KFKI-70-24 HEP

A. Frenkel

INTRODUCTION TO THE CURRENT- CURRENT THEORY OF THE WEAK INTERACTION

Hungarian Academy of Sciences

CENTRAL RESEARCH INSTITUTE FOR PHYSICS

BUDAPEST

C O N T E N T S

T A B L E S

F I G U R E S

P R E F A C E

These notes are based on a serie of seminars given by the author at the Institute of Mathematics of Brussels University during the first semester of the academic year 1969/1970.

The purpose of these lectures was to review the status of the modern current-current theory of weak interactions, and to compare its predictions with the experimental results in the field of elementary particle physics. The audience was composed partly of theoreticians working in the field of the strong interactions, and partly of experimentalists working in the field of the weak interactions. The author hopes that these notes will be useful as a review of the theory of weak interactions for research workers active in the above mentioned branches of elementary particle physics, and as an introduction to this theory for graduate students interested in the subject. No preliminary training in the theory of the weak interaction itself is required by the reader, but the knowledge of the elements of of relativistic field theory /e.g. of quantum electrodynamics without the renormalization technique/ and of elementary particle physics is assumed.

No detailed bibliography is given in these notes. Instead we refer to basic works where extensive references can be found. Concerning the numerical values of the various parameters of the theory of the weak interaction, we give mean values and errors, but no systematic effort has been made to use always the "last" or the "best" values, except for the basic coupling constants g and g_V. As is well known, the values of these parameters often change under the influence of new experiments, and for the last and/or best values the reader should consult the proceedings of the appropriate conferences, where he will be referred to the original works.

This stay at the Brussels University was supported by a grant from the Solvay International Institute for Physics which is gratefully acknowledged.

It is a pleasure for the author to express his sincere gratitude to Professors J. Gehéniau and J. Reignier for the kind hospitality extended to him during this period.

Thanks are also due to Prof. G. Marx and to Dr. P. Hraskó for a critical reading of the manuscript and to Dr. C. Schomblond, whose notes taken at the seminars considerably facilitated the production of the present version. The help of Dr. J. Bijtebier in editing these notes is also acknowledged.

The present preprint has been edited at Budapest, but has been printed from the same manuscript as the corresponding Bulletin of the Nuclear Physics Department, edited at the University of Brussels. Thus the two texts should be identical except for minor editorial changes.

I. INTRODUCTION

In the description of weak interaction phenomena the current-current theory plays a central role. In its original form, due to Fermi, the theory served to deal with the nuclear β decay. As well known, in β decay the directly observable decay products are the β particle, (an electron or a positron), and the daughter (or "recoil") nucleus, N . If the β decay were a two body decay $N \to N' + \beta$, then in the rest system of the parent nucleus N, the energy of the β particle would have a fixed value for given parent and daughter nuclei. The measurement of the energy of the β particles revealed that this is not so, and that the β particles have an energy spectrum. To save the law of energy conservation, in 1931 Pauli suggested that the β decay is a three body decay $N \to N' + \beta + \nu$. The invisible third particle, baptized by Fermi the neutrino, was supposed to be a neutral particle with very small, eventually zero mass (because the measurements have shown that the upper limit of the $E_{N'} + E_{\nu}$ values is very close or equal to M_N). The discovery of the neutron in 1932 led to the hypothesis that the elementary processes which manifest themselves in the wide variety of the nuclear β^- and β^+ decays are the $n \to pe^- \bar{\nu}$ and $p \to ne^+ \nu$ transitions, respectively. Of course the β decay of a free proton is forbidden by energy conservation, but in a nucleus the binding energy also enters into the game.

To conserve angular momentum, the spin of the neutrino must be half-integer, and the simplest hypothesis was that it is 1/2. Taking into account all these facts, in 1934 Fermi proposed to induce the nuclear β transitions by a local interaction of the $\psi_n, \psi_p, \psi_e, \psi_\nu$ fields. In other words, he supposed that the interaction Lagrangean may be, e.g.

$$L_F^V(x) = f_V\left(\bar{\psi}_n(x)\, \gamma_\lambda\, \psi_p(x)\, \bar{\psi}_\nu(x)\, \gamma^\lambda\, \psi_e(x) + \text{h.c.}\right) \qquad /1/$$

This Lagrangean is of the current-current form. Indeed, it is the product of the vector current $\bar{\psi}_n \gamma_\lambda \psi_p$ of the nucleons with the vector current $\bar{\psi}_\nu \gamma^\lambda \psi_e$ of the leptons. The comparison of the β spectrum for unpolarized β decay, calculated with $L_F^V(x)$ in lowest order in f_V , with the experimental spectrum showed a very good agreement in all those cases in which the nuclear structure of the involved nuclei was sufficiently known and therefore its influence could be taken into account, or could be legitimately neglected. However, it turns out that practically the same spectrum (with 0,1 % deviations) is given also by the more general Lagrangean

$$L_F(x) = \sum_i \left(f_i \bar{\psi}_n \Gamma_i \psi_p \bar{\psi}_\nu \Gamma'_i \psi_e + h.c.\right) \quad , \quad /i = S,P,V,A,T/ \quad ,$$

$$\Gamma_s = \Gamma'_s = 1, \quad \Gamma_p = \Gamma_{p'} = \gamma_5 \, , \quad \Gamma_V = \gamma_\lambda \, , \quad \Gamma'_V = \gamma^\lambda \qquad /2/$$

$$\Gamma_A = -i \, \gamma_\lambda \, \gamma_5, \quad \Gamma'_A = -i \, \gamma^\lambda \, \gamma_5, \quad \Gamma_T = \frac{1}{2i}\left(\gamma_\lambda \, \gamma_\nu - \gamma_\nu \, \gamma_\lambda\right) \equiv \sigma_{\lambda\nu} \, , \quad \Gamma'_T = \sigma^{\lambda\nu} \, .$$

This is due to the peculiar kinematical situation in the nuclear β decay, expressed by the relations $M_N \gg M_N - M_{N'} \approx m_e$. To find the coupling constant of these current-current interactions constructed with scalar (S), pseudoscalar (P), vector (V), axial (A), and tensor (T) currents, measurements of angular correlation and polarization are needed. These difficult experiments received fundamental importance in 1956, when, from the analysis of the $K \to 2\pi$, $K \to 3\pi$ decays, Lee and Yang came to the conclusion that parity is not conserved in these decays. The kaon decays are so slow compared to the characteristic 10^{-22} sec time interval of the strong interactions, that it was supposed that they can be classified as weak interactions. If so, the possibility of parity violation in nuclear β decay should be envisaged. This expectation was soon confirmed in the celebrated Co^{60} experiment of Wu. The number of the coupling constants increased considerably, because now current-pseudocurrent couplings had also to be included into the Lagrangen. However, this complication turned out to be salutary, because the parity violating terms happened to be of the same strength as the parity-conserving ones, and without them a good agreement with the experimental distributions would not be possible. Namely, from many concording experiments the V and A currents were found to be necessary and sufficient to construct the Lagrangean in the following way:

$$L_F^{V,A}(x) = \bar{\psi}_n \, \gamma_\lambda \left(f_V - f_A \, i \, \gamma_5\right) \psi_p \left(\bar{\psi}_e \, \gamma^\lambda (1 - i \, \gamma_5) \, \psi_\nu\right)^+ + h.c. \qquad /3/$$

Until the early fifties the bulk of the experimental information of the weak interaction came from nuclear physics. The spectacular development of elementary particle physics in the last two decades changed this situation. The already mentioned discovery of the parity violation; the discovery of the two kinds of neutrinos; the establishment of the isospin and strangeness selection rules of the weak interaction; the possibility of the application of the SU(3) algebra to the weak interaction; the discovery of the CP violation - all these results were found in elementary particle physics, and led to a further development of the current - current theory of the weak interaction. In these short notes it is out of question to follow the historical development in detail. Therefore from the very beginning we shall work with the most modern form of the current-current theory, established by Gell-Mann and Cabibbo in 1964. Occasionally we shall explain how and why this form of the theory was adopted, but our order of presentation will not necessarily follow the historical order.

The principal question we shall deal with is the following: to what extent the modern current-current theory can be considered as the general theory of the weak interaction, what are the successes, the failures and open problems of this theory? Here again a complete review of the status of the theory is impossible for us; nevertheless, it is hoped that the general picture will be clear.

The weak interaction Lagrangean of Gell-Mann and Cabibbo can be written in the form

$$L(x) = \frac{g}{\sqrt{2}} \frac{1}{2} \left(J^{\lambda}(x)\, J^{+}_{\lambda}(x) + J^{+}_{\lambda}(x)\, J^{\lambda}(x) \right) \quad ,$$
$$J_{\lambda}(x) = J_{H\lambda}(x) + j_{\lambda}(x) \quad . \qquad /4/$$

The full weak current $J_{\lambda}(x)$ is the sum of the weak current of the hadrons $J_{H\lambda}(x)$, the explicit form of which is unknown apart from some important SU(3) transformation properties to be specified later, and of the weak current of the leptons $j_{\lambda}(x)$, which is supposed to be explicitly known:

$$j_{\lambda}(x) = \frac{1}{2}\left[\bar{\psi}_e(x),\ \gamma_{\lambda}\left(1 - i\gamma_5\right)\psi_{\nu_e}\right] + \frac{1}{2}\left[\bar{\psi}_{\mu}(x),\ \gamma_{\lambda}\left(1 - i\gamma_5\right)\psi_{\nu_{\mu}}(x)\right],$$
$$\left[\bar{\psi}_1\ ,\ \Gamma\psi_2\right] = \left(\bar{\psi}_{1\alpha}\ \psi_{2\beta} - \psi_{2\beta}\ \bar{\psi}_{1\alpha}\right)\Gamma_{\alpha\beta}. \qquad /5/$$

The cumbersome symmetrizations in eq. /4/ and /5/ are necessary when some properties of the theory under CP and SU(3) transformations are investigated. For our purposes they may be ignored in practical calculations.

The lepton current $j_{\lambda}(x)$ contains a vector part $v_{\lambda}(x)$ and an axial part $a_{\lambda}(x)$:

$$j_{\lambda} = v_{\lambda} + a_{\lambda}$$
$$v_{\lambda} \equiv \frac{1}{2}\left[\bar{\psi}_e,\ \gamma_{\lambda}\ \psi_{\nu_e}\right] + \frac{1}{2}\left[\bar{\psi}_{\mu},\ \gamma_{\lambda}\ \psi_{\nu_{\mu}}\right] \quad , \qquad /6/$$
$$a_{\lambda} \equiv -\frac{1}{2}\left[\bar{\psi}_e,\ i\gamma_{\lambda}\ \gamma_5\ \psi_{\nu_e}\right] - \frac{1}{2}\left[\bar{\psi}_{\mu},\ i\gamma_{\lambda}\gamma_5\ \psi_{\nu_{\mu}}\right].$$

The current $j_\lambda = v_\lambda + a_\lambda$ is unfortunately called in the literature a "V-A current", not a V+A current. Later we shall see that the weak hadron current also has a "V-A structure", i.e. it can be written as $J_{H\lambda} = V_\lambda + A_\lambda$. As shown in the appendix, the $(1 - i\gamma_5)$ factor appearing in the lepton current leads to the fact that only left handed neutrinos (neutrinos of negative helicity) and right handed antineutrinos (antineutrinos of positive helicity) can interact.

In the weak lepton current /5/ two neutrino fields are present. Let us call the neutrino emitted in the nuclear β decay the neutrino of the electron, ν_e, and the neutrino emitted in the $\pi^+ \to \mu^+ + \nu_\mu$ decay the neutrino of the muon, ν_μ. Their antiparticles are denoted by $\bar{\nu}_e$, $\bar{\nu}_\mu$. There exists ample experimental evidence ([1] pp 389 - 391; [3] pl) that $\nu_\mu \neq \nu_e$, $\bar{\nu}_e \neq \nu_e$, $\bar{\nu}_\mu \neq \nu_\mu$, and that in all interactions the electronic lepton number L_e and the muonic lepton number L_μ are separately conserved. The assignment of these quantum numbers to the leptons is given in table 1. For all the other particles $L_e = L_\mu = 0$. The conservation laws are of course, respected by the Lagrangean (4).

Table 1

Assignment of lepton numbers

	ν_e	$\bar{\nu}_e$	e^-	e^+	ν_μ	$\bar{\nu}_\mu$	μ^-	μ^+
L_e	1	-1	1	-1	0	0	0	0
L_μ	0	0	0	0	1	-1	1	-1

Concerning the masses of the neutrinos, the experimental upper limits are $m_e \leq 60$ eV, $m_{\nu_\mu} \leq 1{,}6$ MeV. As usual we shall assume that $m_{\nu_e} = m_{\nu_\mu} = 0$.

We shall now discuss an important open problem-(for optimists), or failure, (for pessimists) of the current-current theory. As well known, a four fermion interaction is non-renormalizable, and no higher order corrections can be calculated in such a theory. Unfortunately, this is the case with our current-current theory, as one can see from its purely leptonic $j^\lambda j^+_\lambda$ part. Also, all the plausible expressions for the hadron current in terms of hadron fields lead to non-renormalizable structures. In these notes we shall always deal with such processes, which have non-vanishing matrix element of first order in $L(x)$. Thus our Lagrangean has to be considered as an effective Lagrangean giving first-order approximations to an unknown or unmanageable theory. Moreover, it is easy to see that this first-order approximation cannot be used for the description of very high energy $(E \gtrsim 300 \text{ GeV})$ processes. Indeed, both the $\sigma(\nu_\mu + e^- \to \mu^- + \nu_e)$ partial cross section and the total cross section can be calculated in first order in g with our Lagrangean /4/. For the latter cross section the optical theorem must be used. The result is that in the centre of mass system for $E_\nu \gtrsim 300$ GeV the partial cross section exeeds the total cross section. This phenomenon is called the "unitarity catastrophe". In spite of all these problems, the success of this "bad" first-order theory in the description of a wide set of experimental facts is so impressive, that it can certainly be considered as very good low-energy approximation to any future theory of the weak interaction.

Using the decomposition $J_\lambda = J_{H\lambda} + j_\lambda$ of the full weak current, the Larangean (4) can be re written as follows:

$$L = L_{\ell\ell} + L_{H\ell} + L_{HH} \quad ;$$

$$L_{\ell\ell} = \frac{g}{\sqrt{2}} \frac{1}{2} \left(j^\lambda j_\lambda^+ + j_\lambda^+ j^\lambda \right)$$

$$L_{H\ell} = \frac{g}{\sqrt{2}} \frac{1}{2} \left(J_H^\lambda j_\lambda^+ + j_\lambda J_H^{\lambda+} + J_{H\lambda}^+ j^\lambda + j_\lambda^+ J_H^\lambda \right)$$

$$L_{HH} = \frac{g}{\sqrt{2}} \frac{1}{2} \left(J_H^\lambda J_{H\lambda}^+ + J_{H\lambda}^+ J_H^\lambda \right) \qquad /7/$$

In first order in g $L_{\ell\ell}$ describes purely leptonic processes with four leptons, e.g. $\mu \to \ell\nu\bar{\nu}$ decay and $\ell + \nu \to \nu_\ell + \ell$ scattering. $L_{H\ell}$ describes semileptonic processes in which hadrons and a lepton pair $\ell\nu_\ell$ are involved, e.g. $\nu_\ell + N \to \ell + N'$ scattering and $n \to pe\nu$ decay. We notice that with $J_{H\lambda} = \frac{\sqrt{2}}{g} \bar{\psi}_n \gamma_\lambda (f_V - f_A i\gamma_5) \psi_p$ we get back the symmetrized Fermi Lagrangean /3/. Finally, L_{HH} describes the non-leptonic weak interaction, where only hadrons are present, e.g. $K \to 2\pi$ decay, weak $p + n \to n + p$ scattering.

In order to proceed easily later, we give here the general expression for the matrix element of a semi-leptonic process of the type

$$H \to H' + \ell + \bar{\nu}_e \qquad /8/$$

where H and H' stand for two groups of hadrons, while ℓ denotes e^- or μ^-. The transition matrix element for this process in the Heisenberg picture reads:

$$\langle H'\,\ell\,\bar{\nu}_\ell \text{ out}|H \text{ in}\rangle = \langle H'\text{out}|c_{\text{out}}(\ell)\, d_{\text{out}}(\nu_\ell)|H \text{ in}\rangle \qquad /9/$$

Combining the relation

$$d_{\text{out}}(\bar{\nu}_\ell) = d_{\text{in}}(\bar{\nu}_\ell) - i\int dy\; i\,\frac{\partial\bar{\psi}_{\nu_\ell}(y)}{\partial y_\lambda}\,\frac{v(\bar{\nu}_\ell)}{(2\pi)^{3/2}}\,e^{i\bar{\nu}y} \qquad /10/$$

with the equation of motion for the neutrino field

$$\frac{\partial\bar{\psi}_{\nu_\ell}(y)}{\partial y_\lambda}\gamma_\lambda = \frac{g}{\sqrt{2}}\frac{1}{2}\left[\bar{\psi}_\ell(y)\,\gamma_\lambda(1-i\gamma_5)\,J^\lambda(y) + J^{\lambda+}(y)\,\bar{\psi}_\ell(y)\,\gamma_\lambda(1-i\gamma_5)\right]_- \qquad /11/$$

induced by the weak interaction Lagrangean (4), we arrive at the expression

$$\langle H'\,\ell\,\bar{\nu}_\ell \text{ out}|H \text{ in}\rangle = \frac{-ig}{\sqrt{2}}\frac{1}{2}\int dy \Big\langle H'\text{out}|c_{\text{out}}(\ell)\big(\bar{\psi}_\ell(y)\,\gamma_\lambda(1-i\gamma_5)\,J^{\lambda+}(y) +$$

$$+ J^{\lambda+}(y)\,\bar{\psi}_\ell(y)\,\gamma_\lambda(1-i\gamma_5)|H \text{ in}\Big\rangle \frac{v(\bar{\nu}_\ell)}{(2\pi)^{3/2}}\,e^{i\bar{\nu}_\ell y} \qquad /12/$$

As we explained above, we have to restrict ourselves to first order calculation in g. Since our matrix element /12/ is already proportional to g, all the operators and states in this equation may be considered as free from the point of view of the weak interaction. For simplicity we shall also neglect the electromagnetic interaction. Then the lepton fields in eq. /12/ become free fields, and those in the lepton current $j^{\lambda+}(y)$ are easily seen to give no contribution in our case. Taking into account that $\left[J_H^{\lambda+}(y), \bar{\psi}_\ell(y)\right]_- = 0$, because by definition the hadronic current at $t = y^o$ does not contain lepton operators at the same time t, we find with

$$\langle H'\text{out}|c_{\text{out}}(\ell)\,\bar{\psi}_\ell(y) = \Big\langle H' \text{ out}\Big|\frac{\bar{u}(\ell)}{(2\pi)^{3/2}}\,e^{iy\ell} \qquad /13/$$

the result

$$\langle H'\,\ell\,\nu_\ell \text{ out}|H \text{ in}\rangle = \frac{-ig}{\sqrt{2}}\int dy\,\langle H'\text{out}|J_H^{\lambda+}(y)|H \text{ in}\rangle\;.$$

$$\cdot\,\frac{\bar{u}(\ell)}{(2\pi)^{3/2}}\,\gamma_\lambda(1-i\gamma_5)\,\frac{v(\nu_\ell)}{(2\pi)^{3/2}}\,e^{i(\nu_\ell+\ell)y} \qquad /14/$$

The "in" and "out" labels refer now to the strong interaction only, because we consistently neglect the electromagnetic and higher-order weak interactions.

