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TOWARD A GENERAL THEORY OF LOGICS
Part I.
ON UNIVERSAL ALGEBRAICAL CONSTRUCTION OF LOGICS

Hungarian Academy of Sciences

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TOWARD A GENERAL THEORY OF LOGICS

PART I.

ON UNIVERSAL ALGEBRAICAL CONSTRUCTION OF LOGICS

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ABSTRACT

This study is concerned with the construction of a mathematical base for a general theory of logics. This general theory of logics is a frame in which different kinds of logics or complex systems of logics can be constructed, investigated, interrelated, etc.

The known alternatives of classical and nonclassical logics do fit into this frame. This frame is developed completely inside universal algebra, that is the objects treated in our general theory of logics, as well as their standard properties /e.g. completeness, reducibility, etc./ are completely algebraic. To illustrate the general methodology the following tree of logics is constructed /grown from the "root" logic/: typeless logic, logic of type t and the hierarchy of logics with built-in shorthands /e.g. variable symbols/.

The commonly used alternative of classical first-order logic is a node of this tree.

РЕЗЮМЕ

В данной работе разрабатывается математический аппарат, являющийся основой для разработки общей теории логик. Общая теория логик позволяет исследовать и создать разные логики. Она охватывает как классические, так и неклассические логики. Она состоит с одной стороны из некоторого скелета логики, а с другой - из методов, позволяющих построить на основе скелета желаемую логику. В работе предложен такой скелет логики, разработанный при помощи методов в теории универсальных алгебр. Для иллюстрации методов, позволяющих синтезировать логики, построены бестиповая логика, логика типа t и вложенная логика типа t. Последние составляют бесконечную иерархию логик. Одна из них совпадает с классической логикой первого порядка. Разработанные методы позволяют исследовать логические свойства, такие,например,как компактность, полнота, интерполяционные свойства как чисто алгебраические.

KIVONAT

Ebben a tanulmányban megkonstruáljuk azt a matematikai bázist, amelyen kifejleszthetővé válik a logikák általános elmélete.

A logikák általános elmélete egy olyan keret, melyen belül különböző logikák és kapcsolataik vizsgálhatók, konstruálhatók, stb. Ebbe az általános keretbe beillenek pl. a klasszikus és nem klasszikus logikák egyaránt. Ez a keret áll egyrészt valamilyen logika-vázból és olyan módszerekből, melyekkel a vázból kialakitható valamilyen kivánt logika. A tanulmány ezt a vázat adja meg univerzális algebrai eszközökkel.

A módszerek illusztrálására megkonstruáljuk a tipusfüggetlen, a t-tipusu és a beépitett t-tipusu logikákat. Az utóbbiak tulajdonképpen egy végtelen logikahierarchiát alkotnak. Ezek egyike a szokásos klasszikus elsőrendü logikával megegyezik. Ez a közelitésmód lehetővé teszi, hogy logikai tulajdonságokat tisztán algebrai tulajdonságokként kezeljünk /pl. kompaktság, teljesség, interpolációs tulajdonságok, stb./.

CONTENTS

I. INTRODUCTION

A general frame is introduced in which logics can be constructed as purely universal-algebraic systems. Some basic concepts are developed in this general frame, such as the completeness of a calculus or the reducibility of a logic to another, etc. Then these general tools are applied to construct different versions of classical first order logic, and to study their interrelationships. To this end the theory of cylindric algebras is applied. The interaction between mathematical logic and algebra is bidirectional since we use our typeless logic to prove that $\mathbb{SP}\,\mathsf{Lf} \neq \mathbb{HSP}\,\mathsf{Lf}$ where Lf is the class of locally finite cylindric algebras. (As far as we know this is a new result. Later we also found a purely algebraic proof of this inequality, however the logical proof is far much more straightforward.)

Each logic discussed is constructed as a purely algebraic system, and its algebraic properties are investigated. We tired to concentrate on those algebraic properties which are of essential logical importance. For example from some of these properties different kinds of interpolation properties of the logics can be derived. Strong emphasys is taken on the <u>naturalness</u> (in the universal algebraic sense) of the constructions and the properties.

To the commonly used version of classical first order logic an equivalent logic is constructed with a much more harmonic algebraic structure. Moreover this logic is shown to be recursively reducible to a logic with an even more clear cut structure and even more smooth behaviour. (We have named this logic typeless logic.) The investigations of the relations between typeless logic and the commonly used first order logic give a better understanding of the structure of substitution and questions related to variable symbols.

A methodology is also hinted how to dig to the core of a logic through repeated reductions, in other to grow a rich, structured tree of logics from this core. This growing of a tree can be controlled by adequacy criterias to a system of problem domains.

Now we discuss some technicalities about how to read this paper. We use the notations of the book of Henkin-Monk--Tarski [1]. Since this notation is generally accepted in the literature of algebraic logic we simply sum it up in a list at the end of the article and in the main test do not introduce the individual notations before using them.

The results and concepts of the theory of cylindric algebras used in this paper are summed up in Section III. without proof.

II. DEFINITION OF A GENERAL CONCEPT OF LOGIC

2.0. As it is known, the aim of a logic is to enable its user to formulate statements about certain phenomena and to represent the relation between the statements and the phenomena by truthvalues. To fulfill this task logic should have a language and some tool- to interrelate the elements of the language and the phenomena under consideration.

Definition 2.1.: By a logic we understand a pair $\langle \mathcal{F}, K \rangle$ where $\mathcal{F}$ is a wordalgebra* and $K \subseteq \mathcal{H}o\, \mathcal{F}$, that is K is a set of homomorphisms defined on $\mathcal{F}$.
To substitute the set K with a unique homomorphism we need the following operation:

Definition 2.2.: If G is a set of functions whith a common domain, that is $(\forall f \in G)\ \mathrm{Do}\, f = D$, then we define the product of G as $\prod G \stackrel{d}{=} \langle\langle f_x \rangle_{f \in G} \rangle_{x \in D}$ (see fig.1.)

We now introduce some concepts related to the concept of logic:

$$k \stackrel{d}{=} \prod K$$

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*/ The definition of word algebra is given in Section 2.5.

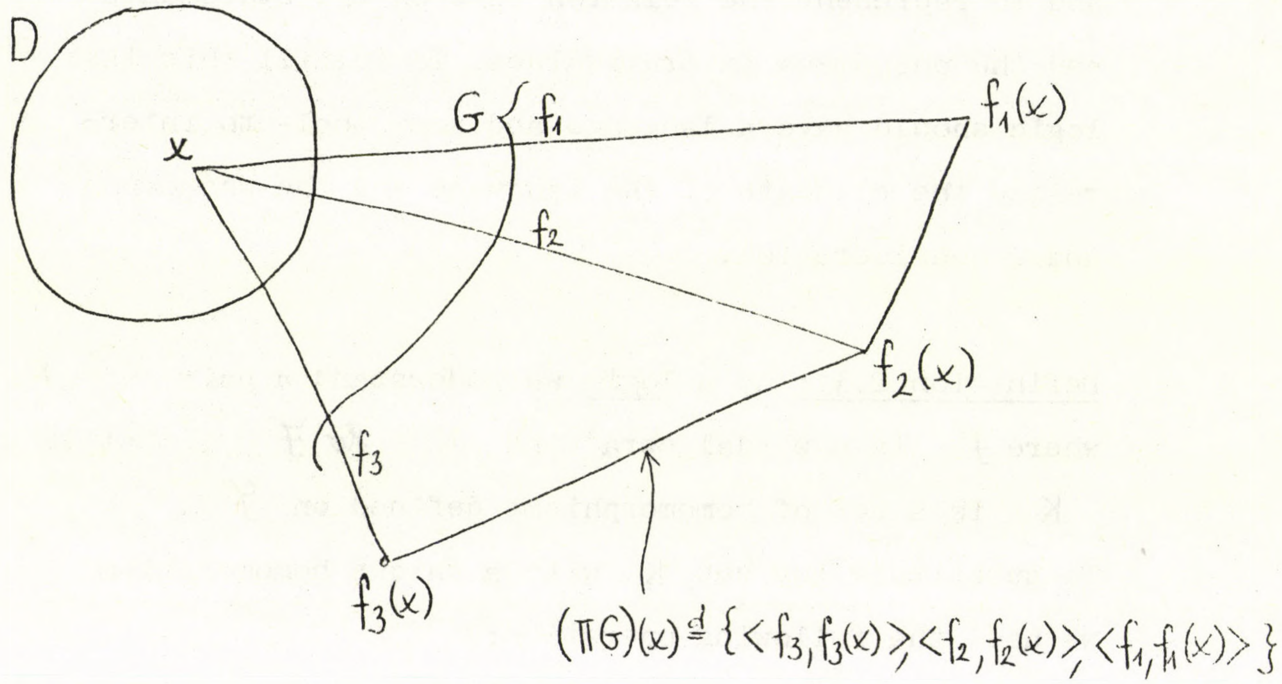


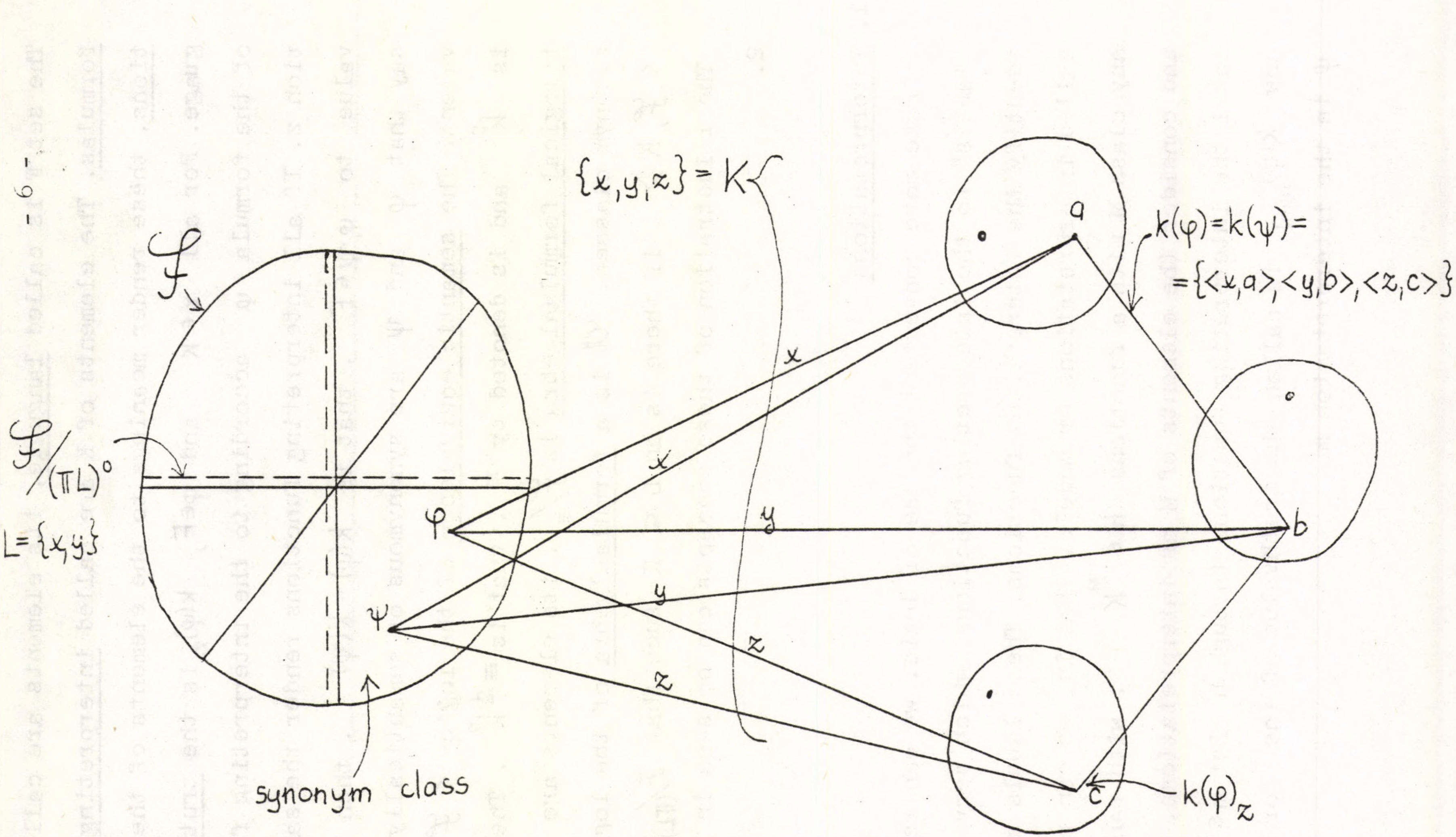
Fig. 1.

The set F is called language, its elements are called formulas. The elements of K are called interpreting functions, these render meanings to the elements of the language. For all $z \in K$ and $\varphi \in F$, $k(\varphi)_z$ is the truthvalue of the formula φ according to the interpreting function z. If all interpreting functions render the same value to $\varphi, \psi \in F$, that is $k(\varphi) = k(\psi)$, then we say that φ and ψ are synonymous or semantically equivalent. The semantic equivalence of the logic $\langle \mathcal{F}, K \rangle$ is k^0 and is denoted by $\equiv$, that is $\equiv \stackrel{d}{=} k^0$. The tautological formulaalgebra is $\mathcal{F}/_\equiv$, its elements are the synonym classes. $\mathfrak{A}$ is a formulaalgebra of the logic $\langle \mathcal{F}, K \rangle$ if there is an $L \subseteq K$ such that $\mathcal{F}/_{(\Pi L)^0} \cong \mathfrak{A}$.

