KFKI-71-3



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# GENERAL RELATIVITY IN SPINORIAL FORM AND PETROV TYPES

Hungarian Academy of Sciences

CENTRAL RESEARCH INSTITUTE FOR PHYSICS

BUDAPEST

KFKI-71-3

# GENERAL RELATIVITY IN SPINORIAL FORM AND PETROV TYPES

/TEXT OF A SEMINARY TALK GIVEN AT THE VIENNA UNIVERSITY, NOVEMBER 20, 1970 /

by

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Abstract

An introduction to spinor methods in general relativity is presented. Gravitational equations are reformulated in spinor terms. The way of classifying gravitational fields with respect to the algebric type of the curvature tensor is discussed. The use of Petrov classification is demonstrated on the example of the peeling-off theorem.

Резюме

В общую теорию относительности вводится применение метода спиноров. Сообщаются гравитационные уравнения, написанные при помощи спиноров. Рассматривается способ классификации гравитационных полей на основе алгебраического типа тензора кривизны. Показано применение классификации Петрова в случае теоремы отслаивания.

Kivonat

Bevezetést adunk a spinor-módszereknek az általános relativitáselméletben való alkalmazásához. Megadjuk a gravitációs egyenletek spinor kifejezésekkel felirt alakját. Tárgyaljuk a gravitációs terek osztályozását a görbületi tenzor algebrai tipusának alapján. A lehántási elmélet példáján bemutatjuk a Petrov-féle osztályozás alkalmazását.

At the beginning permit me to quote the words of the late Herman Weyl taken from his book "Classical Groups" /p.273/: "In every Euclidean field we can construct the spin representation; the Euclidean nature of the field is essential ... In some way Euclid's geometry must be deeply connected with the existence of the spin representation". /He calls any flat space, irrespectively of the signature of the metric: Euclidean space./

These results of Weyl have generally been interpreted to prohibit the existence of spinors in curved space. On the other hand, we encounter many recent results in General Relativity based on the use of spinors. I only want to mention a work which is probably the most significant. Newman and Penrose had developed a spinor method by which a lot of important discoveries about the nature of gravitational field have been obtained [1/] /E.T. Newman and R. Penrose, J.Math.Phys. 3, 566 /1962//.

As an introduction to our main topic, I would like to show how is it possible that - despite of the above quoted statements of Weyl - spinors could have been so successfully fitted into the theory of General Relativity. Our second point will be the thorough reformulation of Einstein's gravitational equations in terms of spinors. This was originally done in 1960 by Roger Penrose [2/] and I think this was the real turning-point since that the significance of spinors in the theory became generally recognized.

We shall finally see the elegance and ease achieved by use of spinors on the example of the Petrov classification of gravitational fields. The original Petrov route of classifying will be replaced by the more perspeciouos spinor methods. All these matters are mainly based on the above mentioned paper of Penrose, excepting only the following discussion of the wayout from Weyl's spectic conclusions.

Let us consider the space-time, as a curved four dimensional manifold, for which it is possible to introduce in each point a tetrad of linearly independent basic vectors $\underline{X}$; $\underline{Y}$; $\underline{Z}$ and $\underline{T}$ and the normalization can be chosen such that

$$\underline{X}^2 = \underline{Y}^2 = \underline{Z}^2 = -1 , \qquad \underline{T}^2 = 1$$

further the vectors are orthogonal to each other. All these orthogonality properties can be summarized in a single equation:

$$g_{\mu\nu} = -X_\mu X_\nu - Y_\mu Y_\nu - Z_\mu Z_\nu + T_\mu T_\nu \quad ,$$

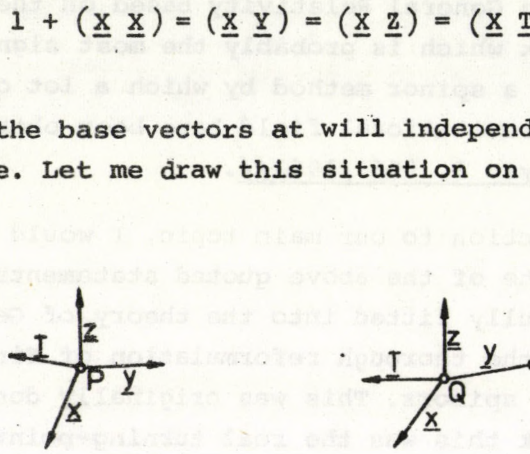
We can easily check that this equation contains all the orthogonality properties. For example we multiply by $X^\nu$ :

$$g_{\mu\nu} X^\nu = -(\underline{X}\,\underline{X}) X_\mu - (\underline{X}\,\underline{Y}) Y_\mu - (\underline{X}\,\underline{Z}) Z_\mu + (\underline{X}\,\underline{T}) T_\mu \; .$$