Using the well-known relation of the translation invariance for a local operator $O(y)$

$$<H'out|O(y)\ H\ in> = <H'out|O(O)|H\ in>\ e^{i(p_{H'}-p_H)y} \qquad /15/$$

we find our final expression for the transition matrix element $H \to H'\ell\bar{\nu}_e$

$$<H'\ell\ \bar{\nu}_\ell\ out|H\ in> = \frac{-ig}{\sqrt{2}}(2\pi)^4\ <H'out|J_H^{\lambda+}(O)|H\ in>\ .$$

$$\cdot\ \frac{\bar{u}(\ell)}{(2\pi)^{3/2}}\gamma_\lambda(1-i\gamma_5)\frac{v(\bar{\nu}_\ell)}{(2\pi)^{3/2}}\delta\left(p_{H'}-p_H-\ell-\bar{\nu}_\ell\right). \qquad /16/$$

The matrix element of the $H \to H'\bar{\ell}\ \nu_\ell$ processes, where $\bar{\ell}$ denotes a μ^+ or e^+, can be calculated in the same manner and turns out to be:

$$<H'\ell^+\ \nu out|H\ in> = \frac{-ig}{\sqrt{2}}(2\pi)^4\ <H'out|J_H^{\lambda}(O)|H\ in>\ .$$

$$\cdot\ \frac{\bar{u}(\nu_\ell)}{(2\pi)^{3/2}}\gamma_\lambda(1-i\gamma_5)\frac{v(\ell^+)}{(2\pi)^{3/2}}\delta\left(p_{H'}-p_H-\ell^+-\nu_\ell\right). \qquad /17/$$

Eq. /16/ and /17/ contain the hadronic matrix elements of the weak current operator of the hadrons. Since the explicit form of this operator is unknown - and even if it were known the lack of a strong interaction theory would prevent us from calculating its matrix elements exactly - the only thing we can do is to write down the general form of its various matrix elements allowed by Lorentz invariance and by other symmetry principles, and then to find such relations between them as can be tested experimentally. We shall deal with this problem in some detail and we shall see that the current-current theory is at least in qualitative agreement with all the available experimental data.

Let us now turn to the purely leptonic processes. For them instead of the hadronic matrix element $<H'|J_H^{\lambda}(O)|H>$ in eq. /17/ we shall have a matrix element of the lepton current between a lepton state and its neutrino. This matrix element is explicitely known, and thus the whole matrix element is calculable.The evaluation of the decay rates and cross section is then straightforward. To fix our notations and normalizations, we shall give the relevant formulae for the decay rates of a particle A into r particles in the appendix. The only purely leptonic weak process for which detailed experimental data are available is the muon decay. We shall see that the data are in perfect agreement with the current-current theory. We mention also that an experiment on the $\bar{\nu}_e + e^- \to \bar{\nu}_e + e^-$ scattering is in progress in the USA, but no confirmed results are available as yet.

The question of the applicability of the current-current theory to the non-leptonic weak interaction is completely open. It is obvious that the method used in the case of the pure and semi-leptonic processes fails in this case, since neither the weak current of the hadrons, nor the Lagrangean of the strong interaction are known. Nevertheless, with modern techniques (current algebra, partially conserved axial current (PCAC) hypothesis) interesting, qualitatively correct results could be reached in non leptonic kaon and hyperon decays. However, the current-current structure is not relevant to these results. A very clear account of the status of the non leptonic weak decays is given in [2], where the current-current theory is abandoned when comparison with the experiments is made. On the other hand, a recent analysis of the non-leptonic hyperon decays based on the current-current theory was given in Phys. Rev. 175, 2180, 1968 by Nussinov and Preparata. In both cases the results are qualitatively (30 %-100 % errors) in agreement with the experimental data and may depend on auxiliary hypotheses. Thus no definite conclusion can be made concerning the applicability of the current-current theory to the non leptonic weak interaction.

In these notes we shall concentrate on the successes of the current--current theory, and the problems and/or failures will be only shortly commented. Accordingly, the material will be presented in the following order. In chapter II we shall discuss the $\pi \rightarrow \ell\nu, \mu \rightarrow \nu\bar{\nu}$ and $n \rightarrow p\, e\, \nu$ decays. As we shall see, their investigation allows to establish the basic properties of the strangeness-conserving weak interaction. In chapters III and IV the current-current theory of the leptonic decays of the hadrons will be developed. In III the isotriplet vector current (IVC) theory of Gell-Mann, in IV the octet current theory of Cabibbo will be presented and compared with experimental data. In both cases the concept of the universality of the weak interaction, developed by Gell-Mann, will be formulated. In chapter V some of the open problems will be briefly discussed. Finally, technical material will be gathered in the appendix.

II. THE STRANGENESS CONSERVING WEAK DECAYS

§1. The $\pi \to \mu\nu$ decay

The most conspicuous decay mode of the $\pi^{\pm}$ meson is the $\pi \to \mu\nu$ decay. The pion being a spinless particle, the only quantities to measure are the full decay rate $\Gamma(\pi \to \mu\nu)$ and the polarization of the leptons. In the rest system of the pion the kinematics is particularly simple. Namely, we have

$$\underline{\nu} = -\underline{\mu}$$

$$m_\pi = \mu^o + \nu^o = \mu^o + |\underline{\nu}| = \mu^o + |\underline{\mu}| = \mu^o + \sqrt{\mu^{o2} - m_\mu^2}$$

$$\mu^o = \frac{m_\pi^2 + m_\mu^2}{2m_\pi}, \quad \nu^o = |\underline{\nu}| = |\underline{\mu}| = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} \qquad /18/$$

Let us calculate $\Gamma(\pi \to \mu^- \bar{\nu}_\mu)$ in the current - current theory. Eq. /16/ gives

$$\langle \mu^- \bar{\nu}_\mu \text{ out}|\pi^- \text{in}\rangle = \frac{-ig}{\sqrt{2}}(2\pi)^4 \langle O|J_H^{\lambda+}(O)|\pi^- \text{in}\rangle \frac{\bar{u}^s(\mu)}{(2\pi)^{3/2}} \gamma_\lambda (1-i\gamma_5) \frac{v(\nu)}{(2\pi)^{3/2}} \delta^{(4)}(p_\pi - \mu - \nu) \equiv$$

$$\equiv F\left(8m_\pi \nu^o \mu^o\right)^{-1/2} \delta^{(4)}(p_\pi - \mu - \nu) \quad . \qquad /19/$$

We suppress the helicity index of the antineutrino spinor because the $(1-i\gamma_5)$ factor forces the antineutrino to be always of positive helicity (see the Appendix). Then in the rest system of the pion the μ^- must also have helicity + 1 because of angular momentum conservation.

Let us investigate the hadronic matrix element $\langle O|J_H^{\lambda+}(O)|\pi^- \text{in}\rangle$. We shall begin with a simple example: if we suppose that $J_H^{\lambda+}(x)_{\text{eff}} = -\partial^\lambda \varphi_{\pi^- \text{in}}(x)$ then using eq. of the Appendix we find:

$$\langle O|J_H^{\lambda+}(O)|\pi^- \text{in}\rangle = \frac{i\, p_\pi^\lambda}{(2\pi)^{3/2} \sqrt{2p_\pi^o}} \qquad /20/$$

We see that the matrix element is not exactly a four vector, because of the energy-dependent extra factor $(2\pi)^{-3/2}(2p_\pi^o)^{-1/2}$. The appearance of this factor is due to our definition of the emission and absorbtion operators, given in the Appendix.

Let us now find the most general form of the matrix element. The weak interaction is parity-violating, thus a matrix element of $J_H^{\lambda+}$ could contain both a vector part (V) and an axial vector part (A). However, our matrix element depends only on the pion four momentum p_π^λ, which is a vector, and no axial vector can be constructed from it. Thus the most general form is conveniently written as

$$\langle 0|J_H^{\lambda +}(0)|\pi^- in\rangle = \frac{i\, p_\pi^\lambda}{(2\pi)^{3/2}\sqrt{2p\pi^o}}\, f\left(p_\pi^2\right) \, , \qquad /20a/$$

where $f\left(p_\pi^2\right)$ is an arbitrary scalar function of p_π^2, called the form factor of the $\langle 0|J_H^{\lambda +}(0)|\pi^- \; in\rangle$ matrix element. Its deviation from the value $f\left(p_\pi^2\right) = 1$ is a measure of the deviation of the axial vector part of $J_H^{\lambda +}(x)$ from the simple $-\partial^\lambda \varphi_{\pi^- in}(x)$ expression.

In the $\pi^- \to \mu^- \bar{\nu}_\mu$ decay the pion is on its mass shell, i.e. $p_\pi^2 = m_\pi^2$. We define $g\, f\left(m_\pi^2\right) \equiv F_\pi$ as the coupling constant of the $\pi^- \to \mu^- \bar{\nu}_\mu$ decay*. Then from eq. /19/ and /20a/ we find

$$F = (2\pi)^4 \frac{F_\pi}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \sqrt{4\mu^o \nu^o}\, p_\pi^\lambda \frac{\bar{u}^s(\mu)}{(2\pi)^{3/2}} \gamma_\lambda (1 - i\gamma_5) \frac{v(\nu)}{(2\pi)^{3/2}} \qquad /21/$$

The $\pi \to \mu\nu$ decay rate in the pion rest system according to eq. /55/ of the Appendix reads

$$d\Gamma\left(\pi^- \to \mu^- \bar{\nu}_\mu\right) = \frac{1}{2\pi} \frac{1}{2m} \sum_s |F|^2 \delta^{(4)}\left(p_\pi - \mu - \nu\right) \frac{d\underline{\mu}}{2\mu^o} \frac{d\underline{\nu}}{2\nu^o} \qquad /22/$$

and a straightforward trace calculation yields

$$\sum_s |F|^2 = \frac{|F_\pi|^2}{\pi} m_\mu^2 (\mu\, \nu) \qquad /23/$$

In the pion rest frame

$$(\mu\, \nu) \equiv \mu^o \nu^o - \underline{\mu}\, \underline{\nu} = \frac{1}{2}\left(m_\pi^2 - m_\mu^2\right) \qquad /24/$$

while the invariant phase space integral gives

$$\int \delta\left(p_\pi - \mu - \nu\right) \frac{d\underline{\mu}}{2\mu^o} \frac{d\underline{\nu}}{2\nu^o} = \frac{\pi}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) . \qquad /25/$$

Finally we arrive at the result

$$\Gamma\left(\pi^- \to \mu^- + \bar{\nu}_\mu\right) = \frac{1}{2\pi} \frac{1}{2m_\pi} \frac{|F_\pi|^2}{\pi} m_\mu^2 \left(m_\pi^2 - m_\mu^2\right) . \frac{\pi}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) \qquad /26/$$

From the measured $\pi^- \to \mu^- \bar{\nu}_\mu$ decay rate, which practically equals the total $\Gamma(\pi^-)$ decay rate, we obtain the absolute value of F_π in $\hbar = c = 1$ units:

$$\Gamma(\pi^-) = \left(38{,}42 \pm 0{,}02\right) \frac{10^6}{\text{sec}} \qquad |F_\pi| = \left(14{,}97 \pm 0{,}02\right) \frac{10^{-10}}{\text{MeV}} \qquad /27/$$

From the μ^- lifetime we shall obtain the value of $|g|$ and then we shall know also $|f(m_\pi^2)| = |F_\pi : g|$. More important, however, is the fact that in our current-current theory we can also calculate the $\pi^- \to e^- + \bar{\nu}_e$ decay

*The relation of $f\left(m_\pi^2\right)$ to the coupling constant f_π is given in eq./218/.

rate, and the only difference from the expression for the $\pi^- \to \mu^- + \bar{\nu}_\mu$ decay given in /26/ will be that instead of m_μ we will have m_e. This is a direct consequence of the so called "μ-e universality" of the weak interaction, expressed by the invariance of the weak lepton current under the substitution $\mu \rightleftarrows e$. We note that the quantum electrodynamics also has this property of μ - e universality, the electric current of the leptons being $\frac{1}{2}\left[\bar{\psi}_e, \gamma_\lambda\, \psi_e\right] + \frac{1}{2}\left[\bar{\psi}_\mu, \gamma_\lambda\, \psi_\mu\right]$.

Thus in our current-current theory the ratio $R(\pi^-)$ of the $\pi^- \to \mu^- \bar{\nu}_\mu$ and $\pi^- \to e^- \bar{\nu}_e$ decay rates turns out to be independent on the coupling constants and is equal to

$$R(\pi^-) \equiv \frac{\Gamma(\pi^- \to e^- \bar{\nu}_e)}{\Gamma(\pi^- \to \mu^- \bar{\nu}_\mu)} = \frac{m_e^2}{m_\mu^2} \cdot \frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2} \cdot \frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2} =$$

$$= 2{,}35.10^{-5} \cdot \frac{1}{0{,}43} \cdot \frac{1}{0{,}43} = 1{,}28.10^{-4} \qquad /28/$$

The experimental value is $R(\pi^-)\big|_{exp} = (1{,}24 \pm 0{,}03) \cdot 10^{-4}$, in good agreement with the theoretical value. We see that the smallness of $R(\pi^-)$ is due to the smallness of the ratio of the matrix elements, and not to the ratio of the phase spaces, which is 1:0,43. Indeed, the matrix element $\Sigma|F|^2$ turned out to be proportional to the lepton mass squared. It is easy to see that this is due to the fact that in our current-current theory we have V and A currents. Namely, the expression for F (see eq./21/) contains the factor

$$F_\pi\, p_\pi^\lambda\, u^s(\ell)\, \gamma_\lambda\, (a - ib\gamma_5)\, v(\nu_\ell) = F_\pi\, \bar{u}^s(\ell)(\hat{\ell} + \hat{\nu})(a - ib\gamma_5)\, v(\nu_\ell) =$$

$$= F_\pi\, m_\ell\, \bar{u}^s(\ell)(a - ib\gamma_5)\, v(\nu_\ell) \,, \qquad /29/$$

proportional to the lepton mass m_ℓ. In eq. /29/ we have slightly generalised the V - A coupling $\gamma_\lambda(1 - i\gamma_5)$ of the lepton current to the V, A coupling $\gamma_\lambda(a - ib\gamma_5)$, to stress that our result do not depend on the specific V - A character, but only on the V, A character. If instead of a V, A theory we would have a scalar (S), pseudoscalar (P) current-current theory, the matrix element for the decay would be proportional to

$$g\langle 0|J_H^+(0)|\pi, in\rangle\, \bar{u}^s(\ell)(a' - ib'\gamma_5)\, v(\nu_\ell) \qquad /30/$$

where J_H^+ is a scalar + pseudoscalar weak hadron current. The most general form of its matrix element reads:

$$\langle 0|J_H^+(0)|\pi^-, in\rangle = \frac{1}{(2\pi)^{3/2}}\, \frac{1}{\sqrt{2p^0}}\, f'(p_\pi^2) \,, \qquad /31/$$

and $g\,f'\left(m_\pi^2\right) \equiv F'_\pi$ is a new coupling constant. Now we obtain instead of /29/ a factor

$$F'_\pi\, \overline{u}^s(\ell)\left(a'-ib\gamma_5\right)\, v(\nu_\ell) \qquad /32/$$

without the lepton mass. Then the ratio of the electron and muon rates gives

$$R(\pi^-) = \frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2} \cdot \frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2} \approx 5{,}5 \qquad /33/$$

in bad contradiction with the experiment. Thus if the μ-e universality of the weak lepton current is accepted, the experimental $R(\pi^-)$ ratio indicates that the V, A coupling strongly dominates over the S, P coupling, the S, P coupling may be even completely absent. We stress also that nothing can be said from this experiment about the possible presence or absence of a tensor (T) current. Indeed, the $\langle 0|J_H^{\lambda\mu +}(0)|\pi^-,in\rangle$ matrix element will be proportional to $g^{\lambda\mu}$ and $p_\pi^\lambda\, p_\pi^\mu$, i.e. will be symmetric in λ,μ . On the other hand, the lepton current will contain the antisymmmetric tensor $\sigma_{\lambda\nu} = (\gamma_\lambda\, \gamma_\mu - \gamma_\mu\, \gamma_\lambda)/2i$ and thus the contribution of the tensor coupling to the $\pi \to \mu\nu$ decay will be zero ewen if the tensor currents are present in the Lagrangean. The symmetric tensor $\left(\gamma_\lambda\, \gamma_\mu + \gamma_\mu\, \gamma_\lambda\right)\left(a-ib\gamma_5\right)$equals $2g_{\lambda\mu}\left(a-ib\gamma_5\right)$ and gives an effective S, P coupling with a hadron current $J_H = J_H^{\lambda\mu}\, 2g_{\lambda\mu}$.

Let us still investigate the polarization in the V, A theory. Writing $\gamma_\lambda(a-ib\gamma_5)$ in the lepton current, we obtain, after a straightforward calculation, the following expression for the average helicity of the lepton ℓ in the $\pi^- \to \ell^-\, \overline{\nu}_\ell$ decay

$$\langle h_{\ell^-}\rangle \equiv \frac{\Gamma_\uparrow - \Gamma_\downarrow}{\Gamma_\uparrow + \Gamma_\downarrow} = \frac{2\mathrm{Re}\; ab^*}{|a|^2 + |b|^2} \qquad /34/$$

here $\Gamma_\uparrow(\Gamma_\downarrow)$ stands for the decay rate $\Gamma(\pi^- \to \ell^-\, \overline{\nu}_e)$ with ℓ^- of positive /negative/ helicity. Thus for the $\gamma_\lambda(1-i\gamma_5)$ theory $\langle h_{\ell^-}\rangle = +1$, for the $\gamma_\lambda(1+i\gamma_5)$ theory $\langle h_{\ell^-}\rangle = -1$,and for pure V or pure A theory $\langle h_{\ell^-}\rangle = 0$. The experimental result, $\langle h_{\mu^-}\rangle = 1{,}17 \pm 0{,}32$, clearly favourizes the V-A lepton current.

Let us end this discussion by the remark that the $K^- \to \ell^-\, \overline{\nu}_\ell$ decay can be treated in our theory exactly in the same manner as the π^- decay. Of course,a new coupling constant $F_k = g\, \varphi(m_k^2)$ will take the place of F_π , and m_k will turn up instead of m_π . The ratio of the electronic decay rate to the muonic in the V, A theory equals $2{,}75.10^{-5}$, while in the S, P theory it is 1,1. The experimental result, $(2 \pm 0{,}65).\,10^{-5}$ again shows the correctness of the V, A coupling. From the experimental $K^- \to \ell^-\, \overline{\nu}_\ell$ decay rate we find $|F_k|$:

$$\Gamma\left(K^- \to \mu^- \bar{\nu}_\mu\right) = \left(51{,}64 \pm 0{,}23\right) \frac{10^6}{\text{sec}} \qquad |F_k| = \left(4{,}124 \pm 0{,}010\right) \frac{10^{-10}}{\text{MeV}} \qquad /35/$$

The polarization measurement carried out on the $K^- \to \mu^- \bar{\nu}_\mu$ decay also confirms the prediction of the V-A lepton current.