The illustration of these concepts can be seen in fig. 2.

2.1. Interpretations

To make more convenient the use of logic, we can render "labels" to the interpreting functions, which serve to identify the interpreting functions. These labels are called interpretations or models. That is, we can pick any class M with a functions $h \in {}^M K$, with range K and consider the elements of M as interpretations, which label the interpreting functions through h. Let $m \in M$, now $k(\varphi)_{h(m)}$ is called the truthvalue of the formula φ in the interpretation m.



$\{x,y,z\} = K$
$k(\varphi)=k(\psi)=$
$=\{\langle x,a\rangle,\langle y,b\rangle,\langle z,c\rangle\}$
$\mathcal{F}$
$\mathcal{F}/(\Pi L)^{\circ}$
$L=\{x,y\}$
φ
ψ
x
y
z
a
b
c
synonym class
$k(\varphi)_z$

Fig. 2.

2.2. Theories of a logic, relations between logics

The set of theories of the logic $\langle \mathcal{F}, K \rangle$ is: $\{ (\Pi L)^0 : L \subseteq K \}$. So in this approach the theories are special congruence. More intuitively a theory of $\langle \mathcal{F}, K \rangle$ is the semantical equivalence of a logic $\langle \mathcal{F}, L \rangle$, where $L \subseteq K$. If given theory R we often identify it with the logic $\langle \mathcal{F}, \bigcup \{ L \subseteq K : R = (\Pi L)^0 \} \rangle$. That is certain congruences are theories and certain logics are theories too.

L is axiomatisable in $\langle \mathcal{F}, K \rangle$ iff $L = \{ f \in K : f^0 \supseteq (\Pi L)^0$.

We note that

a) L is axiomatisable in a logic iff $\langle \mathcal{F}, L \rangle$ is a theory of that logic.

b) The theories (as congruences) form a closed-set system. Given a subset G of ${}^2F$ the smallest theory containing G is the theory generated by G.

L is recursively axiomatisable in $\langle \mathcal{F}, K \rangle$ if there is a recursive subset G of ${}^2F$ such that $L = \{ f \in K : f^0 \supseteq G \}$. If the logic $L_1 = \langle \mathcal{F}, K_1 \rangle$ is a theory of $L_2 = \langle \mathcal{F}, K_2 \rangle$, then L_1 is reducible to L_2; if moreover K_2 is recursively axiomatisable in L_1 then L_2 is recursively reducible to L_1. Reducibility is a close relation between logics: If L_2 is reducible to L_1 then any logic which is a theory of L_2 is a theory of L_1 too. That is if a theorem states something about all the theories of a logic than a proof of this theorem for L_1 is also a proof of it to L_2. So if we prove the reducibility of L_1 to L_2 then all such proofs about L_1 become superfluous. Theorems of this kind are e.g.: the compactness theorem, the Löwenheim-Skolem th., the ultraproduct-th., and also the completeness theorem can be reformulated in such a form.

2.3. Shorthands

There is another means to make the use of a logic more convenient (the first one was the use of interpretations). We can introduce shorthands for the formulas, that is instead of the elements of F we can use their names. Of course, just as it was the case with the interpretations, different purposes may require different kinds of shorthands for the same logic.

N will usually stand for the set of the choosen names (or shorthands) and $\Vdash \in {}^{N}F$ stand for the function "is a name of" (or "is a shorthand for").

For example well known shorthands are:

$(\varphi \vee \psi) \Vdash \neg(\neg\varphi \wedge \neg\psi)$; $(\varphi \wedge \psi) \Vdash \langle \wedge, \langle \varphi, \psi \rangle \rangle$; and

$\forall_i \Vdash \neg\exists_i\neg$ for any $\varphi, \psi \in F$.

The illustration of the concepts discussed in this section can be seen on fig.3.

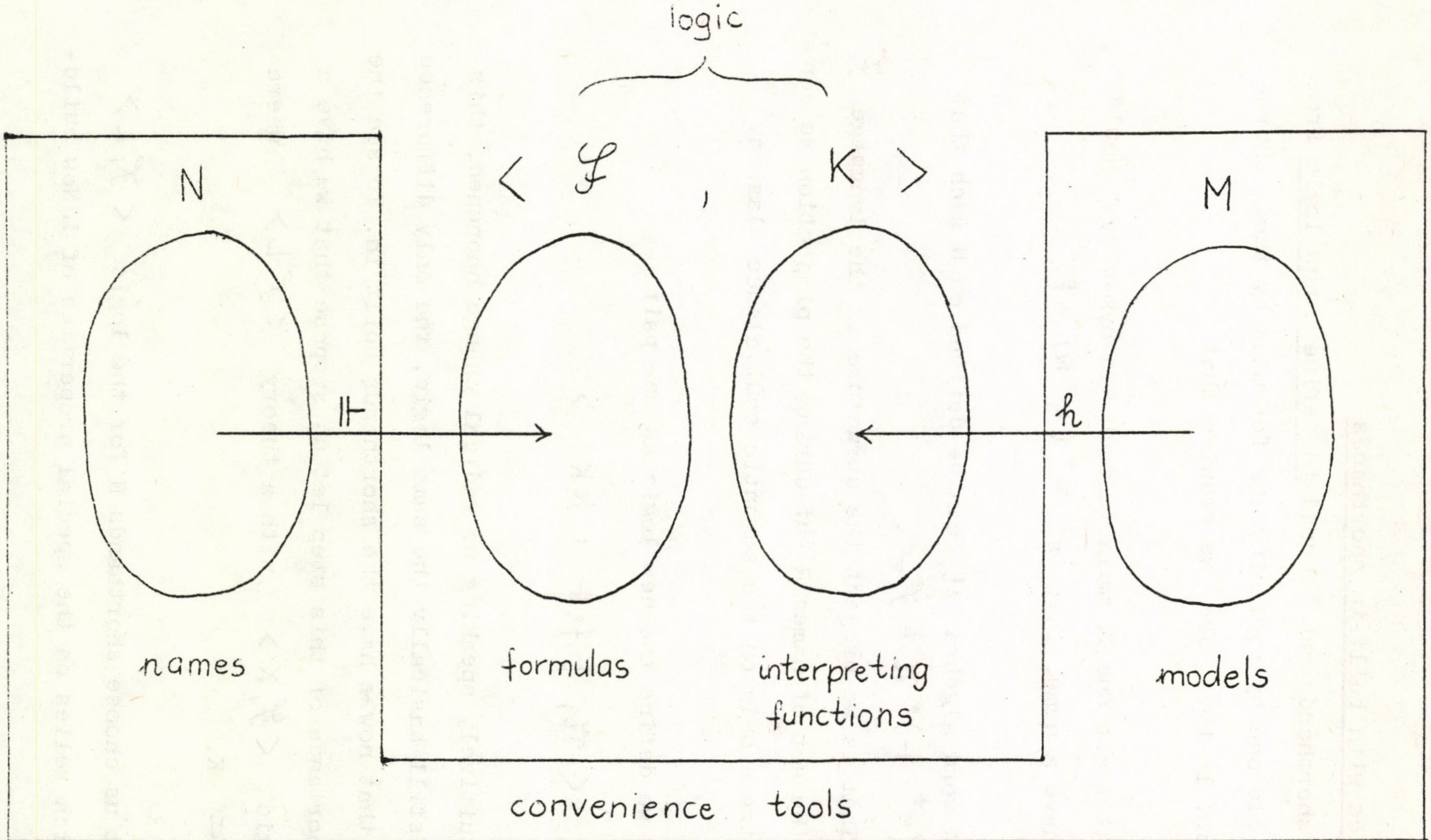

logic
⟨ $\mathcal{L}$, K ⟩
N
M
⊩
h
names
formulas
interpreting functions
models
convenience tools

Fig. 3.

2.4. Logics with built-in shorthands

The shorthands can be used to define a new logic from the old one by replacing the formulas by their shorthands. In this case we require that

1/ At least one formula in each synonym-class should have a name. That is $\equiv^* (\Vdash^* N) = F$.

2/ A word algebra $\mathfrak{N}$ can be defined on N such that $(\equiv^* \circ \Vdash)^* \mathfrak{N} = \mathcal{F}/_{\equiv}$.

That is we project the structure of the language $\mathcal{F}$ to the set of names N but during the projection we concentrate only on the semantic equivalence classes.

Now we define the new logic as the pair

$$\langle \mathfrak{N}, \{ f \circ \Vdash \ : \ f \in K \} \rangle \ .$$

Intuitively speaking no radical change happened, this is still basically the same logic, the only difference is that now we have the shorthands built in. To see the importance of this step let us suppose that we have a logic $\langle \mathcal{F}, K \rangle$ with a theory $\langle \mathcal{F}, L \rangle$ where $L \subset K$.

Let us choose shorthands N for the logic $\langle \mathcal{F}, L \rangle$ which relies on the special properties of L. Now building in the shorthands N we also build in the structure of L that is the logic $\langle \mathfrak{N}, \{ f \circ \Vdash : f \in L \} \rangle$ compared to the logic $\langle \mathcal{F}, L \rangle$ has the special structure of L built in.

One of the central aims of this paper is to investigate this process. The theory of the above processes have great importance in artificial intelligence related to the representation problem. See for example [2], [3]. More generally these questions seem to play an important role in the foundations of the theory of adequacy of languages.

In the following we give an example of these processes worked out to the case of classical first order logics. During this we arrive at different logics each of which has its special advantages. In the same time a frame is worked out in which logics can be constructed according to arbitrarily choosen purposes.

Now we show an example of the fact that a general theory of logics can be developed in the framework outlined so far. Different new disciplines (e.g. artificial intelligence) are calling for such a general theory of logics, on the other hand the special theory of logics are elaborated enough to give birth to such a general theory which is not at all trivial.

In the following section we give a result from the general theory of logics as an example. We also apply this result in section 4.4.

2.5. On compactness and complete calculuses in general theory of logics

One of our basic tool in the algebraic investigation of logics is the well-known universal algebraic concept "word-algebra" or "absolutely free algebra".

Definition 2.3.: The definition of the word-algebra is: First we fix a t-type algebra $\mathfrak{W}$ which can be thought of as a "pre-word-algebra":

a/ the universe W is the set of all n-typles of the elements of $X \cup Do\,t$ that is:

$$W \stackrel{d}{=} (X \cup Do\,t) \cup (X \cup Do\,t)\times(X \cup Do\,t) \cup (X \cup Do\,t)\times((X \cup Do\,t)\times(X \cup Do\,t))$$

b/ for all $\varrho \in Do\,t$

$$Op^{(\mathfrak{W})}(\varrho)(x_1,\dots,x_{t(\varrho)-1}) \stackrel{d}{=} \langle \varrho, \langle x_1,\dots,x_{t(\varrho)-1}\rangle\rangle$$

in the case $t(\varrho)-1=0$

$$Op^{(\mathfrak{W})}(\varrho) \stackrel{d}{=} \varrho \in W$$

Now, the absolutely free algebra or word algebra of type t generated by X is $\mathfrak{Fr}_{X,t}$, where

$$\mathfrak{Fr}_{X,t} \stackrel{d}{=} \mathfrak{Sg}^{(\mathfrak{W})} X$$

The universe of $\mathfrak{Fr}_{X,t}$ is denoted by $Fr_{X,t}$

<u>Definition 2.4.:</u> A set $S \subseteq {}^{2}\mathfrak{F}_{X,t}$ is called a defining relation.

<u>Definition 2.5.:</u> $\Gamma^{(S)}_{I,t}\mathcal{A}$ is the class of homomorphisms over the class of t-type algebras $\mathcal{A}$ with generators I and with def. relation S:

$$\Gamma^{(S)}_{I,t}\mathcal{A} \overset{d}{=} \{ g \in \mathcal{H}o\,\mathfrak{F}_{I,t} : g^{*}\mathfrak{F}_{I,t} \in \mathbb{S}\mathcal{A},\ g^{0} \supseteq S \}$$

, in case S = 0 we omit the superscript:

$$\Gamma^{(0)}_{I,t}\mathcal{A} \overset{d}{=} \Gamma_{I,t}\mathcal{A}$$

<u>Theorem 2.1.:</u> Let $\langle \mathfrak{F}_{I,t}, \Gamma_{I,t}\mathcal{A} \rangle$ be an arbitrary logic, that is $\mathcal{A}$ is an arbitrary class of algebras. If $\mathbb{S}\mathbb{P}\mathcal{A}$ is a variety[*] then the compactness theorem is valid for $\langle \mathfrak{F}_{I,t}, \Gamma_{I,t}\mathcal{A} \rangle$.

To prove the theorem we need the following five purely algebraic lemmas. From now on $\mathcal{A}$ is an arbitrary class of similar t-type algebras and S is an arbitrary defining relation.

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*/ A class of algebras is called variety, if it can be defined by a set of equations. There is a universal algebraic result, that $\mathcal{A}$ is a variety iff $\mathbb{H}\mathbb{S}\mathbb{P}\mathcal{A} = \mathcal{A}$.

