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The general solution of the field equations of stationary vacuum gravitational fields possessing geodesic eigenrays with nonvanishing shear is obtained. Nontrivial solutions exist only if the eigenrays do not rotate. The resulting metrics fall into two classes: either there is a functional dependence among the field quantities /this class belongs to the Papapetrou solutions/, or the quantity γ^o, which in the shear-free case has been interpreted as the central mass, is uniquely determined. This latter class consists of two space-times. The curvature invariants vanish in the $r \to \infty$ limit for both solutions; however, the metrics exhibit singular behavior in this limit.

1. INTRODUCTION

The class of gravitational fields possessing geodesic rays[1] is very conveniently treated by the spin coefficient technique developed by Newman and Penrose in 1962[2]. Newman and Tamburino[1] have shown how the metrics can be obtained in explicit form using spin coefficients. They calculated metrics for which the rays have nonvanishing shear, and, surprisingly, they learned that this class cannot be considered as a generalization from the nonshearing case. A later result of Unti and Torrence[3] indicated that the class of metrics with shearing geodesic rays is rather poor, in the sense that solutions exist only if the rays are either hypersurface orthogonal or cylindrical.

If the space-time contains a Killing vector field, the gravitational equations can be reformulated in a 3-dimensional space V_3 associated with the trajectories of the Killing motion[4, 5]. /For notations see reference 5. This paper will hereafter be referred to as P./ For stationary space-times /time-like Killing field/ an SU/2/ spin coefficient method has been developed in P. The field equations in SU/2/ spinor base can be solved exactly if the eigenrays are geodesics of V_3. /The notion of eigenrays will be elucidated below./ As shown in P, the gravitational fields with nonshearing geodesic eigenrays are of Petrov type D; they have been thoroughly studied in previous papers. Therefore, we can anticipate new results only for metrics with eigenrays of nonvanishing shear.

In this paper all stationary vacuum metrics with shearing geodesic eigenrays are constructed in explicit form. The findings resemble in many respects those of Newman and Tamburino. The dropping of the nonshearing condition leads to a rather restricted set of metrics which does not contain the shear-free Kerr solution as a limiting case. Nevertheless, there are some solutions with shearing eigenrays. To demonstrate this, we first list some results of P.

Let the coordinate $x^o = t$ be chosen as the arc of the trajectories of motion. The line element is then of the form

$$ds^2 = -f^{-1} ds^2 + f(dt + \omega_i \, dx^i)^2 \quad , \qquad /1/$$

with all functions independent of t. ds $\left(ds = \sqrt{g_{ik} dx^i dx^k}\right)$ stands for the line element of the 3-dimensional background space V_3.

One can introduce in V_3 a complex basic vector "triad" $z_p^i = (\ell^i, m^i, \bar{m}^i)$; $p = 0, +, -$ with the orthogonality properties $\ell_i \, \ell^i = m_i \, \bar{m}^i = 1$, $\ell_i \, m^i = m_i \, m^i = 0$. The direction of the real unit vector ℓ^i is conveniently fixed by the relation

$$G_+ \equiv G_i \, m^i = 0 \quad . \qquad /2/$$

Here, G_i is a complex 3-vector determined by the gravitational field as follows:

$$G_i \stackrel{d}{=} \frac{\varepsilon_{,i}}{2\mathrm{Re}\varepsilon}; \quad \varepsilon_{,i} = f_{,i} + i\epsilon_{ijk} \, \omega^{j;k} \sqrt{g} \, f^2 \quad . \qquad /3/$$

Equation /2/ defines a congruence of curves with the tangent vector ℓ^i. These curves are called the eigenrays of the gravitational field. We now take the coordinate $x^1 = r$ to be the arc length of the eigenrays. Thus we have for the base vectors

$$\ell^i = \delta^i_1, \quad m^i = \omega\delta^i_1 + \xi^a \, \delta^i_a \quad /a=2,3/ \quad . \qquad /4/$$

The coordinate transformations

$$r' = r + r^o(x^a) \quad , \qquad /5/$$

$$x^{a'} = x^{a'}(x^b) \quad b = 2,3, \qquad /6/$$

are still permissible.