The theoretical results for the $\pi^+ \to \ell^+ \nu_\ell$, $K^+ \to \ell^+ \nu_\ell$ decays are the same as for the π^-, K^- decays by CPT invariance, with the obvious difference that F_π^* and F_k^* stand instead of F_π and F_k (these coupling constants are, however, real if T or CP invariance holds) and that $\langle h_{\ell^+}\rangle = -\langle h_{\ell^-}\rangle$.

§2. The $\mu \to e\,\nu\,\bar{\nu}$ decay

This is the only observed decay mode of the muon. The transition matrix element

$$\langle \nu_\mu\, e^-\, \bar{\nu}_e \text{ out} | \mu^- \text{in}\rangle = \langle \nu_\mu \text{ out} | C_{\text{out}}(e)\, d_{\text{out}}(\bar{\nu}) | \mu^- \text{in}\rangle \qquad /36/$$

for the μ^- decay can be calculated using eq./10/and /11/ for $\bar{\nu}_e$. Then we get

$$\langle \nu_\mu\, e^-\, \bar{\nu}_e \text{ out} | \mu^- \text{ in}\rangle = \frac{-ig}{\sqrt{2}} \frac{1}{2} \int dy \Big\langle \nu_\mu \text{ out} | C_{\text{out}}(\ell) \Big(\bar{\psi}_e(y) \gamma_\lambda (1 - i\gamma_5)\, J^{\lambda +}(y) +$$

$$+ J^{\lambda +}(y)\, \bar{\psi}_e(y) \gamma_\lambda (1 - i\gamma_5) \Big) \Big| \mu^- \text{in} \Big\rangle \frac{v(\bar{\nu})}{(2\pi)^{3/2}}\, e^{i\bar{\nu}y} \qquad /37/$$

We again neglect electromagnetic and higher order weak interactions. Then $J_H^{\lambda+}$ gives no contribution, $C_{\text{out}}(e)$ with $\bar{\psi}_e(y)$ gives a factor $(2\pi)^{-3/2}\,\bar{u}(e)\,e^{iey}$, while the matrix element of the lepton current is

$$\langle \nu_\mu \text{out} | j^{\lambda +}(y) | \mu^- \text{in}\rangle = \frac{\bar{u}(\nu)}{(2\pi)^{3/2}} \gamma_\lambda \left(1 - i\gamma_5\right) \frac{u(\mu)}{(2\pi)^{3/2}}\, e^{iy(\nu - \mu)} \qquad /38/$$

Integration over y leads to the final result

$$\langle \nu_\mu\, e^- \bar{\nu} \text{ out} | \mu^- \text{in}\rangle = \frac{-i\,g}{\sqrt{2}} (2\pi)^4 \frac{\bar{u}(e)}{(2\pi)^{3/2}} \gamma_\lambda \left(1 - i\gamma_5\right) \frac{v(\bar{\nu})}{(2\pi)^{3/2}} \cdot$$

$$\cdot \frac{\bar{u}(\nu)}{(2\pi)^{3/2}} \gamma_\lambda \left(1 - i\gamma_5\right) \frac{u(\mu)}{(2\pi)^{3/2}}\, \delta^{(4)}(\mu - e - \nu - \bar{\nu}) \equiv \qquad /39/$$

$$\equiv F_{s_\mu, s_e} \left(16 \mu^o\, e^o\, \nu^o\, \bar{\nu}^o\right)^{-1/2} \delta^{(4)}(\mu - e - \nu - \bar{\nu})$$

We shall work in the rest system of the muon. Let μ^- be polarized along the positive z axis $\left(S_\mu = \uparrow\right)$ and let the electron have helicity $S_e\left(S_e = +1 \text{ or } -1\right)$. After straightforward trace calculation we find

$$\left|F_{\uparrow,S_e}\right|^2 \left(16 m_\mu e^o \nu^o \bar{\nu}^o\right)^{-1} = 2 \frac{|g|^2}{(2\pi)^4}\left[1-\beta_e \cos\Theta_{e\nu} + S_e\left(\cos\Theta_{e\nu} - \beta_e\right)\right]\left[1+\cos\Theta_{\bar{\nu}}\right] \quad /40/$$

Here $\beta_e = |\underline{e}| : e^o$ is the velocity of the electron, and the angles are shown in Fig. 1. For $S_\mu = \downarrow$ we would obtain $1 - \cos\Theta_{\bar{\nu}}$ instead of $1 + \cos\Theta_{\bar{\nu}}$. Thus for unpolarized μ^- decay and unmeasured electron helicity we obtain:

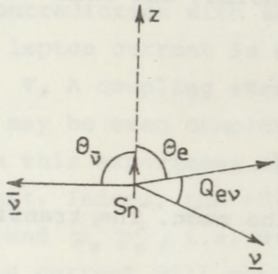

Fig. 1.
Angles in the decay of a polarized muon.

$$\frac{1}{2}\sum_{S_\mu,S_e}\left|F_{S_\mu,S_e}\right|^2 \left(16 m_\mu e^o \nu^o \bar{\nu}^o\right)^{-1} = 4 \frac{|g|^2}{(2\pi)^4}\left(1 - \beta_e \cos\Theta_{e\nu}\right) \quad /41/$$

and the differential decay probability according to eq.A-55 reads

$$d\Gamma\left(\mu^- \to e^- \ \nu \ \bar{\nu}\right) = \frac{4|g|^2}{(2\pi)^5}\left[1-\beta_e \cos\Theta_{e\nu}\right] \delta^{(4)}\left(\mu-e-\nu-\bar{\nu}\right) d\underline{e}\, d\underline{\nu}\, d\underline{\bar{\nu}} \quad /42/$$

The muon mass is very large as compared with the electron mass (and the $\bar{\nu}_e$, $\bar{\nu}_\mu$ masses) ; thus the electron is almost always extreme relativistic. Neglecting the electron mass when calculating the lifetime of the muon, we find after straightforward integration over the full phase space that

$$\Gamma(\mu^-) = \frac{1}{\tau(\mu^-)} = \frac{|g|^2 m_\mu^5}{192\pi^3} \quad /43/$$

Radiative corrections are calculable for this case, and they give a small contribution. Namely,

$$\Gamma_{corr}(\mu^-) = \frac{|g|^2 m_\mu^5}{192\pi^3}\left[1 - \frac{e^2}{8\pi^2}\left(\pi^2 - \frac{25}{4}\right)\right] = \frac{|g|^2 m_\mu^5}{192\pi^3}\left[1 - 4{,}2.10^{-3}\right] . \quad /44/$$

From the experimental value of the muon lifetime we calculate the value of $|g|$ (with electromagnetic corrections taken into account):

$$\tau(\mu^-) = \left(2{,}1983 \pm 0{,}0008\right).10^{-6}$$

$$|g| = \left(1{,}43506 \pm 0{,}00026\right).10^{-49} \text{ erg cm}^3 = \left(1{,}1659 \pm 0{,}0002\right) \frac{10^{-5}}{\text{GeV}^2} =$$

$$= \frac{10^{-5}}{M_p^2}\left(1{,}02636 \pm 0{,}00019\right) \approx 1{,}02\,\frac{10^{-5}}{M_p^2} \qquad /45/$$

from eq. /27/ and /35/ we then find:

$$\left|f(m_\pi^2)\right| = \left(128{,}4 \pm 0{,}15\right)\text{MeV} \qquad \left|f(m_k^2)\right| = \left(35{,}37 \pm 0{,}08\right)\text{MeV} \qquad /46/$$

Let us now calculate the momentum distribution of the unpolarized electrons in the case of polarized μ^- decay. In the approximation $m_e = 0$ we find,

$$d\Gamma_{S_\mu=\uparrow}(\cos\Theta_e,\ x) = \frac{|g|^2\, m_\mu^5}{192\pi^3}\left[(3-2x) + (1-2x)\cos\Theta_e\right] x^2\, dx\, d\cos\Theta_e \qquad /47/$$

Here $x \equiv |\underline{e}| / |\underline{e}^{max}|$, i.e. $x = 2|\underline{e}| / m_\mu$ for $m_e = 0$. The measured momentum distribution is in good agreement with this formula. For the electron energy distribution in unpolarized μ^- decay we find,

$$d\Gamma(x) = \frac{|g|^2\, m_\mu^5}{96\,\pi^3}\,(3 - 2x)x\, dx \quad . \qquad /48/$$

The distribution functions $(3 - 2x)\,x^2$ and $(1-2x)x^2$ are shown in Fig. 2. and Fig. 3, respectively, together with the radiative corrections to them.

Let us look also at the helicity of the electron. From eq. /40/ we easily find that for a given $\underline{e}$, $\underline{\nu}$ configuration

$$\langle h_e+\rangle = \frac{\cos\Theta_{e\nu} - \beta_e}{1-\beta_e\cos\Theta_{e\nu}} = -\beta_e\,\frac{1-\beta_e^{-1}\cos\Theta_{e\nu}}{1-\beta_e\cos\Theta_{e\nu}} \qquad /49/$$

For $\beta_e \approx 1$ we have $\langle h_e-\rangle \approx \beta_e \approx -1$. Thus, except for the rare slow electrons, the electron helicity is ≈ -1 in μ^- decay, and the positron helicity $\langle h_e+\rangle = -\langle h_e-\rangle \approx +1$ in μ^- decay. The experimental result is $\langle h_{e+}\rangle = 1{,}03 \pm 0{,}10$ in agreement with our V - A lepton current.

We see that all the available experimental results on the muon decay are accounted for by the V - A theory. Nevertheless, a 30 % tensor or scalar impurity can still be introduced without getting into contradiction with these experimental data; but these new couplings would lead to bad results in pion decay and in β decay, thus we do not introduce them into the theory.

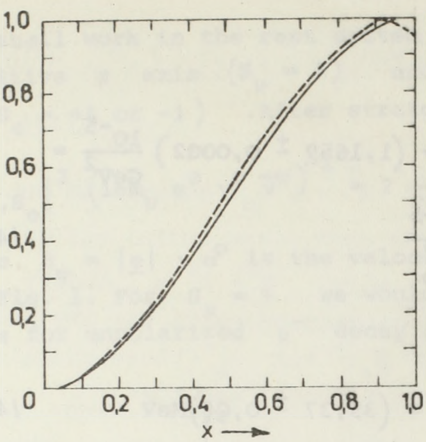

Fig. 2.
The isotropic part of the muon decay spectrum.

——— the distribution function $(3-2x)x^2$
------ radiative correction included

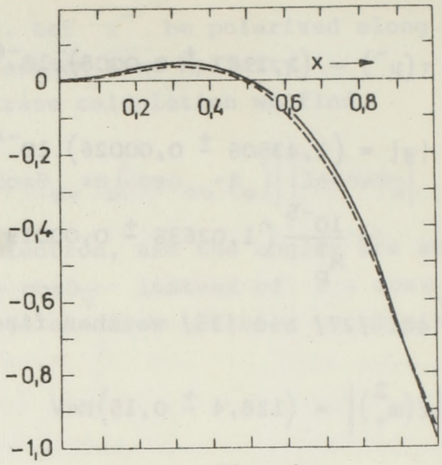

Fig. 3.
The $\cos\theta_e$ part of the muon decay spectrum.

——— the distribution function $(1-2x)x^2$
------ radiative correction included

§3 The $n \to pe\,\overline{\nu}$ decay

The transition matrix element for this decay reads (see eq. /16/) :

$$\langle pe^- \,\overline{\nu}_e \text{ out}|n \text{ in}\rangle = \frac{-i\,g}{\sqrt{2}}\,(2\pi)^4 \langle p\,\text{out}|J_H^{\lambda +}(0)|n\text{ in}\rangle \;. \qquad /50/$$

$$\frac{\overline{u}(e)}{(2\pi)^{3/2}}\,\gamma_\lambda(1 - i\gamma_5)\,\frac{v(\nu)}{(2\pi)^{3/2}}\,\delta^{(4)}(n-p-e-\nu)$$

If the neutron and the proton are on their mass shell, which is the case in the neutron decay, the most general expression for $\langle p\,\text{out}|J^+_{H\lambda}(0)|n\text{ in}\rangle$, compatible with Lorentz invariance is

$$\langle p\text{out}|J^+_{H\lambda}(0)|n\text{ in}\rangle = \langle n\text{ in}|J_{H\lambda}(0)|p\text{ out}\rangle^* =$$

$$= \frac{\overline{u}(p)}{(2\pi)^{3/2}}\Big\{c_V^*\Big[\gamma_\lambda\,F_1^*(q^2) - i\sigma_{\lambda\nu}\,\frac{(p-n)^\nu}{M_p + M_n}\,F_2^*(q^2) + (p-n)_\lambda\,F_3^*(q^2)\Big] -$$

$$-c_A\Big[\gamma_\lambda\,H_1^*(q^2) - i\sigma_{\lambda\nu}\,\frac{(p-n)^\nu}{M_p + M_n}\,H_2^*(q^2) + (p-n)_\lambda\,H_3^*(q^2)\Big]\,i\gamma_5\Big\}\,\frac{u(n)}{(2\pi)^{3/2}} \;. \qquad /51/$$

The six form factors F_i, H_i $(i = 1,2,3)$ are scalar functions of the momentum transfer squared $q^2 = (p - n)^2$. Lorentz invariance leaves these functions to be completely arbitrary. One of our main problems is precisely the determination, both experimental and theoretical, of these functions. The constant weight factors c_V and c_A have been introduced for later convenience.

The form factors with $i = 1,2$ are multiplied by the factor $(p-n)_\mu/(M_p + M_n)$ which is of the order of 10^{-3} in the physical region of the neutron decay. Since the hadron masses are the natural units of the energy - momentum for a hadronic matrix element of the hadron current, we may hope that it will be a good $(\sim 10^{-3})$ approximation to retain only $F_1^*(q^2)$ and $H_1^*(q^2)$. In the same spirit we shall neglect also the q^2 dependence of these form factors in the physical region $m_e \leq q^2 \leq (M_n - M_p)^2$, which is again very small compared to any hadron mass squared. Thus we shall work with the values of these form factors at the point $q^2 = 0$, very close to the physical region. Introducing the vector and axial vector coupling constants g_V and g_A of the nuclear β decay and the usual notation λ for their ratio by the definitions

$$g_V \equiv g c_V^* F_1^*(0) , \qquad g_A \equiv g c_A^* H_1^*(0) , \qquad \lambda \equiv \frac{g_A}{g_V} = \frac{c_A^* H_1^*(0)}{c_V^* F_1^*(0)}, \qquad /52/$$

we arrive at the result

$$\langle p e^- \bar{\nu}_e \text{ out}|n \text{ in}\rangle = \frac{-i\, g_V}{\sqrt{2}} (2\pi)^4 \frac{\bar{u}(p)}{(2\pi)^{3/2}} \gamma_\lambda (1 - i\lambda\gamma_5) \frac{u(n)}{(2\pi)^{3/2}} \cdot$$

$$\cdot \frac{\bar{u}(e)}{(2\pi)^{3/2}} \gamma_\lambda (1 - i\gamma_5) \frac{v(\nu)}{(2\pi)^{3/2}} \delta^{(4)}(n-p-e-\nu) \equiv$$

$$\equiv \left(16 n^o p^o e^o \nu^o\right)^{-1/2} F \delta^{(4)}(n-p-e-\nu) , \qquad /53/$$

and

$$d\Gamma(n \to p e \nu) = \frac{1}{2\pi} |F|^2 \frac{\delta^{(4)}(n-p-e-\nu)}{16 n^o p^o e^o \nu^o} d\underline{p}\, d\underline{e}\, d\underline{\nu} . \qquad /54/$$

We point out that the same **result is found** with the Fermi-Lagrangean (3) if the strong and electromagnetic **interactions** are neglected and if $f_V = g_V^*$, $f_A = g_A^*$. (The coupling constants are real if T invariance effects are neglected.)

In the rest frame of the neutron the following approximations are useful.

After integrating over $\underline{p}$ eq. /54/ becomes (with $\nu \equiv \nu^o$, $e \equiv |\underline{e}|$)

$$d\Gamma(n \to p e \nu) = \frac{|F|^2}{2\pi} \frac{\delta\left(m_n - \sqrt{e^2 + \nu^2 + 2e\,\nu\cos\Theta_{e\nu}} - e^o - \nu\right)}{16 m_n p^o e^o \nu} \nu^2 d\nu\, d\Omega_{e\nu}\, e^2 de\, d\Omega_e \qquad /55/$$

The implicit dependence of the Dirac delta on e gives a factor

$$\left| -1 - \frac{\nu + e\cos\Theta_{e\nu}}{m_n - e^o - \nu} \right| = \left| -1 + O(10^{-3}) \right| , \qquad /56/$$

because

$$\max\{\nu,\ e,\ e^{o}\} < m_n - m_p \approx 10^{-3}\ m_n\ . \qquad /57/$$

Thus neglecting in eq. /56/ $O(10^{-3})$, we find

$$d\Gamma(n \to pe\nu) = \frac{1}{2\pi} |F|^2\ \nu^2\ e^2\ d\Omega_{e\nu}\ de\ d\Omega_e\ . \qquad /58/$$

In the neutron rest frame the kinetic energy of the proton T_p is always much less than $m_n - m_p$. Indeed,

$$T_p \leq T_p^{max} \equiv p^o_{max} - m_p = \frac{(m_n - m_p)^2 - m_e^2}{2m_n} \approx 10^{-3}\left(m_n - m_p\right)\ . \qquad /59/$$

Thus in the energy balance

$$e^o + \nu = m_n - m_p - T_p \qquad /60/$$

we can neglect T_p and write

$$e^o + \nu = m_n - m_p \qquad /61/$$

Below we shall always work with the approximate equations /58/ and /61/.

Let us now give the relevant theoretical formulae to be compared with experiment.

i/ Let the neutron be polarized in the direction of the positive Z axis. Then for unpolarized proton and electron after straightforward trace calculation we find

$$\frac{\sum\limits_{S_e, S_p} |F|^2}{16 m_n\ p^o\ e^o\ \nu} = \frac{|g_V|^2}{(2\pi)^4}\left[1 - \frac{|\lambda|^2 - 1}{1 + 3|\lambda|^2}\ \beta_e \cos\Theta_{e\nu} - \frac{2\mathrm{Re}\ \lambda^*(\lambda - 1)}{1 + 3|\lambda|^2}\ \beta_e \cos\Theta_e + \right.$$

$$\left. + \frac{2\mathrm{Re}\ \lambda^*(\lambda+1)}{1 + 3|\lambda|^2} \cos\Theta_\nu + \frac{2\mathrm{Im}\,\lambda}{1 + 3|\lambda|^2}\ \frac{e_x \nu_y - e_y \nu_x}{e^o\ \nu}\right]\ . \qquad /62/$$

The angles in eq /62/ are shown in Fig. 4.

ii/ If the neutron were polarized in the opposite direction, the last three terms in eq. /62/ would change sign. Hence for unpolarized neutron, proton and electron we find

$$\frac{\sum\limits_{S_n S_p S_e} \frac{1}{2} |F|^2}{16 m_n\ p^o\ e^o\ \nu} = \frac{|g_V|^2}{(2\pi)^4}\left(1 + 3|\lambda|^2\right)\left[1 - \frac{|\lambda|^2 - 1}{1 + 3|\lambda|^2}\ \beta_e \cos\Theta_{e\nu}\right] \qquad /63/$$

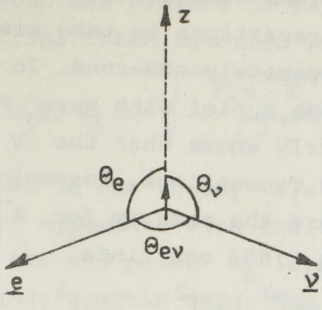

Fig. 4.