<u>Lemma 2.1.:</u> $(\Pi\, \Gamma^{(S)}_{I,t} \mathcal{A})^0 = Cr^{(S)}_{I,t}\, \mathcal{A}$

Proof: $\langle x,y\rangle \in (\Pi\Gamma^{(S)}_{I,t}\mathcal{A})^0$ iff $(\forall g \in \Gamma^{(S)}_{I,t}\mathcal{A})\; g(x)=g(y)$ iff

$(\forall g \in \mathcal{H}o\,\mathfrak{F}r_{I,t})\,[\,(g^*\mathfrak{F}r_{I,t} \in \mathbb{S}\mathcal{A}\ \&\ g^0 \supseteq S) \Rightarrow \langle x,y\rangle \in g^0\,]$

iff $(\forall R \in Co\,\mathfrak{F}r_{I,t})\,[\,(\mathfrak{F}r_{I,t}/R \in \mathbb{I}\mathbb{S}\mathcal{A}\ \&\ R \supseteq S) \Rightarrow \langle x,y\rangle \in R\,]$

iff $\langle x,y\rangle \in Cr^{(S)}_{I,t}\,\mathcal{A}$

<u>Lemma 2.2.:</u> For all $g \in \Gamma^{(S)}_{I,t}\, \mathbb{S}\mathbb{P}\mathcal{A}$ there is a $G \subseteq \Gamma^{(S)}_{I,t}\mathcal{A}$ such that

$$g^0 = (\Pi G)^0$$

Proof: Since $g \in \Gamma^{(S)}_{I,t}\, \mathbb{S}\mathbb{P}\mathcal{A}$ there exists a J index set and $f \in {}^J\mathcal{A}$ for which $g^*\mathfrak{F}r_{I,t} \cong|\subseteq \underset{j\in J}{\mathbb{P}} f_j$.
If i stands for the isomorphism, for all $j \in J$ we have $\varepsilon_j \circ i \circ g \in \Gamma^{(S)}_{I,t}\mathcal{A}$ and since $(\forall x \in \mathfrak{F}r_{I,t})\; Do\, g(x) = J$, we have:

$$g^0 = (\Pi\{\varepsilon_j \circ i \circ g : j \in J\})^0 .$$

<u>Lemma 2.3.:</u> For all set I and congruence R

$$\mathfrak{F}r_{I,t}/R \in \mathbb{S}\mathbb{P}\mathcal{A} \text{ iff } (\exists L \subseteq \Gamma_{I,t}\mathcal{A})(\Pi L)^0 = R$$

Proof: 1/ $\mathfrak{F}r_{I,t}/R \in \mathbb{S}\mathbb{P}\mathcal{A} \Longrightarrow (\exists L \subseteq \Gamma_{I,t}\mathcal{A})(\Pi L)^0 = R$

$R^* \in \mathcal{H}o(\mathfrak{F}r_{I,t}, \mathfrak{F}r_{I,t}/R) \subseteq \Gamma_{I,t}\, \mathbb{S}\mathbb{P}\mathcal{A}$ since $\mathfrak{F}r_{I,t}/R \in \mathbb{S}\mathbb{P}\mathcal{A}$.

By this Lemma 2.2. gives $(\exists L \subseteq \Gamma_{I,t}\mathcal{A})(\Pi L)^0 = R$

This with the fact that for all equivalence relation r, $(r^*)^0 = r^0$ completes the proof of 1/ .

2/ $(\exists L \subseteq \Gamma_{I,t}\mathcal{A})\,(\Pi L)^0 = R \implies \mathfrak{F}_{I,t}/R \in \mathbb{SP}\mathcal{A}$

$L \subseteq \Gamma_{I,t}\mathcal{A} \implies \Pi L \in \mathcal{H}om(\mathfrak{F}_{I,t}, \mathbb{P}_{g\in L}\, g^*\mathfrak{F}_{I,t})$,

since for all $g \in L$, $g^*\mathfrak{F}_{I,t} \in \mathcal{A}$ we have $\mathfrak{F}_{I,t}/(\Pi L)^0 \in \mathbb{SP}\mathcal{A}$

Now $(\Pi L)^0 = R$ completes the proof of 2/.

<u>Lemma 2.4.:</u> $Cr^{(S)}_{I,t}\,\mathcal{A} = Cr^{(S)}_{I,t}\,\mathbb{SP}\mathcal{A}$

Proof: $\langle x,y\rangle \in Cr^{(S)}_{I,t}\,\mathbb{SP}\mathcal{A}$ iff (from Lemma 2.1.)

$(\forall g \in \Gamma^{(S)}_{I,t}\mathbb{SP}\mathcal{A})\langle x,y\rangle \in g^0$ iff (from Lemma 2.2.)

$(\forall G \subseteq \Gamma^{(S)}_{I,t}\mathcal{A})\langle x,y\rangle \in (\Pi G)^0$ iff (from Lemma 2.1.) $\langle x,y\rangle \in Cr^{(S)}_{I,t}\mathcal{A}$

<u>Lemma 2.5.:</u> $Cr_{I,t}\,\mathcal{A} = Cr_{I,t}\,\mathbb{HSP}\mathcal{A}$

Proof: is well known and can be found in [1]

The proof of theorem 2.1.:

Let us suppose that $\mathbb{SP}\mathcal{A} = \mathbb{HSP}\mathcal{A}$, and $K \overset{d}{=} \Gamma_{I,t}\mathcal{A}$.
Let I be an arbitrary set and $\equiv\, \subseteq R \in Co\,\mathfrak{F}_{I,t}$. Since $\mathfrak{F}_{I,t}/\equiv\, = \mathfrak{F}_{I,t}\mathcal{A} \in \mathbb{SP}\mathcal{A}$ we have that $\mathfrak{F}_{I,t}/R \in \mathbb{HSP}\mathcal{A}$. This, by the hypothesis, gives that $\mathfrak{F}_{I,t}/R \in \mathbb{SP}\mathcal{A}$ and so by Lemma 2.3. $(\exists L \subseteq K)(\Pi L)^0 = R$ that is R is a theory of $\langle \mathfrak{F}_{I,t}, K\rangle$. This means that the set $\{R \in Co\,\mathfrak{F}_{I,t} : R \supseteq \equiv\}$ coincides with the set of theories (on $\langle \mathfrak{F}_{I,t}, K\rangle$). Since it is well known [1] that the set of congruences containing a fixed congruence is an inductive closed-set system, we have proved the compactness th. for this logic.

We note that the above theorem states e.g. the compactness of the propositional logic since the latter has the form: $< \mathfrak{F}_{I,t}\ ,\ \Gamma_{I,t}\{<2,\cap,\smallsetminus>\} >$.

Now we turn our attention to the calculuses of a logic defined in such a general setting.

By a <u>calculus</u> of a logic we understand an algorythm listing elements of the tautological equivalence $\equiv$. A calculus is <u>complete</u> if it lists all the elements of $\equiv$.

Now we outline a general method to obtain complete calculuses for logics of the form $< \mathfrak{F}_{I,t}\ ,\ \Gamma^{(S)}_{I,t}\ \mathcal{V} >$ where $\mathcal{V}$ is a variety with a recursively given set of defining equations.

Now it is easy to see that $\equiv\ =\ G^{(S)}_{I,t}\ \mathcal{V}$ (for $G^{(S)}_{I,t}\ \mathcal{V}$ see the list of definitions.) Let Σ be the set of equations defining $\mathcal{V}$. The set of variable symbols occuring in Σ is disjoint from I (and all the other sets used). We consider the elements of I as constant symbols. Note that the equations in S consist exclusively of symbols in I and Dot. Let our algorythm start from the equations $\Sigma \cup S$ and use the usual equation rewriting rules see e.g. [4]. By the well known equational completeness theorem of universal algebra (see also [4]) this algorythm is a complete calculus , that is it lists all the pairs in $\equiv$.

Note that this approach simplifies the logical completeness considerations since the equational completeness theorem cited above has a very simple and straightforward proof.

In case of a compact logic by a complete calculus all the consequences of a recursively enumerable set of synonim-pairs can be listed.

For example the completeness of the propositional calculus can be proved in this manner in very few steps [5] (since the variety of Boolean algebras can be defined by three equations).

III. SOME PROPERTIES AND CLASSES OF CYLINDRIC ALGEBRAS

3.0. During the investigations of the kinds of logics we are going to introduce the theory of cylindric algebras will be applied.

By a structure of type $t \in {}^{I}\omega$ we understand a pair $\mathfrak{A} = \langle A, Op^{(\mathfrak{A})} \rangle$ where A is the universe of the structure and $Op^{(\mathfrak{A})}$ is a function with domain I. For any $\varrho \in I$ the value $Op^{(\mathfrak{A})}(\varrho)$ is a $t(\varrho)$-ary relation on A.

A structure is an algebra if all of its relations are functions everywhere defined on A.

We fix a type ℓ which shall be used through out the paper:

$$\ell \stackrel{d}{=} \{ \langle \wedge, 3 \rangle, \langle \neg, 2 \rangle, \langle \exists_i, 2 \rangle, \langle =_{ij}, 1 \rangle : i, j \in \omega \}$$

From now on we restrict our discussion to algebras of type ℓ.

So, before going into more detail we introduce notations for algebras of type ℓ. If $\mathfrak{A}$ is an algebra of type ℓ then:

$$Op^{(\mathfrak{A})}(\wedge) \stackrel{d}{=} \cdot^{(\mathfrak{A})}$$

$$Op^{(\mathfrak{A})}(\neg) \stackrel{d}{=} -^{(\mathfrak{A})}$$

$$Op^{(\mathfrak{A})}(\exists_i) \stackrel{d}{=} c_i^{(\mathfrak{A})}$$

$$Op^{(\mathfrak{A})}(=_{ij}) \stackrel{d}{=} d_{ij}^{(\mathfrak{A})}$$

We usually omit the index $(\mathfrak{A})$.

Let us introduce the <u>dimension-sensitivity</u> function $\Delta^{(\mathfrak{A})}$:

$$\Delta^{(\mathfrak{A})}(x) \stackrel{d}{=} \{ i : c_i^{(\mathfrak{A})} x \neq x \}$$

Since we devote ourselves to algebras of type ℓ, we set

$$\mathfrak{F}_X \stackrel{d}{=} \mathfrak{F}_{X,\ell} \quad ,$$

and we call $\mathfrak{F}_X$ the word algebra generated by X.

3.1. <u>Some important classes of ℓ-type algebras</u>

The variety of <u>cylindric algebras (CA)</u>, ([1],1.1.1.)

Let us introduce the following shorthands:

$$x+y \stackrel{d}{=} -(-x\cdot -y)$$

$$0 \stackrel{d}{=} y\cdot -y$$

$$1 \stackrel{d}{=} -0$$

Now we can define CA the class of cylindric algebras: For any ℓ -type algebra $\mathfrak{A}$

$\mathfrak{A} \in CA$ if for all $x,y,z \in A$ and $i,j,n \in \omega$ the following equations hold:

(C0) $\langle A, \cdot^{(\mathfrak{A})}, -^{(\mathfrak{A})} \rangle$ is a Boolean algebra, that is

a) $x \cdot y = y \cdot x$

b) $x \cdot (y+z) = (x \cdot y) + (x \cdot z)$

c) $x \cdot 1 = x$

Note that the symbols +, 0, and 1 are only shorthands for expressions and are not operation symbols of the algebra $\mathfrak{A}$

(C1) $c_i 0 = 0$

(C2) $c_i x \cdot x = x$

(C3) $c_i (x \cdot c_i y) = c_i x \cdot c_i y$

(C4) $c_i c_j x = c_j c_i x$

(C5) $d_{ii} = 1$

(C6) $i \neq j,n \Rightarrow d_{j,n} = c_i (d_{j,i} \cdot d_{i,n})$

(C7) $i \neq j \Rightarrow c_i (d_{ij} \cdot x) \cdot c_i (d_{ij} \cdot -x) = 0$

The class of locally finite cylindric algebras (Lf), ([1], 1.11.1.)

$$Lf \stackrel{d}{=} \{ \mathfrak{A} \in CA : (\forall x \in A)\ |\Delta^{(\mathfrak{A})} x| < \omega \} .$$

The class of <u>full cylindric set algebras</u> (Th), ([1], 1.1.5.)

The full cylindric set algebra $\mathcal{L}_A$ induced by the set A is defined by:

$$\cdot^{(\mathcal{L}_A)}(X,Y) \stackrel{d}{=} X \cap Y$$
$$-^{(\mathcal{L}_A)}(X) \stackrel{d}{=} {}^{\omega}A \setminus X$$
$$c_i^{(\mathcal{L}_A)}(X) \stackrel{d}{=} \{ s \in {}^{\omega}A : (\exists z \in X)(\forall j \in \omega \setminus \{i\})\, s_j = z_j \}$$
$$d_{ij}^{(\mathcal{L}_A)} \stackrel{d}{=} \{ s \in {}^{\omega}A : s_i = s_j \}$$

The operations $c_i^{(\mathcal{L}_A)}$ and $d_{ij}^{(\mathcal{L}_A)}$ are illustrated in fig.4.

$$Th \stackrel{d}{=} \{ \mathcal{L}_A : A \neq 0 \}$$

The class of <u>cylindric set algebras</u> ($\mathcal{H}a$), ([1], 1.1.5.)

$$\mathcal{H}a \stackrel{d}{=} \mathsf{S}\, Th$$

Note, that, as it is easily seen, any cylindric set algebra is the subalgebra of exactly one full cylindric set algebra.