From the linear independence of the basis vectors it follows that

$$1 + (\underline{X}\,\underline{X}) = (\underline{X}\,\underline{Y}) = (\underline{X}\,\underline{Z}) = (\underline{X}\,\underline{T}) = 0 \; .$$

We can rotate the base vectors at will independently in the various points of the space-time. Let me draw this situation on the following symbolical figure:




Consider now point P. We can express each of the new $\underline{X}'$; $\underline{Y}'$; $\underline{Z}'$ and $\underline{T}'$ base vectors obtained by an arbitrary rotation, in terms of the old ones:

$$\underline{X}' = \alpha_{11}\,\underline{X} + \alpha_{12}\,\underline{Y} + \alpha_{13}\,\underline{Z} + \alpha_{14}\,\underline{T}$$
$$\ldots$$

To see the structure of the tetrad rotation group, we observe that the metric can equally well be put down in terms of the new tetrad, such that

$$g_{\mu\nu} = -X_\mu X_\nu - Y_\mu Y_\nu - Z_\mu Z_\nu + T_\mu T_\nu =$$

$$= -X'_\mu X'_\nu - Y'_\mu Y'_\nu - Z'_\mu Z'_\nu + T'_\mu T'_\nu$$

The transformations which preserve the above quadratic form are just the Lorentz transformations. Identifying $\underline{X}$, $\underline{Y}$, etc. with the orthogonal coordinates $x$, $y$ etc. respectively, all properties of Lorentz transformations can be transferred to the tetrad rotation group. We must not, however, confuse the tetrad transformations with the coordinate transformations in the curved space, affecting the base vectors in the well known way:

$$X'_\mu = \frac{\partial x^\nu}{\partial x'^\nu} X_\nu \quad ,$$

Now we construct a 2x2 Hermitian matrix from the basis vectors as follows:

$$\underline{\underline{M}} = \frac{1}{\sqrt{2}} \begin{bmatrix} \underline{T} + \underline{Z} & \underline{X} - i\underline{Y} \\ \underline{X} + i\underline{Y} & \underline{T} - \underline{Z} \end{bmatrix}$$

That means, $\underline{\underline{M}}$ is a matrix, the elements of which are vectors. What are the transformation properties of $\underline{\underline{M}}$ ? For coordinate transformations we obviously have:

$$\underline{\underline{M}}'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} \underline{\underline{M}}_\nu$$

Too see the tetrad transformation rules for $\underline{\underline{M}}$ , we recall that replacing the elements of $\underline{\underline{M}}$ by the rectilinear coordinates $x$, $y$ etc., we arrive at a complex two-dimensional representation of the Lorentz group:

$$\underline{\underline{M}}'_{AB'} = \Lambda_A^{\ R} \underline{\underline{M}}_{RS'} \bar{\Lambda}^{S'}_{\ B'} \qquad \begin{matrix} A, \ldots = 0,1 \\ B', \ldots = 0', 1' \ . \end{matrix}$$

Here $\Lambda$ is a 2x2 unimodular matrix, and the row indices of $\underline{\underline{M}}$ which transform with the adjoint of $\Lambda$ , are distinguished by a prime. In general, any two component quantity $\xi_A$ which transforms as

$$\xi'_A = \Lambda_A^{\ B} \xi_B$$

is called a /first rank/ spinor. All the rules of this kind of spinor calculus are easily inferred from the usual flat-space one using the correspondence between base vectors and rectilinear coordinates. We summarize these rules in the following: Primed spinors transform like

$$\bar{\xi}_{A'} = \bar{\Lambda}_{A'}{}^{B'}\, \xi_{B'}$$

the components of primed spinors can be taken the complex conjugate of the unprimed components. Spinor indices are raised and lowered by the anti-symmetric metric spinors

$$\epsilon_{AB} = \epsilon_{A'B'} = \epsilon^{AB} = \epsilon^{A'B'} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

such that, e.g., $\xi^A \eta_A$ is an invariant.