The quantity[6] $\epsilon = m_{i;j}\bar{m}^i \ell^j$ is made zero by a complex rotation of m^i. There is still a freedom in the choice of the triad:

$$\ell'^i = \ell^i ,$$
$$m'^i = e^{iC^o} m^i . \qquad /7/$$

Here, C^o is an arbitrary real function of the coordinates x^a.

Geodesic eigenrays are characterized by vanishing of the complex rotation coefficient $\kappa = m_{i;j}\ell^i\ell^j$. The field equations for geodesic eigenrays are taken from P:

$$D\omega = \bar{\rho}\omega + \sigma\bar{\omega} \qquad /8a/$$

$$D\xi^a = \bar{\rho}\xi^a + \sigma\bar{\xi}^a \qquad /8b/$$

$$D\sigma = (\rho + \bar{\rho})\sigma \qquad /8c/$$

$$D\rho = \rho^2 + \sigma\bar{\sigma} + G_o\bar{G}_o \qquad /8d/$$

$$DG_o = (2\rho + G_o - \bar{G}_o)G_o \qquad /8e/$$

$$D\tau = \rho\tau - \bar{\sigma}\bar{\tau} + \bar{G}_o G_- \qquad /8f/$$

$$\delta\bar{\omega} - \bar{\delta}\omega = \bar{\tau}\bar{\omega} - \tau\omega + \bar{\rho} - \rho \qquad /8g/$$

$$\delta\bar{\xi}^a - \bar{\delta}\xi^a = \bar{\tau}\bar{\xi}^a - \tau\xi^a \qquad /8h/$$

$$\delta\rho - \bar{\delta}\sigma = -2\sigma\tau + \bar{G}_+ G_o \qquad /8i/$$

$$\delta\tau + \bar{\delta}\bar{\tau} = \rho\bar{\rho} - \sigma\bar{\sigma} + 2\tau\bar{\tau} - G_o \bar{G}_o + G_- \bar{G}_+ \qquad /8j/$$

In our coordinate system the scalar differential operators appearing in the field equations are of the form $D = \partial/\partial r$; $\delta = \omega\partial/\partial r + \xi^a \partial/\partial x^a$. From eq. /8c/ it follows that the phase of the complex shear σ is independent of r. Therefore, by the triad freedom /7/ we make σ real and positive. Thus the triad is completely fixed.

2. THEOREM ON SHEARING GEODESIC EIGENRAYS

When trying to integrate the field equations /8/ for nonvanishing σ one is lead to many separate cases, most of which do not contain any solution since the calculation ends at some prohibitive relation. It is desirable to recognize such cases directly, without the lengthy integration procedure, from the field equations. Information can be gained immediately from the field equations by an operation used by Newman and Penrose in their proof of the Goldberg-Sachs theorem[2/]: one takes appropriately chosen derivatives of the Newman-Penrose equations and eliminates second order terms by the commutators of the scalar differential operators. Consecutive effectuation of a procedure of this kind provides the proof of our main theorem:

<u>Geodesic eigenrays in a curved vacuum stationary space-time cannot have coexisting shear and curl. If the eigenrays do shear / $\sigma \neq 0$ /, then they also diverge / $\rho + \bar{\rho} \neq 0$ /, and one has</u>

$$\rho\bar{\rho} - \sigma\bar{\sigma} - G_o \bar{G}_o = 0 \quad . \qquad /9/$$

The proof takes a more concise form by the use of the operators

$$\delta_{\pm} \stackrel{\mathrm{d}}{=} R(\delta \pm i\bar{\delta}) \quad . \qquad /10/$$