The class of <u>locally independently-finite cylindric set algebras</u> ($\mathcal{L}v$), ([5])

Let $\mathfrak{A} \in \mathcal{H}a$. $a \in A$ is an independently-finite element (in the followings <u>i-finite</u> element), if $|\Delta^{(\mathfrak{A})} a| < \omega$ and $s \in a$ iff $(\exists z \in a)(\forall i \in \Delta a)\, s_i = z_i$ (see fig.5.)

$$\mathcal{L}v \stackrel{d}{=} \{ \mathfrak{A} \in \mathcal{H}a \cap Lf : (\forall a \in A)\ a \text{ is i-finite} \} .$$

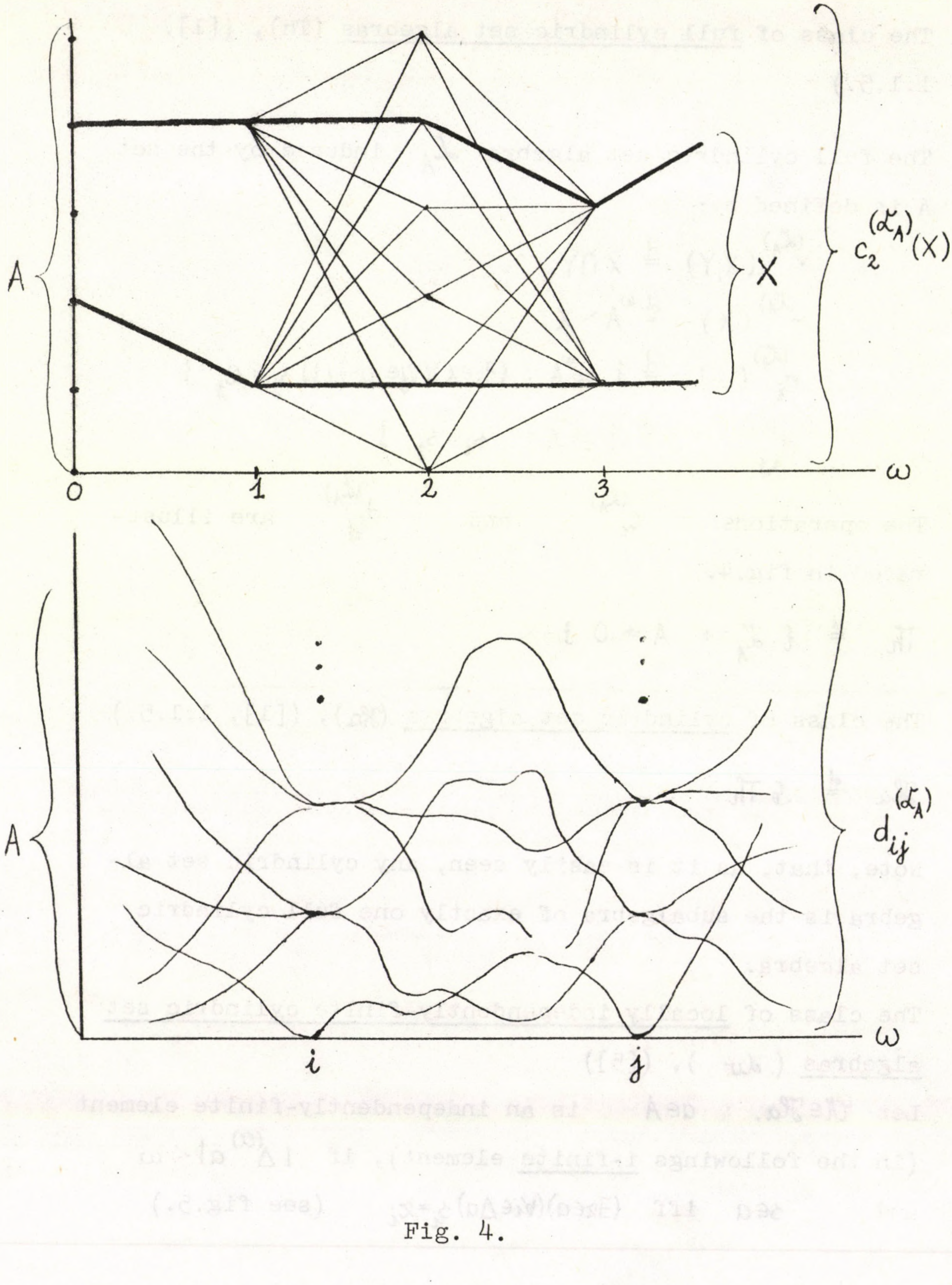


Fig. 4.

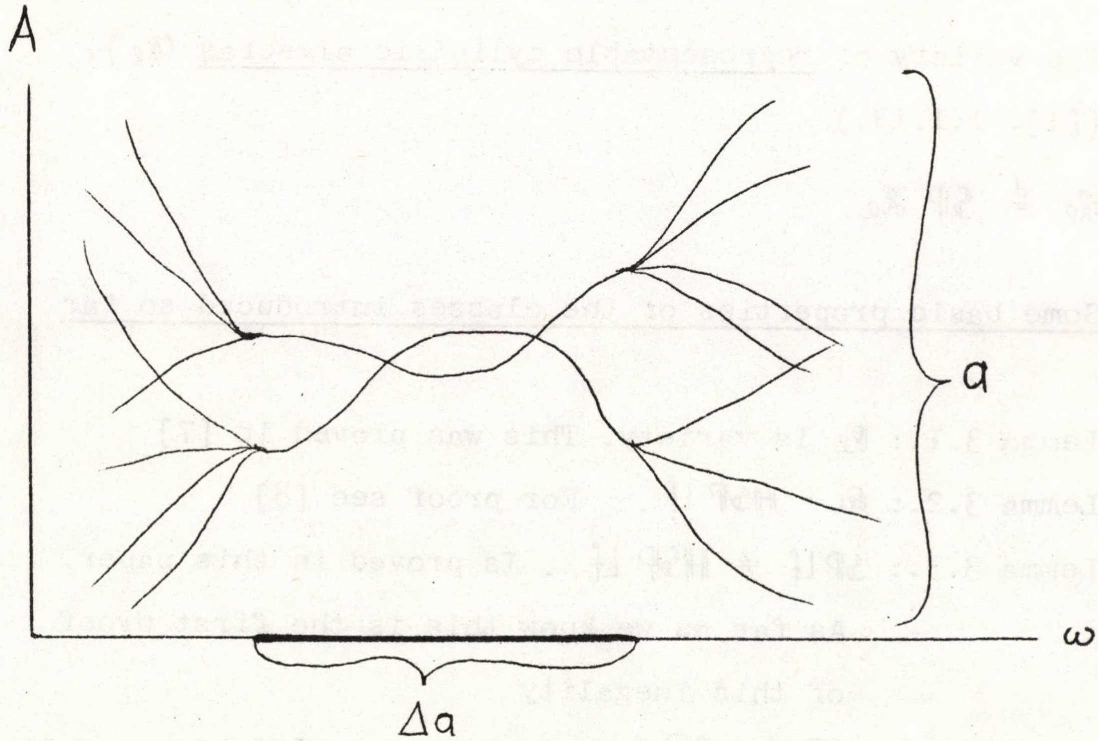


Fig. 5.

The class $\mathcal{L}w$ plays a central role in our algebraic investigations of logics. We gave a more detailed account of this, and also of the algebraic behaviour of $\mathcal{L}w$ in [6].

The variety of <u>representable cylindric algebras</u> ($\mathcal{R}e$), ([1], 1.1.13.)

$\mathcal{R}e \stackrel{d}{=} \mathbb{S}\mathbb{P}\,\mathcal{H}a$

3.2. <u>Some basic properties of the classes introduced so far</u>

Lemma 3.1.: $\mathcal{R}e$ is variety. This was proved in [7]

Lemma 3.2.: $\mathcal{R}e = \mathbb{H}\mathbb{S}\mathbb{P}\,\mathbb{L}f$. For proof see [8]

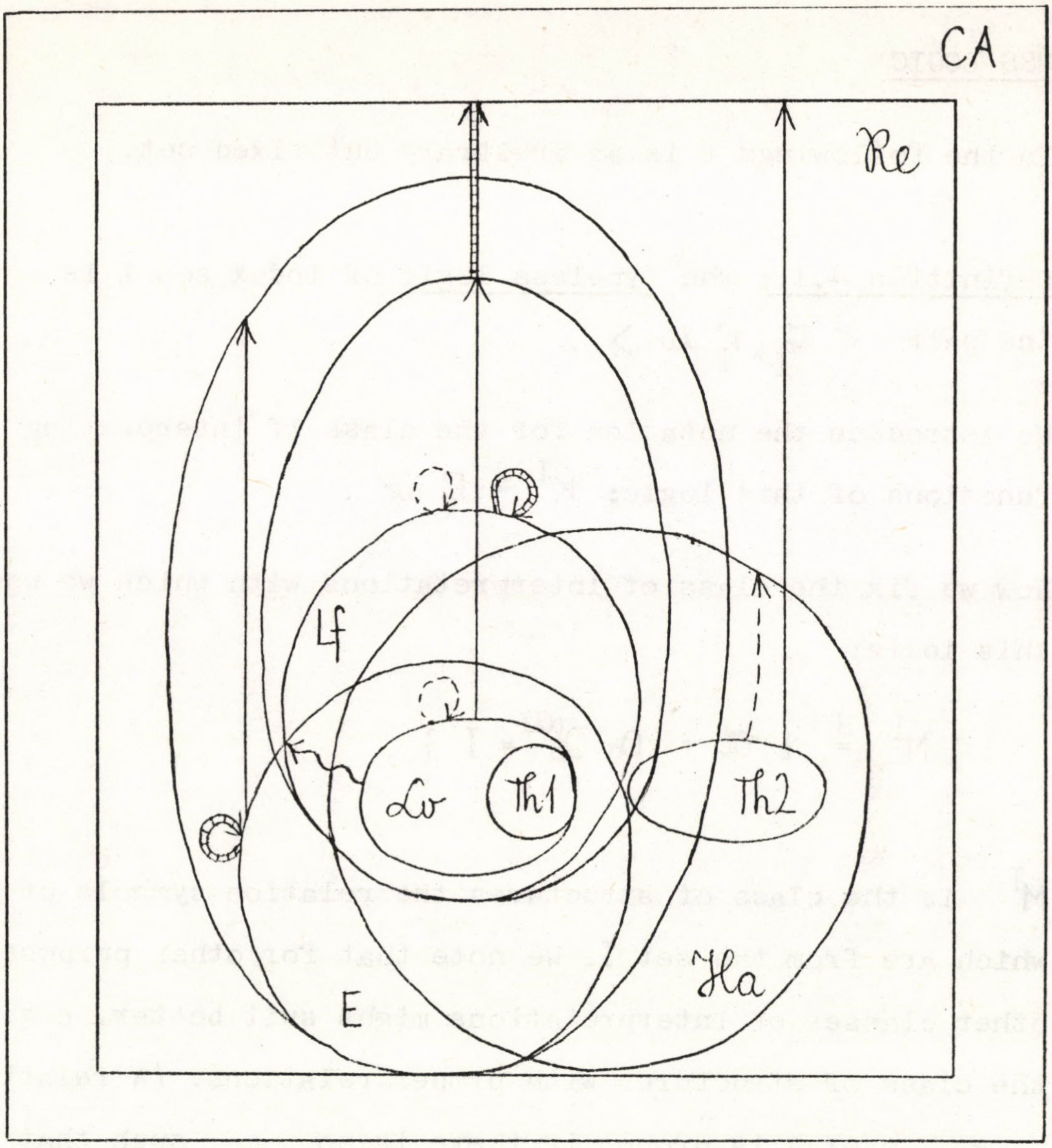
Lemma 3.3.: $\mathbb{S}\mathbb{P}\,\mathbb{L}f \neq \mathbb{H}\mathbb{S}\mathbb{P}\,\mathbb{L}f$. Is proved in this paper. As far as we know this is the first proof of this inegality

Lemma 3.4.: $\mathbb{S}\mathbb{P}\,\mathbb{L}f = \mathbb{S}\mathbb{P}\,\mathcal{L}w$. Was proved in [5]. Our results in logic are based on this equality.

Summing up the relations between classes of cylindric algebras:

$$\mathcal{L}w \subsetneq \mathbb{L}f \cap \mathcal{H}a$$
$$\mathbb{S}\mathbb{P}\,\mathcal{L}w = \mathbb{S}\mathbb{P}\,\mathbb{L}f \neq \mathbb{S}\mathbb{P}\,\mathcal{H}a = \mathbb{S}\mathbb{P}\,\mathbb{T}h = \mathcal{R}e$$
$$\mathbb{H}\mathbb{S}\mathbb{P}\,\mathcal{L}w = \mathbb{H}\mathbb{S}\mathbb{P}\,\mathbb{L}f = \mathbb{H}\mathbb{S}\mathbb{P}\,\mathcal{H}a = \mathbb{H}\mathbb{S}\mathbb{P}\,\mathbb{T}h = \mathcal{R}e$$

The connections between the different classes of cylindric algebras can be seen on fig.6.



$\mathbb{I}$ - closure

$\mathbb{S}$ - closure

$\mathbb{H}$ - closure

$\mathbb{SP}$ - closure

variety

non-variety

$E \overset{d}{=} \{$ simple cylindric algebras with the trivial cylindric algebra $\}$

$Th1 \overset{d}{=} \{ \mathcal{L}_A : |A| = 1 \}$

$Th2 \overset{d}{=} Th \setminus Th1 = \{ \mathcal{L}_A : |A| > 1 \}$

Fig.6.

IV. TYPELESS LOGIC

4.0. In the followings I is an arbitrary but fixed set.

Definition 4.1.: The typeless logic of index set I is the pair $\langle \mathfrak{F}_I , \Gamma_I \mathcal{L}\sigma \rangle$.

We introduce the notation for the class of interpreting functions of this logic: $K^I \overset{d}{=} \Gamma_I \mathcal{L}\sigma$.

