_0\}$
...
...
$h_{er}(\exists_2(\exists_1(\exists_0(g \wedge =_{02}) \wedge =_{10}) \wedge =_{12})) = \{s : s_1 < s_0\}$

Fig. 9.

VI. THE FIRST ORDER LOGIC OF TYPE t

Throughout this chapter $t\in{}^{I}(\omega\smallsetminus 1)$, that is t is a type and I is its domain or index set.

We remind the reader that defining relations and related concepts were discussed at the beginning of the chapter on typeless logic. Sometimes we use t as if it were a defining relation, in that case the superscript t stands for the superscript $(\{\langle\exists_i\varrho,\varrho\rangle : \varrho\in I, i\in\omega\smallsetminus t_\varrho\})$. That is t is used to stand for the dimension restricting defining relation induced by t.

VI.1 Definition: By the /first order/ <u>logic of type t</u> we understand the cuple $\langle \mathfrak{F}_I, \Gamma_I^{(t)}\mathcal{L}\sigma\rangle$. ▲

VI.2 Theorem: The logic of type t is a recursively axiomatisable theory of the typeless logic of index set I.

Proof: The set of axioms $\{\exists_i\varrho\to\varrho : \varrho\in I, i\in\omega\smallsetminus t_\varrho\}$ defines $\Gamma_I^{(t)}\mathcal{L}\sigma$ in the logic of index set I. It is easily seen that this set is recursive if I is recursive.

We could introduce a new class of interpretations, e.g. the structures of type t, but the old ones will do for our purposes. We introduce the shorthand K_t for the class of interpreting functions: $K_t \stackrel{d}{=} \Gamma_I^{(t)}\mathcal{L}\sigma$. $\equiv_t$ stands for the semantical equivalence of the logic of type t, that is $\equiv_t \stackrel{d}{=} (\Pi K_t)^{\circ}$.

VI.3 Theorem: $\equiv_t = Gr_I^{(t)} CA$

Proof:

$$\equiv_t \underset{\text{by def.}}{=} (\Pi\Gamma_I^{(t)}\mathcal{L}\omega)^{\circ} \underset{\text{by V.6}}{=} Gr_I^{(t)}\mathcal{L}\omega \underset{\text{by V.9., II7.4}}{=} Gr_I^{(t)} Lf \underset{\text{because t is dimension restriction}}{=} Gr_I^{(t)} CA$$

by def. by V.6 by V.9., II7.4 because t is dimension restriction ▲

VI.4 Corollary: $\mathfrak{F}_I/_{\equiv_t} = \mathfrak{F}_I^{(t)} CA$ ▲

Now we have that the semantical equivalence is the t-dimension restricted free congruence over the variety CA, and the tautological formulaalgebra is the t-dimension restricted free algebra over CA.

VI.5 Theorem: The class of formulaalgebras is identical with Lf, that is $Lf = \mathbb{I}\,\{\mathfrak{F}_I/_{(\Pi L)^{\circ}}$: I is arbitrary and there is a t such that $L \subseteq \Gamma_I^{(t)}\mathcal{L}\omega\}$

Proof: 1./ Any formulaalgebra $\mathfrak{A}$ is the homomorphic image of some tautological formulaalgebra $\mathfrak{F}_I/_{\equiv_t}$. Since $\mathfrak{F}_I/_{\equiv_t} = \mathfrak{F}_I^{(t)} CA \in Lf$,the formulaalgebra $\mathfrak{A}$ is also a locally finite cylindric algebra.

2./ Let $\mathfrak{B} \in Lf$, then there is a t and I such that $\mathfrak{F}_I^{(t)} Lf \succcurlyeq \mathfrak{B}$. Now there is a $g \in Ho(\mathfrak{F}_I, \mathfrak{B})$ such that $g \in \Gamma_I^{(t)} Lf \subseteq \Gamma_I^{(t)} \mathbf{SP}\,\mathcal{L}\omega$. By V.7 there is an $L \subseteq \Gamma_I^{(t)}\mathcal{L}\omega$ for which $g^{\circ} = (\Pi L)^{\circ}$. Now $\mathfrak{B} \cong \mathfrak{F}_I/_{g^{\circ}} = \mathfrak{F}_I/_{(\Pi L)^{\circ}}$, that is $\mathfrak{B}$ is a formulaalgebra. ▲

So the quasivariety generated by the formulaalgebras with type is the class of typeless formulaalgebras. We shall see that the above theorem gives a logical importance to Lf saying that Lf is just the class of formulaalgebras of classical first order logic.