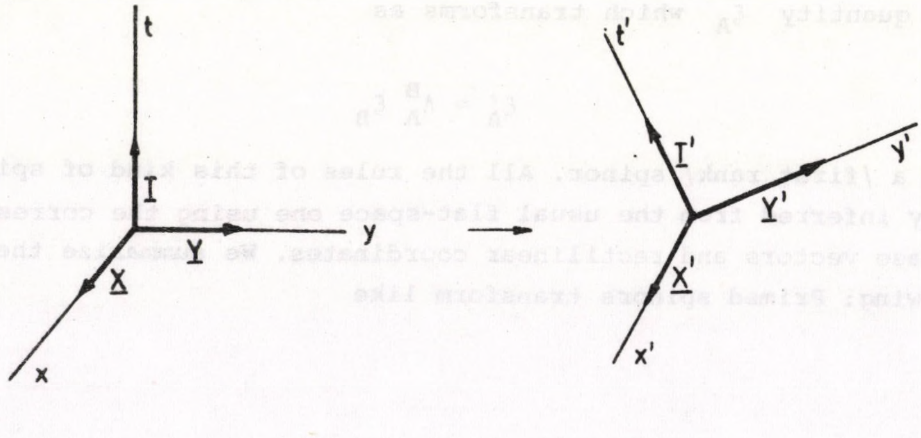
The action of a combined tetrad- and coordinate transformation on the matrix $\underline{M}$ is written as follows;

$$M'_{\mu AB'} = \frac{\partial x^\nu}{\partial x'^{\mu}} \Lambda_A{}^R\, M_{\nu RS'}\, \bar{\Lambda}_{B'}{}^{S'}$$

In terms of the orthogonality properties of the base, we can verify that $\underline{M}_\mu$ satisfies

$$M_{\mu AC'}\, M_{\nu B}{}^{C'} + M_{\mu AC'}\, M_{\mu B}{}^{C'} = g_{\mu\nu}\, \epsilon_{AB} \quad .$$

This relation shows that, in our calculus, $\underline{M}$ takes over the role of the Pauli matrices. However, as is well-known, in the usual flat-space spinor calculus, the form of Pauli matrices is independent of the coordinate system. To see the relation between the flat-space calculus and the generalized one which we want to use, let us confine ourselves to Minkowskian coordinates. We may ask if it is possible to combine the two kinds of transformations such that, as a result, the form of $\underline{M}$ be unchanged. As you know, for Lorentz transformations this can be achieved. This is easily seen if we move the base vectors and coordinate axes together:



It is clear, that if we rotate both the axes and the base together, then the new $\underline{X}'$, $\underline{Y}'$,... vectors will have the same components in the new coordinates as the old basis in the old coordinates, therefore the form of $\underline{\underline{M}}$ is preserved. This is just the well-known property of Pauli matrices in the usual flat-space spinor calculus. Therefore in the following we shall always write $\underline{\underline{M}}_\mu = \underline{\underline{\sigma}}_\mu$. Of course, in curved space we cannot stick to the rule that the components of $\underline{\underline{\sigma}}_\mu$ be coordinate invariant, but, by dropping this requirement we will be allowed to perform coordinate and spin /tetrad/ transformations quite independently. In the following, we may forget about tetrads and the construction of $\underline{\sigma}$'s. We must bear in mind only that there exists a set of the Hermitian matrices $\underline{\underline{\sigma}}_\mu$ satisfying

$$\sigma_{\mu AC'}\,\sigma_{\nu B}{}^{C'} + \sigma_{\nu AC'}\,\sigma_{\mu B}{}^{C'} = g_{\mu\nu}\,\epsilon_{AB}$$

in terms of which any tensor can be expressed equivalently as the set of its spinor components. For example:

$$\phi_{\mu\nu} \longleftrightarrow \phi_{AB'CD'} \equiv \phi_{\mu\nu}\,\sigma^{\mu}_{AB'}\,\sigma^{\nu}_{CD'}\,.$$

The spinor algebra has some special features which arise essentially from the identity

$$\epsilon_{A[B}\,\epsilon_{CD]} = 0$$

/Bracket denotes antisymmetrization/ This identity is easily checked if we recall that the indices can take only two distinct values /0 and 1/, yet the antisymmetrized quantities vanish unless all the enclosed indices are distinct. The most important consequence of this identity is that antisymmetric tensors are equivalent to symmetric second-rank spinors. We can prove this if we contract the above identity with the spinor equivalent of $F_{\mu\nu} = -F_{\nu\mu}$.

$$F^{AB'CD'} = F_{\mu\nu}\,\sigma^{\mu AB'}\,\sigma^{\nu CD'}\,,$$

and subtract the complex conjugate of the equation so obtained. As a result we have

$$F_{AB'CD'} = \frac{1}{2}\left(\epsilon_{AC}\,\overline{\phi}_{B'D'} + \epsilon_{B'D'}\,\phi_{AC}\right)$$

where

$$\phi_{AC} = F_{AR'C}{}^{R'}\,.$$

We can construct the spinor representation of the curvature tensor along the same lines. The curvature tensor has, as it can be proven by direct computation, the symmetry properties