Now we fix the class of interpretations with which we use this logic:

$$M^I \overset{d}{=} \{ \mathfrak{A} : Do\, Op^{(\mathfrak{A})} = I \} .$$

M^I is the class of structures the relation symbols of which are from the set I. We note that for other purposes other classes of interpretations might suit better, e.g. the class of structures with proper relations. (A relation r over A is proper if there is no q such that $r = q \times A$)

The labelling function $h \in {}^{M^I}K^I$ is defined as follows:

Definition 4.2.: For any $\mathfrak{A} \in M^I$, $h_{\mathfrak{A}} \in Hom(\mathfrak{F}_I , \mathcal{L}_A)$ such that for all $\varrho \in I$

$$h_{\mathfrak{A}}(\varrho) \overset{d}{=} \{ s \in {}^{\omega}A : (\exists n \in \omega) \langle s_0, \dots, s_n \rangle \in Op^{(\mathfrak{A})}(\varrho) \} .$$

Intuitively speaking $h_{\mathfrak{A}}$ correlates with each formula φ the set of evaluations (a subset of ${}^{\omega}A$) which satisfie φ in the interpretation $\mathfrak{A}$.

Theorem 4.1.: M^I together with h form a class of interpretations for the typeless logic of index set I, that is

$$h^* M^I = K^I .$$

Proof: 1) $h^* M^I \subseteq K^I$. We know that $h^*_{\mathfrak{A}} \mathfrak{Fr}_I \in \mathcal{Ha}$ and it is easy to see that for any $\varrho \in I$ the element $h_{\mathfrak{A}}(\varrho) = \{ s : (\exists n)\, n | s \in Op_{\varrho}^{(\mathfrak{A})}\}$ is i-finite. Since $h^*_{\mathfrak{A}} I$ generates $h^*_{\mathfrak{A}} \mathfrak{Fr}_I$ we have $h^*_{\mathfrak{A}} \mathfrak{Fr}_I \in \mathcal{Lw}$.

This complets the proof of 1).

2) $h^* M^I \supseteq K^I$. Let $g \in \Gamma_I \mathcal{Lw} = K^I$. To g we construct an $\mathfrak{A} \in M^I$ such that $h_{\mathfrak{A}} = g$. $\mathfrak{B} \overset{d}{=} g^* \mathfrak{Fr}_I \in \mathcal{Lw}$. According to the note following the definition of $\mathcal{Ha}$ there is a unique A such that $\mathfrak{B} \subseteq \mathcal{L}_A$. Now we define a structure $\mathfrak{A}$ on A: for all $\varrho \in I$ we pick an $n \in \omega$ such that $\Delta^{(\mathfrak{B})} g(\varrho) \subseteq n$ and then we fix $Op^{(\mathfrak{A})}(\varrho) \overset{d}{=} \{ \langle s_0, \dots, s_{n-1} \rangle : s \in g(\varrho) \}$. Now we show that $h_{\mathfrak{A}} = g$. For all $\varrho \in I$:

$$h_{\mathfrak{A}}(\varrho) = \{ s \in {}^{\omega}A : (\exists m \in \omega)\, m | s \in Op^{(\mathfrak{A})}(\varrho) \} =$$

$$= \{ s \in {}^{\omega}A : (\exists z \in g(\varrho)) \langle z_0, \dots, z_{n-1} \rangle = \langle s_0, \dots, s_{n-1} \rangle \} = g(\varrho)$$

from the def. of $\mathfrak{A}$ from $\mathfrak{B} \in \mathcal{Lw}$ and and $\Delta^{(\mathfrak{B})} g(\varrho) \subseteq n$.

And since $\mathfrak{B} \subseteq \mathcal{L}_A$ implies that $h_{\mathfrak{A}}, g \in \mathcal{Hom}(\mathfrak{Fr}_I, \mathcal{L}_A)$ we have: $h_{\mathfrak{A}} = g$. That is for any $g \in K^I$ there is an $\mathfrak{A} \in M^I$ such that $h_{\mathfrak{A}} = g$ and so $h^* M^I \supseteq K^I$.

Now we start to investigate the algebraic properties of typeless logic.

We prove that the semantic equivalence of typeless logic is the free congruence over $\mathcal{L}\omega$, the tautological formulaalgebra is the free algebra over $\mathcal{L}\omega$ and the class of formulaalgebras is $\mathbb{S}\mathbb{P}\mathcal{L}\omega$.

<u>Definition 4.3.:</u> The semantical equivalence of the typeless logic of index set I : $\equiv^I \overset{d}{=} (\Pi K^I)^0$

<u>Definition 4.4.:</u> The class of typeless formulaalgebras:

$\mathfrak{G}^{(F)} \overset{d}{=} \mathbb{I}\{\mathfrak{Fr}_I/_{(\Pi L)^0} : I$ is an arbitrary set, $L \subseteq K^I\}$

<u>Theorem 4.2.:</u> a) $\equiv^I = Cr_I\, \mathcal{L}\omega$

b) $\mathfrak{Fr}_I/_{\equiv^I} = \mathfrak{Fr}_I\, \mathcal{L}\omega$

c) $\mathfrak{G}^{(F)} = \mathbb{S}\mathbb{P}\, \mathcal{L}\omega$

Proof: a) $Cr_I \mathcal{L}\omega = (\Pi \Gamma_I \mathcal{L}\omega)^0$ by Lemma 2.1.

b) follows from a)

c) 1/ $\mathfrak{G}^{(F)} \subseteq \mathbb{S}\mathbb{P}\mathcal{L}\omega$. According to the definition of $\mathfrak{G}^{(F)}$ for any $\mathfrak{A} \in \mathfrak{G}^{(F)}$ there is an I and $L \subseteq \Gamma_I \mathcal{L}\omega$ such that $\mathfrak{A} \cong \mathfrak{Fr}_I/_{(\Pi L)^0}$. From this and Lemma 2.3. follows that $\mathfrak{A} \in \mathbb{S}\mathbb{P}\,\mathcal{L}\omega$.

2) $\mathfrak{S}^{(F)} \supseteq \mathbf{SP}\,\mathit{Lr}$. For any $\mathfrak{A} \in \mathbf{SP}\,\mathit{Lr}$ there is a set I such that $\mathfrak{F}_I \succcurlyeq \mathfrak{A}$. (e.g. $\mathfrak{F}_A \succcurlyeq \mathfrak{A}$.) By Lemma 2.3. this implies that $(\exists L \subseteq \Gamma_I \mathit{Lr})\,\mathfrak{A} \cong \mathfrak{F}_{I/(\mathrm{ML})^\circ}$ that is $\mathfrak{A} \in \mathfrak{S}^{(F)}$

We note that the same is true for the propositional logic if we replace (Lr) by $\{<2,\cap,\setminus>\}$. For the algebraic purposes the definition of Lr is not algebraic enough. So we try to replace it with more algebraic classes. E.g. the fact that the tautological formulaalgebra of the propositional logic is the free Boolean algebra is more algebraic as our Theorem 4.2. since the class of Boole algebras is a variety. In the followings we succeed in replacing Lr by Lf as well as Re, both having purely algebraic definitions. (The presently known algebraic definition of Re is more complicated than that of Lr, however it has the advantage that Re is a variety and a set of equations is known for it.)

<u>Theorem 4.3.:</u> a) $\models^I = C_I\,\mathit{Lf}$

b) $\mathfrak{F}_I/_{\equiv^I} = \mathfrak{F}_I\,\mathit{Lf}$

c) $\mathfrak{S}^{(F)} = \mathbf{SP}\,\mathit{Lf}$

Proof: by Lemma 3.4. and Lemma 2.4.

<u>Theorem 4.4.:</u> a) $\models^I = C_I\,\mathit{Re}$

b) $\mathfrak{F}_I/_{\equiv^I} = \mathfrak{F}_I\,\mathit{Re}$

Proof: by Lemma 3,2, and Lemma 2.5. and Theorem 4.3.

We remark that this theorem does not generalise part c) of theorem 4.3. This generalisation ($\mathfrak{F}^{(F)} = \mathsf{SP}\,\mathcal{R}e = \mathcal{R}e$) is easily seen to be equivalent with the equality $\mathsf{SP}\,\mathfrak{L}\mathfrak{f} = \mathsf{HSP}\,\mathfrak{L}\mathfrak{f}$.

In the Section 4.4. the compactness theorem is shown to fail to the typeless logic from which $\mathsf{SP}\,\mathfrak{L}\mathfrak{f} \neq \mathsf{HSP}\,\mathfrak{L}\mathfrak{f}$ immediately follow-s by Theorem 2.1. So the generalisation of part c) of Theorem 4.3. fails. However Section 4.4. does also contain an important positive result as well, Theorem 4.4. b) is used to show that typeless logic has various forms of the "interpolation property" (this has e.g. definition-theoretical corollaries).

4.1. Calculuses for typeless logic

According to the definition in 2.5. a calculus of $\langle \mathfrak{F}_I, K^I \rangle$ lists the set $\equiv^I$. It is easy to find such a calculus by using that $\equiv^I = Cr_I\,\mathcal{R}e$ and a system of equations defining $\mathcal{R}e$ is known [1]. Thus starting from the equations defining $\mathcal{R}e$ and by using the usual transformations on equations an algorithm can deduce any element of $\equiv^I$. The calculus can also lists the consequences of any finite set of formulas. The correspondence $\equiv^I = Cr_I\,\mathcal{R}e$ can be a tool not only to construct new calculuses but also to check calculuses to be complete.

We have to check that the relation listed by the calculus is a congruence and contains the equations defining $\mathfrak{Re}$.

4.2. Shorthands for typeless logic

We remind the reader that in Sectio-n 2.3. we discussed the use of shorthands and fixed some definitions. For the typeless logic of index set I we can introduce the usual shorthands, e.g. $\vee, \to, \forall_i$ etc. However we cannot introduce shorthands for substitutions that is variables. We would like to have:

$$h_{\mathfrak{A}}(\Vdash(\varrho v_{i_0} \dots v_{i_n})) = \{ s \in {}^{\omega}A : (\exists m \in \omega)\, m / \langle s_{i_0}, \dots, s_{i_n}, s_{n+1}, \dots \rangle \in \varphi^{(\mathfrak{A})}(\varrho) \}$$

We can not define this because $\Delta^{(\mathfrak{F}_I/\equiv^I)}(\varrho/\equiv^I) = \omega$.

4.3. Examples

1/ Let $I \stackrel{d}{=} \{\varrho\}$ and for each $n \in \omega$ the structure ${}_n\mathfrak{A}$:

$${}_n\mathfrak{A} \stackrel{d}{=} \langle \omega, \{ \langle \varrho, \{ \langle a_1, \dots, a_n \rangle \in {}^n\omega : a_1 < a_2 < \dots < a_n \} \rangle \} \rangle$$

It is easy to see that for any $k, n \in \omega$

$$h_{{}_n\mathfrak{A}}(\exists_0 \dots \exists_k \varrho) = {}^{\omega}\omega \quad \text{iff} \quad k \geqslant n \quad .$$

From this example it follows that $\Delta^{(\mathfrak{F}_I/\equiv^I)}(\varrho/\equiv^I) = \omega$.

2/ We would like to produce a formula φ such that

$h_{\mathfrak{A}}(\varphi) = \{ s \in {}^{\omega}\omega : s_1 < s_0 \}$, where $\mathfrak{A} \stackrel{d}{=} {}_2\mathfrak{A}$ of the example 1/. We shall see that $\varphi = \exists_2(\exists_1(\exists_0(\varrho \wedge =_{02}) \wedge =_{10}) \wedge =_{12})$

has just the required thruthvalue in $\mathfrak{A}$.

Fig.7. illustrates the above examples.

4.4. Some properties of typeless logic

Theorem 4.5.: The compactness theorem holds for the typeless logic iff $I = 0$.

Proof: 1/ Let $I \neq 0$ and $\varrho \in I$.

Consider the set of formulas

$$\Sigma \stackrel{d}{=} \{ \exists_0 \neg \varrho , \exists_1 \varrho , \exists_2 \varrho , \ldots , \exists_i \varrho , \ldots \}$$

We shall see that Σ has no models, and in the same time any finite subset Θ of Σ has models.

A/ Let Θ be a finite subset of Σ , and n the greatest integer such that $\exists_n \varrho \in \Theta$ /If there is no such number then $\Theta \subseteq \{ \exists_0 \neg \varrho \}$ and so obviously has a model./

Now we construct a model $\mathfrak{A}$ for Θ

$$A \stackrel{d}{=} \omega$$

$$Op^{(\mathfrak{A})}(\varrho) \stackrel{d}{=} \{ s \in {}^{n}A : (\exists i > 0)\ s_i > s_0 \}$$

and for any $\varrho \neq \xi \in I$ the relation $Op^{(\mathfrak{A})}(\xi)$ is arbitrary.