Here, R is the "luminosity distance" satisfying[2/] $DR = -\frac{\rho + \bar{\rho}}{2} R$. By definition /10/ the operators D and $\delta_{\pm}$ commute as follows:

$$D\delta_{\pm} - \delta_{\pm}D = -i(a \mp \sigma)\delta_{\mp} \; . \qquad /11/$$

We now observe that the only field equation from which new first order relation can be obtained is /8e/. The action of the δ operator on this equation, followed by elimination of the second derivatives, yields

$$\sigma\left(\bar{\delta}\ \ln G_o + \bar{\delta}\ \ln\sigma - G_- - 2\tau\right) = 0 \; . \qquad /12/$$

Considering the case when $\sigma \neq 0$, we have to prove that $2ia \overset{d}{=} \rho - \bar{\rho} = 0$ or, which is equivalent, that there exists a coordinate system in which $\omega = 0$.

Acting on equation /12/ with the D operator and subtracting the δ derivatives of appropriate radial equations such that the second order terms are canceled again:

$$3\gamma^2\bar{\delta}\rho + \left(\gamma^2+\sigma^2\right)\delta\sigma + \left(\gamma^2-\sigma^2\right)\left(\overline{\delta\rho}+2\overline{\sigma\tau}\right) = 0 \; ; \; \left(\gamma \overset{d}{=} |G_o|\right) . \qquad /13/$$

Equation /13/ can be made homogenous by substitution of /8i/:

$$\gamma\left(3\bar{\delta}\rho + 2\delta\sigma + \overline{\delta\rho}\right) + 2\sigma\delta\gamma = 0 \quad . \qquad /13'/$$

In the $\delta_{\pm}$ notation, /13'/ can easily be split up into components which are mutually orthogonal in the complex plane:

$$\gamma\delta_{\pm}(\rho+\bar{\rho}) - i\gamma\delta_{\mp} a \pm i\delta_{\mp}(\sigma\gamma) = 0 \; . \qquad /14/$$

Denoting $\Delta^2 = \sigma^2 + \gamma^2 - a^2$, we obtain from the commutators /11/, and from /8/

$$\left[D - (\rho+\bar{\rho})\right]\ \delta_{\pm}(\rho+\bar{\rho}) = 4\Delta\delta_{\pm}\Delta - i(a\mp\sigma)\ \delta_{\mp}(\rho+\bar{\rho})$$

$$\left[D - (\rho+\bar{\rho})\right]\ \delta_{\pm}\underline{v} = \underline{v}\ \delta_{\pm}(\rho+\bar{\rho}) - i(a\mp\sigma)\ \delta_{\mp}\underline{v}; \quad \underline{v} = \begin{pmatrix}\Delta\\ a\\ \sigma\\ \gamma\end{pmatrix} . \qquad /15/$$

Repeated application of the operator $D-\rho+\bar{\rho}$ on equation /14/ and use of /15/ yields the following series of first order equations:

$$(3a \pm 2\sigma)\ \delta_{\pm}(\rho+\bar{\rho}) + 4i\Delta\delta_{\mp}\Delta - 2i(a\mp\sigma)\ \delta_{\mp}a = 0 \qquad /16/$$

$$\left[(2a\pm 4\sigma)(a\mp\sigma) - 4\Delta^2\right]\delta_{\mp}(\rho+\bar\rho) + i(3a \pm 4\sigma)\,4\Delta\delta_{\pm}\Delta = 0 \qquad /17/$$

$$\left[a(a^2-\sigma^2) - \Delta^2(5a\mp 4\sigma)\right]\delta_{\mp}(\rho+\bar\rho) + i\left[(2a\mp\sigma)(a\pm\sigma) - \Delta^2\right]4\Delta\delta_{\pm}\Delta = 0\,. \qquad /18/$$

/17/ and /18/ are homogeneous in $\delta_{\pm}(\rho+\bar\rho)$ and $\delta_{\pm}\Delta$ with the determinants

$$D_{u \atop \ell} = 4\Delta^4 - \Delta^2(8\sigma^2-5a^2\mp 2a\sigma) + (a^2-\sigma^2)(a^2-4\sigma^2\pm 2a\sigma)\,. \qquad /19/$$