Shorthands for the logic of type t

Now we can introduce a shorthand for substitutions: for any $\varrho \in I$ we define, that $n \overset{d}{=} t_\varrho - 1$, $y \overset{d}{=} n+1+\sum_{j=0}^{n} i_j$,

$$\varrho v_{i_0} \dots v_{i_n} \vdash \exists_{y+0} \dots \exists_{y+n} (\exists_0 \dots \exists_n (\varrho \wedge =_{0,y+0} \wedge \dots \wedge =_{n,y+n}) \wedge =_{y+0,i_0} \wedge \dots \wedge =_{y+n,i_n}$$

It is easy to see that this definition is correct.

VI.6 Theorem: $(h_{\mathfrak{A}} \circ \Vdash)(\varrho v_{i_0} \dots v_{i_n}) = \{ s \in {}^{\omega}A : \langle s_{i_0}, \dots, s_{i_n} \rangle \in \mathfrak{A}_\varrho \}$

Proof: The proof is easy and is similar to that of example 2./ ▲

Remark: The above theorem can also be proved as an immediate corollary of III2.2L of [2] which sais: for any $\mathfrak{B} \in \mathsf{Lw}$, $x \in B$ and one-one transformation μ on ω :

$$S^{(\mathfrak{B})}_{[\mu_0/\nu_0, \dots, \mu_n/\nu_n]} x = \{ s : (\exists z \in x)(\langle z_{\mu_0}, \dots, z_{\mu_n} \rangle = \langle s_{\nu_0}, \dots, s_{\nu_n} \rangle \ \& \ (\forall m \notin \{\mu_0, \dots, \mu_n\}) z_m = s_m \}$$

It is easily seen, that $(\Vdash(\varrho v_{i_0} \dots v_{i_n}))/_{\equiv_t} = S^{(\mathfrak{F}_I/\equiv_t)}_{[0/i_0, \dots, n/i_n]} \varrho/_{\equiv_t}$,

and by this the theorem follows from the lemma.

As it was mentioned at the end of chap. IV, we can define a new logic by appropriately choosing a subset of the names of the formulas. We shall choose the wordalgebra generated by P_t, where $P_t \overset{d}{=} \{ \varrho v_{i_0} \ldots v_{i_{t_\varrho - 1}} : \varrho \in I \}$.

Now P_t is a set of sequences and $\Vdash$ is everywhere defined in $\mathfrak{F}r_{P_t}$ and also $\Vdash^* \mathfrak{F}r_{P_t} = \mathfrak{F}r_I$, moreover $\Vdash \in \mathcal{H}o(\mathfrak{F}r_{P_t}, \mathfrak{F}r_I)$

VI.6. Definition: We define the t-type (logic) with <u>built in substitution</u> as the pair

$$L_t = \langle \mathfrak{F}r_{P_t}, \{ \mathfrak{F}r_{P_t} \restriction (f \circ \Vdash) : f \in K_t \} \rangle \quad \blacktriangle$$

It is easily seen that this is a logic indeed.

We define a labeling function for the logic L_t. The interpretations are the structures of type t, we denote their class by M_t. So the labeling function k is defined as for all $\mathfrak{A} \in M_t$, $k_{\mathfrak{A}} \in \mathcal{H}om(\mathfrak{F}r_{P_t}, \mathfrak{L}_A)$ such that for all $\varrho \in I$

$$k_{\mathfrak{A}}(\varrho v_{i_0} \ldots v_{i_{t_\varrho - 1}}) \overset{d}{=} \{ s \in {}^{\omega}A : \langle s_{i_0}, \ldots, s_{i_{t_\varrho - 1}} \rangle \in \mathfrak{A}_\varrho \}.$$

VI.7. Theorem: For all t-type structure $\mathfrak{A}$, $\mathfrak{F}r_{P_t} \restriction (h_{\mathfrak{A}} \circ \Vdash) = k_{\mathfrak{A}}$, and so $k^* M_t = \{ \mathfrak{F}r_{P_t} \restriction (f \circ \Vdash) : f \in K_t \}$.
(See fig. 10.)