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}$$

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$$

$$R_{\alpha[\beta\gamma\delta]} = 0$$

Using the above construction for representing antisymmetric tensors with symmetric second rank spinors, we obtain

$$R_{AE'BF'CG'DH'} = \Psi_{ABCD}\,\epsilon_{E'F'}\,\epsilon_{G'H'} + \epsilon_{AB}\,\epsilon_{CD}\,\bar{\Psi}_{E'F'G'H'} +$$

$$+ \epsilon_{CD}\,\Phi_{ABG'H'}\,\epsilon_{E'F'} + \epsilon_{AB}\,\bar{\Phi}_{E'F'CD}\,\epsilon_{G'H'} +$$

$$+ 2\Lambda\left(\epsilon_{AC}\,\epsilon_{BD}\,\epsilon_{E'F'}\,\epsilon_{G'H'} + \epsilon_{AB}\,\epsilon_{CD}\,\epsilon_{E'H'}\,\epsilon_{G'F'}\right).$$

Here the totally symmetric fourth-rank spinor $\Psi_{ABCD}$ is defined thus:

$$\Psi_{ABCD} = C_{\alpha\beta\gamma\delta}\,\sigma^{\alpha}_{AE'}\,\sigma^{\beta E'}_{B}\,\sigma^{\gamma}_{CG'}\,\sigma^{\delta G'}_{D}\ .$$

$C_{\alpha\beta\gamma\delta}$ is the trace-free part of the curvature tensor and it is called <u>Weyl tensor or conform tensor</u>. This latter name stems from a symmetry property of $C_{\alpha\beta\gamma\delta}$ which momentarily is of no interest for us. In vacuum the Einstein equations reduce to $R_{\mu\nu} = 0$ and thus the traces of $R_{\alpha\beta\gamma\delta}$ all vanish, therefore in empty space we have $C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}$. This means that in vacuum the curvature tensor corresponds to the spinor $\Psi_{ABCD}$.

In the above equation the mixed fourth rank spinor $\Phi_{ABC'D'}$ represents the trace-free part of Ricci tensor $R_{\alpha\beta}$ as follows;

$$2\Phi_{ABC'D'} = \left(R_{\mu\nu} - \frac{1}{4}\,g_{\mu\nu}\,R\right)\sigma^{\mu}_{AC'}\,\sigma^{\nu}_{BD'}\ .$$

Finally, $\Lambda$ is related to the curvature invariant $R$ by

$$\Lambda = \frac{R}{24}\ .$$

The spinor representation of curvature tensor gives us the clue for classifying the gravitational fields with respect to the algebraic properties of this tensor. But before doing this, I want to accomplish the reformulation of gravitational equations in spinor language. First of all we obviously need some rules of spinor analysis. As it is easily seen, the partial derivatives of spinors /e.g. $\partial_\mu \xi_A$/ do not transform like covariant quantities. Just as for tensors, we must introduce the covariant derivative of a spinor. Again I do not want to go into the details of building up spinor analysis, I only would like to point out, that similarly to tensor analysis, we stipulate the following properties for covariant spinor derivatives 3/:

1. Linearity: $\nabla_\mu(\xi... + \eta...) = \nabla_\mu \xi... + \nabla_\mu \eta...$
2. Leibnitz rules: $\nabla_\mu(\xi...\eta...) = \xi...\nabla_\mu \eta + (\nabla_\mu \xi...)\eta...$
3. Be partial derivative on scalars: $\nabla_\mu \varphi = \varphi,_\mu$
4. Be real operation: $\overline{\nabla_\mu \xi...} = \nabla_\mu \bar{\xi}...$
5. The covariant derivative of the fundamental quantites $\sigma_{\mu AB'}$ and $\epsilon_{AC}$ should vanish:

$$\nabla_\nu \sigma_{\mu AB'} = \nabla_\nu \epsilon_{AC} = 0 \quad .$$

There is an other version of spinor analysis, where the vanishing of $\nabla_\nu \epsilon_{AC}$ is not required. This version was intended to include the electromagnetic four-potential into the geometry, but has the drawback that spinor indices under covariant derivatives cannot be raised and lowered. R. Penrose has shown 2/ that electromagnetic phenomena can be treated satisfactorily simply even if $\nabla_\nu \epsilon_{AB} = 0$ holds.