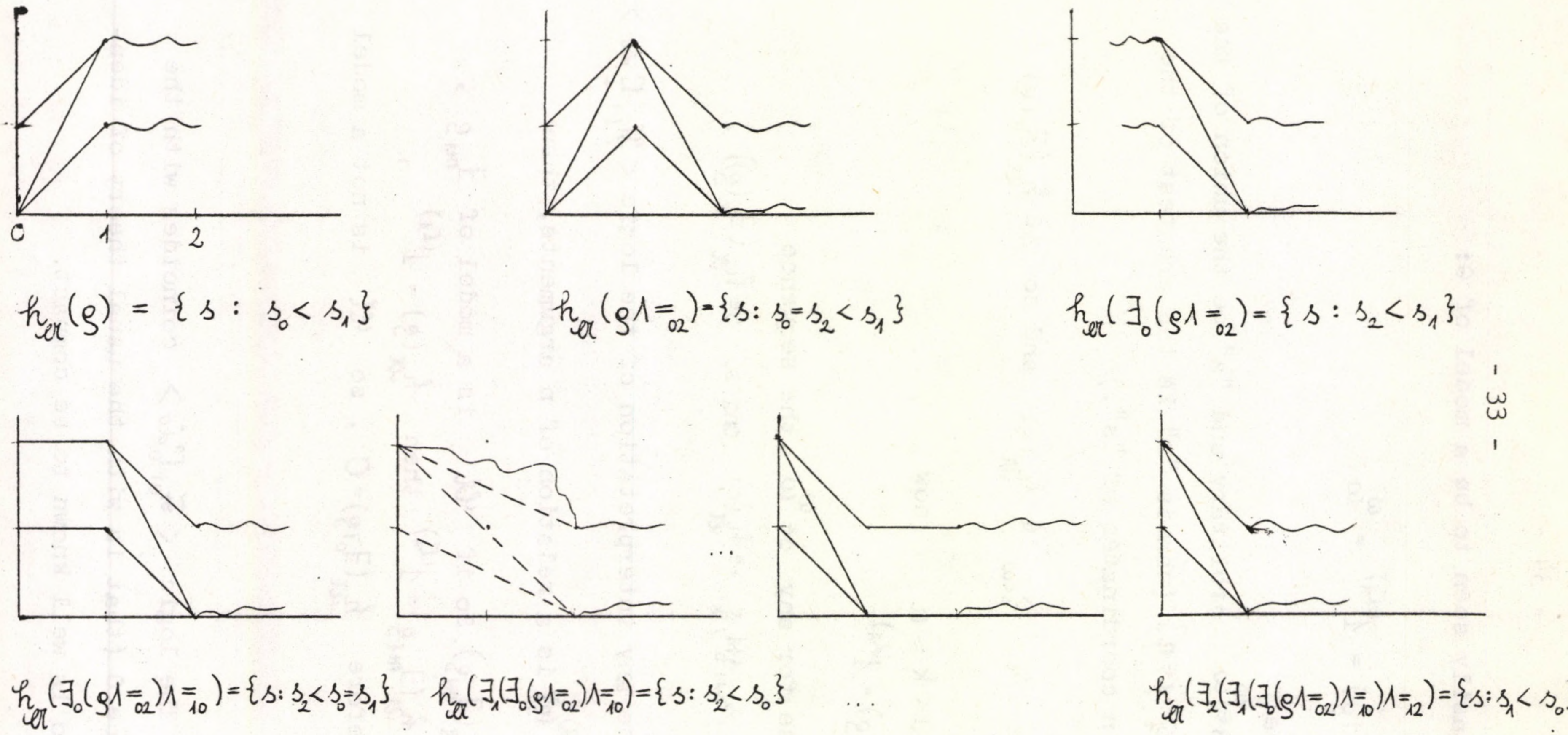

0
1
2
$h_{er}(\varrho) = \{ s : s_0 < s_1 \}$
$h_{er}(\varrho \wedge =_{02}) = \{ s : s_0 = s_2 < s_1 \}$
$h_{er}(\exists_0(\varrho \wedge =_{02})) = \{ s : s_2 < s_1 \}$
$h_{er}(\exists_0(\varrho \wedge =_{02}) \wedge =_{10}) = \{ s : s_2 < s_0 = s_1 \}$
$h_{er}(\exists_1(\exists_0(\varrho \wedge =_{02}) \wedge =_{10})) = \{ s : s_2 < s_0 \}$
...
$h_{er}(\exists_2(\exists_1(\exists_0(\varrho \wedge =_{02}) \wedge =_{10}) \wedge =_{12})) = \{ s : s_1 < s_0 \}$

Fig. 7.

$\mathfrak{A}$ is easily seen to be a model of Θ:

a) $h_{\mathfrak{A}}(\exists_0 \neg \varrho) = 1^{(\mathcal{L}_A)} = {}^{\omega}\omega$

because

let $s \in {}^{\omega}\omega$ arbitrary and "a" be the union of the set $\{s_i : i \in n\}$ (that is "a" is the greatest of the first n coordinates of "s".)

$\langle a, s_1, \dots, s_i, \dots \rangle_{i<\omega} \notin h_{\mathfrak{A}}(\varrho)$ and so $s \in h_{\mathfrak{A}}(\exists_0 \neg \varrho)$

b) Let $0 < k \leq n$ now

$h_{\mathfrak{A}}(\exists_k \varrho) = 1^{(\mathcal{L}_A)}$

because for any $s \in {}^{\omega}\omega$ the sequence

$\langle s_0, s_1, \dots, s_{k-1}, s_0+1, s_k, \dots \rangle \in h_{\mathfrak{A}}(\varrho)$ and so $s \in h_{\mathfrak{A}}(\exists_k(\varrho))$.

B/ Let $\mathfrak{A}$ be any interpretation of the logic $\langle \mathfrak{F}_I, \Gamma_I \mathcal{L}v \rangle$

If $Op^{(\mathfrak{A})}(\varrho)$ is a relation of n arguments, then

$h_{\mathfrak{A}}(\varrho) = h_{\mathfrak{A}}(\exists_{n+1} \varrho)$. So if $\mathfrak{A}$ is a model of $\exists_{n+1} \varrho$, that is $h_{\mathfrak{A}}(\exists_{n+1} \varrho) = 1^{(\mathcal{L}_A)}$ then $h_{\mathfrak{A}}(\varrho) = 1^{(\mathcal{L}_A)}$, and therefore $h_{\mathfrak{A}}(\exists_0 \neg \varrho) = 0$, so $\mathfrak{A}$ is not a model of $\exists_0 \neg \varrho$.

2/ The logic $\langle \mathfrak{F}_0, \Gamma_0 \mathcal{L}v \rangle$ coincides with the logic of type 0 (that is with the usual theory of identity) and so is well known to be compact.

<u>Corollary 4.1.:</u> $\mathbb{SP}\, Lf \neq \mathbb{HSP}\, Lf$ (and so of course $\mathbb{SP}\, \mathcal{L}\omega \neq \mathbb{HSP}\, \mathcal{L}\omega$)

Proof: Follows directly from theorem 2.1. and theorem 4.5.

This result is made more interesting by the fact (see [1] 2.6.52) that tha smallest universal class containing Lf coincides with $\mathbb{HSP}\, Lf$. That is $\mathbb{SP}\, Lf$ is not even universal. (This however does not mean that $\mathbb{SP}\, Lf$ would not be elementary see 2.6.53. [1]) The above corollary also implies that the <u>smallest free class</u> of algebras containing Lf is not a variety (and is not even universal) by theorem 1 of [9].

A class of algebras is free it is containins free algebras for arbitrary defining relations. Malcev proves that the property of being free coincides with the property of being closed for $\mathbb{SP}$.(See [9])

The above one is a logical proof for $\mathbb{SP}\, Lf \neq \mathbb{HSP}\, Lf$, we have also found a purely algebraic proof for this fact which is however somewhat more involved.

This algebraic proof can be found in the Appendix.

Now we turn our attention to the interpolation (and definition theoretic) properties of typeless logic.

First we define the interpolation property for logics with cylindric formulaalgebras.

Definition 4.5.:

Let $L = \langle \mathfrak{F}_I, K \rangle$ be a logic with $\mathfrak{F}_{I/\equiv} \in CA$.
Let φ and ψ be two formulas of L and k the set of symbols from I occuring in φ, and $\mathfrak{J}$ the same set for ψ.
That is:

$$\varphi, \psi \in \mathfrak{F}_I$$
$$\varphi \in \mathfrak{F}_k$$
$$\psi \in \mathfrak{F}_{\mathfrak{J}}$$
$$k \cup \mathfrak{J} \subseteq I$$

Let ψ be a consequence of φ that is $\varphi \leq \psi$.
L satisfies the interpolation property (IP) if:
We can find a formula χ whith symbols common in φ and ψ that is $\chi \in \mathfrak{F}_{k \cap \mathfrak{J}}$ such that:

a) strong IP: $\varphi \leq \chi \leq \psi$

b) normal IP:

There is a finite set of natural numbers $\{i_1, \dots, i_n\} \subseteq \omega$ for which

$$\varphi \leq \chi \leq \exists_{i_1} \dots \exists_{i_n} \psi$$

c) weak IP:

There is a finite $\{i_1, \dots, i_n\} \subseteq \omega$ for which

$$\forall_{i_1} \dots \forall_{i_n} \varphi \leq \chi \leq \exists_{i_1} \dots \exists_{i_n} \psi$$

see fig.8.

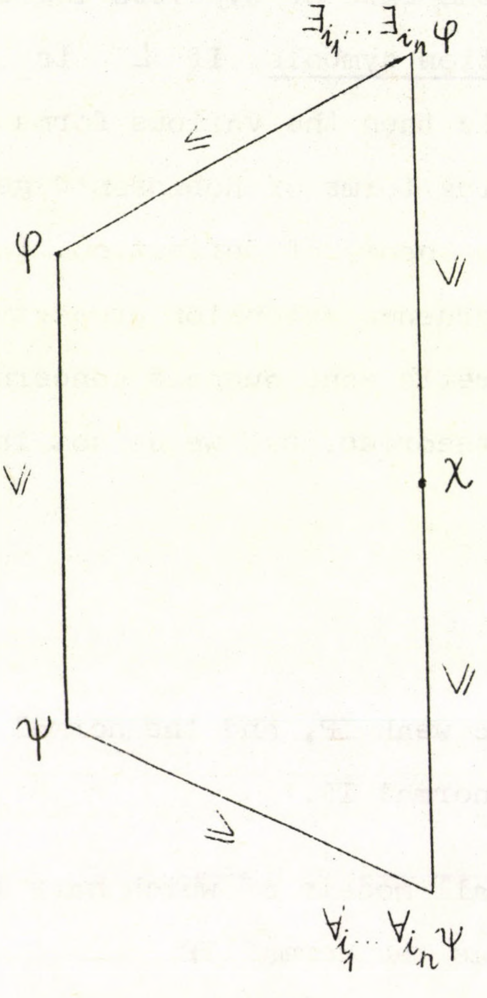

$\exists_{i_1} \dots \exists_{i_n} \varphi$
φ
χ
ψ
$\forall_{i_1} \dots \forall_{i_n} \psi$

Fig. 8.

$\mathcal{L}$ has the restricted interpolation property (RIP) if it satisfies the conditions of IP for every disjoint $\mathcal{J}$ and k .

We remark that since in the case of typeless logic the set I is the set of relation symbols, if $\mathcal{L}$ is any theory of a typeless logic then the various forms of the IP implie the corresponding forms of Robinson's general consistency result in the theory of definition. (see [10]) They also implie the congruence extension property which has some nice proof theoretic consequences concerning the independence of certain theories, but we do not investigate this line here.

Theorem 4.6.:

a) Typeless logic has the weak IP, and the normal RIP but it does not have the normal IP.

b) Any typeless theory, all models of which have the same finite cardinality, has the normal IP.

Proof: The proof is based on the corresponding results on free cylindric algebras (see [11]) and on Theorem 4.4.b) in this paper.

Some open problems:

Does typeless logic have the strong RIP?

Does any typeless theory with fixed finite cardinality of models the strong IP or at least the strong RIP?

(These problems are strongly related to the corresponding problems in the theory of CA-s.)

V. THE FIRST ORDER LOGIC OF TYPE t

5.0. Throughout this Section $t \in {}^{I}(\omega \smallsetminus 1)$ that is t is a type and I is its domain or index set.

We remind the reader that defining relations and related concepts were discussed in Section 2.5. Sometimes we use t as if it were a defining relation, in that case the superscript (t) stand for the superscript $(\{\langle \exists_i \varrho, \varrho\rangle : \varrho \in I,\ i \in \omega \smallsetminus t(\varrho)\})$. That is t is used to stand for the dimension restricting defining relation induced by t.

Definition 5.1.: By the (first order) logic of type t we understand the cuple

$$\langle \mathfrak{F}_I ,\ \Gamma_I^{(t)} \mathcal{L}\sigma \rangle .$$

Theorem 5.1.: The logic of type t is a recursively axiomatisable theory of the typeless logic of index set I.

Proof: The set of axioms $\{ \exists_i \varrho \rightarrow \varrho : \varrho \in I,\ i \in \omega \smallsetminus t(\varrho) \}$ defines $\Gamma_I^{(t)} \mathcal{L}\sigma$ in the logic of index set I. It is easily seen that this set is recursive if I is recursive.

We could introduce a new class of interpretations, e.g. the structures of type t, but the old ones will do for our purposes. We introduce the shorthand K_t for the class of interpreting functions: $K_t \stackrel{d}{=} \Gamma_I^{(t)} \mathcal{L}\sigma$. $\equiv_t$ stands for the semantical equivalence of the logic of type t, that is $\equiv_t \stackrel{d}{=} (\Pi K_t)^0$.

Now we prove that the semantical equivalence of the logic of type t is the t-dimension restricted free congruence over the veriety CA, and the tautological formulaalgebra is the t-diemnsion restricted free algebra over CA.

The quasiveriety generated by the formulaalgebras with type is also shown to coincide with the class of typeless formulaalgebras. We shall see that the above theorem gives a logical importance to Lf saying that Lf is just the class of formulaalgebras of the classical first order logic.

<u>Theorem 5.2.:</u> a) $\equiv_t = Cr_I^{(t)} CA$

b) $\mathfrak{F}r_I/_{\equiv_t} = \mathfrak{F}r_I^{(t)} CA$

c) The class of formulaalgebras is identical with Lf, that is $\mathsf{Lf} = \mathbf{I}\{ \mathfrak{F}r_I/_{(\Pi L)^0} :$ I is arbitrary and there is a t such that $L \subseteq \Gamma_I^{(t)} \mathcal{L}\sigma \}$.

