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ON THE GEODESICS OF CERTAIN SYMMETRIC SPACES

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ABSTRACT

Geodesics of Riemannian spaces admitting certain types of Killing's motions are considered. It is shown that to the geodesics there correspond curves in a lower-dimensional space which reveal a striking resemblance to the force laws of general relativity. The forces emerging are of the electromagnetic and of the potential type. Some connections with Kaluza's five-dimensional theory and with the theory of dynamical groups are discussed. The so-called totally covariant calculus for such spaces is also developed.

INTRODUCTION

It was shown by Kaluza soon after the discovery of general relativity that one can retain the general features of an electrovac theory from a five-dimensional formalism [1]. It was essentially this work which stimulated the numerous unified theories in the twenties and the early thirties. The five-dimensional Riemannian spaces involved in these theories generally posses some symmetry properties. In Kaluza's theory, for example, this symmetry is a Killing motion, and it is shown that the equation

$$R_{\alpha\beta} = 0$$

- $R_{\alpha\beta}$ being the five-dimensional Ricci tensor - can be split into two equations

$$\bar{R}_{ik} = T_{ik} - \frac{1}{2} q_{ik} T^r_r ,$$

$$F^r_{i;r} = 0 .$$

Here F_{ik} is the electromagnetic tensor, T_{ik} is its energy-momentum tensor, $\bar{R}_{ik}$ is the Ricci tensor in four dimensions, and F_{ik} is related to the Killing vector of the five-dimensional space.

Attention has been mainly directed to such field equations, while the features of the geodesics of these varieties have not been investigated in detail. However, it can be shown that to the five-dimensional geodesics there correspond curves in four dimensions and that the first curvature of these curves is equal to an electromagnetic force.

The aim of this paper is to extend this result and show that in a Riemannian space of arbitrary dimensions, admitting several Killing motions, geodesics can be projected onto a variety of lower dimensions and that the first curvature of the curves obrained in this manner is an electromagnetic force. An additional term which may be present in the curvature is a force of the potential type, being a gradient of a scalar, although it can always be removed by means of a confrom transformation of the metric of the lower dimensional space. The variety obtained is generally not a subspace of the original one, though it is a Riemannian space whose metric is fixed by that of the large space. This structure of the geodesics is easily revealed if the Killing motions of the embedding space form an Abelian group.

The first section of the paper recalls the proof of a theorem which states that to require a Riemannian space to admit several Killing motions forming an Abelian group is nothing else but to require the existence of a special coordinate system in which the metric tensor is independent of some variables.

In the second section a general covariant treatment of such Riemannian spaces is developed and the concept of a parameter space corresponding to the motions is introduced. In order to handle both the parameter and the covariant indices in a uniformly covariant way, para-covariant and totally covariant differentation are also defined. This formalism is then applied to the special coordinate system of section one.

The third section is devoted to the decomposition of the equations of geodesics. The special coordinate system is used here to show that curves corresponding to geodesics are actually force laws in a lower dimensional variety.

Finally we conclude that the procedure if looked at in the other way round is related to the group theoretical approach to the equations of dynamics.

I.

We first recall and give the proof of a theorem arising in the theories of Riemannian spaces and partial differential equations. In a form adapted to our problem, this states that:

The necessary and sufficient conditions of the existence of a coordinate system /C.S./ in which the metric tensor $g_{\alpha\beta}$ of an n dimensional Riemannian space V_n is independent of the coordinates $x^1, \ldots, x^r$ (r<n) are that there exist r linearly independent Killing vectors, $K_{A|}^{\alpha}$ x, and that these vectors form an Abelian group of motions, i.e. the generators

$$G_{A|} = K_{A|}{}^{\alpha}\partial_{\alpha} \text{ xx} \qquad /1/$$

fulfil the relations

$$G_{A|}\, G_{B|} - G_{B|}\, G_{A|} = 0 \ . \qquad /2/$$

By linear independence of the vectors $K_{A|}^{\alpha}$ we mean that the relation

$$\lambda^{R|}\; K_{R|}^{\alpha} = 0$$

in any point of the space can only be satisfied by the trivial set of the scalars

$$\lambda^{A|} = 0 \ .$$

This guarantees, for instance, that none of the vectors $K_{A|}^{\alpha}$ can vanish at any particular point.

Before proving the theorem we recall that by definition a Killing vector $K_{A|\alpha}$ satisfies Killing's equation

$$K_{A|\alpha;\beta} + K_{A|\beta;\alpha} = 0 \text{ xxx}. \qquad /3/$$

Expressing the Christoffel symbols by $g^{\alpha\beta}$ and the derivatives of $g_{\alpha\beta}$, (3) can be cast in the form

$$K_{A|,\beta}^{\rho}\, g_{\rho\alpha} + K_{A|,\alpha}^{\rho}\, g_{\rho\beta} + K_{A|}^{\rho}\, g_{\alpha\beta,\rho} = 0 \ . \qquad /4/$$

It should also be remarked that in consequence of the symmetry of the Christoffel symbols $\{{}_{\beta}{}^{\alpha}{}_{\gamma}\}$ in the two subscripts, (2) can be written in the following way:

$$G_{A|}\, G_{B|} - G_{B|}\, G_{A|} =$$
$$= K_{A|}^{\rho} K_{B|;\rho}^{\rho}\, \partial_{\sigma} - K_{B|}^{\rho} K_{A|;\rho}^{\rho}\, \partial_{\sigma} = 0 \ .$$

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x The stroke in $K_{A|}^{\alpha}$ means that the preceeding subscript or supercript is not a covariant index but is simply the name of the quantity. Such indices will also be detoned by underlining.

xx Greek indices will always run from 1 to n , while capital Latin ones from 1 to r . For any types of index we adopt the usual summation convention.

xxx The semicolon denotes covariant differentation with respect to a variable indicated by a subscript, providing the subscript does not preceed a stroke or is not underlined; the sign ∂_{α} or a comma instead of the semicolon denote the ordinary partial derivative.