Here u and ℓ label the determinants of the equations with the upper and lower signs, respectively. The simultaneous vanishing of both D_u and D_ℓ means $\gamma = 0$ /flat space/. We may still have either of the determinants vanishing. Using now the upper and lower signs and labels for the alternate possibilities, we may write

$$D_{u \atop \ell} = 0 \qquad /20/$$

and

$$\delta_{\pm}(\rho+\bar\rho) = \delta_{\mp}\Delta = 0. \qquad /21/$$

From /21/ we derive the first order relations

$$4\Delta\delta_{\pm}\Delta - i(a\mp\sigma)\delta_{\mp}(\rho+\bar\rho) = 0$$

$$\Delta\delta_{\mp}(\rho+\bar\rho) - i(a\pm\sigma)\delta_{\pm}\Delta = 0\,. \qquad /22/$$

Hence we immediately have

$$\delta_{\pm}\Delta = \delta_{\mp}(\rho+\bar\rho) = 0 \qquad /23/$$

if the determinant of eq.s /22/ is different from zero.
The condition

$$\det \equiv 4\Delta^2 + a^2 - \sigma^2 = 0\,, \qquad /24/$$

when substituted back into /17/, leads to the same result.
We thus conclude that for $\gamma \neq 0$ eq.s /17/ and /18/ always yield

$$\delta_+(\rho+\bar\rho) = \delta_-(\rho+\bar\rho) = \delta_+\Delta = \delta_-\Delta = 0 \quad ; \tag{25}$$

or, in terms of the δ operator,

$$\delta(\rho+\bar\rho) = \delta\Delta = 0 \quad . \tag{26}$$

Assuming for the moment that $\rho+\bar\rho$ is nonzero, the commutator /eq. /72b/ of P /

$$\delta\bar\delta - \bar\delta\delta = \bar\tau\delta - \tau\bar\delta + (\bar\rho-\rho)D \tag{27}$$

when acting on $\rho+\bar\rho$ and Δ gives $a = 0$. We now show that $\rho+\bar\rho$ actually cannot vanish. Equations /25/ together with /14/ and /16/ ensure that $\delta\sigma = \delta\gamma = \delta a = 0$.

Let us denote the phase of G_o by χ . Our starting relation /12/ is then written in the form

$$i\bar\delta\chi - G_- = 2\tau \quad . \tag{28}$$

Letting the operator δ act on /28/, taking the real part and subtracting /8j/, we get

$$(\rho-\bar\rho)^2 + 2(\rho\bar\rho - \gamma^2 - \sigma^2) = 0 \ . \tag{29}$$

This equation, when compared with /8d/, tells us that $\rho+\bar\rho$ is nonvanishing and that we have $\rho\bar\rho - \gamma^2 - \sigma^2 = 0$. This completes the proof of our central theorem.

The "radial" equations /8a,...e/ containing the operator D are now readily integrated to yield

$$\frac{\sigma}{\sigma^o} = \frac{\gamma}{\gamma^o} = -\rho = \frac{1}{2r} \tag{30}$$

$$G_o = -\frac{\gamma^o}{2r}\,\frac{r\gamma^o - iQ}{r\gamma^o + iQ} \tag{31}$$

$$\xi^a = \frac{1}{\sqrt{2r}}\left(A^a\, r^{\sigma^o/2} + iB^a\, r^{-\sigma^o/2}\right). \qquad /32/$$

Here, Q, A^a, B^a are real integration "constants", depending only on $x^2 = x$ and $x^3 = y$; σ^o and γ^o are positive numbers subject to

$$\sigma^{o2} + \gamma^{o2} = 1 \ . \qquad /33/$$

The coordinate freedom /5/ has been used in /30/ to fix the origin of r. From $\delta\rho=0$ we get $\omega=0$ in this coordinate system. The r dependence of the complex scalar quantity ε appearing in /3/ can be obtained from the definition of G_o /see P /:

$$G_o = \frac{D\varepsilon}{2\mathrm{Re}\,\varepsilon} \ . \qquad /34/$$

Now G_o is given by /31/, and thus ε takes the form

$$\varepsilon \equiv f + i\varphi = \frac{f^o}{r^{\gamma^o} + iQ} + i\varphi^o \ , \qquad /35/$$

with f^o, φ^o real functions of x, y.