Properties 1-5 imply that the covariant derivative of a first rank spinor has the form

$$\nabla_\mu \xi_B = \partial_\mu \xi_B - \Gamma_{\mu B}{}^{C} \xi_C \quad ,$$

where the spinor affine connection, $\Gamma_{\mu B}{}^{C}$ is written

$$\Gamma_{\mu B}{}^{C} = \frac{1}{2} \sigma_\rho{}^{CR'} \left( \sigma_{BR'}{}^{\tau} \Gamma_{\mu\tau}{}^{\rho} + \partial_\mu \sigma_{BR'}{}^{\rho} \right) .$$

The derivation rules for higher rank spinors follow from property 2.

We are now in position to translate differential relations of general relativity to spinor language. We have two important identities. The Ricci identity prescribes how the second covariant derivatives acting on a vector $v_\lambda$ commute:

$$\nabla_\mu \nabla_\nu v_\lambda - \nabla_\nu \nabla_\mu v_\lambda = R^\rho{}_{\lambda\mu\nu}\, v_\rho$$

The curvature tensor, by construction, satisfies the Bianchi identity:

$$\nabla_{[\lambda} R_{\mu\nu]\rho\sigma} = 0 \ .$$

The spinor translations of these identities read as follows;

$$\left.\begin{array}{l} \nabla_{H(E'} \nabla^H{}_{F')}\, \xi_D = \phi_{DBE'F'}\, \xi^B \\ \nabla_{(A}{}^{P'} \nabla_{B|P'|}\, \xi_{C)} = -\Psi_{ABCD}\, \xi^D + 2\Lambda\, \xi_{(A}\, \epsilon_{B)C} \end{array}\right\} \text{Ricci}$$

$$\left.\begin{array}{l} \nabla^D{}_{E'}\, \Psi_{ABCD} = \nabla_{(C}{}^{F'}\, \Phi_{AB)E'F'} \\ \nabla^{BF'}\, \Phi_{ABE'F'} = -3\nabla_{AE'}\, \Lambda \end{array}\right\} \text{Bianchi}$$

Here the "spinor derivatives" are defined by

$$\nabla_{AE'} \equiv \sigma_{AE'}{}^{\mu}\, \nabla_\mu$$

and the parentheses denote symmetrization with the appropriate normalization factors, e.g. $\xi_{(AB)} = \frac{1}{2!}\left(\xi_{AB} + \xi_{BA}\right)$. Subscripts for which the symmetrization does not hold are separated by strokes.

We have now four differential identities upon which the Einstein equations are to be imposed:

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = -\kappa\, T_{\mu\nu}$$

These equations mean algebraic conditions on the quantities $\phi_{ABC'D'}$ and $\Lambda$. For example, in empty space we have

$$\phi_{ABC'D'} = \Lambda = 0 \ .$$

Thus in vacuum the field equations take the form:

$$\nabla^D{}_{E'}\, \Psi_{ABCD} = 0 + \text{Ricci identities} \ .$$

We now see that the vacuum gravitational fields are massless fourth rank spinor fields /that means, this field is of spin 2/ which satisfy a covariant spinor wave equation.

The spinorial form of the gravitational equations is the starting point of the Newman - Penrose equations [1/]. We have time not enough to give a detailed account of the Newman-Penrose method. I would like only to mention, that in the spin space, just like in the usual four dimensional space-time, it is possible to introduce a basis. This now consists of two independent spinors. All spinor quantities are expressed in terms of their projections onto this basic dyad. The Newman-Penrose equations are essentially the dyad projections of the above spinor relations. They are extensively used for treating problems in General Relativity.

Now let us turn our attention to the classification of gravitational fields. This task was first undertaken by A.Z. Petrov [4/]. We shall not follow his rather complicated method but rather we use spinor technique which enabled Penrose [2/] to refine and simplify the original Petrov classification.

As I have shown previously, the decomposition of the curvature tensor in terms of irreducible spinors yields three quantities which correspond to the conformal tensor, Ricci tensor and curvature scalar, respectively. The latter quantities are locally fixed by the Einstein equations for any given distribution of matter. The only quantity which essentially describes the gravitational field is the conformal tensor or the corresponding spinor $\psi_{ABCD}$. In addition, for empty space this tensor is the curvature itself. Therefore we shall consider the local properties of this quantity.