This operator equation still does not seem to be covariant, because the operator $K_{A|}{}^{\rho}{}_{;\alpha}\, \partial_\rho$ is not a covariant vector. However, if we demand (2) to be valid for any function in a particular C.S., then, applying it to the function $f = x^\alpha$, we get

$$\left(G_{A|}\, G_{B|} - G_{B|}\, G_{A|}\right) x^\alpha = K_{A|}{}^{\rho}\, K_{B|}{}^{\alpha}{}_{;\rho} - K_{B|}{}^{\rho}\, K_{A|}{}^{\alpha}{}_{;\rho} = 0 \;,$$

which is covariant and will guarantee that (2) is fulfilled in any C.S.

The proof of the theorem goes as follows. Assume first that there exists a C.S. in which $g_{\alpha\beta}$ depends only on $x^{r+1}, \ldots, x^n$, and define r linearly independent vectors* by

$$K_{A|}{}^{\alpha} = \delta^\alpha_A \qquad /5/$$

These will satisfy (4) according to the assumption

$$g_{\alpha\beta,A} = 0 \;.$$

Using (1) and (5) one can readily verify that (2) is also satisfied.

The proof of the reversed statement is a bit more involved. Assume that there exist r linearly independent vectors $K_{A|}{}^{\alpha}$ satisfying (2) and (3). First we will show that one can choose a C.S. in which

$$K_{A|}{}^{\alpha} = \delta^\alpha_A \;.$$

There exists a C.S. in which the non-vanishing vector $K_{1|}{}^{\alpha}$ has the form [4]

$$K_{1|}{}^{\alpha} = \delta^\alpha_1 \;. \qquad /6/$$

Using (6), we utilize (2) for the case $A = 1$, $B = 2$ by applying it to the function x^α, which gives

$$K_{2|}{}^{\alpha}{}_{,1} = 0 \;.$$

Thus the equation

$$\sum_{\rho=2}^{n} K_{2|}{}^{\rho}\, \psi_{,\rho} = 0$$

has $n-2$ independent solutions, $\psi^{\underline{k}}(x^2, \ldots, x^n)$, $(k = 3, \ldots, n)$, all independent of x^1. Performing a coordinate transformation

$$x'^1 = x^1 \;,$$

$$x'^2 = h(x^2, \ldots, x^n) \;,$$

$$x'^k = \psi^{\underline{k}}(x^2, \ldots, x^n) \;; \quad (k = 3, \ldots, n) \;,$$

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* By vectors we naturally mean vector fields having continuous derivatives of at least the first order.

where h is chosen so to avoid the vanishing of the Jacobian, we have

$$K_{1|}{}'^{\alpha} = \delta^{\alpha}_{1} \qquad \text{/i.e. (6) is unchanged/}$$

and

$$K_{2|}{}'^{k} = \sum_{\rho=2}^{n} K_{2|}{}^{\rho}\, \psi^{k}_{\ ,\rho} = 0 \ ; \quad (k = 3,\dots,n) \ .$$

$K_{2|}{}'^{2}$ cannot be zero as it would not then be linearly independent of $K_{1|}{}'^{\alpha}$. A transformation of the kind

$$x''^{1} = x'^{1} - \int_{0}^{x'^{2}} \frac{K_{2|}{}'^{1}}{K_{2|}{}'^{2}}\, dx'^{2} \ ,$$

$$x''^{2} = \int_{0}^{x'^{2}} \frac{1}{K_{2|}{}'^{2}}\, dx'^{2} \ ,$$

$$x''^{k} = x'^{k} \ ; \quad (k = 3,\dots,n)$$

leaves (6) unchanged and leads to the desired form of $K_{2|}{}''^{\alpha}$

$$K_{2|}{}''^{\alpha} = \delta^{\alpha}_{2} \ .$$

By repeating of this procedure we finally end up with a C.S. where

$$K_{A|}{}^{\alpha} = \delta^{\alpha}_{A} \ . \qquad /7/$$

We still have to show that if (7) is valid $g_{\alpha\beta}$ is independent of x^{A}. By assumption $K_{A|}{}^{\alpha}$ is a Killing vector; thus it fulfils (4), which, using (7), will give

$$g_{\alpha\beta,A} = 0 \ ,$$

which was to be proved.

The C.S. in which (7) is valid is called a special coordinate system /S.C.S./. Similar systems play an essential role in the group theoretical classification of the space-times of general relativity [3].

The most general transformations leaving (7) unchanged have the form

$$x'^{A} = x^{A} + f^{A}\left(x^{r+1},\dots,x^{n}\right) \ ,$$

$$x'^{j} = f^{j}\left(x^{r+1},\dots,x^{n}\right) \ ; \quad (j = r+1,\dots,n) \ .$$

We call such a transformation a special coordinate transformation /S.C.T./. The subgroup

$$x'^{A} = x^{A} + f^{A}\left(x^{r+1},\dots,x^{n}\right)$$

$$x'^{j} = x^{j} \ ; \quad (j = r+1,\dots,n)$$

of the S.C.T.-s called the group of gauge transformations /see the end of sect. II/, while a transformation of the form

$$x'^A = x^A ,$$

$$x'^j = f^j(x^{r+1},\dots,x^n) ; \quad (j = r+1,\dots,n)$$

is a restricted coordinate transformation.

A Riemannian space admitting r linearly independent Killing vectors forming an Abelian group of motions will be denoted by V_n^r

II.