Proof: $\Vdash \in \mathcal{H}o(\mathfrak{F}r_{P_t}, \mathfrak{F}r_I)$ and $h_{\mathfrak{A}} \in \mathcal{H}om(\mathfrak{F}r_I, \mathfrak{L}_A)$ implies that $\mathfrak{F}r_{P_t} \restriction (h_{\mathfrak{A}} \circ \Vdash) \in \mathcal{H}om(\mathfrak{F}r_{P_t}, \mathfrak{L}_A)$. Because $(\forall \varrho \in I)(h_{\mathfrak{A}} \circ \Vdash)(\varrho v_{i_0} \ldots v_{i_{t_\varrho - 1}}) = k_{\mathfrak{A}}(\varrho v_{i_0} \ldots v_{i_{t_\varrho - 1}})$, the functions $k_{\mathfrak{A}}$ and $\mathfrak{F}r_{P_t} \restriction (h_{\mathfrak{A}} \circ \Vdash)$ are identical. $\blacktriangle$

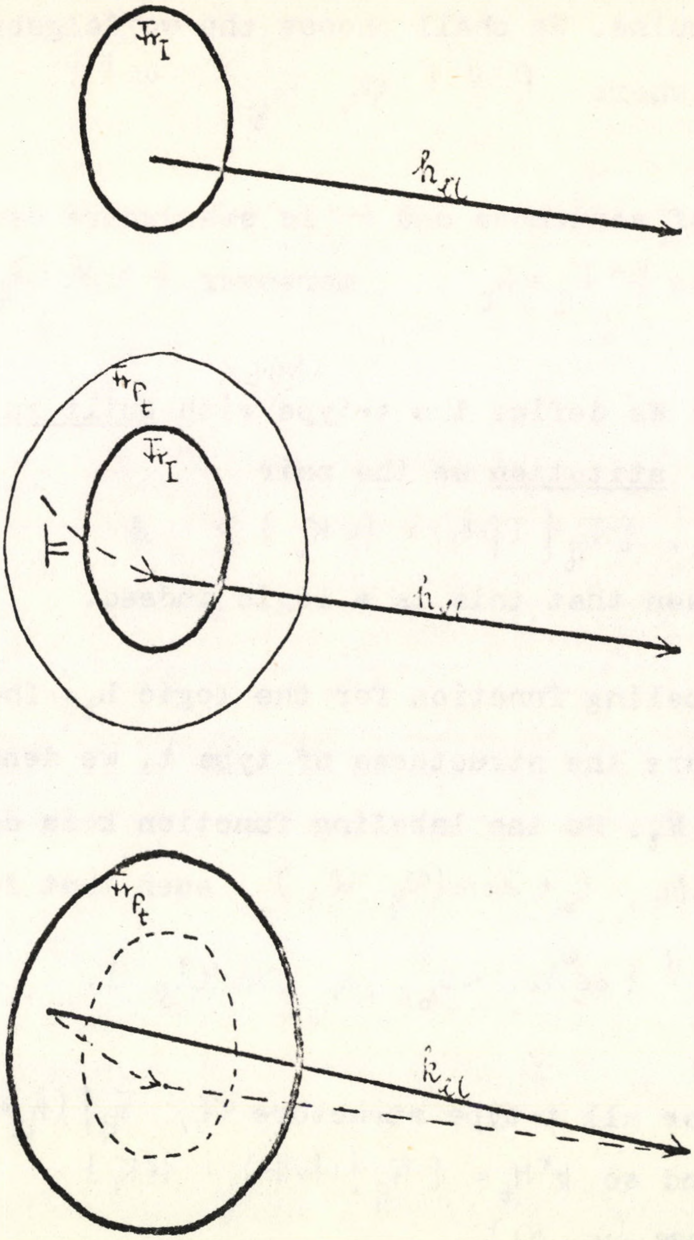

Fig. 10.