As Pirani has remarked [3/], one of the most elegant results of the spinor algebra is the decomposition of any symmetric spinor into the symmetrized product of first-rank spinors. Let $\varphi_{ABC\ldots}$ be an arbitrary symmetric /that means, irreducible/ spinor with $p$ indices. Consider the expression

$$\varphi(\xi) = \varphi_{ABC\ldots}\, \xi^A \xi^B \xi^C \ldots$$

where $\xi^A$ is a first rank spinor. Writing $z = \xi^1/\xi^o$ and separating the factor $(\xi^o)^p$ in $\varphi$ we get

$$\varphi(\xi) = (\xi^o)^p\, P(z)$$

where $P$ is a complex polinomial expression in $z$. We can factorize $P$ as

$$P(z) = (\alpha_1 z - \alpha_o)(\beta_1 z - \beta_o) \ldots$$

such that for $\varphi$ we may write

$$\varphi(\xi) = \left(\alpha_A \xi^A\right)\left(\beta_B \xi^B\right)\ldots$$

Since $\xi^A$ is arbitrary,

$$\varphi_{ABC\ldots} = \alpha_{(A} \beta_B \ldots \pi_{P)}$$

Thus we have obtained the canonical decomposition of the symmetric $\varphi_{ABC\ldots}$ into the symmetrized product of first rank spinors. The spinors $\alpha_A$, $\beta_B$, ... are called principal spinors and are determined up to a /complex/ scalar factor. They need not be all distinct.

Any first rank spinor $\alpha_A$ determines a real litelike vector $\ell_\mu$ by

$$\ell_\mu = \alpha_A \sigma_\mu^{AB'} \bar{\alpha}_{B'}$$

That the vector $\ell_\mu$ is litelike, follows from the fundamental properties of the $\underline{\sigma}$ matrices /see above/ from which the necessary relation

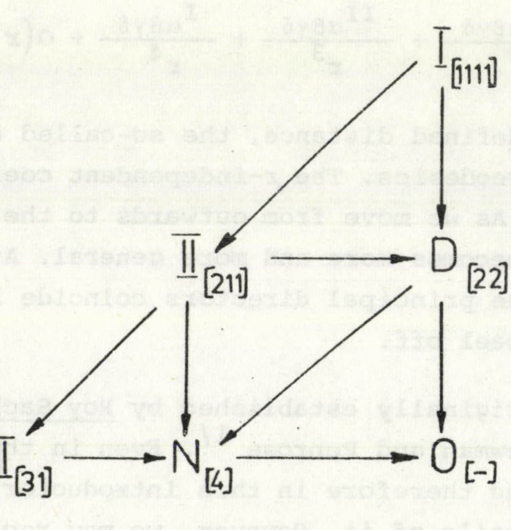
$$\sigma_{AB'\mu} \sigma_{CD'}{}^{\mu} = \epsilon_{AC} \epsilon_{B'D'}$$

can be obtained.

Thus we see that the canonical decomposition of irreducible spinor of rank p gives at least one but at most p real null directions. Let us now consider the curvature spinor $\Psi_{ABCD}$. Obviously there are six different possibilities. The type and name of the various classes is shown on the following table:

| Type | Name | Symbolically |
|---|---|---|
| $\Psi_{ABCD} = \alpha_{(A} \beta_B \gamma_C \delta_{D)}$ | I | [1111] |
| $\Psi_{ABCD} = \alpha_{(A} \alpha_B \beta_C \gamma_{D)}$ | II | [211] |
| $\Psi_{ABCD} = \alpha_{(A} \alpha_B \beta_C \beta_{D)}$ | D | [22] |
| $\Psi_{ABCD} = \alpha_{(A} \alpha_B \alpha_C \beta_{D)}$ | III | [31] |
| $\Psi_{ABCD} = \alpha_{(A} \alpha_B \alpha_C \alpha_{D)}$ | N | [4] |
| O | O | [-] |