In this section we examine the structure of a V_n^r in detail. We shall assume that the matrix $\| k_{AB|} \|$ of the scalar products

$$k_{AB|} = K_{A|\rho} K_{B|}{}^{\rho}$$

of the Killing vectors is not singular. The inverse $\|k^{AB|}\|$ of $\|k_{AB|}\|$ can be used to raise the capital Latin subscript of a $K_{A|\alpha}$:

$$K^{A|}{}_{\alpha} = k^{AR|} K_{R|\alpha} \quad .$$

Note that $K^{A|}{}_{\rho}$ is generally not a Killing vector.

In a V_n^r there exists a set of n - r independent scalar functions $\varphi^{\underline{k}}(x^1,\dots,x^n)$ satisfying

$$K_{A|}{}^{\rho} \varphi^{\underline{k}}{}_{,\rho} = 0 \qquad ^x \qquad /8/$$

In fact, the requirement (8) is fulfilled by the functions $\varphi^{\underline{k}} = x^k$ in the S.C.S. of a V_n. Obviously, given such a set any function $F(\varphi^{\underline{r+1}},\dots,\varphi^{\underline{n}})$ will also satisfy (8). Introducting the notation

$$\gamma^{\underline{k}}_{\rho} = \varphi^{\underline{k}}{}_{,\rho} \quad , \qquad /9/$$

we have

$$K_{A|}{}^{\rho} \gamma^{\underline{k}}_{\rho} = K^{A|\rho} \gamma^{\underline{k}}_{\rho} = 0 \quad . \qquad /10/$$

Owing to the independence of the functions $\varphi^{\underline{k}}$ the vectors $\gamma^{\underline{k}}_{\rho}$ are linearly independent. Thus, according to (10), the vectors

$$K_{A|\rho}, \; \gamma^{\underline{k}}_{\rho} \qquad /11/$$

form a set of n linearly independent vectors in a V_n^r . We may then introduce the "inverse" γ- s:

$$K_{A|\rho} \gamma^{\rho}_{\underline{k}} = 0 \quad ,$$

$$\gamma^{\underline{i}}_{\rho} \gamma^{\rho}_{\underline{k}} = \delta^{\underline{i}}_{\underline{k}} \quad ; \qquad /12/$$

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x From now on small Latin indices /except r and n/ will run from r+1 to n .

these $n(n-r)$ equations uniquely define the $n(n-r)$ quantities $\gamma^{\alpha}_{\underline{a}}$

To characterize unambiguously a vector U^{α} we can use any one of the four sets of scalars

$$U^{\underline{a}} = \gamma^{\underline{a}}_{\rho}\, U^{\rho} \quad , \qquad U^{\underline{a}} = \gamma^{\underline{a}}_{\rho}\, U^{\rho} \quad ,$$
$$U^{A|} = K^{A|}{}_{\rho}\, U^{\rho} \; ; \qquad U_{A|} = K_{A|\rho}\, U^{\rho} \; ; \qquad /13/$$

$$U_{\underline{a}} = \gamma^{\rho}_{\underline{a}}\, U_{\rho} \quad , \qquad U_{\underline{a}} = \gamma^{\rho}_{\underline{a}}\, U_{\rho} \quad ,$$
$$U^{A|} = K^{A|\rho}\, U_{\rho} \; ; \qquad U_{A|} = K_{A|}{}^{\rho}\, U_{\rho} \; .$$

Similar quantities for tensors of higher order can also be formed, e.g.

$$T^{A|\alpha\beta}{}_{\underline{a}} = K^{A|}{}_{\rho}\, \gamma^{\sigma}_{\underline{a}}\, T^{\rho\alpha\beta}{}_{\sigma}$$

The tensor

$$\varepsilon^{\alpha}_{\beta} = \gamma^{\alpha}_{\underline{r}}\, \gamma^{\underline{r}}_{\beta} \qquad /14/$$

is a projector, i.e.

$$\varepsilon^{\alpha}_{\rho}\, \varepsilon^{\rho}_{\beta} = \varepsilon^{\alpha}_{\beta}$$

in accordance with (12). Equations (10) and (12) give

$$\varepsilon^{\alpha}_{\rho}\, K_{A|}{}^{\rho} = \varepsilon^{\rho}_{\alpha}\, K_{A|\rho} = 0$$

and

$$\varepsilon^{\alpha}_{\rho}\, \gamma^{\rho}_{\underline{i}} = \gamma^{\alpha}_{i} \; ; \quad \varepsilon^{\rho}_{\alpha}\, {}^{\underline{k}}_{\rho} = \gamma^{\underline{k}}_{\alpha} \quad . \qquad /15/$$

We can express $\varepsilon^{\alpha}_{\beta}$ by means of the vectors $K_{A|\alpha}$ in the following way:

$$\varepsilon^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} - K^{R|\alpha}\, K_{R|\beta} = \delta^{\alpha}_{\beta} - K_{R|}{}^{\alpha}\, K^{R|}{}_{\beta} \; . \qquad /16/$$

This relation is proved by multiplying it by and then contracting it with the n linearly independent vectors (11) and finally taking (15) into account.