We are now faced with the following remnants of the original field equations /8/:

$$\delta\varepsilon = 0 \qquad /36a/$$

$$\mathrm{Im}\left[\left(\delta - \bar{\tau}\right)\bar{\xi}^a\right] = 0 \qquad /36b/$$

$$2\sigma\tau = \bar{G}_+ G_o \qquad /36c/$$

$$\tau\left(\sigma^2 - \gamma^2\right) = 0 \ . \qquad /36d/$$

In accordance with eq. /36d/, the metrics split up into classes with either $\tau=0$ or $\gamma=\sigma$. This bifurcate logics of the field equations must be dealt with by treating both of the classes separately in the following sections.

3. METRICS WITH $\tau = 0$

For this class from eq. /36c/ we get $\bar{G}_{+}=0$ or $\bar{\delta}\varepsilon = 0$. Thence $\varphi^{o} = 0$ and the quantities f^{o}, Q are in fact constants. This gives rise to a functional relationship between the quantities f and φ.

On substituting the expression /32/ for ξ^{a} into $\mathrm{Im}\,\delta\bar{\xi}^{a} = 0$ /equation /36b/ /, the operators $A \overset{d}{=} A^{a}\partial a$ and $B \overset{d}{=} B^{a}\partial a$ commute:

$$[A, B] = 0 \quad . \qquad /37/$$

The only coordinate freedom in V_3 is now /6/. The quantities A^{a}, B^{a} behave as two component vectors under the transformations /6/. Since at regular space-time points A^{a} and B^{a} are linearly independent, we can make them tangential to the coordinate curves x and y, respectively:

$$A^{a} = \delta^{a}_{2} \quad , \quad B^{a} = \delta^{a}_{3} \quad . \qquad /38/$$

With this choice of the coordinates we have

$$\xi^{2} = \frac{1}{\sqrt{2r}}\, r^{\sigma^{o}/2} \quad , \quad \xi^{3} = \frac{i}{\sqrt{2r}}\, r^{-\sigma^{o}/2} \quad . \qquad /39/$$

The 3-vector ω_{i} in the line element /1/ will now be evaluated. The relation required at this point is taken from P

$$\omega_{i,j} - \omega_{j,i} = \epsilon_{ijk}\, \varphi^{;k} \sqrt{g}\, f^{-2} \quad . \qquad /40/$$

The coordinate t can be shifted without disturbing /1/: $t' = t + F(r, x^{a})$. By this transformation we are able to put $\omega_{r} = 0$, and we still have

$$t' = t + t^{o}(x, y) \quad . \qquad /41/$$

Equation /40/ takes the form

$$\omega_{x,r} = \omega_{y,r} = 0 \quad ,$$
$$/42/$$
$$\omega_{x,y} - \omega_{y,x} = 2\gamma^{o}Q/f^{o}$$

From /41/ and /42/ we obtain

$$\omega_{i} = \left(0,\ 0,\ -2\,\frac{\gamma^{o}Q}{f^{o}}\, x\right) . \qquad /43/$$

With the aid of formulas /35/, /39/ and /43/ we may put down the line element for the class with $\tau = 0$:

$$ds^2 = -\frac{r^{2\gamma^o} + Q^2}{f^o\, r^{\gamma^o}}\left(dr^2 + r^{1-\gamma^o}dx^2 + r^{1+\gamma^o}dy^2\right) + \frac{f^o\, r^{\gamma^o}}{r^{2\gamma^o} + Q^2}\left(dt - 2\,\frac{\gamma^o x}{f^o}\,Q\,dy\right)^2 . \qquad /44/$$

This space-time is stationary and axially symmetric. In addition, as observed above, the invariant f is a function of φ . Therefore the line element /44/ represents a particular Papapetrou solution[7].