Proof: a) $\equiv_t \overset{\text{by def.}}{=} (\Pi \Gamma_I^{(t)} \mathcal{L}\sigma)^0 \overset{\text{by Lemma 2.1.}}{=} Cr_I^{(t)} \mathcal{L}\sigma \overset{\text{by Lemma 2.4., Lemma 3.4.}}{=} Cr_I^{(t)} \mathsf{Lf} \overset{\text{because t is dimension restriction}}{=} Cr_I^{(t)} CA$

by def. by Lemma 2.1.

by Lemma 2.4., Lemma 3.4.

because t is dimension restriction

b) follows from a)

c/1) Any formulaalgebra $\mathfrak{A}$ is the homomorphic image of some tautological formulaalgebra $\mathfrak{F}_I/\equiv_t$. Since $\mathfrak{F}_I/\equiv_t = \mathfrak{F}_I^{(t)} CA \in Lf$, the formulaalgebra $\mathfrak{A}$ is also a locally finite cylindric algebra ([1], 2.3.3.)

c/2) Let $\mathfrak{B} \in Lf$, then there is a t and I such that $\mathfrak{F}_I^{(t)} Lf \succcurlyeq \mathfrak{B}$. Now there is a $g \in Ho(\mathfrak{F}_I, \mathfrak{B})$ such that $g \in \Gamma_I^{(t)} Lf \subseteq \Gamma_I^{(t)} SP\, \mathfrak{Lr}$. By Lemma 2.2. there is an $L \subseteq \Gamma_I^{(t)} \mathfrak{Lr}$ for which $g^0 = (\Pi L)^0$. Now $\mathfrak{B} \cong \mathfrak{F}_I/g^0 \cong$ $\cong \mathfrak{F}_I/(\Pi L)^0$ that is $\mathfrak{B}$ is a formulaalgebra.

5.1. Shorthands for the logic of type t

Now we can introduce a shorthand for substitutions: To do this for any $\varrho \in I$ we introduce notations n and y:

$$n \overset{d}{=} t(\varrho)-1 \quad \text{and} \quad y \overset{d}{=} n+1+\sum_{j=0}^{n} i_j \, .$$

Let $\Vdash$ be the smallest relation, for which:

a) for any $\varrho \in I$ and $i_0, \ldots, i_n$

$$\varrho v_{i_0} \ldots v_{i_n} \Vdash \exists_{y+0} \ldots \exists_{y+n} (\exists_0 \ldots \exists_n (\varrho \wedge =_{0,y+0} \wedge \ldots \wedge =_{n,y+n}) \wedge =_{y+0,i_0} \wedge \ldots \wedge =_{y+n,i_n})$$

b) if $\alpha \Vdash \beta$ holds, then for any sequence x,y

$$x \alpha y \Vdash x \beta y$$

(that is, the relation $\Vdash$ is "context-free")

c) $\Vdash$ is transitive

(that is the relation $\Vdash$ is a "derivation-rule")*

It is easy to see, that $\Vdash$ is a function. So choosing N such that $\Vdash^{*} N \subseteq Fr_I$ holds, $N \restriction \Vdash$ is a correct "is a name of"-function.

The following theorem states, that $\Vdash$ "gives just that meaning" to the formula $\varrho v_{i_0} \dots v_{i_n}$ which is in accordance with our intuition concerning the variables.

<u>Theorem 5.3.:</u> $(h_{\mathfrak{A}} \circ \Vdash)(\varrho v_{i_0} \dots v_{i_n}) = \{ s \in {}^{\omega}A : \langle s_{i_0}, \dots, s_{i_n} \rangle \in Op^{(\mathfrak{A})}(\varrho) \}$

Proof: The proof is easy and is similar to that of example 2)

We remark, that the above theorem can also be proved as an immediate corollary of III.2.2L of [5] which sais: for any $\mathcal{B} \in \mathcal{L}\omega$, $x \in B$ and one-one-transformation μ on ω :

$$s^{(\mathcal{B})}_{[\mu_0/\nu_0, \dots, \mu_n/\nu_n]} x = \{ s : (\exists z \in x)(\langle z_{\mu_0}, \dots, z_{\mu_n} \rangle = \langle s_{\nu_0}, \dots, s_{\nu_n} \rangle \& (\forall m \notin \{\mu_0, \dots, \mu_n\} z_m = s_m \}$$

It is easily seen, that

$$(\Vdash(\varrho v_{i_0} \dots v_{i_n}))/\equiv_t = s^{(\mathfrak{F}_I/\equiv_t)}_{[0/i_0, \dots, n/i_n]} \varrho/\equiv_t$$

and by this the theorem follows from the lemma.

---

*/ Since $\Vdash$ is a "text-function", it would be possible (and perhaps more convenient) to define $\Vdash$ by tools used in mathematical linguistic (in the present case, e.g. by a context-free grammar).

As it was mentioned in Section 2.4., we can define a new logic by appropriately choosing a subset of the names of the formulas. We shall choose the word algebra generated by P_t , where $P_t \stackrel{d}{=} \{ \varrho v_{i_0} \dots v_{i_{t(\varrho)-1}} : \varrho \in I \}$.

Now P_t is a set of sequences and $\Vdash$ is everywhere defined in $\mathfrak{F}_{P_t}$ and also $\Vdash^* \mathfrak{F}_{P_t} = \mathfrak{F}_I$ moreover $\Vdash \in \mathcal{H}o(\mathfrak{F}_{P_t}, \mathfrak{F}_I)$.

5.2. The t -type logic with built-in substitution

Definition 5.2.: We define the t -type logic with built-in substitution as the pair

$$L_t \stackrel{d}{=} \langle \mathfrak{F}_{P_t}, \{ \mathfrak{F}_{P_t} 1 (f \circ \Vdash) : f \in K_t \} \rangle$$

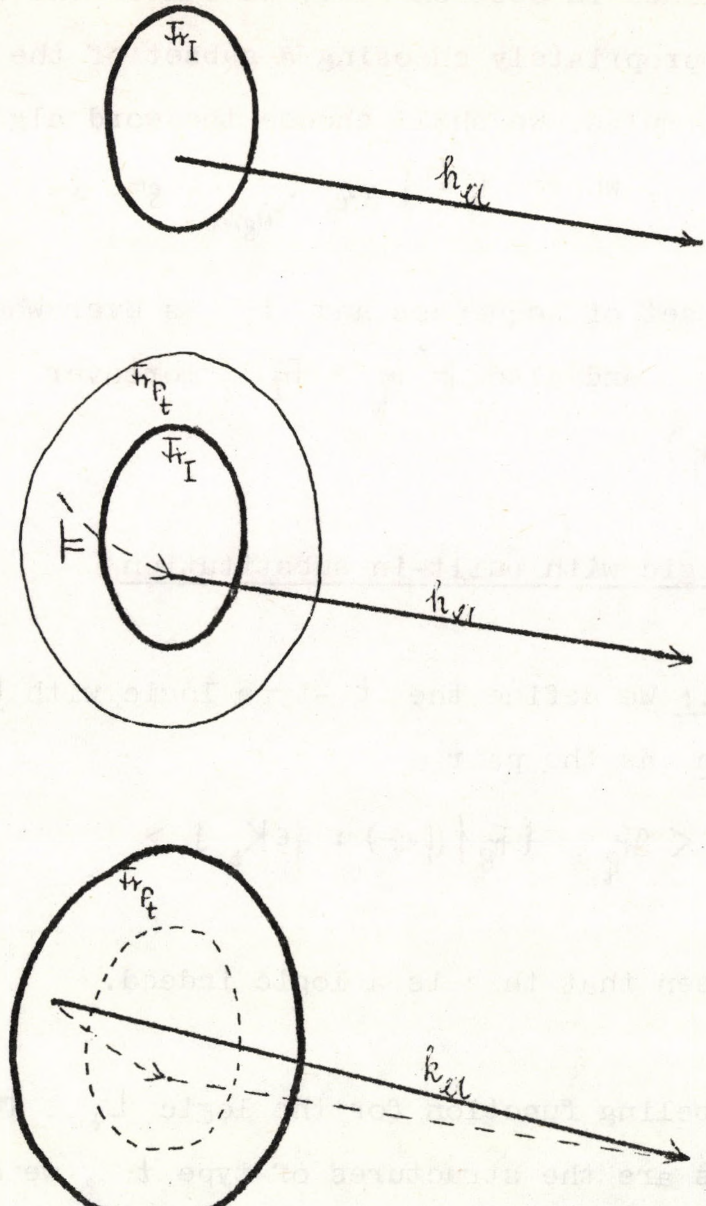
It is easily seen that this is a logic indeed.

We define a labeling function for the logic L_t . The interpretations are the structures of type t , we denote their class by M_t . The labeling function k is defined as follows:

for all $\mathfrak{A} \in M_t$, $k_{\mathfrak{A}} \in \mathcal{H}om(\mathfrak{F}_{P_t}, \mathcal{L}_A)$ such that for all $\varrho \in I$

$$k_{\mathfrak{A}}(\varrho v_{i_0} \dots v_{i_{t_\varrho - 1}}) \stackrel{d}{=} \{ s \in {}^{\omega}A : \langle s_{i_0}, \dots, s_{i_{t_\varrho - 1}} \rangle \in Op^{(\mathfrak{A})}(\varrho) \}$$

For the connection between the t -type logic and the t-type logic with built-in substitution see fig.9.



Fr_I
$h_{\mathcal{L}\mathcal{L}}$
$Fr_{\mathcal{L}_t}$
Fr_I
π
$h_{\mathcal{L}\mathcal{L}}$
$Fr_{\mathcal{L}_t}$
$\hat{k}_{\mathcal{L}\mathcal{L}}$

Fig. 9.

<u>Theorem 5.4.:</u> For all t-type structure $\mathfrak{A}$, $Fr_{P_t} \upharpoonright (h_{\mathfrak{A}} \circ \Vdash) = k_{\mathfrak{A}}$, and so

$$k^* M_t = \{ Fr_{P_t} \upharpoonright (f \circ \Vdash) : f \in K_t \}.$$

Proof: $\Vdash \in Ho(\mathfrak{F}r_{P_t}, \mathfrak{F}r_I)$ and $h_{\mathfrak{A}} \in Hom(\mathfrak{F}r_I, \mathcal{L}_A)$ implies that $Fr_{P_t} \upharpoonright (h_{\mathfrak{A}} \circ \Vdash) \in Hom(\mathfrak{F}r_{P_t}, \mathcal{L}_A)$.

Because $(\forall \varrho \in I)\,(h_{\mathfrak{A}} \circ \Vdash)\, \varrho v_{i_0} \dots v_{i_{t(\varrho)-1}} = k_{\mathfrak{A}}(\varrho v_{i_0} \dots v_{i_{t(\varrho)-1}})$, the functions $k_{\mathfrak{A}}$ and $Fr_{P_t} \upharpoonright (h_{\mathfrak{A}} \circ \Vdash)$ are identical.

<u>Theorem 5.5.:</u> The logic L_t is recursively equivalent with $\langle \mathfrak{F}r_I, K_t \rangle$ that is there is a recursive function k from Fr_{P_t} into Fr_I and another function g from Fr_I into Fr_{P_t} such that for any t-type $\mathfrak{A}$

$$k_{\mathfrak{A}} = h_{\mathfrak{A}} \circ k \quad \text{and} \quad h_{\mathfrak{A}} = k_{\mathfrak{A}} \circ g$$

Proof: The proof is easy.

We remark, that the above theorem states that the logic L_t coincides with the classical first order logic of type t, and so the logic $\langle \mathfrak{F}r_I, K_t \rangle$ also coincides with the classical logic of type t if we use the appropriate shorthands. So we proved that classical first order logic is recursively reducible to typeless logic or in other words is a recursively axiomatisable theory of typeless logic. The advantage of $\langle \mathfrak{F}r_I, K_t \rangle$ to classical logic is that we can use $\langle \mathfrak{F}r_I, K_t \rangle$ on two levels:

one is the level of shorthands ($\mathfrak{Fr}_{P_t}$) where we have all the ease of expression we have in classical logic, and the other level is the level of $\mathfrak{Fr}_I$ which makes the algebraic properties much more translucent and clear cut then that of L_t as it is shown in the followings.

Let $\approx_t$ and $\mathcal{K}_t$ stand for the semantical equivalence and class of interpreting functions of L_t respectively. Now we fix some defining relations on $\mathfrak{Fr}_{P_t}$.

$$R_t \stackrel{d}{=} D_t \cup H_t \quad \text{, where}$$

$$D_t \stackrel{d}{=} \{ \langle \exists_j \varrho v_{i_0} \dots v_{i_{t(\varrho)-1}}, \varrho v_{i_0} \dots v_{i_{t(\varrho)-1}} \rangle : \varrho \in I, i \in {}^{\omega}\omega, j \in \omega \setminus \{i_0, \dots, i_{t(\varrho)-1}\} \}$$

$$H_t \stackrel{d}{=} \{ \langle =_{i_m, j} \wedge \varrho v_{i_0} \dots v_{i_{t(\varrho)-1}}, =_{i_m, j} \wedge \varrho v_{i_0} \dots v_{i_{m-1}} v_j v_{i_{m+1}} \dots v_{i_{t(\varrho)-1}} \rangle : \varrho \in I, i \in {}^{\omega}\omega, j \in \omega \}$$

<u>Theorem 5.6.:</u> $\Gamma_I^{(R_t)} d\sigma = \mathcal{K}_t$

Proof: The proof can be found in [5].

<u>Theorem 5.7.:</u> a) $\approx_t \; = \; Cr_I^{(R_t)} CA$

b) $\mathfrak{Fr}_{P_t} / \approx_t \; = \; \mathfrak{Fr}_I^{(R_t)} CA$

c) The class of the formulaalgebras of classical first order logic is Lf.