Angles in the decay of a polarized neutron.

iii/ For the electron with helicity $S_e = \pm 1$ and for unpolarized neutron and proton the calculation gives

$$\frac{\sum_{S_n S_p} \frac{1}{2}|F|^2}{16 m_n p^o e^o \nu} = \frac{1}{2}\left(1+3|\lambda|^2\right)\frac{|g_V|^2}{(2\pi)^4}\left[1 - \frac{|\lambda|^2-1}{1+3|\lambda|^2}\beta_e \cos\theta_{e\nu} + S_e\left(\frac{|\lambda|^2-1}{1+3|\lambda|^2}\cos\theta_{e\nu} - \beta_e\right)\right] \qquad /64/$$

Let us now compare the theoretical predictions with experiment.

1/ Dominance of the V, A coupling in the hadron current

From eq. /63/ we see that the e-ν angular correlation in the unpolarized neutron decay is determined by the coefficient

$$\xi = \frac{|\lambda|^2 - 1}{1 + 3|\lambda|^2} \qquad /65/$$

For pure V hadron current $\lambda = 0$, and then $\xi = -1$. In table 2 we show the theoretical value of ξ for pure V, A, S and T hadron currents. It is easy to see that in the nonrelativistic (static) limit $\underline{p} \to \underline{o}$ both the V and S currents give $\chi_p^+ \chi_n$ (Fermi transition "F") , the A and T currents give $\chi_p^+ \underline{\sigma} \chi_n$ (Gamow-Teller transition "G - T"), while the P current gives no contribution. Indeed, $\bar{u}_p(\underline{O})\gamma_5 u_n(\underline{O}) = 0$. To the free neutron decay

Table 2.

Theoretical values of ξ for pure S,V,A and T hadron currents

Hadron current	S	V	A	T
ξ	+1	-1	+1/3	-1/3

Table 3.

Experimental values for ξ

Decay	Character	ξ
$He^6 \xrightarrow{e^-} Li^6$	G-T	+0,3343 ± 0,0030
$Ne^{23} \xrightarrow{e^-} Na^{23}$	G-T	+0,33 ± 0,03
$Ar^{35} \xrightarrow{e^+} Cl^{35}$	Mostly F	-0,97 ± 0,14

both F and G - T transitions contribute. In nuclei the nuclear structure often forbids one of these transitions to take place. Thus, in nuclei, F and G - T transitions may be separately observed. In table 3 we give the experimental results for ξ in some nuclei with pure F or G - T transitions. Comparison with table 2 clearly shows that the V and A couplings are much stronger than possible S and T couplings, respectively. (For β^+ decay the theoretical values for ξ are the same as for β^- decay. Let us notice also that in nuclei instead of eq. /65/ one finds

$$\xi = \frac{\frac{1}{3}|\langle\underline{\sigma}\rangle|^2 \, |g_A|^2 - |g_V|^2 \, |\langle 1\rangle|^2}{|g_V|^2 \, |\langle 1\rangle|^2 + |g_A|^2 \, |\langle\underline{\sigma}\rangle|^2} \, , \qquad /66/$$

where $\langle 1\rangle$ and $\langle\underline{\sigma}\rangle$ are shorthand notations for the F and G - T nuclear matrix elements, respectively. For the free neutron $|\langle 1\rangle|^2 = 1$, $|\langle\underline{\sigma}\rangle|^2 = 3$, and we get back eq. /65/. In nuclei these values of $|\langle 1\rangle|^2$ and $|\langle\underline{\sigma}\rangle|^2$ correspond to the so called superallowed F and G - T decays.)

2/ <u>The energy distribution of the electrons. The Fermi spectrum</u>

To calculate this energy distribution for unpolarized neutron, proton and electron we have to insert the expression for $\sum \frac{1}{2} |F|^2$ in eq. /63/ at the place of $|F|^2$ in eq. /58/ and then to integrate over all the variables except e^0. The result is:

$$d\Gamma(x) = \left(1 + 3|\lambda|^2\right) \frac{|g_V|^2}{(2\pi)^3} 4m_e^5 \left(W_o - x\right)^2 \sqrt{x^2-1}\, x \, dx \qquad /67/$$

where $W_o = \left(m_n - m_p\right) : m_e = 2{,}53$ is the end point energy. Indeed, the dimensionless energy variable $x \equiv e^o : m_e$ changes from $x = 1$ to $x = W_o$ according to eq. /61/.

Eq. /67/ refers to the decay of a free unpolarized neutron. The corresponding formula for the allowed unpolarized $N \to N' + \beta + \nu$ decay can be obtained from eq. /67/ by changing $1 + 3|\lambda|^2$ to $|\langle 1\rangle|^2 + |\langle\sigma\rangle|^2 \, |\lambda|^2$ and W_o from $\left(m_n - m_p\right) : m_e$ to $\left(m_N - m_{N'}\right) : m_e$. Moreover, the influence of the extended charge distribution of the nucleus on the motion of the β particle may be quite important and must be taken into account by multiplying the function $\left(W_o - x\right)^2 \sqrt{x^2-1}\, x$ by an appropriate Coulomb correction factor, for which detailed tables exist. With all these changes we obtain a theoretical expression which can be tested not only for the neutron decay, but also for the wide variety of allowed nuclear β decays. The energy distribution (the so-called Fermi distribution or "Fermi spectrum")

$$F\left(x, W_o\right) = \left(W_o - x\right)^2 \sqrt{x^2 - 1}\, x \qquad /68/$$

is shown on Fig. 5. In general one prefers to represent the experimental data on the Curie plot /Fig. 6./. The Curie function is defined as

$$K(x, W_o) = \sqrt{\frac{F(x, W_o)}{x\sqrt{x^2 - 1}}} \qquad /69/$$

and from eq. /68/ we see that in our theory $K(x,W_o) = W_o - x$. The experimental results are in excellent agreement with the theory. We remark that the shape of the spectrum near the end point W_o strongly depends on the assumption that the neutrino mass is exactly zero. The best experimental upper limit coming from the end point behaviour is $m_e \leqslant 60$ eV.

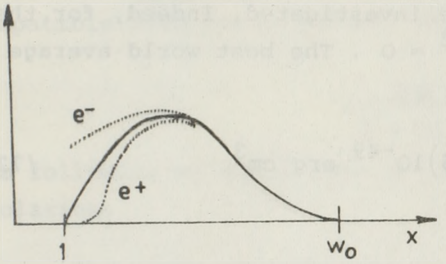

Fig. 5.
The Fermi spectrum.
—— the distribution function $F(x,W_o)$
.... the distortions due to the Coulomb correction

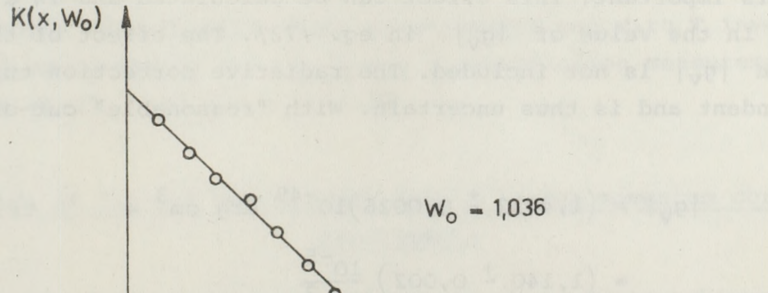

Fig. 6.
The Curie plot for the H^3 nucleus.

3/ Determination of $|g_V|$ and $|\lambda|$.

Integration of eq. /67/ over x gives the neutron lifetime $\tau(n)$:

$$\frac{1}{\tau(n)} = \Gamma(n) = 4\left(1 + 3|\lambda|^2\right) \frac{|g_V|^2}{(2\pi)^3} m_e^5 \int_1^{W_o} F\left(x, W_o\right) dx \qquad /70/$$

where

$$\int_1^{W_o} F\left(x, W_o\right) dx = \frac{W_o}{4} \ln\left(W_o + \sqrt{W_o^2 - 1}\right) + \frac{\sqrt{W_o^2 - 1}}{60}\left(2W_o^4 - 9W_o^2 - 8\right) . \qquad /71/$$

To obtain the value of $|g_V|$ and $|\lambda|$ separately, β decays with superallowed pure Fermi transitions must be investigated. Indeed, for them $|\lambda|^2 \to \frac{1}{3} |\langle\underline{\sigma}\rangle|^2$ $|\lambda|^2 = 0$, since $|\langle\underline{\sigma}\rangle|^2 = 0$. The best world average for $|g_V|$ is

$$|g_V| = \left(1{,}4138 \pm 0{,}0026\right) 10^{-49} \text{ erg cm}^3 \qquad /72/$$

and then from the neutron lifetime

$$|\lambda| = 1{,}23 \pm 0{,}01 \qquad /73/$$

The effect of the static Coulomb field of a (heavy) nucleus on the β particle is important. This effect can be calculated and is already taken into account in the value of $|g_V|$ in eq. /72/. The effect of the radiative correction on $|g_V|$ is not included. The radiative correction turns out to be cut-off dependent and is thus uncertain. With "reasonable" cut-off one finds

$$|g_V| = \left(1{,}4032 \pm 0{,}0026\right) 10^{-49} \text{ erg cm}^3 =$$

$$= \left(1{,}140 \pm 0{,}002\right) \frac{10^{-5}}{\text{GeV}^2}$$

$$= \left(1{,}00357 \pm 0{,}00176\right) . \frac{10^{-5}}{M_p^2} \approx 1{,}00 \frac{10^{-5}}{M_p^2} \qquad /74/$$

4/ Parity P and time reversal T experiments. The sign of λ.

The last term in eq. /62/ violates time reversal invariance. To see this, let us write down the general form of this term for the neutron polarized in an arbitrary direction $\underline{P}_n$. We find

$$\frac{2\mathrm{Im}\,\lambda}{1+3|\lambda|^2}\,\frac{\beta_e}{\nu}\,\underline{P}_n\left[\underline{e}\times\underline{\nu}\right] . \qquad /75/$$

Under T $\underline{P}_n \to -\underline{P}_n$, $\underline{e}\to-\underline{e}$, $\underline{\nu}\to-\underline{\nu}$, hence /75/ changes sign and thus the distribution /62/ is not invariant under T unless $\mathrm{Im}\lambda = 0$. Measurements of the $\underline{P}_n\left[\underline{e}\times\underline{\nu}\right]$ correlation show that $\mathrm{Im}\lambda$ is surely small and is compatible with zero. Indeed, the experimental result is

$$\frac{2\mathrm{Im}\,\lambda}{1+3|\lambda|^2} = 0{,}01 \pm 0{,}01 \qquad /76/$$

In the following we shall take λ real, i.e. we neglect the possible small T violation.

The terms proportional to $\cos\Theta_e$ and $\cos\Theta_\nu$ in eq. /62/violate P. Indeed, with the neutron polarized in a direction P_n, $\cos\Theta_e \to (\underline{P}_n\cdot\underline{e})/e$, $\cos\Theta_\nu \to (\underline{P}_n\cdot\underline{\nu})/\nu$. Under P $\underline{P}_n\to\underline{P}_n$, $\underline{e}\to-\underline{e}$, $\underline{\nu}\to-\underline{\nu}$, hence these expressions change sign. The experimental results on the $(\underline{P}_n\cdot\underline{e})$ and $(\underline{P}_n\cdot\underline{\nu})$ correlations, presented in Table 4, are seen to be compatible with the value $|\lambda| = 1{,}23$ found from the life-time measurements and with T invariance (i.e. with λ real) supported by the $\underline{P}_n\left[\underline{e}\times\underline{\nu}\right]$ correlation measurement, only if we choose $\lambda = +1{,}23$ and not $\lambda = -1{,}23$.

Table 4.

The sign of λ and the neutron spin - lepton momentum correlation experiments

asymmetry parameters	$\lambda = +1{,}23$	$\lambda = -1{,}23$	experiment
for e^- $\frac{-2\lambda(\lambda-1)}{1+3\lvert\lambda\rvert^2}$	-0.09	-0.99	$-0.11\pm0{,}02$
for $\bar{\nu}$ $\frac{2\lambda(\lambda+1)}{1+3\lvert\lambda\rvert^2}$	+0.99	+0.09	$+0.88\pm0.15$

Finally we remark that the first two terms of the distribution /62/ conserve both P and T. In his original theory of β decay Fermi supposed that P and T are conserved and he worked with the Lagrangean (2) with $f_S = f_P = f_T = 0$ and with f_A/f_V real, instead of with the Langrangean(3). It is easy to see that for β-decay with unpolarized particles both theories lead to the same result, namely to eq. /63/. Only after the discovery of the parity violation in the K^o decay in 1956 by T.D. Lee and C.N. Yang one looked at the polarized Co^{60} decay, and then the fact of the parity violation in nuclear β decay was established.

5/ Experiments on the helicity of the β^--particle.

The average helicity of the electron for unpolarized neutron and proton can be easily found from eq. /64/. For given $\underline{e}$ and $\underline{\nu}$ we obtain

$$\langle h_{e^-}\rangle = \frac{\xi\cos\Theta_{e\nu} - \beta_e}{1 - \beta_e\,\xi\cos\Theta_{e\nu}} = -\beta_e\,\frac{1 - \beta_e^{-1}\,\xi\cos\Theta_{e\nu}}{1 - \beta_e\,\xi\cos\Theta_{e\nu}} \qquad /77/$$

This formula is similar to that for $\langle h_{e^-}\rangle$ in the μ^- decay, given in eq. /49/. But now β_e can often be $\ll 1$, and thus it is not true that $\langle h_{e^-}\rangle \approx -\beta_e$ for almost every $\underline{e}$, $\underline{\nu}$ configuration. On the other hand, the approximate formula /56/ now holds, and we can easily integrate our distribution /64/ over $\cos\Theta_{e\nu}$, which was not the case for the corresponding distribution in μ^- decay. Integration over all the neutrino variables and electron angles gives

$$d\Gamma_{S_e}(x) = 2\left(1 + 3|\lambda|^2\right)\frac{|g_V|^2}{(2\pi)^3}\,m_e^5\left(1 - S_e\,\beta_e\right)F\left(x, W_o\right)dx\ . \qquad /78/$$

Thus, for fixed x (i.e. for fixed $\beta_e \equiv \sqrt{x^2-1}/x$) we find

$$\langle h_{e^-}\rangle = \frac{(1-\beta_e) - (1+\beta_e)}{(1-\beta_e) + (1-\beta_e)} = -\beta_e \qquad /79/$$

If in the lepton current we would allow the general V, A coupling $\gamma_\lambda\left(a - ib\,\gamma_5\right)$, we would obtain for β^- and β^+ decay

$$\langle h_{e^-}\rangle = -\beta_e\,\frac{2\mathrm{Re}\,a\,b^*}{|a|^2 + |b|^2} = -\langle h_{e^+}\rangle\ . \qquad /80/$$

The experimental results shown in Table 5 give strong support to a pure V - A lepton current $\gamma_\lambda\left(1 - i\gamma_5\right)$.

Table 5.

Experimentally determined helicities of the $e^{\pm}$ particles

Decay	Character	$\langle h_e \rangle : \beta_e$
$B^{12} \xrightarrow{e^-} C^{12*}$	G - T	-0.98 ± 0,06
$Ga^{68} \xrightarrow{e^+} Zn^{68}$	G - T	+0.99 ± 0,09
$O^{14} \xrightarrow{e^+} N^{14}$	F	+0,97 ± 0,19

§4. Conclusions

The available experimental data on $\pi \to \ell\nu$, $K \to \ell\nu$ $\mu \to e\nu\nu$, $n \to pe\nu$ and nuclear β decays are compatible with the hypothesis that the weak interaction inducing these decays can be described by the current-current Lagrangean /4/ with vector and axial vector currents. For the hadron current the predominance of the A coupling over the P coupling is supported by the experiments on the $\Gamma(\pi \to e\nu) : \Gamma(\pi \to \mu\nu)$ and $\Gamma(K \to e\nu) : \Gamma(K \to \mu\nu)$ ratios if the μ-e universality of the lepton current (see page 11) is taken for granted*. The predominance of the V, A couplings over the S, T couplings is supported by the ν-e angular correlation measurements in β decay. The $\gamma_\lambda(1-i\gamma_5)$ structure ("V - A structure") of the lepton current is confirmed by helicity measurements in the β decay and in $\pi \to \mu\nu$ decay. All the measurements on the μ decay are also compatible with a pure V - A lepton current; however, the data in this decay would allow for 20 % -30 % admixture from S,P and T couplings. Neglecting these unwanted couplings which would not allow the description of the μ, π and n decays in the framework of a unique theory, one finds from the muon lifetime the absolute value of the coupling constant g . The detailed experimental analysis of the nuclear β decay and of the free neutron decay has shown that the matrix element of the weak hadron current between nucleon states can be approximated, at least at low momentum transfer q^2, by an effective $\gamma_\lambda(1-i\gamma_5)$ coupling, if, instead of g , a new coupling constant g_V is used (Fermi approximation). $\lambda = +1,23$ from these experiments. It is remarkable that g_V is practically equal to g (we suppose that they have the same sign), i.e. that the nucleons take part in the weak interaction practically with the same strength as the leptons. For the axial vector part the "renormalization" of the hadron current is stronger, $g \to g_V\lambda$, but still it is only about 20%.

*The predominance of V over S and T is supported by the experimental results in $K \to \pi\ell\nu$ decays. See e.g. [8], Chapter 5.

In chapter IV we shall see that the theory of Cabibbo and Gell-Mann explains the fact that g_V is smaller than g, saying that the missing strength of the hadron coupling is held by the strangeness changing part of the weak hadron current $J_{H\lambda}$. Thus a universal theory of the weak interaction, describing all the leptonic and semi-leptonic decays will emerge, with a Lagrangean containing only a few free parameters. The problems concerning the application of this theory to the non-leptonic weak decays will be briefly described in chapter V.

III. THE LEPTONIC DECAYS OF THE HADRONS. THE ISOTRIPLET VECTOR CURRENT (IVC) THEORY

§1. The IVC hypothesis

In chapters III and IV we shall discuss the status of the theory of the weak leptonic decays of hadrons, i.e. of processes where the decaying particle is a hadron which decays into a lepton pair $\ell\nu$ accompanied or not by one or several other hadrons. Such decays may be strangeness conserving, as the $\pi \to \ell\nu$ and $n \to pe\nu$ decays, or strangeness-changing, as the $K \to \ell\nu$ decays. In chapter II we have seen that these three mentioned decays could be described by the current-current theory with V and A currents. In this chapter we shall see how this theory applies to the leptonic decays of the hadrons in general.