4. METRICS WITH $\sigma = \gamma$

Equation /33/ precludes the flat space limit by determining uniquely the constants σ^o and γ^o :

$$\sigma^o = \gamma^o = \frac{1}{\sqrt{2}} .$$

The remaining field equations /36a, b, c/ can be worked out after calculating the quantity $G\_ = \bar{\delta}\ln f$. /36c/ immediately yields τ . From /36a, b/ we get

$$\delta\varepsilon = 0 \rightarrow \begin{cases} A\varphi^o = 0 & AQ = (B - QA)\ln f^o \\ B\varphi^o = Af^o & BQ = Q(B - QA)\ln f^o \end{cases} \qquad /46a/$$

$$\mathrm{Im}\left[(\delta - \bar{\tau})\,\xi^a\right] = 0 \rightarrow \quad 2[A, B] = 2Q(A\ln f^o)A - (A\ln f^o)B - (B\ln f^o)A . \qquad /46b/$$

The problem becomes somewhat simpler if instead of A and B we use the operators

$$\alpha = \sqrt{f^o}\,A ,$$

$$\beta = \sqrt{f^o}\,(B - QA) . \qquad /47/$$

Thus for the field equations we have:

$$[\alpha, \beta] = -(\alpha Q)\alpha \qquad /48a/$$

$$\beta Q = 0 \qquad /48b/$$

$$\alpha Q = \beta \ln f^o \qquad /48c/$$

$$\alpha f^o = \beta \varphi^o \qquad /48d/$$

$$\alpha \varphi^o = 0 \,. \qquad /48e/$$

Letting the commutator /48a/ act on Q, and taking account of /48b, c/, we find that

$$\beta\left(f^{o-2} \beta f^o\right) = 0 \,. \qquad /49/$$

This relation, when compared with /48b/, tells us that $f^{o-2}\beta f^o$ is a functional of Q if Q is not constant. But let us consider first the case when Q is constant. For such metrics the operators α and β commute; therefore the coordinates can be chosen to have

$$\alpha = \partial_2 \,, \qquad \beta = \partial_3 - Q\partial_2 \,. \qquad /50/$$

Equations /48c, d, e/ are easily integrated to yield

$$f^o = P(x + Qy) \,, \qquad \varphi^o = Py \,. \qquad /51/$$

Here, P is a constant of integration and the origin of coordinates has been shifted to make the constant terms vanish. There exists another solution with f^o and φ^o constant, but this latter metric has vanishing τ and thus belongs to the class which has been discussed in section 3.

The most important field quantities obtained by use of /51/ are:

$$\varepsilon \equiv f + i\varphi = P \,\frac{x + ir^{\gamma^o} y}{r^{\gamma^o} + iQ}$$

$$\xi^2 = \left(2r\, f^o\right)^{-1/2}\, r^{\gamma^o/2} \quad , \quad \xi^3 = i\left(2r\, f^o\right)^{-1/2}\, r^{-\gamma^o/2}$$

$$g_{ij} = \begin{vmatrix} 1 & & \\ & f^o\, r^{1-\gamma^o} & \\ & & f^o\, r^{1+\gamma^o} \end{vmatrix} \quad . \qquad /52/$$

The calculation of the 4-dimensional line element terminates with the evaluation of ω_i , using formula /35/. As a result we have

$$d\tilde{s}^2 = - \frac{f^o}{f}\left(r^{1-\gamma^o}\, dx^2 + r^{1+\gamma^o}\, dy^2\right) + 2dr\left(dt - 2\gamma^o\, Qy\, dx\right) +$$

$$+ f\left(dt - 2\gamma^o\, Qy\, dx\right)^2 \quad . \qquad /53/$$

A glance at the line element convinces us that for Q vanishing, $\partial/\partial y$ is a Killing vector. Investigation of the curvature invariants Ψ_A /cf. P / shows that this space-time has true singularity at $r = 0$ and becomes flat in the limit $r\to\infty$, $f<\infty$. However, the behavior of the metric is rather awkward; it remains regular for $r\to\infty$ only if x or y also goes properly to infinity.