By means of (16) one can decompose the fundamental form of a V^{r}_{n}:

$$g_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta} = g_{\alpha\rho}\left(\varepsilon^{\rho}_{\beta} + K^{R|\rho}\, K_{R|\beta}\right) dx^{\alpha}\, dx^{\beta} =$$
$$= g_{\rho\sigma}\left(\varepsilon^{\sigma}_{\alpha} + K^{T|\sigma}\, K_{T|\alpha}\right) \varepsilon^{\rho}_{\beta}\, dx^{\alpha}\, dx^{\beta} + K^{R|}{}_{\alpha}\, K_{R|\beta}\, dx^{\alpha}\, dx^{\beta} = \qquad /17/$$
$$= g_{\rho\sigma}\, \varepsilon^{\rho}_{\alpha}\, \varepsilon^{\sigma}_{\beta}\, dx^{\alpha}\, dx^{\beta} + K^{R|}{}_{\alpha}\, K_{R|\beta}\, dx^{\alpha}\, dx^{\beta} \; ,$$

where in the last step (16) has been used. Introducing

$$g_{\underline{ik}} = \gamma^{\rho}_{\underline{i}} \gamma^{\sigma}_{\underline{k}} g_{\rho\sigma} , \qquad /18/$$

(17) can be rewritten

$$g_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta} = g_{\underline{rs}}\, \gamma^{\underline{r}}_{\alpha} \gamma^{\underline{s}}_{\beta}\, dx^{\alpha}\, dx^{\beta} + K^{R|}{}_{\alpha} K_{R|\beta}\, dx^{\alpha}\, dx^{\beta} \quad .$$

The quantities $g_{\underline{ik}}$ form a nonsingular matrix; one can readily show that in consequence of (10) and (12) we have

$$g_{\underline{ir}}\, g^{\underline{rk}} = \delta^{k}_{i}$$

if

$$g^{\underline{ik}} = \gamma^{\underline{i}}_{\rho} \gamma^{\underline{k}}_{\sigma} g^{\rho\sigma} .$$

The $g_{\underline{ik}}$ -s are the components of a metric tensor of a V_{n-r} and play an important part in the derivation of force laws in section III.

We turn now to questions of tensor analysis. From the partial derivatives of a quantity B one can form the expressions

$$B_{,A|} = K_{A|}{}^{\rho} B_{,\rho} \;; \quad B_{,\underline{a}} = \gamma^{\rho}_{\underline{a}} B_{,\rho} \quad . \qquad /19/$$

The first is called the inner derivative of B in the direction of $K_{A|}$, while the second is the "p-derivative of B with respect to $\varphi^{\underline{a}}$ ". B is said to be A - cyclic if $B_{,A|} = 0$.

We will show that $g_{\underline{ik}}$ is A - cyclic for any A. To this end we calculate

$$g_{\underline{ik},A|} = \left(g_{\mu\nu}\, \gamma^{\mu}_{\underline{i}} \gamma^{\nu}_{\underline{k}}\right)_{,\rho} K_{A|}{}^{\rho} =$$

$$= \left([\mu\rho,\nu] + [\nu\rho,\mu]\right) \gamma^{\mu}_{\underline{i}} \gamma^{\nu}_{\underline{k}} K_{A|}{}^{\rho} + \qquad /20/$$

$$+ g_{\mu\nu}\, \gamma^{\mu}_{\underline{i}} \gamma^{\nu}_{\underline{k},\rho} K_{A|}{}^{\rho} + g_{\mu\nu}\, \gamma^{\mu}_{\underline{k}} \gamma^{\nu}_{\underline{i},\rho} K_{A|}{}^{\rho} ,$$

where $[\alpha\beta,\gamma]$ is the Christoffel symbol of the first kind. In order to evaluate $g_{\mu\nu} \gamma^{\mu}_{i} \gamma^{\nu}_{\underline{k},\rho} K_{A|}{}^{\rho}$ we differentiate the second relation of (12) and use (14) to get

$$\varepsilon^{\nu}_{\tau}\, \gamma^{\tau}_{\underline{k},\rho} = -\gamma^{\tau}_{\underline{k}}\, \gamma^{\underline{r}}_{\tau,\rho}\, \gamma^{\nu}_{\underline{r}} .$$

which together with (16) gives

$$\gamma^{\nu}_{\underline{k},\rho} = K^{R|\nu} K_{R|\tau}\, \gamma^{\tau}_{\underline{k},\rho} - \gamma^{\tau}_{\underline{k}}\, \gamma^{\underline{r}}_{\tau,\rho}\, \gamma^{\nu}_{\underline{r}} \quad .$$

This, multiplied by $g_{\mu\nu}\,\gamma^{\mu}_{\underline{i}}\,K_{A|}{}^{\rho}$ and contracted for ρ and ν, yields

$$\gamma^{\nu}_{\underline{k},\rho}\,\gamma^{\mu}_{\underline{i}}\,g_{\mu\nu}\,K_{A|}{}^{\rho}\,. \qquad /21/$$

Since by definition (9) $\gamma^{\underline{r}}_{\tau,\rho} = \gamma^{\underline{r}}_{\rho,\tau}$, and that as a consequence of (10) $K_{A|}{}^{\rho}\,\gamma^{\underline{r}}_{\rho,\tau} = -K_{A|}{}^{\rho}{}_{,\tau}\,\gamma^{\underline{r}}_{\rho}$, we may write (21) in the form

$$\gamma^{\nu}_{\underline{k},\rho}\,\gamma^{\mu}_{\underline{i}}\,g_{\mu\nu}\,K_{A|}{}^{\rho} = \gamma^{\tau}_{\underline{k}}\,\gamma^{\underline{r}}_{\rho}\,\gamma^{\nu}_{\underline{r}}\,\gamma^{\mu}_{\underline{i}}\,g_{\mu\nu}\,K_{A|}{}^{\rho}{}_{,\tau} =$$

$$= \gamma^{\tau}_{\underline{k}}\,\gamma^{\mu}_{\underline{i}}\,g_{\mu\rho}\,K_{A|}{}^{\rho}{}_{,\tau}\ ^{*}$$

If we substitute into (19) the latter expression and the expression obtained from it by interchanging $\underline{i}$ and $\underline{k}$, we get

$$g_{\underline{ik},A|} = \gamma^{\mu}_{\underline{i}}\,\gamma^{\nu}_{\underline{k}}\,\left(K_{A|\mu;\nu} + K_{A|\nu;\mu}\right)\,,$$

which on the assumption that $K_{A|\alpha}$ is a Killing vector gives

$$g_{\underline{ik},A|} = 0\,, \qquad /22/$$

which was to be proved.