First of all we shall examine the structure of the weak hadron current operator $J_{H\lambda}(x)$ which enters the general expression of the transition matrix elements of the leptonic hadron decays, which is given in eq./16/ and /17/, if H is now a one - hadron state. The fact that both strangeness-conserving and strangeness-changing leptonic hadron decays are observed, shows that $J_{H\lambda}(x)$ must have a strangeness-changing and strangeness-conserving part. If we would consider the weak interactions in any order of g, then the separation of $J_{H\lambda}(x)$ into two such parts at a given time would not be maintained at a later time: the weak interaction would add a strangeness violating part to the strangeness-conserving one and vice-versa. The same reasoning may be applied to the separation of the weak hadron current and also of the lepton current into a vector and axial vector part. Only in first order in g have these separations a time independent meaning. Since we shall always work in this approximation, we can write $J_{H\lambda}(x)$ in the form:

$$J_{H\lambda}(x) = c_V V_\lambda^{s=o}(x) + c_A A_\lambda^{s=o}(x) + d_V V_\lambda^{s\neq o}(x) + d_A A_\lambda^{s\neq o}(x) \qquad /81/$$

where c_V, c_A, d_V and d_A are coefficients, not necessarily real. At first sight the introduction of these coefficients may seem to be superfluous; they could be included into the operators V_λ and A_λ, which are themselves unknown. However, we shall see later that these operators are supposed to obey commutation laws which normalize them. Then their coefficients give the weights of these normalized operators in the full hadronic current /81/.

Important properties of the operators V_λ and A_λ in eq. /81/ have been specified by M. Gell-Mann and N. Cabibbo.

In the present chapter we shall discuss the isotriplet vector current hypothesis of Gell-Mann, which refers to the transformation property of the $V_\lambda^{s=o}$ current under rotations in the isotopic spin space. To come to the basic idea of Gell-Mann, let us consider the matrix element of $V_\lambda^{s=o}$ between nucleon states.

The comparison of eq. /51/ with eq. /81/ shows that

$$\langle p(p_2)|V_\lambda^{s=o}(o)^+|n(p_1)\rangle$$

$$= \frac{\bar{p}(p_2)}{(2\pi)^{3/2}}\left[\gamma_\lambda F_1^*(q^2) - i\sigma_{\lambda\nu}\frac{(p_2-p_1)^\nu}{M_n+M_p}F_2^*(q^2) + \frac{(p_2-p_1)_\lambda}{M_n+M_p}F_3^*(q^2)\right]\frac{n(p_1)}{(2\pi)^{3/2}}\;; \qquad /82/$$

from now on we drop the "in" and "out" labels of the state vectors, and denote the neutron state of momentum p by $|n(p)\rangle$, and the neutron spinor by $n(p)$.

The operator $V_\lambda^{s=o}(x)^+$ increases the value of the electric charge by one unit, while $V_\lambda^{s=o}(x)$ lowers it by one unit. Its matrix element $\langle n|V_\lambda^{s=o}(o)|p\rangle$ appearing in the nuclear β^+ decay, can be easily calculated from eq./82/:

$$\langle n(p_2)|V_\lambda^{s=o}(o)|p(p_1)\rangle = \langle p(p_1)|V_\lambda^{s=o+}(o)|n(p_2)\rangle^* =$$

$$= \frac{\bar{n}(p_2)}{(2\pi)^{3/2}}\left[\gamma_\lambda F_1(q^2) - i\sigma_{\lambda\nu}\frac{(p_2-p_1)^\nu}{M_n+M_p}F_2(q^2) + \frac{(p_2-p_1)_\lambda}{M_n+M_p}F_3(q^2)\right]\frac{p(p_1)}{(2\pi)^{3/2}} \qquad /83/$$

It is well known that for any hadron the relation

$$Q = I_z + \frac{1}{2}Y\,, \qquad /Y = B + S/ \qquad /84/$$

holds. Then for $\Delta S = 0$ transitions we have $\Delta Q = Q_{H'} - Q_H = \Delta I_z = \pm 1$ since $\Delta B = 0$ /the lepton current with $\Delta Q = \pm 1$ and $\Delta B = 0$ ensures the total Q and B conservation/. Namely, for $V_\lambda^{s=o}(x)^+$ we have $\Delta I_z = I_{zH'} - I_{zH} = +1$ and for $V_\lambda^{s=o}(x)$ $\Delta I_z = -1$. Thus we see that if these operators have definite transformation properties under the isospin group, then the simplest possibility for them is to be the $q = +1$ and $q = -1$ spherical components of an irreducible isotriplet isovector operator $V_{q,\lambda}^{(1)}(x)$ $(q = -1, 0, +1)$. This means that there exist then three hermitean operators $V_{i,\lambda}(x)$ $(i = 1,2,3)$ which satisfy with the three hermitean generators I_i of the isospin group the commutation relations

$$\left[I_s, V_{k,\lambda}(x)\right] = i\varepsilon_{sk\ell}\, V_{\ell,\lambda}(x)\,, \qquad /85/$$

i.e. the very relations which the I_i satisfy with each other:

$$\left[I_s, I_k\right] = i\, \varepsilon_{sk\ell}\, I_\ell \qquad /86/$$

The precise connection of the $V_\lambda^{s=o}$, $V_\lambda^{s=o+}$ operators with the $V_{i,\lambda}$ operators and the expression of these latter through the spherical components may be defined in the following way

$$V_\lambda^{s=o}(x) = V_{1,\lambda}(x) - i\, V_{2,\lambda}(x) = \sqrt{2}\, V^{(1)}_{-1,\lambda}(x)$$

$$V_\lambda^{s=o}(x)^+ = V_{1,\lambda}(x) + i\, V_{2,\lambda}(x) = -\sqrt{2}\, V^{(1)}_{+1,\lambda}(x)$$

$$V_{3,\lambda}(x) = V^{(1)}_{o,\lambda}(x) \qquad /87/$$

The natural question arises whether the third component $V_{3,\lambda}(x)$ has a physical meaning or not. In 1958 Gell-Mann suggested that it has. Namely, he supposed that the currents $V_{i,\lambda}(x)$ are just the density operators for the generators I_i . Then by definition

$$I_i(t) = \int d\underline{x}\; V_{i,o}(x;t) \qquad /88/$$

If the isospin group were an exact symmetry group, then the generators would not depend on the time. We know , however, that in Nature electromagnetic and weak* interactions violate this symmetry. Then the eq. /85/ and /86/ are supposed to hold as equal time-commutation relations:

$$\left[I_s(t), V_{k,\lambda}(\underline{x},t)\right] = i\, \varepsilon_{sk\ell}\, V_{\ell,\lambda}(\underline{x},t) \qquad /89/$$

$$\left[I_s(t), I_k(t)\right] = i\, \varepsilon_{sk\ell}\, I_\ell(t) \qquad /90/$$

We notice that from eq. /88/ and /89/ eq. /90/ follows, while /88/ and /90/ do not imply /89/.

The physical meaning of the $V_{3,\lambda}(x)$ operator is then obvious. It enters the electric current operator of the hadrons $J^{e\ell}_{H\lambda}(x)$ according to the well known formula

$$J^{e\ell}_{H\lambda}(x) = V_{3,\lambda}(x) + \frac{1}{2}\, Y_\lambda(x)\ , \qquad /91/$$

---

* see footnote on page 34

the space integral of which, with $\lambda = 0$ and $I_3 \equiv I_z$, gives eq./84/. The Lagrangean of the electromagnetic interaction in these notations reads

$$L^{e\ell}(x) = e\, A^{\lambda}(x)\left[J^{e\ell}_{H\lambda}(x) + j^{e\ell}_{\lambda}(x)\right] \quad ; \qquad /92/$$

let us also remark that the hypercharge current is an isoscalar operator, i.e.

$$\left[I_s(t),\ Y_{\lambda}(x,t)\right] = 0 \qquad /93/$$

It is important to point out that the eq./88/ and /90/ normalize the currents $V_{i,\lambda}(x)$ and also fix their sign. Indeed, if we multiply the currents by a common factor K, these equations remain true if, and only if, $K = +1$. The reason for the introduction of the weight factor c_V in eq. /81/ is now clear. It leaves open the possibility that the $s = 0$ vector part of the weak hadron current is not exactly equal, but only proportional to $V_{1,\lambda}(x) - i\, V_{2,\lambda}(x)$. In the original formulation of the isotriplet vector current /IVC/ hypothesis, Gell-Mann suggested that $C_V = 1$, in analogy with eq./91/ for the electric current, where the coefficient of $V_{3\ \lambda}(x)$ is equal to 1. At present, however, both theoretical and experimental considerations indicate that C_V is probably slightly smaller than 1. We shall return to this important question when we shall discuss the hypothesis of the universality of the weak current at the end of chapter IV.

The /IVC/ hypothesis has far reaching consequences, expressed by the Wigner-Eckart (W-E) theorem. This theorem for the SU(2) isospin group can be written in the form

$$\langle I', I_z', \alpha' | T^{(k)}_q | I,\ I_z, \alpha\rangle = \left(I,\ I_z;\ k, q | I' I_z'\right)\left(\alpha' || T^{(k)} || \alpha\right) \qquad /94/$$

In this equation I, I_z, (I', I_z') stand for the total isospin quantum number and its third component in the initial (final) states, α and α' denote all the other quantum numbers specifying those states, $T^{(k)}_q$ is the q-th component of an irreducible tensor operator belonging to the SU(2) representation of dimension $n=2k+1$, $(I, I_z; kq | I' I_z')$ is an SU(2) Clebsh Gordan coefficient, while $(\alpha' || T^{(k)} || \alpha)$ denotes the reduced matrix element, which depends an all the variables which occur in the matrix element itself, except the magnetic quantum numbers I_z, I_z', q.

The W-E theorem connects the weak nucleon form factors $F_i(q^2)$ of the weak $s = 0$ vector current in eq./83/ with the isovector form factors of the nucleons $F_i^{\ V}(q^2)$, defined by the relation

$$\langle p(p_2)|V_{3,\lambda}(0)|p(p_1)\rangle =$$

$$= \frac{\bar{p}(p_2)}{(2\pi)^{3/2}} \left[\gamma_\lambda F_1^V(q^2) - i\sigma_{\lambda\nu} \frac{(p_2-p_1)^\nu}{2M_p} F_2^V(q^2) + \frac{(p_2-p_1)_\lambda}{2M_p} F_3^V(q^2)\right] \frac{p(p_1)}{(2\pi)^{3/2}} \qquad /95/$$

it also connects, through eq. /91/, the $F_i^V(q^2)$ with the electromagnetic form factors of the proton $F_i^p(q^2)$ and of the neutron $F_i^n(q^2)$, which are defined as follows $(N = p,n)$:

$$\langle N(p_2)|J_{H\lambda}^{e\ell}(0)|N(p_1)\rangle =$$

$$= \frac{\bar{N}(p_2)}{(2\pi)^{3/2}} \left[\gamma_\lambda F_1^N(q^2) - i\sigma_{\lambda\nu} \frac{(p_2-p_1)^\nu}{2M_N} F_2^N(q^2) + \frac{(p_2-p_1)_\lambda}{2M_N} F_3^N(q^2)\right] \frac{N(p_1)}{(2\pi)^{3/2}} \qquad /96/$$

The eq. /91/ contains also the operator $\frac{1}{2} Y_\lambda(x)$ which defines the isoscalar form factors of the nucleons $F_i^s(q^2)$ through the relation $(N = p,n)$

$$\langle N(p_2)|\tfrac{1}{2} Y_\lambda(0)\, N(p_1)\rangle =$$

$$= \frac{\bar{N}(p_2)}{(2\pi)^{3/2}} \left[\gamma_\lambda F_1^s(q^2) - i\sigma_{\lambda\nu} \frac{(p_2-p_1)^\nu}{2M_N} F_2^s(q^2) + \frac{(p_2-p_1)_\lambda}{2M_N} F_3^s(q^2)\right] \frac{N(p_1)}{(2\pi)^{3/2}} \qquad /97/$$

The W-E theorem holds exactly only if the operators and the states involved are exact multiplets. In our case this is not so, because even if the operators satisfy the group properties exactly at a given time, the physical proton and nucleon states do not form an exact isodoublet as shown e.g. by the fact that $M_p \neq M_n$. Below we shall work, however, in the exact SU(2) limit, i.e. we shall neglect the small, few percent SU(2) breaking effects. In this limit we have of course $M_p = M_n \equiv M_N$. Then straightforward application of the W-E theorem and of eq. /91/ gives the following results:

$$F_i(q^2) = 2F_i^V(q^2) = F_i^p(q^2) - F_i^n(q^2)$$

$$2F_i^s(q^2) = F_i^p(q^2) + F_i^n(q^2) \qquad /98/$$

To illustrate how such relations are found, we write

$$\langle n(p_2)|V_\lambda^{s=0}(0)|p(p_1)\rangle =$$

$$= \langle n(p_2)|\sqrt{2}\, V_{-1,\lambda}^{(1)}(0)|p(p_1)\rangle = \sqrt{2}\left(\tfrac{1}{2}, \tfrac{1}{2}; 1, -1\middle|\tfrac{1}{2}, -\tfrac{1}{2}\right)\left(\tfrac{1}{2}\, p_2\middle|\middle|V_\lambda^{(1)}(0)\middle|\middle|\tfrac{1}{2}\, p_1\right) =$$

$$= \frac{2}{\sqrt{3}}\left(\tfrac{1}{2}\, p_2\middle|\middle|V_\lambda^{(1)}(0)\middle|\middle|\tfrac{1}{2}\, p_1\right),$$

$$\langle p(p_2)|V_{3,\lambda}(0)|p(p_1)\rangle =$$

$$= \langle p(p_2)|V_{0,\lambda}^{(1)}(0)|p(p_1)\rangle = \left(\tfrac{1}{2}, \tfrac{1}{2}; 1, 0\middle|\tfrac{1}{2}, \tfrac{1}{2}\right)\left(\tfrac{1}{2}\, p_2\middle|\middle|V_\lambda^{(1)}(0)\middle|\middle|\tfrac{1}{2}\, p_1\right) =$$

$$= \frac{1}{\sqrt{3}}\left(\tfrac{1}{2}\, p_2\middle|\middle|V_\lambda^{(1)}(0)\middle|\middle|\tfrac{1}{2}\, p_1\right) \qquad /99/$$

i.e.

$$\langle n(p_2)V_\lambda^{s=o}(o)|p(p_1)\rangle = 2\langle p(p_2)|V_{3\lambda}(o)|p(p_1)\rangle \qquad /100/$$

Eq. /83/ and /95/, taken in the SU(2) limit with $M_p = M_n = M_N$, imply then the relation $F_i(q^2) = 2F_i^v(q^2)$. We stress that the relations /98/ are derived by us only for $q^2 \le 0$, which is the physical region of q^2 if $M_p = M_n$.

The following properties of the form factors $F_i^v(q^2)$, $F_i^s(q^2)$ are known on theoretical grounds:

i/ for i = 1, 2 these functions are real, for i = 3 they are purely imaginary. These properties follow from the fact that $V_{3,\lambda}(x)$ and $Y_\lambda(x)$ are hermitean operators.

ii/ Using eq./15/ we easily find the matrix elements of our current operators at an arbitrary point $x \neq 0$. We know also that in the SU(2) limit all our vector currents are conserved:

$$\partial^\lambda V_{i,\lambda}(x) = 0 \; , \quad \partial^\lambda Y_\lambda(x) = 0 \qquad /101/$$

Application of the current conservation to eq./95/ and /97/ gives:

$$F_3^v(q^2) = 0 \quad , \quad F_3^s(q^2) = 0 \qquad /102/$$

iii/ If we integrate the eq./95/ and /97/, written for $x \neq 0$ over $\underline{x}$ with $\lambda = 0$ and take into account that $I_3|p\rangle = \frac{1}{2}|p\rangle$, $\frac{1}{2}Y|p\rangle = \frac{1}{2}|p\rangle$ we find

$$F_1^v(0) = \tfrac{1}{2} \; , \; F_1^s(0) = \tfrac{1}{2} \qquad /103/$$

Some further properties of the form factors are known from experiments:

iv/ From magnetic moment measurements we know that

$$F_2^p(0) \equiv \mu_p = 1{,}793 \quad , \quad F_2^n(0) \equiv \mu_n = -1{,}913 \qquad /104/$$

where μ_p and μ_n are the anomalous magnetic moments of the proton and of the neutron, measured in $e\hbar/2M_pc$ units.

v/ At Stanford the proton and neutron electromagnetic form factors have been measured for a wide range of $q^2 < 0$. The phenomenological formulae for the form factors can be written, within the experimental errors, as follows:

$$\frac{G_M^p(q^2)}{1+\mu_p} = \frac{G_M^n(q^2)}{\mu_n} = G_E^p(q^2) = \frac{1}{\left(1 - 1,25\,\frac{q^2}{M_p^2}\right)^2} \, , \qquad /105/$$

$$G_E^n(0) = 0 \quad , \quad \left.\frac{d\,G_E^n(q^2)}{dq^2}\right|_{q^2=0} = 0,563\,M_p^{-2} \qquad /106/$$

In eq. /105/ and /106/ the Sachs form factors

$$G_M^N \equiv F_1^N + F_2^N$$
$$/N = p,\, n/ \qquad /107/$$
$$G_E^N \equiv F_1^N + \frac{q^2}{4M_N^2} F_2^N$$

have been introduced.

According to eq./98/ the information i-v can be transferred to the weak form factors

i′ $F_1^*(q^2) = F_1(q^2)$, $F_2^*(q^2) = F_2(q^2)$, $F_3^*(q^2) = -F_3(q^2)$

ii′ $F_3(q^2) = 0$

iii′ $F_1(0) = 1$

iv′ $F_2(0) = \mu_p - \mu_n = 3,706$

$$\text{v}' \quad \frac{G_M^p \mp G_M^n - \frac{4M_N^2}{q^2}\left(G_E^p \mp G_E^n\right)}{1 - \frac{4M_N^2}{q^2}} = 2F_1^{\,v,s}$$

$$\frac{G_M^p \mp G_M^n - \left(G_E^p \mp G_E^n\right)}{1 - q^2/4M_N^2} = 2F_2^{\,v,s} \; ; \; F_{1,2}(q^2) = 2F_{1,2}^{\,v}(q^2) \qquad /108/$$

The relations /108/ have been derived through the W-E theorem in the exact isospin symmetry limit for the range $q^2 < 0$. In Nature, however, this symmetry is violated by the electromagnetic interaction*. The violation is thus controlled by the fine structure constant $\alpha = \frac{1}{137}$ and is expected to be at most of a few percent. Nevertheless, the precision of the experiments in nuclear β decay would make it desirable to take into account this small

---

*and also by the weak interaction. However if we work in first order in 9 with the Lagrangean (4), then the weak current $J_\lambda(x)$ is free from the weak interaction.

effect. Unfortunately, no reliable theoretical method for the calculation of the departure from an exact internal symmetry is known. It is obvious only that when calculating the phase space for the neutron decay one must use the observed, unequal neutron and proton masses because with $M_p = M_n$ the energy conservation would forbid the decay, and that the relations i - v must be continued from the $q^2 < 0$ range of the $e + N \rightarrow e + N$ scattering to the $m_e^2 < q^2 \lesssim (20\ \mathrm{MeV})^2$ range of the nuclear β decay. The true analytic expressions of the electromagnetic form factors are however unknown, and it is hopeless to try to obtain reliable corrections of a few percent from the analytic continuation of the approximate expressions /105/, /106/. Moreover, the symmetry-breaking corrections to the functional form of the form factors are also uncalculable*. In practice the following procedure is adopted when testing the IVC hypothesis: both the q^2 dependence of the form factors and the SU(2) symmetry breaking effects are neglected, and the values of the form factors at $q^2 = 0$ are used. In nuclear β decay and also in the $\pi^{\pm} \rightarrow \pi^o e^{\pm} \nu_e(\bar{\nu}_e)$ decay the physical region of q^2 is so small compared to a hadron mass squared, that it is hoped that the error caused by this approximation is at most a few percent. In particular, we find then from eq. /52/, /45/ and /74/:

$$g_V = gc_V^* F_1^*(0) = gc_V^* , \qquad |c_V| = 0 , \qquad /109/$$

The same procedure will be applied also to the $\Sigma \rightarrow \Lambda \ell \nu$ decay, where, of course, the neglect of the q^2 dependence is less reliable.