Consider now the case when the quantity Q does depend on the coordinates. We want to integrate the simultaneous equations /48/ without committing ourselves to any particular coordinate system. According to eq. /49/ we may write

$$f^{o-2}\, \beta f^{\,o} = q(Q) \, . \qquad /54/$$

Next we act with /48a/ on f^o :

$$\beta\left[f^{o-3}\left(\alpha f^o + \frac{q'}{q}\, \beta\, f^o\right)\right] = 0 \;\to\; \alpha\, f^o = q\, p(Q)\, f^{o3} - \frac{q'}{q}\, \beta\, f^o \qquad /55/$$

/Prime stands for d/dQ./ From /55/ and /48d/ it follows that

$$\beta\left(\varphi^o - \frac{1}{2}\, p\, f^{o2} + \frac{q'}{q}\, f^o\right) = 0 \;\to\; \varphi^o = \frac{1}{2}\, p\, f^{o2} - \frac{q'}{q}\, f^o + s(Q) \, . \qquad /56/$$

Here, the functionals p, q and s are arbitrary for the moment.

The latter expression for φ^o is now placed in equation /48e/:

$$\left[\frac{1}{2}\, p'\, f^{o2} - \left(\frac{q'}{q}\right)'\, f^o + s' + \left(pf^o - \frac{q'}{q}\right)^2\, f^o\right]\beta\, f^o = 0. \qquad /57/$$

/48c/ shows that βf^{o} cannot vanish. This condition means that f^{o} still varies once Q is fixed. Therefore the parenthesized quantity of eq. /57/ is equal to zero and can be regarded a polynomial equation in f^{o}. We make the coefficients of the various f^{o} powers vanish:

$$p = 0 \quad , \quad s' = 0 \, , \left(\frac{q'}{q}\right)' - \left(\frac{q'}{q}\right)^2 = 0 \quad . \qquad /58/$$

It is possible to set $s = 0$ because φ is defined only up to an arbitrary constant term. Straightforward integration yields

$$q = \frac{1}{aQ + b} \, , \qquad /59/$$

with a, b real constants. We are now in position to put down the relation connecting f^{o}, Q and /cf. eq. /56/ /:

$$\varphi^{o} = \frac{a}{aQ + b} f^{o} \, , \qquad /60/$$

Upon the above considerations, the quantities f^{o} and Q are appropriate candidates for independent coordinates. Nevertheless, some simplification of the final results is achieved by the following choice of the coordinate system:

$$Q = \frac{x}{y} \quad , \quad f^{o} = \frac{ax + by}{y^2} \qquad /61/$$

Calculation of the metric can be performed in a similar way to that used in previous examples. The results are summarized in the following:

$$\varphi^{o} = a/y \quad ; \qquad f = \frac{ax + by}{x^2 r^{-\gamma^{o}} + y^2 r^{\gamma^{o}}} \quad ;$$

$$d\tilde{s}^2 = - \frac{f^{o}}{f}\left(r^{1-\gamma^{o}}\, dx^2 + r^{1+\gamma^{o}}\, dy^2\right) + 2dr\left(dt - \gamma^{o} \frac{x^2}{y} dy\right) + \qquad /62/$$

$$+ f\left(dt - \gamma^{o} \frac{x^2}{y} dy\right)^2 \; .$$

The curvature invariants vanish for $r \to \infty$ except in such directions in which f becomes unbounded. Curvature singularities exist at $r = 0$, $y = 0$, $ax + by = 0$, and in the exceptional $r \to \infty$ limit.

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