According to the remarks following (8) any function of the $\varphi^{\underline{k}}$ -s will equally satisfy (8). Thus as well as the coordinates we may also transform the parameter functions $\varphi^{\underline{k}}$. A transformation

$$\varphi'^{\underline{k}} = \varphi'^{\underline{k}}\left(\varphi^{\underline{r+1}},\dots,\varphi^{\underline{n}}\right)$$

is called a parameter or "p"-transformation if the Jacobian

$$\left\|\frac{\partial\varphi'^{\underline{k}}}{\partial\varphi^{\underline{\ell}}}\right\|$$

is of rank $n-r$. The definition of the p-tensors is straight-forward; for example, we call $V_{\underline{k}}$ a covariant p-vector if for a p-transformation it transforms like

$$V'_{\underline{k}} = \frac{\partial\varphi^{\underline{r}}}{\partial\varphi'^{\underline{k}}}\,V_{\underline{r}}\,.$$

Note that for a p-transformation the ordinary coordinates /o-coordinates/ remain unchanged, and thus the ordinary tensors /o-tensors/ behave like p-scalars; and in turn for an o-transformation the p-tensors are to be treated as o-scalars. From definition (9) it can be readily shown that $\gamma^{\underline{i}}_{\alpha}$ is a contravariant p-vector and a covariant o-vector. From (12) is follows that $\gamma^{\beta}_{\underline{k}}$ is a contravariant o- and covariant p-vector. It can also be verified that the inner derivation in any direction will not alter the p-behaviour of a p-tensor. The ordinary partial derivative of a p-tensor with respect to a variable x^{α}, however, is not a p-tensor, and therefore we need a p-

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* In the last step we have used $\gamma^{\underline{r}}_{\rho}\,\gamma^{\nu}_{\underline{r}}\,g_{\mu\nu}\,\gamma^{\mu}_{\underline{i}} = \varepsilon^{\nu}_{\rho}\,g_{\mu\nu}\,\gamma^{\mu}_{\underline{i}} = g_{\mu\rho}\,\gamma^{\mu}_{\underline{i}}$, which is a result of (14) and (16)

covariant rule of differentation. Consider the o-covariant derivative of the o-vector $U^{\alpha} = \gamma^{\alpha}_{\underline{r}} U^{\underline{r}}$, where $U^{\underline{a}}$ is an arbitrary p-vector

$$U^{\alpha}{}_{;\beta} = \left(\gamma^{\alpha}_{\underline{r},\beta} + \{{}^{\alpha}_{\beta\rho}\}\gamma^{\rho}_{\underline{r}}\right)U^{\underline{r}} + \gamma^{\alpha}_{\underline{r}} U^{\underline{r}}{}_{,\beta} \quad .$$

Multiplying this equation by $\gamma^{\underline{a}}_{\alpha}$ and summing for α, we get a p-contravariant and o-covariant vector, as $U^{\alpha}{}_{,\beta}$ is a p-scalar:

$$U^{\rho}{}_{;\beta}\,\gamma^{\underline{a}}_{\rho} = U^{\underline{a}}{}_{,\beta} + \left(\gamma^{\underline{a}}_{\sigma}\,\gamma^{\rho}_{\underline{r}}\;\{{}^{\sigma}_{\rho\beta}\} - \gamma^{\sigma}_{\underline{r}}\,\gamma^{\underline{a}}_{\sigma,\beta}\right) U^{\underline{r}} \qquad /23/$$

We call the R.H.S. of (23) the p-covariant derivative of $U^{\underline{a}}$ with respect to x^{β} . The quantity in brackets can be regarded as the analogue of the ordinary Christoffel symbols of the second kind, and may be denoted by

$$\{{}^{\underline{a}}_{\underline{b}\,\beta}\} = \gamma^{\underline{a}}_{\sigma}\,\gamma^{\rho}_{\underline{b}}\,\{{}^{\sigma}_{\beta\,\rho}\} - \gamma^{\sigma}_{\underline{b}}\,\gamma^{\underline{a}}_{\sigma,\beta} \quad . \qquad /24/$$

By the p-covariant derivative of a p-covariant vector $Z_{\underline{a}}$ with respect to x^{β} we mean

$$Z_{\underline{a},\beta} - \{{}^{\underline{r}}_{\underline{a}\,\beta}\}\, Z_{\underline{r}} \quad .$$

This can be shown to be a p-covariant and o-covariant vector.

We can define now the totally covariant derivative of a tensor of arbitrary p- and o-covariant character with respect to x^{β}; this is the partial derivative of the tensor plus terms with appropriate signs containing o-Christoffel symbols for o-indices and the symbols (24) for p-indices. For example totally covariant derivative of $V^{\underline{a}\alpha}{}_{\eta}$ with respect to x^{β} is

$$V^{\underline{a}\alpha}{}_{\eta;\beta} = V^{\underline{a}\alpha}{}_{\eta,\beta} + \{{}^{\underline{a}}_{\underline{r}\,\beta}\}\, V^{\underline{r}\alpha}{}_{\eta} + \{{}^{\alpha}_{\rho\,\beta}\}\, V^{\underline{a}\rho}{}_{\eta} - \{{}^{\rho}_{\eta\,\beta}\}\, V^{\underline{a}\alpha}{}_{\rho} \, .$$

From now on the semicolon will always denote the totally covariant derivative.

Obviously the totally covariant derivative of a p- or an o-scalar coincides with its o- or p-covariant derivative, respectively.