§2. Experimental tests of the IVC hypothesis

1/ The $B^{12} - C^{12*} - N^{12}$ isotriplet. Weak magnetism

It was pointed out by Gell-Mann that a convincing test of the IVC hypothesis can be carried out with the three $J^P = 1^+$ nuclei B^{12}, C^{12*}, N^{12}, which are the q = -1, 0, +1 components of an isotriplet. These nuclei decay to the 0^+ ground state of the isosinglet C^{12} via β^-, γ and β^+ emission respectively /see fig.7/. Gell-Mann drew attention to the fact that the relevant correction term of the first order in q_λ/M_N to the unpolarized spectrum of the B^{12} and N^{12} decays is of exatly the same structure as the matrix element which induces the $C^{12*} \rightarrow C^{12} + \gamma$ transition. Indeed, in a $1^+ \rightarrow 0^+$ β decay, the only zero order term in the matrix element is the well

*For conserved current these corrections are of second order in the symmetry breaking at $q^2 = 0$. This is the Ademollo-Gatto theorem.

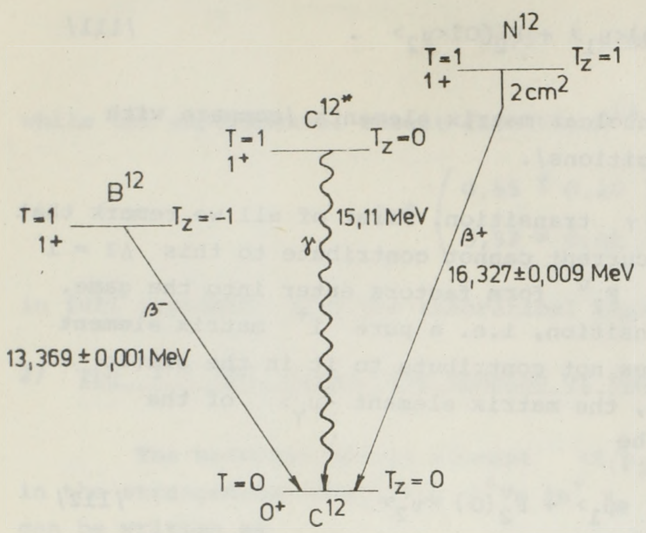

Fig. 7.

The decays of the B^{12} - C^{12*} - N^{12} isotriplet

known G-T term $H_1(0)\langle\sigma\rangle \equiv \lambda\langle\sigma\rangle$. This term leads to the Fermi spectrum $\lambda^2|\langle\sigma\rangle|^2\ F(x, W_o)$. The first order correction to shape of the spectrum comes from the interference of the $\lambda\langle\sigma\rangle$ term with two terms of order q_λ/M_N . The first comes from the $F_1(0)\ \gamma_\lambda$ structure when the recoil of the daughter nucleus is taken into account, the second from the $F_2(0)\ \sigma_{\lambda\nu}\ q^\nu/2M_N$ structure without recoil. After a lengthy but straightforward calculation the shape correction factor which multiplies the $\lambda^2|\langle\sigma\rangle|^2\ F(x, W_o)$ spectrum turns out to be

$$1 \pm \frac{8}{3}\ \frac{\langle\mu\rangle}{\lambda\langle\sigma\rangle}\ \frac{m_e}{M_p}\ x\ , \qquad x \equiv \frac{e_o}{m_e} \qquad /110/$$

where + (-) refers to the B^{12} (C^{12*}) decay. In eq./110/ $\langle\mu\rangle$ stands for the sum of the contributions of the $F_1(0)$ and $F_2(0)$ form factors, modified by the nuclear structure:

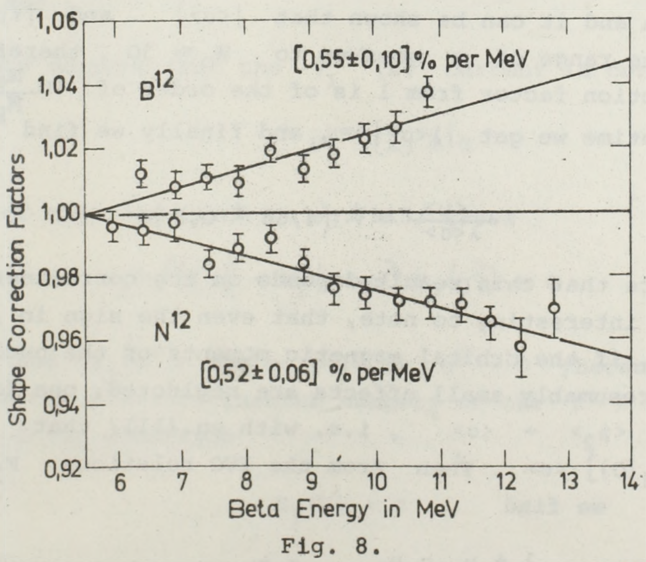

Fig. 8.

The measured shape correction factors for B^{12} and N^{12}

$$\langle\mu\rangle \equiv F_1(0)\langle\mu_1\rangle + F_2(0)\langle\mu_2\rangle \quad . \qquad /111/$$

$\langle\mu_1\rangle$ and $\langle\mu_2\rangle$ are appropriate nuclear matrix elements /compare with $H_1(0) \to H_1(0)\langle\sigma\rangle$ in the G-T transitions/.

Let us now investigate the γ transition. First of all we remark that the isoscalar part of the electric current cannot contribute to this $\Delta I = 1$ transition. Thus only the F_1^V and F_2^V form factors enter into the game. Moreover, we have and $1^+ \to 0^+$ transition, i.e. a pure 1^+ matrix element is needed. The F_1^V form factor does not contribute to it in the static limit, while the F_2^V does. Namely, the matrix element $\langle\mu_\gamma\rangle$ of the $C^{12*} \to C^{12} + \gamma$ decay turns out to be

$$\langle\mu_\gamma\rangle = F_1^V(0)\ \langle\mu_1\rangle + F_2^V(0)\ \langle\mu_2\rangle \qquad /112/$$

From the IVC relation $F_{1,2}(q^2) = 2F_{1,2}^V(q^2)$ we immediately find that $\langle\mu\rangle = 2\langle\mu_\gamma\rangle$; thus measuring $|\langle\mu_\gamma\rangle|$ in the $C^{12*} \to C^{12} + \gamma$ decay, value of $|\langle\mu\rangle|$ in eq. /110/ can be predicted, assuming that the IVC hypothesis is correct.

To test the shape correction formula, still $\lambda\langle\sigma\rangle$ is needed. A good experimental value of $|\lambda\langle\sigma\rangle|$ can be obtained from the lifetime of the B^{12}. The point is that the contribution of the shape correction factor to the lifetime can be safely neglected. Indeed, the nuclear structure of B^{12} is sufficiently known and it can be shown that $|\langle\sigma\rangle|$ and $|\langle\mu\rangle|$ are of the order of 1. the range of x goes up to $W_o \approx 30$, therefore the deviation of the correction factor from 1 is of the order of $40\frac{m_e}{M_p} \approx 0{,}02$. Thus from the B^{12} lifetime we get $|\lambda\langle\sigma\rangle|$, and finally we find

$$\frac{\langle\mu\rangle}{\lambda\langle\sigma\rangle} = \pm\left(4{,}68 \pm 0{,}5\right) \qquad /113/$$

We stress once more that this result depends on the correctness of the IVC hypothesis. It is interesting to note, that even the sign in /113/ can be predicted. Namely, if the orbital magnetic moments of the nucleons and other complicated but presumably small effects are neglected, one finds that $\langle\mu_1\rangle \to \langle\sigma\rangle$, $\langle\mu_2\rangle \to \langle\sigma\rangle$, i.e. with eq./111/ that $\langle\mu\rangle \to \left[F_1(0) + F_2(0)\right]\langle\sigma\rangle$. Then from the IVC relations $F_1(0) = 1$, $F_2(0) = \mu_p - \mu_n$ we find

$$\frac{\langle\mu\rangle}{\lambda\langle\sigma\rangle} \approx \frac{1 + \mu_p - \mu_n}{\lambda} = \frac{4{,}7}{1{,}23} = 3{,}75 \qquad /113'/$$

Comparison with eq./113/ shows that the + sign must be chosen. Thus the IVC prediction for the shape factor /110/, with eq. /113/ taken into account becomes

$$1 \pm (0{,}57 \pm 0{,}06)\, 10^{-2} \frac{e_o}{MeV} \qquad /114/$$

while the experimental result from the B^{12} and N^{12} spectrum gives /see fig.8/

$$1 \pm \begin{pmatrix} 0{,}55 \pm 0{,}10 \\ 0{,}52 \pm 0{,}06 \end{pmatrix} 10^{-2} \frac{e_o}{MeV} ,$$

in full agreement with the theoretical expectation.

2/ <u>The $\Sigma \to \Lambda e\nu$ decay. The damping of the Fermi transition</u>

The hadronic matrix element $\langle \Lambda(p_2)|J_{H\lambda}(x)|\Sigma^+(p_1)\rangle$ which appears in the strangeness-conserving $\Sigma^+ \to \Lambda e^+ \nu_e$ decay contains a vector part which can be written as

$$\langle \Lambda(p_2)\, V_\lambda^{s=o}(x)|\Sigma^+(p_1)\rangle =$$

$$= e^{i(p_2-p_1)x} \frac{\bar{\Lambda}(p_2)}{(2\pi)^{3/2}} \left[E_1(q^2)\, \gamma_\lambda - i\sigma_{\lambda\nu} \frac{(p_2-p_1)^\nu}{M_\Lambda + M_\Sigma} E_2(q^2) + \frac{(p_2-p_1)_\lambda}{M_\Lambda+M_\Sigma} E_3(q^2) \right] \frac{\Sigma^+(p_1)}{(2\pi)^{3/2}} \qquad /115/$$

The space-time structure of this matrix element is, of course, the same as that of the corresponding nucleon matrix element /83/, but the form factors are different, since the current operator is taken between states belonging to other multiplets.

Let us now suppose that the $V_\lambda^{s=O}(x)$ current is conserved:

$$\partial^\lambda V_\lambda^{s=o}(x) = 0 \qquad /116/$$

The application of this condition to eq./115/ gives

$$E_1(q^2)\left(M_\Lambda^2 - M_\Sigma^2\right) + q^2 E_3(q^2) = 0 \qquad /117/$$

Since $M_\Lambda \neq M_\Sigma$ and $E_3(q^2)$ has no pole at $q^2 = 0$ /because no bound state of zero mass with the discrete quantum numbers of the $\Sigma^+ \tilde{\Lambda}$ system exists/, we find at $q^2 = 0$ the condition

$$E_1(0) = 0 \qquad /118/$$

Eq./118/ is a direct consequence of the current conservation /116/. If the IVC hypothesis holds,/116/ follows from $\partial^\lambda V_{3,\lambda}(x)=0$ in the exact SU(2) limit. We stress that even in that limit $M_\Lambda \neq M_\Sigma$ since Λ and Σ belong to different isomultiplets. The result is different if the masses are equal.

This is the case for the $\langle n|V_\lambda^{S=o}(x)|p\rangle$ matrix element, where from

$$F_1(q^2)\left(M_n^2 - M_p^2\right) + q^2\ F_3(q^2) = 0$$

in the SU(2) limit we find, with $M_n = M_p$ the well known condition $F_3(q^2) = 0$ for $q^2 < 0$.

The physical region of q^2 in the $\Sigma \to \Lambda e\nu$ decay is $m_e^2 \leq q^2 \leq \left(M_\Sigma - M_\Lambda\right)^2 \approx (76\ \mathrm{MeV})^2$, and the approximation $E_1(q^2) = E_1(0) = 0$ is certainly not very good for the whole range. Nevertheless, we expect the Fermi transition to be considerably damped as compared to the G-T transition, where no current conservation effect occurs. The experimental results are again in the favour of this IVC prediction /Fig. 9, 10/.

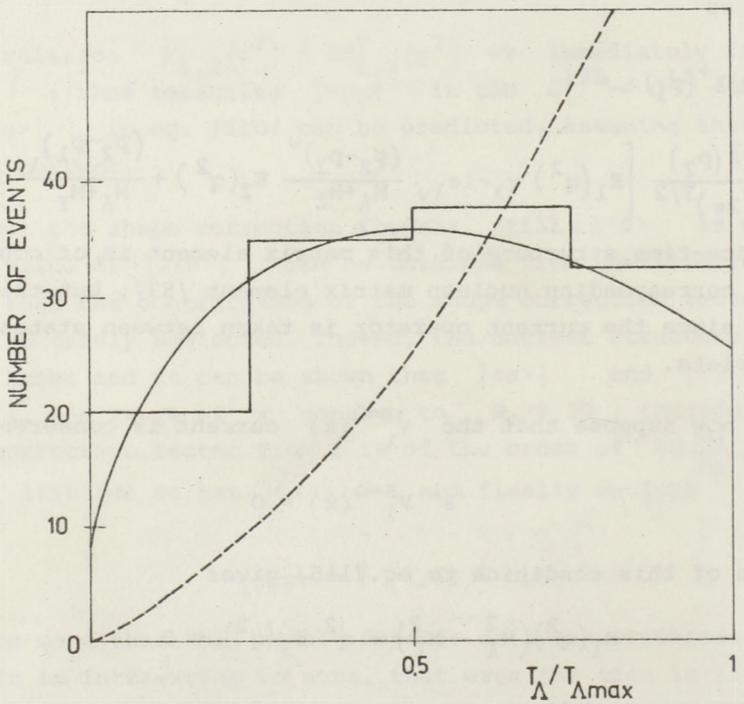

Fig. 9.

The kinetic energy spectrum of the Λ hyperons in in $\Sigma^\pm \to \Lambda e^\pm \nu$ decay

———— pure axial current

- - - - - pure vector current

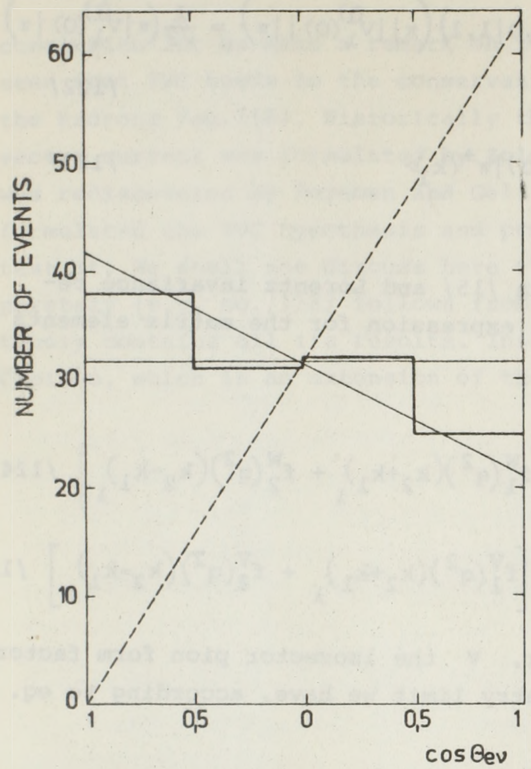

Fig. 10.
The electron - neutrino angular distribution in $\Sigma^{\pm} \to \Lambda e^{\pm}\nu$ decay.

———— pure axial current
- - - - - pure vector current

3/ The β decay of the charged pion

The IVC hypothesis can be successfully applied to the $\pi^{\pm} \to \pi^{o} e^{\pm} \nu_e(\bar{\nu}_e)$ decays too. These rare decay modes have been observed with the rate

$$\frac{\Gamma\left(\pi^{+} \to \pi^{o} e^{+} \nu_e\right)\exp}{\Gamma\left(\pi^{+} \to \mu^{+} \nu_\mu\right)\exp} = \left(1{,}02 \pm 0{,}07\right) 10^{-8} \qquad /119/$$

The hadronic matrix element for the π^{+} decay reads:

$$<\pi^{o}(k_2)|J_{H\lambda}(o)|\pi^{+}(k_1)> = c_V<\pi^{o}(k_2)|V_{\lambda}^{s=o}(o)\pi^{+}(k_1)> \qquad /120/$$

The strangeness-changing current is absent because the decay is strangeness-conserving, and the $A_{\lambda}^{s=o}$ current is absent because no axial vector can be constructed from the two available four momenta k_1 and k_2 .

From the IVC hypothesis we get in the SU(2) limit

$$<\pi^{o}|V_{\lambda}^{s=o}(o)|\pi^{+}> = \sqrt{2}<\pi_{o}|V_{-1,\lambda}^{(1)}(o)|\pi^{+}> =$$

$$= \sqrt{2}\left(1,1;1,-1|1,0\right)\left(\pi||V_{\lambda}^{(1)}(o)||\pi\right) = \left(\pi||V_{\lambda}^{(1)}(o)||\pi\right) \qquad /121/$$

$$<\pi^+|V_{3,\lambda}(0)|\pi^+> = <\pi^+|V^{(1)}_{0,\lambda}(0)|\pi^+> = \left(1,1;1,0|1,1\right)\left(\pi||V^{(1)}_\lambda(0)||\pi\right) = \frac{1}{\sqrt{2}}\left(\pi|V^{(1)}_\lambda(0)|\pi\right) \qquad /122/$$

$$<\pi^0(k_2)|V^{s=0}_\lambda(0)|\pi^+(k_1)> = \sqrt{2}\,<\pi^+(k_2)|V_{3,\lambda}(0)|\pi^+(k_1)> \qquad /123/$$

Using the translational invariance formula /15/ and Lorentz invariance requirements, we find that the most general expression for the matrix elements of $V^{s=0}_\lambda(x)$ and $V_{3,\lambda}(x)$ is

$$<\pi^0(k_2)|V^{s=0}_\lambda(x)|\pi^+(k_1)> = \frac{e^{i(k_2-k_1)x}}{(2\pi)^3\sqrt{4E_1E_2}}\left[f^W_1(q^2)\left(k_2+k_1\right)_\lambda + f^W_2(q^2)\left(k_2-k_1\right)_\lambda\right] \qquad /124/$$

$$<\pi^+(k_2)|V_{3,\lambda}(x)|\pi^+(k_1)> = \frac{e^{i(k_2-k_1)x}}{(2\pi)^3\sqrt{4E_1E_2}}\left[f^V_1(q^2)\left(k_2+k_1\right)_\lambda + f^V_2(q^2)\left(k_2-k_1\right)_\lambda\right] \qquad /125/$$

In eq. /124/ and /125/ W labels the weak, V the isovector pion form factors, and $q_\lambda \equiv \left(k_2-k_1\right)_\lambda$. In the isospin symmetry limit we have, according to eq. /123/ - /125/,

$$f^W_i(q^2) = \sqrt{2}\,f^V_i(q^2) \quad \text{for } q^2 < 0 \qquad /i = 1,2/ \qquad /126/$$

The conservation of the isospin current $V_{3,\lambda}(x)$ leads to $f^V_2(q^2) = 0$ for all $q^2 < 0$, and integration of eq. /125/ over $\underline{x}$ gives $f^V_1(0) = 1$. Then eq. /123/ says that in the isospin symmetry limit

$$f^W_1(0) = \sqrt{2}\,, \qquad f^W_2(q^2) = 0 \qquad /127/$$

Neglecting the small effects of isospin symmetry violation and taking $f^W_1(q^2) = f^W_1(0)$ in the small physical region $m_e^2 \leq q^2 \leq (m_{\pi^+} - m_{\pi^0})^2$ of the $\pi^+ \to \pi^0 e^+ \nu_e$ decay, we see that in the decay rate formula for this decay no unknown quantity remains; in particular $|g\,c_V| = |g_V|$ is known from nuclear β decay. The calculation of the decay rate then gives

$$\frac{\Gamma\left(\pi^+ \to \pi^0 e^+ \nu_e\right)_{theor}}{\Gamma\left(\pi^+ \to \mu^+ \nu\right)_{exp}} = \left(1{,}07 \pm 0{,}003\right)10^{-8} \qquad /128/$$

in good agreement with the purely experimental rate /119/.