Proof: a) $(\Pi\,\Gamma_I^{(R_t)}\mathcal{L}\omega)^0 = \mathfrak{Fr}_I^{(R_t)}\mathcal{L}\omega = \mathfrak{Fr}_I^{(R_t)} Lf = \mathfrak{Fr}_I^{(R_t)} CA$

b) follow-s from a)

c) $\Vdash$ induces an isomorphism between $\mathfrak{Fr}_{P_t}/\approx_t$ and $\mathfrak{Fr}_I$ and the correspondance $h_{\mathfrak{er}}, k_{\mathfrak{er}}$ is in accordance with this isomorphism.

We remark, that about the necessity of the inconvenient* set R_t is proved in [5], that

$$\begin{array}{ccccc} & & \mathfrak{Fr}_I^{(D_t)}CA & & \\ & \subsetneq & & \subsetneq & \\ \approx_t & & & & \mathfrak{Fr}_I^{(R_t)}CA \\ & \subsetneq & & \subsetneq & \\ & & \mathfrak{Fr}_I^{(H_t)}CA & & \end{array}$$

and that for any (ℓ -type) variety $\mathcal{V}$, $\approx_t \neq \mathfrak{Fr}_I\,\mathcal{V}$

To check the completeness of a calculus of L_t we have to check that the calculus lists the equations of CA and the equalities in R_t . If instead of L_t we have $\langle \mathfrak{Fr}_I , K_t \rangle$, then checking the equalities $i > t_\varsigma \Rightarrow c_i\varsigma = \varsigma$ suffices (and of course CA). (Of course we have to check that the relation listed by the calculus is a congruence.) To produce a complete calculus the algorithm could start from the equations of CA and the equalities in R_t (or $c_i\varsigma = \varsigma$ respectively) and use the equation transformation rules just as in the case of the typeless logic.

*/ $\mathfrak{Fr}_I^{(R_t)} CA$ is not an independent algebra over CA with generators I (in the sense of [11]) while $\mathfrak{Fr}_I^{(t)} CA$ is.

5.3. Interpolation properties in some interesting logics

Now we sum up the interpolation properties of four important kinds of logics:

1/ typeless logic,

2/ typeless theories with fixed finite modelcardinalities

3/ first order logic with substitution in general and

4/ the usual logic of type t

Of the above four kinds of logics the properties of the usual logic of type t are well known, but in the light of our theorem 5.2. and 5.5. their proof is more straightforward.

Theorem 5.8.:

		typeless logic	theories of typeless log. with finite characterist.	log. of type t	any first order logic with substitution
	strong	-	?	+	+
IP	normal	-	+	+	+
	weak	+	+	+	+
	strong	?	?	+	+
RIP	normal	+	+	+	+
	weak	+	+	+	+

Proof: the proof is based on the algebraic results of [11] and our results on the formulaalgebras. The second and the fourth column needs however some explanation:

The typeless theories with fixed finite modelcardinality are cylindric algebras of finite characteristics; and for any logic with substitution if the formulaalgebras are CA-s then they are also dimension restricted according to [1] (236p)

In the case of the last two kinds of logics (these with substitution) the results implie not only Robinson's definition-theoretic results but also the corresponding forms of Craig's interpolation theorem.

APPENDIX

Algebraic proof of $\mathbb{S}\mathbb{P}\,Lf \neq \mathbb{H}\mathbb{S}\mathbb{P}\,Lf$

In Section 4.4. we gave a logical proof of this purely algebraic theorem.

But later we succeed to find a purely algebraic proof also, here it follows:

A part of the algebraic proof can be formulated as an independent universal algebraic theorem which we have formulated as:

Theorem A.1.: For any simple algebra $\mathfrak{A}$ and any class K of algebras:

$$\mathfrak{A} \in \mathbb{S}\mathbb{P}\,K \quad \text{iff} \quad \mathfrak{A} \in \mathbb{S}\mathbb{I}\,K$$

Proof: 1/ it is well known that $\mathbb{S}\mathbb{P}\,K \supseteq \mathbb{S}\mathbb{I}\,K$

2/ let $\mathfrak{A} \in \mathbb{S}\mathbb{P}\,K$

There is a sequence $\langle \mathfrak{M}_i \rangle_{i \in J}$ of algebras in K such that $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathbb{P}_{i \in J}\, \mathfrak{M}_i$.

So for any $i \in J$ there is a homomorphic image (by the i-th projection function) $\mathfrak{B}_i$ of $\mathfrak{A}$ such that $\mathfrak{B}_i \in \mathbb{S}K$.

Since $\mathfrak{A}$ is simple any $\mathfrak{B}_i$ is either isomorphic to $\mathfrak{A}$ or trivial (has only one element). From this it follows that $\mathfrak{A}$ is isomorphic to some of the $\mathfrak{B}_i$ - s and so is a member of $\mathbf{S}\mathbf{I}\ K$.

Now we are ready to give an algebraic proof of the following theorem.

Theorem A.2.: $\mathbf{S}\mathbf{P}\,Lf \neq \mathbf{H}\mathbf{S}\mathbf{P}\,Lf$

Proof: By 2.5.24. of [1] there is a simple $\mathfrak{A} \in CA \smallsetminus Lf$. Since $Lf = \mathbf{S}\mathbf{I}\ Lf$ Theorem 3 gives that $\mathfrak{A} \notin \mathbf{S}\mathbf{P}\ Lf$. However according to 2.6.52. of [1] all simple cilindric algebras are in $\mathbf{H}\mathbf{S}\mathbf{P}\ Lf$ and so $\mathfrak{A} \in \mathbf{H}\mathbf{S}\mathbf{P}\ Lf$

Note, that this proof is much more involved than the logical one, because the results (2.5.24. and 2.6.52. of [1]) the above algebraic proof relies on have rather complicated proofs themselves.

LIST OF DEFINITIONS

0	the empty set
1	$\overset{d}{=} \{0\}$
2	$\overset{d}{=} \{0,1\}$
ω	$\overset{d}{=} \{0,1,2,\dots\}$
$D_o f$	domain of the function or relation f
$Rg\, f$	range of f
f_x, fx	x-th value of f: $f_x \overset{d}{=} fx \overset{d}{=} f(x)$
$\langle f(x)\rangle_{x\in A}$	way of defining functions: $\langle f(x)\rangle_{x\in A} \overset{d}{=} \{\langle x, f(x)\rangle : x\in A\}$
$\langle s_0, s_1, \dots, s_n, \dots\rangle_{n<\alpha}$	is a function defined on the ordinal α that is: $\langle s_0, \dots, s_n, \dots\rangle_{n<\alpha} \overset{d}{=} \{\langle n, s_n\rangle : n<\alpha\}$
$X \upharpoonright f$	f domain-restricted to X: $X\upharpoonright f \overset{d}{=} \{\langle x, fx\rangle : x\in X\}$
${}^B A$	power of A to B: ${}^B A \overset{d}{=} \{f : f: B\to A\}$
$Sb\,A$	class of subsets of A: $Sb\,A \overset{d}{=} \{B : B\subseteq A\}$
$r\circ q$	composition of r and q: $r\circ q \overset{d}{=} \{\langle b,a\rangle : (\exists c)(\langle c,a\rangle\in r \,\&\, \langle b,c\rangle \in q)\}$
$r\mid q$	relative product of r and q: $r\mid q \overset{d}{=} \{\langle a,b\rangle : (\exists c)(\langle a,c\rangle\in r \,\&\, \langle c,b\rangle\in q)\}$
f°	the equivalence-relation induced by f: $f^\circ \overset{d}{=} f\mid f^{-1}$
r^*	if $A\subseteq D_o r$, then r^*A is the r-image of A: $r^*A \overset{d}{=} \{y : (\exists x\in A)\langle x,y\rangle\in r\}$

r^*	if $a \in Do\, r$, then $r^* a$ is the r-image of a: $$r^* a \overset{d}{=} \{y : \langle a,y\rangle \in r\} = r^*\{a\}$$
$\mathfrak{Sg}^{(\mathfrak{A})} X$	subalgebra of $\mathfrak{A}$ generated by X, that is: $\mathfrak{Sg}^{(\mathfrak{A})} X$ is the least (by $\subseteqq$) element of the set $\{\mathfrak{B} \subseteq \mathfrak{A} : X \subseteq B\}$
$\mathfrak{A} \subseteqq \mathfrak{B}$	$\mathfrak{A}$ is subalgebra of $\mathfrak{B}$
$\mathfrak{A} \preccurlyeq \mathfrak{B}$	$\mathfrak{A}$ is homomorphic to $\mathfrak{B}$
$\mathfrak{A} \cong \mathfrak{B}$	$\mathfrak{A}$ is isomorphic to $\mathfrak{B}$
$\mathcal{H}o\, \mathfrak{A}$	class of homomorphisms on $\mathfrak{A}$
$\mathcal{H}o(\mathfrak{A},\mathfrak{B})$	set of homomorphisms from $\mathfrak{A}$ <u>onto</u> $\mathfrak{B}$
$\mathcal{H}om(\mathfrak{A},\mathfrak{B})$	set of homomorphisms from $\mathfrak{A}$ <u>into</u> $\mathfrak{B}$
$Co\, \mathfrak{A}$	set of congruence-relations on $\mathfrak{A}$
$\mathfrak{A}/_{\equiv}$	is defined if $\equiv\ \in Co\, \mathfrak{A}$ and than it denotes the factor-structure
$f^* \mathfrak{A}$	is defined only if f is a homomorphism on $\mathfrak{A}$ and then there is a unique $\mathfrak{B}$ such that $f \in \mathcal{H}o(\mathfrak{A},\mathfrak{B})$ now: $f^*\mathfrak{A} \overset{d}{=} \mathfrak{B}$
$\underset{i\in I}{\mathbb{P}}\ \mathfrak{A}_i$	direct product of the algebras $\mathfrak{A}_i$ according to the indexing I
$\mathbb{I}K$	class of algebras isomorphic to the elements of K: $\mathbb{I}K \overset{d}{=} \{\mathfrak{B} : \mathfrak{B} \cong\mid\in K\}$
$\mathbb{H}K$	class of algebras homomorphic to the elements of K: $\mathbb{H}K \overset{d}{=} \{\mathfrak{B} : \mathfrak{B} \preccurlyeq\mid\in K\}$
$\mathbb{S}K$	class of subalgebras of the elements of K: $$\mathbb{S}K \overset{d}{=} \{\mathfrak{B} : \mathfrak{B} \subseteqq\mid\in K\}$$

$\mathbb{P}K$ class of direct products of the elements of K:

$$\mathbb{P}K \overset{d}{=} \{\mathfrak{B} : (\exists \langle \mathfrak{A}_i\rangle_{i\in I})(Rg\,\mathfrak{A} \subseteq K \ \&\ \mathfrak{B} \cong \mathbb{P}_{i\in I}\mathfrak{A}_i\}$$

$Gr^{(S)}_{I,t}K$ free congruence over K with I generators and with defining relation S:

$$Gr^{(S)}_{I,t}K \overset{d}{=} \bigcap\{R\in Co\,\mathfrak{Fr}_{I,t} : S\subseteq R,\ \mathfrak{Fr}_{I,t}/R \in \mathbf{IS}K\}$$

$\mathfrak{Fr}^{(S)}_{I,t}K$ free algebra over K with I generators and with defining relation S:

$$\mathfrak{Fr}^{(S)}_{I,t}K \overset{d}{=} \mathfrak{Fr}_{I,t}/Gr^{(S)}_{I,t}K$$

$Gr^{(t)}_{I,t}K$

$\mathfrak{Fr}^{(t)}_{I,t}K$

$$\left.\begin{array}{l} \overset{d}{=} Gr^{(T)}_{I,t}K \\ \overset{d}{=} \mathfrak{Fr}^{(T)}_{I,t}K \end{array}\right\}$$ where T is the defining relation: $\{\langle\rho : c_i\rho\rangle : \rho\in I,\ i\notin t(\rho)\}$

$s^{i(\mathfrak{A})}_j$ substitution operation in $\mathfrak{A}$, j for i:

$$s^{i(\mathfrak{A})}_j x \overset{d}{=} c_i^{(\mathfrak{A})}(d_{ij}^{(\mathfrak{A})} \cdot^{(\mathfrak{A})} x)$$

$s^{(\mathfrak{A})}_\sigma$ is defined if $\mathfrak{A}\in Lf$ and σ is a finite transformation of ω, and then $s^{(\mathfrak{A})}_\sigma$ is the unary operation defined as follows:

if $\sigma=[\mu_0/\nu_0,\dots,\mu_{k-1}/\nu_{k-1}]$ is the canonical representation of σ ($\mu,\nu\in{}^k\omega$, $\mu_0<\dots<\mu_{k-1}$), if x is any element of A, and if $\pi_0,\dots,\pi_{k-1}$ are in this order the first k ordinals in $\omega\setminus(\Delta^{(\mathfrak{A})}x\cup Rg\,\mu\cup Rg\,\nu)$, then

$$s^{(\mathfrak{A})}_\sigma x \overset{d}{=} s^{\pi_0(\mathfrak{A})}_{\nu_0}\cdots s^{\pi_{k-1}(\mathfrak{A})}_{\nu_{k-1}} s^{\mu_0(\mathfrak{A})}_{\pi_0}\cdots s^{\mu_{k-1}(\mathfrak{A})}_{\pi_{k-1}} x$$

$[i_1/j_1,\dots,i_n/j_n]_A$ a way of defining finite transformations:

$$[i_1/j_1,\dots,i_n/j_n]_A \overset{d}{=} \{\langle i_1,j_1\rangle,\dots,\langle i_n,j_n\rangle\}\cup\{\langle a,a\rangle : a\in A\setminus\{i_1,\dots,i_n\}\}$$

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