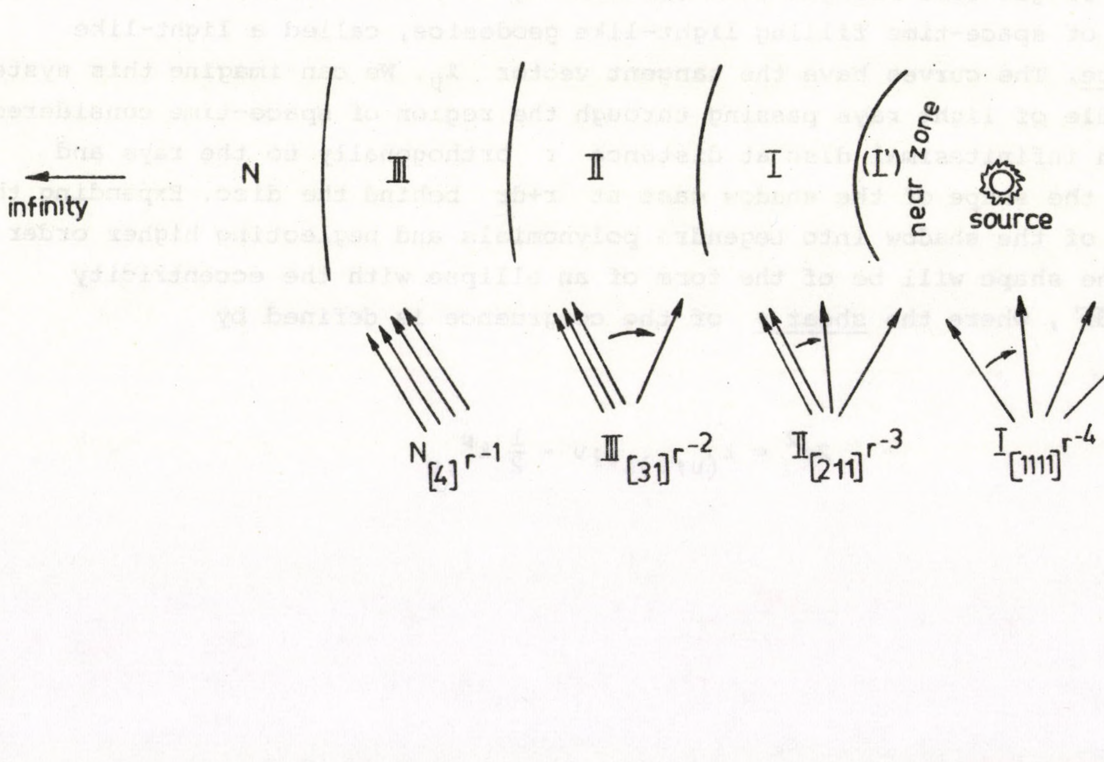
The historical names for the various types will be explained later. The interrelation among Petrov-types can be visualized on the Penrose-diagram:




Penrose diagram

Here the arrows show the direction of increasing specialization.

It is usual to call type I as <u>algebrically general</u> and the remaining types <u>algebrically special</u>.
The physical significance of the Petrov classification very strikingly reveals in the "<u>peeling-off" theorem</u>. This theorem states that the gravitational radiational field of a localized source can be divided into shells with different Petrov types. The situation can be visualized on the following qualitative picture in the ordinary three-dimensional space:



This picture reflects that the curvature tensor of the radiation field can be written in the form

$$R_{\alpha\beta\gamma\delta} = \frac{N_{\alpha\beta\gamma\delta}}{r} + \frac{III_{\alpha\beta\gamma\delta}}{r^2} + \frac{II_{\alpha\beta\gamma\delta}}{r^3} + \frac{I_{\alpha\beta\gamma\delta}}{r^4} + O\left(r^{-5}\right) .$$

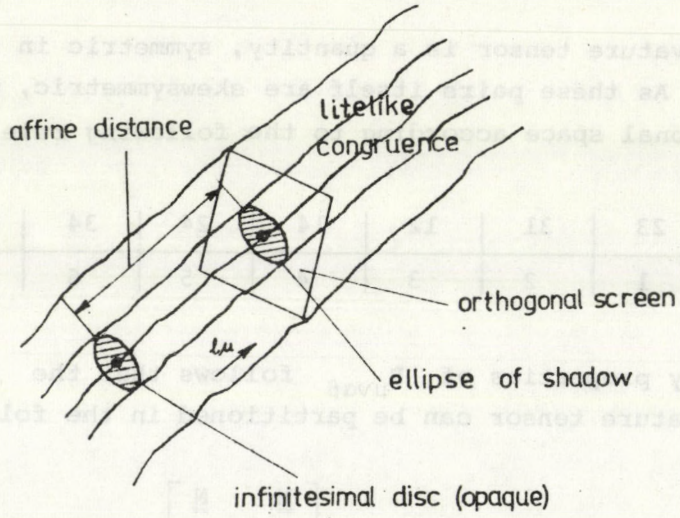
Here $r$ is an invariantly defined distance, the so-called affine distance along the lightlike radial geodesics. The r-independent coefficients belong to the Petrov types indicated. As we move from outwards to the source, the type of the gravitational field becomes more and more general. At large distances, where the type is N, all the principal directors coincide but as we approach the source, they gradually peel off.

This theorem was originally established by Roy Sachs. The proof was considerably shortened by Newman and Penrose [1/]. Even in this form the calculations are rather involved and therefore in this introductory seminary it is of no use to go into the details of it. However, we may recall that the electromagnetic radiation /in its classical version/ behaves in a very similar fashion. The electromagnetic field can be represented by a second rank spinor for which we have two types altogether: the two principal spinors either coincide or not. Accordingly, the radiation zone separates into the algebraically general near zone and the degenerate radiation zone.