The totally covariant derivative of an arbitrary tensor can also be "projected" by means of the γ-s. Thus e.g.,

$$U^{\underline{a}}{}_{;\underline{b}} = U^{\underline{a}}{}_{;\rho}\,\gamma^{\rho}_{\underline{c}} = U^{\underline{a}}{}_{,\rho}\,\gamma^{\rho}_{\underline{c}} + \{{}^{\underline{a}}_{\underline{r}\,\rho}\}\,\gamma^{\rho}_{\underline{b}}\, U^{\underline{r}} \quad ,$$

which is called "the totally covariant derivative of $U^{\underline{a}}$ with respect to $\psi^{\underline{b}}$". The quantities

$$\{{}^{\;a}_{\underline{b}\,\underline{c}}\} = \{{}^{\;a}_{\underline{b}\;\rho}\}\,\gamma^{\rho}_{\underline{c}} \qquad /25/$$

are called p-Christoffel symbols of the second kind. A straightforward but lengthy calculation shows that

$$\{{}^{\;a}_{\underline{b}\,\underline{c}}\} = \frac{1}{2}\, g^{\underline{a}\underline{r}}\left(g_{\underline{r}\underline{b},\underline{c}} + g_{\underline{r}\underline{c},\underline{b}} - g_{\underline{b}\underline{c},\underline{r}}\right) ,$$

where $g_{\underline{a}\underline{b},\underline{c}}$ is defined according to the second relation of (19). It can be derived that

$$g_{\underline{a}\underline{b};\underline{c}} = g_{\underline{a}k;\rho}\,\gamma^{\rho}_{\underline{c}} = g^{\underline{a}\underline{b}}{}_{;\underline{c}} = g^{\underline{a}\underline{b}}{}_{;\rho}\,\gamma^{\rho}_{\underline{c}} = 0 \quad ,$$

which are the analogues of the corresponding o-relations.

We shall now apply the formalism developed above to a S.C.S. of a V^r_n /see section I/. Thus we are assuming (7) to hold, and therefore

$$K_{A|\alpha} = g_{\alpha\rho}\, K_{A|}{}^{\rho} = g_{\alpha A} \quad . \qquad /26/$$

and consequently,

$$k_{AB|} = K_{A|\rho}\, K_{B|}{}^{\rho} = K_{A|B} = K_{B|A} = g_{AB} \quad . \qquad /27/$$

In accordance with the remarks following (8) we may take

$$\varphi^{\underline{k}} = x^{k} \quad . \qquad /28/$$

A S.C.S. together with this choice of the functions $\varphi^{\underline{k}}$ is called a natural system /N.S./ of a V^r_n .

We shall work in a N.S. By definition (9) we have

$$\gamma^{\underline{k}}_{\underline{a}} = \delta^{k}_{a} \quad . \qquad /29/$$

Taken with (12) this gives

$$\gamma^{k}_{\underline{i}} = \delta^{k}_{i} \quad \text{and} \quad \gamma^{A}_{\underline{i}} = -K^{A|}{}_{i} \quad ,$$

which make possible the calculation of $g_{\underline{i}\underline{k}}$ by means of (18):

$$g_{\underline{i}\underline{k}} = g_{ik} - K^{R|}{}_{i}\, K_{R|k}$$

This equation and the relations (26) and (27) enable us to express $g_{\alpha\beta}$ by means of $g_{\underline{i}\underline{k}}$, $K_{A|\alpha}$ and $k_{AB|}$:

$$g_{ab} = g_{\underline{a}\underline{b}} + K^{R|}{}_{a}\, K_{R|b} \quad ,$$

$$g_{aB} = K_{B|a} \quad , \qquad /30/$$

$$g_{AB} = k_{AB|} \quad .$$

These in turn allow us to calculate $g^{\alpha\beta}$:

$$g^{ab} = g^{\underline{ab}}$$
$$g^{aB} = -g^{\underline{ar}}\, K^{B|}_{\;\;r}$$
$$g^{AB|} = \left(k^{AB|} + K^{A|}_{\;\;r}\, K^{B|}_{\;\;s}\, g^{\underline{rs}}\right) . \qquad /31/$$

A N.S. is at the same time a S.C.S., so $g_{\alpha\beta}$ is independent of x^A. Thus as a consequence of (30) and (31), $k_{AB|}$, $k^{AB|}$, $K_{A|}$ and $K^{A|}_{\;\;\alpha}$ together with $g_{\underline{ab}}$ are also independent of these variables. This incidentally is in agreement with (22), which due to (7) can now be written in the form

$$g_{\underline{ab},A|} = g_{\underline{ab},\rho}\, K_{A|}^{\;\;\rho} = g_{\underline{ab},A} = 0$$

It is seen that in (30) and (31) p- and o-indices are mixed up. If we perform the S.C.T.

$$x'^A = x^A + f^A(x^{r+1},\dots, x^n)$$
$$x'^k = x'^k(x^{r+1},\dots, x^n) \qquad /32/$$

then (7) will remain valid. On the other hand (28), and consequently the relations (29), (30) and (31), will become void, i.e. the new system will not be a N.S. However, if simultaneously with the o-transformation (32) we perform a p-transformation

$$\varphi'^{\underline{k}} = x'^k(\varphi^{\underline{r+1}},\dots,\varphi^{\underline{n}})$$

and take (28) into account, then not only (7) but also (28) will be left unchanged, and the new system will be a N.S. again.

Such a pair of simultaneously performed transformations is called a natural transformation /N.T./. Accordingly the gauge transformation

$$x'^A = x^A + f^A(x^{r+1},\dots,x^n)$$
$$x'^k = x^k$$

is a N.T. calling forth

$$K'_{A|a} = K_{A|a} - f^A_{\;\;,a} ,$$

which enlightens the designation "gauge".