VI.8. Theorem: The logic L_t is recursively equivalent with $\langle \mathfrak{F}_I, K_t \rangle$ that is there is a recursive function k from F_{P_t} into F_I and another g from F_I into F_{P_t} such that for any t-type $\mathfrak{A}$:

$$k_{\mathfrak{A}} = h_{\mathfrak{A}} \circ k \quad \text{and} \quad h_{\mathfrak{A}} = k_{\mathfrak{A}} \circ g$$

/see fig. 10./

Proof: The proof is easy. ▲

Remark: The above theorem states that the logic L_t coincides with the classical first order logic of type t, and so the logic $\langle \mathfrak{F}_I, K_t \rangle$ also coincides with the classical logic of type t if we use the appropriate shorthands! So we proved that classical first order logic is recursively reducible to typeless logic or in other words is a recursively axiomatisable theory of typeless logic. The advantage of $\langle \mathfrak{F}_I, K_t \rangle$ to classical logic is that we can use $\langle \mathfrak{F}_I, K_t \rangle$ on two levels: one is the level of shorthands (F_{P_t}) where we have all the ease of expression we have in classical logic, and the other level is the level of F_I which makes the algebraic properties much more translucent and clear cut then that of L_t as it is shown in the followings.

Let $\approx_t$ and $\mathcal{K}_t$ stand for the semantical equivalence and class of interpreting functions of L_t respectively. Now we fix dome defining relations on $\mathfrak{Fr}_{P_t}$:

$$R_t \overset{d}{=} D_t \cup H_t$$

$$D_t \overset{d}{=} \{ < \exists_j g v_{i_0} \dots v_{i_{t_g-1}}, g v_{i_0} \dots v_{i_{t_g-1}} > : g \in I,\ i \in {}^{\omega}\omega,\ j \in \omega \}$$

$$H_t \overset{d}{=} \{ < =_{i_m j} \wedge g v_{i_0} \dots v_{i_{t_g-1}}, =_{i_m j} \wedge g v_{i_0} \dots v_{i_{m-1}} v_j v_{i_{m+1}} \dots v_{i_{t_g-1}} > : g \in I,\ i \in {}^{\omega}\omega,\ j \in \omega \}$$

VI.9. Theorem: $\Gamma_I^{(R_t)} d\sigma = \mathcal{K}_t$

Proof: The proof can be found in [2] ▲

VI.10. Theorem: $\approx_t = Cr_I^{(R_t)} CA$

Proof: $(\Pi \Gamma_I^{(R_t)} d\sigma)^{\circ} = Cr_I^{(R_t)} d\sigma = Cr_I^{(R_t)} Lf = Cr_I^{(R_t)} CA$ ▲

VI.11. Corollary: $\mathfrak{Fr}_{P_t} / \approx_t = \mathfrak{Fr}_I^{(R_t)} CA$ ▲

VI.12. Theorem: The class of the formulaalgebras of classical first order logic is Lf.

Proof: $\Vdash$ induces an isomorphism between $\mathfrak{Fr}_{P_t} / \approx_t$ and $\mathfrak{Fr}_I$ and the correspondence $h_{\mathfrak{M}}, k_{\mathfrak{M}}$ is in accordance with this isomorphism. ▲

Remark: About the necessity of the inconvenient set R_t it is proved in [2], that

$$\begin{tikzcd}
 & Gr_I^{(D_t)}CA \arrow[dr, phantom, "\not\subseteq"] & \\
\approx_t \arrow[ur, phantom, "\not\subseteq"] & & Gr_I CA \arrow[dl, phantom, "\not\subseteq"] \\
 & Gr_I^{(H_t)}CA \arrow[ul, phantom, "\not\subseteq"] &
\end{tikzcd}$$

and that for any /ℓ-type/ variety $\mathcal{V}$, $\approx_t \neq Gr_I \mathcal{V}$.

To check the completeness of a calculus of L_t we have to check that the calculus lists the equations of CA and the equalities in R_t. If instead of L_t we have $\langle \mathfrak{F}_I, K_t \rangle$ then checking the equalities $i \geq t_g \Rightarrow c_i g = g$ suffices /and of course CA/. /Of course we have to check that the relation listed by the calculus is a congruence./ To produce a complete calculus the algorithm could start from the equations of CA and the equalities in R_t /or $c_i g = g$ respectively/ and use the equation transformation rules just as in the case of the typeless logic.

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