To demonstrate further the interplay between the type of curvature and physical processes, I would like to tell you about the famous Goldberg-Sachs theorem. According to this, a vacuum gravitational field is algebrically special if and only if it admits a shearfree light-like geodesic congruence.

To get some insight into the meaning of this theorem, let us consider a family of space-time filling light-like geodesics, called a light-like congruence. The curves have the tangent vector $\ell_\mu$. We can imagine this system as a bundle of light rays passing through the region of space-time considered. We put an infinitesimal disc at distance $r$ orthogonally to the rays and consider the shape of the shadow cast at $r+dr$ behind the disc. Expanding the boundary of the shadow into Legendre polynomials and neglecting higher order terms, the shape will be of the form of an ellipse with the eccentricity $\varepsilon = \sqrt{2\sigma\, dr}$ , where the shear $\sigma$ of the congruence is defined by

$$2\sigma^2 = \ell_{(\mu;\nu)}\ell^{\mu;\nu} - \frac{1}{2}\, \ell^{\mu}{}_{;\mu}$$




It is interesting to classify the known exact solutions of the gravitational equations according to the Petrov type of the curvature tensor. By direct calculation we can prove that the Kerr solution [6] /the stationary field of a rotating body/ is of type D. Since the Kerr metric goes over to the Schwarzschild one in the static /nonrotating/ limit, the Schwarzschild metric is also of type D. There exist plane wave solutions of the vacuum Einstein equations which were found first by Brinkman and were investigated later by W. Kundt. These plane wave solutions are of type N. Quite recently W. Kinnersley succeded to find all type D metrics in explicit form. It is an interesting consequence of his work that it turned out that the Kerr metric exhausts all type D fields which can represent the field of a reasonable source. This fact and other features of the Kerr metric indicate that this solution plays a very fundamental role in nature.

There are also type I solutions known. H. Weyl has completely solved the static axially symmetric gravitational equations in empty space. Weyl's fields with a very few exception are of type I. Recently J. Kóta and I found type I stationary solutions of the vacuum Einstein equations [7], which are not equivalent to any other metrics previously obtained. However, we do not know what kind of source can produce these latter gravitational fields. The metrics exhibit rather pathological behaviour.

Finally I would like to stress that everything what can be done with spinor methods can also be achieved by conventional tensor calculus, although in many cases the use of spinor calculus is much more simple. Thus the Petrov classification in its original form was developed by conventional methods. I find not particularly useful to go here into the details of this rather complicated approach to Petrov types. However, the essence of this approach can be depicted in a way which does not show the covariance of the classification [5].

The curvature tensor is a quantity, symmetric in the first and second pair of indices. As these pairs itself are skewsymmetric, we may represent them in a six dimensional space according to the following rule:

| $\mu\nu$ | 23 | 31 | 12 | 14 | 24 | 34 |
| --- | --- | --- | --- | --- | --- | --- |
| M | 1 | 2 | 3 | 4 | 5 | 6 |

From the symmetry properties of $R_{\mu\nu\alpha\beta}$ follows that the 6x6 matrix representation of the curvature tensor can be partitioned in the following manner;

$$R_{AB} = \begin{bmatrix} \underline{\underline{M}} & \underline{\underline{N}} \\ \underline{\underline{N}} & -\underline{\underline{M}} \end{bmatrix}$$

$\underline{\underline{M}}$ and $\underline{\underline{N}}$ are symmetric and trace-free /3x3/ matrices. We can construct a complex /3x3/ matrix $\underline{\underline{P}}$:

$$\underline{\underline{P}} = \underline{\underline{M}} + i\underline{\underline{N}} \ .$$

**The Petrov classification arises now from the eigenvalue problem of this complex matrix. Depending on the number of the linearly independent eigenvectors the type is III /one independent eigenvector/, II /2/ or I /3/. If any two of the eigenvalues coincide, the type is D. If all three eigenvalues vanish, the type is: N.**

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Printed in the Central Research Institute
for Physics, Budapest, Hungary
Kiadja a KFKI Könyvtár- Kiadói Osztály
O.v.: Dr.Farkas Istvánné
Szakmai lektor: Sebestyén Ákos
Nyelvi lektor: Tóth Kálmán
Példányszám: 90 Munkaszám: 5336
Készült a KFKI házi sokszorositójában
F.v.: Gyenes Imre
Budapest, 1970. január hó.