III.

Now we investigate the nonminimal geodesics of a V^r_n. If we choose a parameter s such that $g_{\rho\sigma}\, \dot{x}^\rho\, \dot{x}^\sigma = e$ *, where $e = \pm 1$, then the equations of a geodesic are

$$\ddot{x}^\alpha + \{{}_{\rho}{}^{\alpha}{}_{\sigma}\}\, \dot{x}^\rho\, \dot{x}^\sigma = 0 \qquad /33/$$

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* The dot denotes the derivative with respect to s.

According to (13) we introduce $n - r$ p-vectors

$$V^{\underline{a}} = \gamma^{\underline{a}}_{\rho} \dot{x}^{\rho}$$

and r scalars

$$c_{A|} = K_{A|\rho} \dot{x}^{\rho} \qquad /34/$$

We wish to show that $c_{A|}$ is constant along the geodesic, i.e.

$$\frac{dc_A}{ds} = 0 \quad . \qquad /35/$$

We have

$$\frac{dc_{A|}}{ds} = K_{A|\rho,\sigma} \dot{x}^{\rho} \dot{x}^{\sigma} + K_{A|\rho} \ddot{x}^{\rho} =$$

$$= K_{A|\rho,\sigma} \dot{x}^{\rho} \dot{x}^{\sigma} - K_{A|\rho} \{{}_{\sigma}{}^{\rho}{}_{\tau}\} \dot{x}^{\sigma} \dot{x}^{\tau} =$$

$$= K_{A|\rho;\sigma} \dot{x}^{\rho} \dot{x}^{\sigma} \quad ,$$

but in consequence of (3) $K_{A|\rho;\sigma}$ is antisymmetric in ρ and σ, and thus (35) is indeed satisfied.

One can invert the relations (34):

$$\dot{x}^{\alpha} = V^{\underline{r}} \gamma^{\alpha}_{\underline{r}} + K^{R|\alpha} c_{R|} \quad . \qquad /36/$$

By means of (34) and (16), the relations (36) are seen to be fulfilled identically.

We substitute $\ddot{x}^{\alpha}$, obtained by differentiating (36) into (33) and use (36) again to express every $\dot{x}^{\alpha}$ by the $V^{\underline{a}}$ -s and $c_{A|}$ -s. Finally we multiply the resulting expression by $\gamma^{\underline{a}}_{\alpha}$ and contract α to get

$$\dot{V}^{\underline{a}} + \{{}_{\underline{r}}{}^{\underline{a}}{}_{\underline{t}}\} V^{\underline{r}} V^{\underline{t}} = -2\gamma^{\underline{a}}_{\rho} \gamma^{\sigma}_{\underline{r}} K^{R|\rho}{}_{;\sigma} c_{R|} V^{\underline{r}} -$$

$$-\gamma^{\underline{a}}_{\rho} K^{R|\rho}{}_{;\sigma} K^{T|\sigma} c_{R|} c_{T|} \quad ; \qquad /37/$$

where $\{{}_{\underline{r}}{}^{\underline{a}}{}_{\underline{t}}\}$ is the Christoffel symbol (25). The first term on the R.H.S. can be rewritten

$$-2\gamma^{\underline{a}}_{\rho} \gamma^{\sigma}_{\underline{r}} K^{R|\rho}{}_{;\sigma} c_{R|} V^{\underline{r}} = 2\gamma^{\underline{a}}_{\rho} g^{\rho\tau} \gamma^{\sigma}_{\underline{r}} K^{R|}{}_{\sigma;\tau} c_{R|} V^{\underline{r}}$$

$$= \gamma^{\tau}_{\underline{t}} g^{\underline{ta}} \gamma^{\sigma}_{\underline{r}} \left(K^{R|}{}_{\sigma,\tau} - K^{R|}{}_{\tau,\sigma} \right) c_{R|} V^{\underline{r}} \quad , \qquad /38/$$

where
$$\gamma^{\underline{a}}_{\rho}\, g^{\rho\tau} = \gamma^{\tau}_{t}\, g^{\underline{ta}} \qquad /39/$$

and (3), together with the obvious relation
$$K^{R|}{}_{\alpha;\beta} - K^{R|}{}_{\beta;\alpha} = K^{R|}{}_{\alpha,\beta} - K^{K|}{}_{\beta,\alpha} \quad ,$$
have been used.