The three successful tests of the IVC hypothesis we discussed above give strong evidence that this hypothesis is correct. Other tests in nuclear β decay have also been made and good results were obtained (see ref. 120 in [1]). Therefore this hypothesis may also be referred to as a theory . In conclusion let us make a remark on the history of the IVC theory. We have seen that IVC leads to the conservation of the weak vector current /CVC/ of the hadrons /eg.116/. Historically the CVC hypothesis for the weak $s = 0$ vector current was formulated by Zeldovich and Gerstein as early as 1955, then was rediscovered by Feynman and Gell-Mann in 1958, and later in 1958 Gell-Mann formulated the IVC hypothesis and proposed the $B^{12} - C^{12*} - N^{12}$ experiment to test it. We shall not discuss here the consequences of the CVC theory separately (e.g. eq./118/ follows from CVC alone) because the successful IVC theory contains all its results. Instead we shall turn to the theory of Cabibbo, which is an extension of the IVC theory to the full hadronic current.

IV. THE LEPTONIC DECAYS OF THE HADRONS. THE OCTET CURRENT THEORY OF CABIBBO

§1. The octet current

In the IVC theory the $V_\lambda^{s=o}$ current bears the strong interaction quantum numbers $(B, S, I, I_z; Y\equiv B+S, Q=I_z+\frac{1}{2}Y)$ of a π^- meson. The $(V_\lambda^{s=o})^+$ operator has then the quantum numbers of a π^+ meson, whereas the $V_{3,\lambda}$ current, which is the third component of the isotriplet current operator /87/, has the quantum numbers of a π^o meson. It is natural to ask whether the $V_\lambda^{s\neq o}$ current and its adjoint have definite strong interaction quantum numbers or not. Up to now we know only that $Q = -1$ for all the components of $J_{H\lambda}$, hence also for $V_\lambda^{s\neq o}$. This follows from the fact that we supposed that a full weak current $J_\lambda = J_{H\lambda} + j_\lambda$ exists, and that j lowers the electric charge by one unit according to eq./5/. Of course then $(V_\lambda^{s\neq o})^+$ has $Q = +1$.

In 1962 Gell-Mann discovered the SU(3) group to be an approximate internal symmetry group of the strong interaction. In this context the 8 pseudoscalar mesons $\pi^-, \pi^+, \pi^o; K^+, K^o; \bar{K}^o, K^-; \eta$ transform according to the irreducible, 8 dimensional ("octet") representation of the SU(3) group. The meson octet is a supermultiplet from the point of view of the SU(2) isospin group, which is a subgroup of the SU(3) group of Gell-Mann. The SU(3) symmetry is broken in Nature, as shown, for instance by the large mass differences between those members of the meson octet which belong to different isomultiplets. Nevertheless, the concept of the SU(3) symmetry proved to be a very useful one in many respects. For details and applications in strong interaction the reader is referred to [2].

In 1963 Cabibbo suggested that the $V_\lambda^{s=o}$, $V_{3,\lambda}$, $(V_\lambda^{s=o})^+$ isotriplet operators be as the π^-, π^o, π^+ members of an irreducible SU(3) octet operator, and that attempts be made to identify the 5 missing components with physically interpretable currents. He proposed to include in this octet the current with the quantum numbers of the K^- meson, which is the only $Q = -1$ member of the pseudoscaler meson octet with non-zero strangeness. Then $(V_\lambda^{s\neq o})^+$ will obviously have the quantum numbers of the K^+ meson. Moreover, the η component of this octet operator is known to be proportional to the hypercharge current Y_λ [2], Finally, the K^o and $\bar{K}^o$ components are not known to take part directly in physical interactions.

When we discussed the IVC hypothesis, we saw that the space integrals of the $V_{i,o}$ currents (i = 1,2,3) gave the generators of the SU(2) group. Similarly, the vector octet of Cabibbo gives the 8 generators of the SU(3) group, in particular the 3 isospin operators and the hypercharge operator. The SU(3) formalism will be developed in §3 of this chapter.

Up to now nothing has been said about the possible internal symmetry properties of the axial hadron currents. Cabibbo supposed that their internal symmetry structure may be the same as that of the corresponding vector currents, i.e. he supposed that the $A_\lambda^{s=o}$, $\left(A_\lambda^{s=o}\right)^+$, $A_\lambda^{s\neq o}$, $\left(A_\lambda^{s\neq o}\right)^+$ currents are also members of an irreducible octet operator of the SU(3) group, namely they are the π^-, π^+, K^- and K^+ components of this octet. Of course the π^o and η components of the axial octet are now axial currents, and have nothing to do with the isospin and hypercharge currents. Together with the two other neutral K^o, $\bar{K}^o$ axial currents of this octet, they are not known to have direct physical meaning. This unfortunate situation will be reflected in the fact that while the form factors of the vector currents $V_\lambda^{s=o}$, $V_\lambda^{s\neq o}$ will be connected by the W-E theorem (applied to SU(3)) not only between themselves but also with the isovector $V_{\pi^o,\lambda}$ and isoscalar $\left(\frac{1}{\sqrt{3}} V_{\eta,\lambda} = \frac{1}{2} Y_\lambda\right)$ electromagnetic form factors, the $A_\lambda^{s=o}$ and $A_\lambda^{s\neq o}$ form factors will be connected by SU(3) only with each other.

Another important difference between the vector octet $V_{i,\lambda}$ and the axial octet $A_{i,\lambda}$ ($i = 1,2...,8$ labels the hermitean components of these currents) is that, while $\partial^\lambda V_{i,\lambda}(x)=0$ for $i = 1,2,3$ in exact SU(2) limit and for $i = 1,2,...,8$ in exact SU(3) limit, no such conservation laws are expected to hold for the axial currents. However, an approximate relation leading to the notion of the partially conserved axial current (PCAC) has been introduced with considerable success for the divergence of the axial octet too (see chapter V).

Let us now turn to the experimental verification of the consequences of the Cabibbo theory. All these consequences can be deduced from the W-E theorem. It is customary, however, to divide the results into to groups: the selection rules, which in fact arise because some of the SU(3) Clebsch--Gordan coefficients are zero, but which can be deduced without the cumbersome SU(3) technics; and the intensity rules, where the full SU(3) apparatus is needed.

In §2 we shall deduce the selection rules not only for the leptonic hadron decays $H \to H'\ell\nu$, but also for the non leptonic decays $H \to H'$. In both cases we shall suppose that the decays are induced by the current-current Lagrangean /4/ and that the hadronic weak current is composed from the vector and axial octets of Cabibbo. Then in §4 we shall look at the intensity rules in the leptonic decays of the hadrons. The applicability of the theory of Cabibbo (in fact even of the current-current theory in general) to non--leptonic decays is dubious, and these decays will be only briefly discussed in the next chapter.

When deriving the selection rules, we shall often refer to the relation

$$\Delta Q = \Delta I_z + \frac{1}{2}\Delta S \quad , \qquad /129/$$

which of course holds both for the $H \to H'\ell\nu$ and $H \to H'$ processes. In eq./129/ and below $\Delta X \equiv X_{H'} - X_{H}$, where X_H stands for a strong interaction quantum number of the hadron (or hadrons) H. Eq./129/ follows from the already mentioned relation

$$Q = I_z + \frac{1}{2}(B + S) \qquad /130/$$

valid for any individual hadron, and from the fact that in all weak process $\Delta B = B_{H'} - B_H = 0$. Indeed, the lepton current does not change the baryon number and the total baryon charge is absolutely conserved. Furthermore, in $H \to H'\ell\nu$ decays $\Delta Q = \pm 1$, since the lepton current does change the electric charge by ∓ 1. In $H \to H'$ decays we have of course $\Delta Q = 0$.

§2. Selection rules for weak hadron decays. (Theory and experiment.)

Let us discuss first the selection rules for the $H \to H'\ell\nu$ decays. The hadronic part of these decays is described by the $\langle H'|J^{+}_{H\lambda}|H\rangle$ matrix element for $\Delta Q = +1$, and by $\langle H'|J^{-}_{H\lambda}|H\rangle$ for $\Delta Q = -1$. For $\Delta S = 0$ we find then from eq./129/ that $\Delta I_z = \Delta Q = \pm 1$. To find the possible values of $\Delta I \equiv I_{H'} - I_H$, we must remember that in the Cabibbo theory the $S = 0$ currents transform like the π^+ and π^- mesons, hence they have $I = 1$. Then from $1 \otimes I_H = (I_H - 1) \oplus I_H \oplus (I_H + 1)$ we find that $\Delta I = -1, 0, 1$. The $\Delta I = 0$ case occurs e.g. in the $n \to pe\nu$ and $\pi^{\pm} \to \pi^{o} e^{\pm} \nu_e(\bar{\nu}_e)$ decays, $\Delta I = -1$ in the $\Sigma^{\mp} \to \Lambda e^{\pm} \nu_e(\bar{\nu}_e)$ decays. The $\Delta I = +1$ $H \to H'\ell\nu$ decays are forbidden by energy conservation.

The $\Delta S \neq 0$ $H \to H'\ell\nu$ decays are induced by the $K^+(B=0,\ S=+1,\ I=\frac{1}{2},\ I_z = +\frac{1}{2};\ Y = +1,\ Q = +1)$ and the $K^-(B=0,\ S=-1,\ I=\frac{1}{2},\ I_z = -\frac{1}{2};\ Y=-1,\ Q=-1)$ components of the Cabibbo current. Thus we have $\Delta S = \Delta Q = +1$, $\Delta I_z = \frac{1}{2}$ for $\langle H'|J^{\lambda+}_{H}|H\rangle$, and $\Delta S = \Delta Q = -1$, $\Delta I_z = -\frac{1}{2}$ for $\langle H'|J^{\lambda}_{H}|H\rangle$. In both cases $\Delta I = \pm\frac{1}{2}$ is possible as seen from the $\frac{1}{2} \otimes I_H = \left(I_H - \frac{1}{2}\right) \oplus \left(I_H + \frac{1}{2}\right)$ relation. The $\Delta I = +\frac{1}{2}$ case occurs e.g. in the $\Delta S = \Delta Q = +1$ $\Lambda \to pe^-\bar{\nu}_e$ decay and in the $\Delta S = \Delta Q = -1$ $K^+ \to \pi^o \ell^+ \nu_e$ decay, while $\Delta I = -\frac{1}{2}$ in the $\Sigma^- \to ne^-\bar{\nu}_e$ and $K^- \to \ell^-\bar{\nu}_e$ decays.

Let us now find the selection rules for the $H \to H'$ decays. In the current-current theory these decays are described by the

$$\frac{g}{\sqrt{2}}\,\frac{1}{2}\,\langle H'|J_H^{\lambda}\,J_{H\lambda}^{+}\;+\;J_{H\lambda}^{+}\,J_H^{\lambda}|H\rangle \qquad /131/$$

matrix element. The selection rules for the Cabibbo theory can then be deduced looking at the direct product of the type

$$\left(\pi^+ + K^+\right)\otimes\left(\pi^- + K^-\right) = \pi^+\otimes\pi^- + \pi^+\otimes K^- + K^+\otimes\pi^- + K^+\otimes K^- \quad . \qquad /132/$$

The first and the last term give $\Delta S=0$ transitions. It is easy to see that $\Delta S = 0$ decays are forbidden by energy conservation (e.g. $N \not\rightarrow p\pi^-$, $\Sigma \not\rightarrow \Lambda\pi$, $\Sigma \not\rightarrow NK$ etc.). Thus we are left with the $\pi^+\otimes K^-$ case which gives $\Delta S = -1$ transitions with $\Delta I_z = \frac{1}{2}$ and $\Delta I = \pm\frac{1}{2}, \pm\frac{3}{2}$, and with the $K^+\otimes\pi^-$ case, which gives $\Delta S = +1$, $\Delta I_z = -\frac{1}{2}$, $\Delta I = \pm\frac{1}{2}, \pm\frac{3}{2}$ transitions. The $\Delta I = -\frac{3}{2}$ transitions cannot occur because no hadron with $I \geq \frac{3}{2}$ quantum number exists among the elementary particles (we do not consider the resonances in these notes).

In table 6 we gathered the possible changes of the strong interaction quantum numbers for the weak hadron decays allowed by the Cabibbo theory and by energy conservation. It is left to the reader to verify in his Particle Data Tables that all the allowed decays are indeed observed with normal rates. We shall deal here with the complementary test of Cabibbo's selection rules: namely, we shall look at the decays which are energetically allowed, but forbidden in the Cabibbo theory.

Table 6.

Selection rules for energetically allowed hadron decays

$H \rightarrow H'\ell\nu$ decays		$H \rightarrow H'$ decays
$\Delta s = 0$	$\Delta S \neq 0$	$\Delta S \neq 0$
$\Delta S = 0$	$\Delta S = \Delta Q = \pm 1$	$\Delta S = \pm 1$
$\Delta I_z = \Delta Q = \pm 1$	$\Delta I_z = \frac{1}{2}\,\Delta Q = \pm\frac{1}{2}$	$\Delta I_z = -\frac{1}{2}\,\Delta S = \mp\frac{1}{2}$
$\Delta I = -1, 0$	$\Delta I = \pm\frac{1}{2}$	$\Delta I = \pm\frac{1}{2}, +\frac{3}{2}$

The $\Delta S = 0$ $H \rightarrow H'\ell\nu$ decays are irrelevant in this respect, since all the selection rules except $\Delta I = -1, 0, +1$ come in this case simply from the general (not necessarily octet) current-current theory and eq./129/. Thus $|\Delta I| \leq 1$ is here the only specific prediction of the Cabibbo theory, but unfortunately it follows also from energy conservation if $\Delta S = 0$.

On the contrary, for the $\Delta S \neq 0$ $H \to H'\ell\nu$ decays we find non trival results. As we see from table 6, $-\Delta S = \Delta Q = \pm 1$ transitions (or, which is the same, $\Delta I_z = \pm \frac{3}{2}$ transitions), $\Delta S = 2$ transitions and $|\Delta I| \geqslant \frac{3}{2}$ transitions are forbidden if the Cabibbo theory holds. The experimental results for the $\Gamma(-\Delta S = \Delta Q) : \Gamma(\Delta S = \Delta Q)$ ratios are:

$$\Gamma(\Sigma^+ \to n e^+ \nu) : \Gamma(\Sigma^- \to n e^- \bar{\nu}) \leqslant 0,4.10^{-2}$$

$$\Gamma(\Sigma^+ \to n \mu^+ \nu) : \Gamma(\Sigma^- \to n \mu^- \bar{\nu}) \leqslant 5.10^{-2} \qquad /133/$$

Moreover, 264 $K^+ \to \pi^+ \pi^- e^+ \nu$ $(\Delta S = \Delta Q)$ events have been found against zero $K^+ \to \pi^+ \pi^+ e^- \bar{\nu}$ $(-\Delta S = \Delta Q)$ event. In K^0 decays also only a small $-\Delta S = \Delta Q$ impurity may be present according to the experimental results. Concerning the $\Delta S \neq 2$ rule in the $H \to H'\ell\nu$ decays, the following branchings ratios have been measured:

$\Delta S = 2$ decays	$\Delta S = 1$ decay
$R(\Xi^0 \to p e^- \bar{\nu}) < 1,3.10^{-3}$	$R(\Xi^- \to \Lambda e^- \bar{\nu}) = (0,63 \pm 0,23).10^{-3}$
$R(\Xi^0 \to p \mu^- \bar{\nu}) < 1,3.10^{-3}$	

/134/

It would be desirable to lower the upper limit for the $\Delta S = 2$ decays. However if we take into consideration the fact that these decays have a larger phase space than the $\Xi^- \to \Lambda e^- \bar{\nu}$ decay, these results are already an indication in favour of the Cabibbo theory.

Finally, for the non leptonic $H \to H'$ decays, the $\Delta S = 2$ $\Xi \to N\pi$ decays are energetically allowed but forbidden by the Cabibbo theory. The experimental $\Gamma(\Delta S = 2) : \Gamma(\Delta S = 1)$ ratios are

$$\Gamma(\Xi^- \to n\pi^-) : \Gamma(\Xi^- \to \Lambda\pi^-) < 1,1.10^{-3}$$

$$\Gamma(\Xi^0 \to p\pi^-) : \Gamma(\Xi^0 \to \Lambda\pi^0) < 0,9.10^{-3} \qquad /135/$$

This result is in favour of the applicability of the Cabibbo theory to the $H \to H'$ decays. However, as seen from table 6, this theory predicts the possibility of $\Delta I = + \frac{3}{2}$ in non-leptonic decays, while all the experimental results show that the $\Delta I = + \frac{3}{2}$ amplitude is strongly damped as compared with the $I = \frac{1}{2}$ amplitude. From the point of view of the Cabibbo theory, this seems to be a dynamical accident. We shall return to this problem in chapter V.