The second term of (37) is reformulated as follows
$$-\gamma^{\underline{a}}_{\rho}\, K^{R|\rho}{}_{;\sigma}\, K^{T|\sigma}\, C_{R|}\, C_{T|} = -\gamma^{\underline{a}}_{\rho}\, g^{\rho\tau}\left(k^{RQ|}\, K_{Q|\tau}\right)_{;\sigma} K^{T|\sigma}\, C_{R|}\, C_{T|} =$$
$$= g^{\underline{ar}}\, \gamma^{\rho}_{\underline{r}}\, k^{RQ|}\, k^{TV|}\, K_{Q|\sigma;\rho}\, K_{V|}{}^{\sigma}\, C_{R|}\, C_{T|} =$$
$$= \frac{1}{2}\, g^{\underline{ar}}\, \gamma^{\rho}_{\underline{r}}\, k^{RQ|}\, k^{TV|}\, k_{QV|,\rho}\, C_{R|}\, C_{T|} = \qquad /40/$$
$$= -\frac{1}{2}\, g^{\underline{ar}}\, \gamma^{\rho}_{\underline{r}}\, k^{RT|}{}_{,\rho}\, C_{R|}\, C_{T|} \quad .$$
Here we have made use of (39) and of
$$\gamma^{\underline{a}}_{\rho}\, K^{R|\rho}{}_{;\alpha} = k^{RQ|}\, \gamma^{\underline{a}}_{\rho}\, K_{Q|}{}^{\rho}{}_{;\alpha} \quad ,$$
which is a consequence of (10). The equation
$$k^{AB|}{}_{,\alpha} = -k^{AR|}\, k^{BQ|}\, k_{RQ|,\alpha} \quad ,$$
which is a consequence of
$$k^{AR|}\, k_{RB|} = \delta^{A}_{B} \quad ,$$
has also been used. Inserting (38) and (40) into (37), one gets
$$\dot{V}^{\underline{a}} + \left\{{\underline{a} \atop \underline{r}\,\underline{t}}\right\} V^{\underline{r}}\, V^{\underline{t}} = g^{\underline{at}}\, \gamma^{\rho}_{\underline{t}}\, \gamma^{\sigma}_{\underline{r}} \left(C_{R|}\, K^{R|}{}_{\sigma,\rho} - C_{R|}\, K^{R|}{}_{\rho,\sigma}\right) V^{\underline{r}} -$$
$$- \frac{1}{2}\, g^{\underline{at}}\, \gamma^{\rho}_{\underline{t}}\, k^{RT|}{}_{,\rho}\, C_{R|}\, C_{T|} \quad . \qquad /41/$$

The significance of this equation becomes especially lucid in a N.S., because as a result of (29) we then have
$$V^{\underline{a}} = \dot{x}^{a} \quad ,$$
with $K^{A|}{}_{\alpha}$ and $k^{AB|}$ independent of x^{A}, and moreover
$$K^{A|}{}_{B} = k^{AR|}\, K_{R|B} = k^{AR|}\, k_{RB|} = \delta^{A}_{B} \quad .$$
Thus (41) can be cast in the form
$$\ddot{x}^{a} + \left\{{\underline{a} \atop \underline{r}\,\underline{t}}\right\} \dot{x}^{r}\, \dot{x}^{t} = g^{\underline{at}} \left(C_{R|}\, K^{R|}{}_{r,t} - C_{R|}\, K^{R|}{}_{t,r}\right) \dot{x}^{r} -$$
$$- \frac{1}{2}\, g^{\underline{at}}\, k^{RT|}{}_{,t}\, C_{R|}\, C_{T|} \qquad /42/$$

The first term of the R.H.S. of (42)has the form of an electromagnetic force /similar terms in classical mechanics are sometimes called gyroscopic forces/. The second force is of the potential type. The "electromagnetic" field

$$F_{rt} = C_{R|} K^{R|}{}_{r,t} - C_{R|} K^{R|}{}_{t,r}$$

is a rotation of the "vector potential" $C_R K^{R|}{}_a$

If we pass from one N.S. to another, the o- and p-indices behave equally. Thus (42) is a covariant equation for such systems. Our result can be put in yet another way: There exists a Riemannian space V_{n-r} with the metric $g_{\underline{ik}}$ in which for a set of constants C_A a curve described the equations (42) can be found that corresponds to a geodesic of the original V_n^r, the $C_{A|}$-s then being given by (34).

One may ask if it is possible to find a Lagrangian which when used in a variation principle will lead to the equations (42). The answer to this question is affirmative, since the function

$$L = \left[\left(e - C_{R|} C_{T|} K^{RT|}\right) g_{\underline{rt}} \dot{x}^r \dot{x}^t\right]^{1/2} + C_{R|} K^{R|}{}_t \dot{x}^t \qquad /43/$$

will meet this requirement. We stress, however, that the choice of the parameter s is such that

$$g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = e \quad ,$$

which in a N.S. reads

$$g_{\underline{rt}} \dot{x}^r \dot{x}^t + K^{RT|} C_{R|} C_{T|} = e \quad .$$

Taking this into account and using (43) in the Euler-Lagrange equations, (42) will readily be regained.

We can also define a new metric tensor

$$\bar{g}_{ab} = \left(e - C_{R|} C_{T|} K^{RT|}\right) g_{\underline{ab}} \quad .$$

Using a parameter s' for which

$$\bar{g}_{rt} \frac{dx^r}{ds'} \frac{dx^t}{ds'} = \text{const.} \quad ,$$

the Euler-Lagrange equations

$$\frac{\partial L}{\partial x^a} - \frac{d}{ds'} \frac{\partial L}{\partial \frac{dx^a}{ds'}} = 0 \quad ,$$

with the L defined by (43), will lead to

$$\frac{d^2x^a}{ds'^2} + \overline{\left\{{a \atop r\ t}\right\}} \frac{dx^r}{ds'} \frac{dx^t}{ds'} = \bar{g}^{at} \left(C_{R|} K^{R|}{}_{r,t} - C_{R|} K^{R|}{}_{t,r}\right) \frac{dx^r}{ds'} \quad ,$$

where $\overline{\left\{{a \atop b\ c}\right\}}$ is formed from $\bar{g}_{ab}$ in the usual way.

It is seen that the potential force can be removed by redefining the metric tensor only at the price of making the new metric dependent on the constants $c_{A|}$.

CONCLUSIONS

Looking at the reasoning of section III. from the other way round, we see that a potential-like term of the first curvature of a curve in a V_m can always be removed by redefining the metric of V_m. The new metric tensor is the product of the old one and the so-called conform factor [4]. We frequently encounter similar procedures in the group theoretical approach to the equations of dynamics. For example, in the Kepler problem it can be shown that the trajectories are the geodesics of a sphere of three dimensions [5].

On the other hand, and electromagnetic /or gyroscopic/ force can only be removed at the expense of increasing the dimensions of the space. However the larger space obtained this way will posses Killing symmetries constituting an Abelian group motions.

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