§3. Current algebra relations and Wigner-Eckart theorem for SU(3)

As we noticed already, for the discussion of the intensity rules in the leptonic decays of the hadrons, the SU(3) formalism must be applied. As well known [4], the hermitean generators of the SU(3) group I_i $(i = 1, 2, \ldots, 8)$ satisfy the commutation rules

$$[I_s, I_k] = i\, f_{sk\ell}\, I_\ell \quad /s,k,\ell = 1,2,\ldots,8/ \;, \qquad /136/$$

where the nonzero components of the totally antisymmetric structure constants $f_{sk\ell}$ are:

$sk\ell$	123	147	156	246	257	345	367	458	678
$f_{sk\ell}$	1	1/2	-1/2	1/2	1/2	1/2	-1/2	$\sqrt{3}/2$	$\sqrt{3}/2$

/137/

The three generators $I_1\, I_2\, I_3$ form an SU(2) subgroup and are identified with the isospin operators, while I_8 is proportional to the hypercharge operator Y :

$$Y = \frac{2}{\sqrt{3}}\, I_8 \qquad /138/$$

The eight generators I_i are the hermitean components of an irreducible SU(3) octet. We shall denote the hermitean components of an SU(3) octet operator in general by T_i^8 , and its spherical components by $T_{(\nu)}^8$. The relations between these components and the correspondence of the spherical components to the physical quantum numbers Y, I, I_z are given in table 7. In the same table we give also the state vectors of the pseudoscalar meson octet $|P_{(\nu)}\rangle$. Thus e.g. $|P_{(7)}\rangle = -|K^o\rangle$ $|B_{(7)}\rangle = |\Xi^o\rangle$. The sign convention is that of de Swart [4]. Other sign conventions are also used in the literature and this may lead to unessential differences in the sign of some amplitudes.

The generators I_i are space integrals of the time components of vector currents $V_{i,\lambda}(x)$:

$$I_i(t) = \int d\underline{x}\; V_{i,o}(\underline{x},t) \;. \qquad /139/$$

In exact SU(3) limit the $I_i(t)$ are of course time independent. If SU(3) is violated, the equal time SU(3) commutation rules

$$[I_s(t),\ I_k(t)] = i\ f_{sk\ell}\ I_\ell(t) \qquad /140/$$

are still supposed to hold. The currents $V_{i,\lambda}(x)$ are, by definition, octet operators of SU(3):

$$[I_s(t),\ V_{k,\lambda}(\underline{x},t)] = i\ f_{sk\ell}\ V_{\ell,\lambda}(\underline{x},t) \qquad /141/$$

Table 7.

SU(3) labels

ν	1	2	3	4	5	6	7	8
Y, I, I_z	$1, \frac{1}{2}, \frac{1}{2}$	$1, \frac{1}{2}, \frac{-1}{2}$	0, 1, 1	0, 1, 0	0, 1, -1	0, 0, 0	$-1, \frac{1}{2}, \frac{1}{2}$	$-1, \frac{1}{2}, \frac{-1}{2}$
$P_{(\nu)}$	$-K^+$	$-K^o$	$-\pi^+$	$-\pi^o$	π^-	η	$-\bar{K}^o$	K^-
$B_{(\nu)}$	$-p$	$-n$	$-\Sigma^+$	Σ^o	Σ^-	Λ	Ξ^o	Ξ^-
$T^8_{(\nu)}$	$\frac{-1}{\sqrt{2}}T^8_{4+i5}$	$\frac{-1}{\sqrt{2}}T^8_{6+i7}$	$\frac{-1}{\sqrt{2}}T^8_{1+i2}$	T^8_3	$\frac{1}{\sqrt{2}}T^8_{1-i2}$	T^8_8	$\frac{-1}{\sqrt{2}}T^8_{6-i7}$	$\frac{1}{\sqrt{2}}T^8_{4-i5}$

The last row reads: $T^8_{(1)} = \frac{-1}{\sqrt{2}}\left(T^8_4 + i\ T^8_5\right)$, etc.

We notice that from eq. /141/ eq. /140/ follows, but not vice-versa. Any $O_{i,\lambda}(x)$ with the property $\int d\underline{x}\ O_{i,\lambda}(\underline{x},t) = 0$ could be added to /141/ and eq. /140/ still would be true, even if $O_{i,\lambda}(x)$ is not an SU(3) octet.

According to Cabibbo, the π^- and K^- components /see table 7/ of the current $O_{i,\lambda}(x)$ are proportional to the weak $S = 0$ and $S \neq 0$ vector currents of the hadrons. Furthermore, the weak axial currents are also supposed to be components of an axial octet $A_{i,\lambda}(x)$:

$$\left[I_s(t),\ A_{k,\lambda}(\underline{x},t)\right] = i\ f_{sk\ell}\ A_{\ell,\lambda}(\underline{x},t) \qquad /142/$$

Eq. /142/ do not normalize the $A_{i,\lambda}(x)$ currents. Such a normalization is provided if we suppose with Gell-Mann that the axial charges

$$I^A_i(t) = \int d\underline{x}\ A_{i,o}(\underline{x},t) \qquad /143/$$

satisfy with the currents the following communitation rules:

$$\left[I_s^{\ A}(t),\ V_{k,\ell}(\underline{x},t)\right] = i\ f_{sk\ell}\ A_{\ell,\lambda}(\underline{x},t)\ , \qquad /144/$$

$$\left[I_s^{\ A}(t),\ A_{k,\ell}(\underline{x},t)\right] = i\ f_{sk\ell}\ V_{\ell,\lambda}(\underline{x},t)\ . \qquad /144'/$$

Let us stress that eq. /144/ and /144 / are new and strong conditions, the consequences of which need further theoretical and experimental verification. Only in a few special theoretical models (e.g. in the quark model) are these relations automatically satisfied. Since $V_{i,\lambda}$ is normalized by eq. /139/ - /141/, $A_{i,\lambda}$ is also normalized by eq. /143/ and /144/. However, the sign of $A_{i,\lambda}$ is not determined by these relations: $-A_{i,\lambda}$ is also a solution if $A_{i,\lambda}$ is. For $V_{i,\lambda}$ even the sign is fixed by the relations /139/ - /141/.

Integration of eq. /142/ and /144'/ for $\lambda = 0$ over $\underline{x}$ yields commutation rules between the charges $I_i(t)$ and $I_i^{\ A}(t)$. Together with eq. /140/ this system of commutators is easily seen to generate an $SU(3) \otimes SU(3)$ group. Indeed, introducing the "chiral charges"

$$I_i^{(\pm)}(t) \equiv \frac{1}{2}\left(I_i(t) \pm I_i^{\ A}(t)\right), \qquad /145/$$

one arrives to the commutations rules

$$\left[I_s^{(+)}(t),\ I_k^{(+)}(t)\right] = i\ f_{sk\ell}\ I_\ell^{(+)}(t)\ ,$$

$$\left[I_s^{(-)}(t),\ I_k^{(-)}(t)\right] = i\ f_{sk\ell}\ I_\ell^{(-)}(t)\ ,$$

$$\left[I_s^{(+)}(t),\ I_k^{(-)}(t)\right] = 0 \qquad /146/$$

The Cabibbo current can now be written in the following way /see table 7/ :

$$J_{H\lambda}(x) = c_V\left(V_{1,\lambda}(x) - i\ V_{2,\lambda}(x)\right) + c_A\left(A_{1,\lambda}(x) - i\ A_{2,\lambda}(x)\right) +$$

$$+ d_V\left(V_{4,\lambda}(x) - i\ V_{5,\lambda}(x)\right) + d_A\left(A_{4,\lambda}(x) - i\ A_{5,\lambda}(x)\right) \qquad /147/$$

Since all the operators $V_{i,\lambda}$, $A_{i,\lambda}$ are now normalized, the coefficients $c_{V,A}$ and $d_{V,A}$ are measurable in principle. We have already seen that $c_V = g_V : g = 0{,}9778 \pm 0{,}0018$ when we discussed the IVC theory. This result followed from the fact that if we neglect $SU(2)$ violation effects, then $F_1(0) = 1$ because of current conservation. For the other coefficients the situation is more complicated, because the axial currents are not conserved, and the SU(3) violation effects may substantially modify the form factors of the $s \neq 0$ vector current. On the other hand, the experimental data on weak decays are not good enough for the conjoint determination of these coefficients and of the form factors. Thus theoretical hypotheses which reduce the number of the free parameters are welcomed. Such a hypothesis is the "universality of the weak current". In its modern form /Gell-Mann, Physics 1, 63 1964/ this hypothesis is based on the observation that it is possible to choose the constants c_V, c_A, d_V, d_A in the hadron current so that the full hadronic + leptonic weak current take the universal form

$$J_\lambda(x) = 2\left(c_{1,\lambda}(x) - i\, c_{2,\lambda}(x)\right) \qquad /148/$$

where the currents $c_{1,\lambda}(x)$, $c_{2,\lambda}(x)$ are such that the charges $c_1(t) \equiv \int d\underline{x}\, c_{1,o}(\underline{x},t)$, $c_2(t) \equiv \int d\underline{x}\, c_{2,o}(\underline{x},t)$ and $c_3(t) \equiv -i\left[c_1(t), c_2(t)\right]$ satisfy SU(2) communitation relations:

$$\left[c_s(t), c_k(t)\right] = i\, \varepsilon_{sk\ell}\, c_\ell(t), \qquad /s,k,\ell = 1,2,3/ \, . \qquad /149/$$

The condition of the universality turns out to be:

$$c_V = c_A,\; d_V = d_A, \quad c_V^2 + d_V^2 = 1 = c_A^2 + d_A^2 \qquad /150/$$

then with $c_V \equiv \cos\Theta$ the hadron current takes the form

$$J_{H\lambda} = 2\left[\cos\Theta\left(V_{1,\lambda}^{(+)} - i\, V_{2,\lambda}^{(+)}\right) + \sin\Theta\left(V_{4,\lambda}^{(+)} - i\, V_{5,\lambda}^{(+)}\right)\right]$$

$$\equiv 2\left[C_{1,\lambda}^{H} - i\, C_{2,\lambda}^{H}\right], \qquad /151/$$

where

$$V_{i,\lambda}^{(+)} \equiv \frac{1}{2}\left(V_{i,\lambda} + A_{i,\lambda}\right) \, . \qquad /152/$$

In eq. /150/ and from now on we shall suppose that $c_{V,A}$ and $d_{V,A}$ are real numbers. This means that the small T violation effects are neglected.

Thus if the weak current is universal, then only the $V_{i,\lambda} + A_{i,\lambda}$ combination appears in the hadron current. By an unfortunate mismatch between the generally accepted nomenclature and notation, this combination is usually called V-A and not V+A coupling. The angle in eq. /151/ is called the angle of Cabibbo. We shall discuss the universality hypothesis in more detail in §5 of this chapter.

The matrix elements of the $J_{H\lambda}$ current between hadron states can now be related with each other through the W-E theorem. Some remarks on the peculiarities of this theorem in the SU(3) case will now be given.

It is well known that the SU(2) group has one and only one irreducible representation of any dimension $n = 2j+1 = 1,2,\ldots,k,\ldots$, and that in the direct product of any two irreducible representations j_1 and $j_2 \le j_1$ the irreducible representations $j_1 - j_2$, $j_1 - j_2 + 1,\ldots, j_1 + j_2$ occur once and only once. For SU(n) groups with $n \ge 3$ the situation is more complicated. Inequivalent irreducible representations of the same dimension may exist, and in a direct product of two irreducible representations the same irreducible representation may enter more than once. For example, in SU(3) two inequivalent irreducible representations of dimension 3 exists, the 3 and 3^*. Also 10 and 10^* are inequivalent. Furthermore, in the direct product $8 \otimes 8$ the 8 occurs twice:

$$8 \otimes 8 = 1 \oplus 8_a \oplus 8_s \oplus 10 \oplus 10^* \oplus 27 \qquad /153/$$

The indices a and s mean that 8_a is constructed with the help of the fully antisymmetric SU(3) tensor $f_{ik\ell}$, $8_a \sim f_{ik\ell}\, 8_k\, 8_\ell$, while $8_s \sim d_{ik\ell}\, 8_k\, 8_\ell$, where $d_{ik\ell}$ is a fully symmetric constant tensor. As a consequence, in the W-E theorem for SU(3) several reduced matrix elements may belong to the same irreducible representations μ_2, μ, μ_1:

$$\left\langle \begin{matrix}\mu_2\\(\nu_2)\end{matrix}\right| T\begin{matrix}\mu\\(\nu)\end{matrix} \left|\begin{matrix}\mu_1\\(\nu_1)\end{matrix}\right\rangle = \sum_\gamma \begin{pmatrix}\mu_1 & \mu & \mu_{2\gamma}\\ \nu_1 & \nu & \nu_2\end{pmatrix} \left(\mu_2 || T || \mu_1\right)_\gamma \quad ; \qquad /154/$$

if the representation μ_2 is contained in the direct product $\mu_1 \otimes \mu$ n times, then γ takes n different "values". In particular, if $\mu_1 = \mu = \mu_2 = 8$, then, according to eq./153/ we have

$$\left\langle \begin{matrix}8\\(\nu_2)\end{matrix}\right| T_\nu^{8} \left|\begin{matrix}8\\(\nu_1)\end{matrix}\right\rangle = \sum_{\gamma=a,s} \begin{pmatrix}8 & 8 & 8_\gamma\\ \nu_1 & \nu & \nu_2\end{pmatrix} \left(8 || T^8 || 8\right)_\gamma \qquad /155/$$

The SU(3) Clebsch-Gordon coefficients may be factorized in the following way:

$$\begin{pmatrix}\mu_1 & \mu & \mu_{2\gamma}\\ \nu_1 & \nu & \nu_2\end{pmatrix} = \left(I_1, I_{z1}; II_z | I_2 I_{z2}\right) \left(\begin{matrix}\mu_1 & \mu\\ Y_1 I_1 & YI\end{matrix}\right|\left.\begin{matrix}\mu_2\\ Y_2\ I_2\end{matrix}\right)_\gamma \qquad /156/$$

The second factor on the right hand side of eq. /156/ is called the isoscalar factor. It does not depend on the quantum numbers I_{1z}, I_z, I_{z2}, these are contained only in the known SU(2) clebsch. The isoscalar factors has been tabulated by de Swart for the most important representations. They are also given in the Particle Properties Tables [7]. For the octet (and also for the decuplet) it is costumary to design ν_1 ν, ν_2 in the Clebsch-Gordan coefficients by the corresponding particle of the meson octet for ν_1 and of the baryon octet for ν and ν_2. To give an example, according to table 7 and eq. /156/ we find for $\nu_1=3$, $\nu=6$ and $\nu_2=3$, i.e. for $Y_1 I_1 I_{z1} = 011$, $YII_z = 000$ and $Y_2 I_2 I_{zz} = 011$:

$$\begin{pmatrix} 8 & 8 & 8\gamma \\ 8 & 6 & 3 \end{pmatrix} \equiv \begin{pmatrix} 8 & 8 & 8\gamma \\ 011 & 000 & 011 \end{pmatrix} \equiv \left(\pi^+ \; \Lambda \; \Sigma_\gamma^{\;+}\right) = \left(11;\; 00|11\right)\left(\pi \; \Lambda|\Sigma\right)_\gamma \qquad /157/$$

The isoscalar factor $(\pi \, \Lambda|\Sigma)_a = 0$, while $(\pi \, \Lambda|\Sigma)_s = \frac{1}{\sqrt{5}}$. Thus

$$\begin{pmatrix} 8 & 8 & 8a \\ 3 & 6 & 3 \end{pmatrix} = 0 \;, \quad \begin{pmatrix} 8 & 8 & 8s \\ 3 & 6 & 3 \end{pmatrix} = \frac{1}{\sqrt{5}} \;. \qquad /158/$$

Let us still note the following symmetry property of the SU(3) clebsches:

$$\begin{pmatrix} \mu_1 & \mu & \mu_{2\gamma} \\ \nu_1 & \nu & \nu_2 \end{pmatrix} = \xi_1 \begin{pmatrix} \mu & \mu_1 & \mu_{2\gamma} \\ \nu & \nu_1 & \nu_2 \end{pmatrix} \;, \quad \xi_1 = \pm 1 \;. \qquad /159/$$

In particular, for $\mu_1 = \mu = 8$ $\xi_1 = +1$ if $\mu_{2\gamma} = 1, 27$ or 8_s , and $\xi_1 = -1$ if $\mu_{2\gamma} = 10, 10^*$ or 8_a.

It is now easy to apply the W-E theorem to the hadronic matrix elements which turn up in the $P \to \ell\nu$, $P_2 \to P_1 \ell\nu$, and $B_2 \to B_1 \ell\nu$ decays. If the universality of the weak interactions is supposed, then $J_{H\lambda}$ is taken from eq. /515/. In practice one often uses a weakened form of the universality hypothesis. Namely, one supposes that

$$c_V^2 + d_V^2 = 1 = c_A^2 = d_A^2 \;, \qquad /160/$$

but one does not require $C_A = C_V$. Then the hadron current may be written in the form

$$J_{H\lambda} = \cos\Theta_V\left(V_{1,\lambda} - i\,V_{2,\lambda}\right) + \cos\Theta_A\left(A_{1,\lambda} - i\,A_{2,\lambda}\right) +$$
$$+ \sin\Theta_V\left(V_{4,\lambda} - i\,V_{5,\lambda}\right) + \sin\Theta_A\left(A_{4,\lambda} - i\,A_{5,\lambda}\right) \;; \qquad /161/$$

now the universality in the sense of eq. /149/ does not hold, but still we have a case where the "total strength" $c_V^2 + d_{V'}^2$, $c_A^2 + d_A^2$ of the vector and axial currents is the same as for the case of the universal current. Below we shall see that the experimental results leave open the possibility for $\Theta_A = \Theta_V$. However, we shall derive our results using the general notation, and then impose the weakened or precise form of universality and calculate $\sin\Theta_{V,A}$ or $\sin\Theta$.

§4. Intensity rules for weak leptonic hadron decays. Theory and experiment

1. The $P \to \ell\nu$ decays

In the exact SU(3) symmetry limit the matrix elements $\langle 0|J^+_{H\lambda}(0)|\pi^-(p)\rangle$ and $\langle 0|J^+_{H\lambda}(0)|K^-(p)\rangle$ may be easily connected through the W-E theorem. The vacuum is supposed to be an SU(3) singlet. Then

$$<0|J^+_{H\lambda}(0)|\pi^-(p)> = c_A \left\langle \begin{matrix} 1 \\ 0,0,0 \end{matrix} \right| -\sqrt{2}\, A^8_{\pi^+,\lambda}(0)|P_{\pi^-}(p)\Big\rangle =$$

$$= \sqrt{2}\, c_A \begin{pmatrix} 8 & & & 8 & & & 1 & & \\ 0 & 1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \left(1||A^8_\lambda(0)||8p\right) =$$

$$= \frac{-1}{2}\, c_A \left(1||A^8(0)||8p\right) \qquad /162/$$

$$<0|J^+_{H\lambda}(0)|K^-(p)> = d_A \left\langle \begin{matrix} 1 \\ 0\;0\;0 \end{matrix} \right| -\sqrt{2}\, A^8_{K^+,\lambda}(0)\Big| P_{K^-}(p)\Big\rangle =$$

$$= -\sqrt{2}\, d_A \begin{pmatrix} 8 & & & 8 & & & 1 & & \\ -1 & \frac{1}{2} & -\frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix} \left(1||A^8_\lambda(0)||8p\right) =$$

$$= -\frac{r}{2}\, d_A \left(1||A^8(0)||8p\right) \qquad /163/$$

On the other hand, in chapter II we have given these matrix elements for the physical (not exactly SU(3) symmetric) case:

$$<0\; J^+_{H\lambda}(0)|\pi^-(p_\pi)> = \frac{1}{(2\pi)^{3/2}} \frac{i\, p_{\pi,\lambda}}{\sqrt{2E_\pi}}\, f\left(m_\pi^2\right) \;, \qquad /164/$$

$$<0|J^+_{H\lambda}(0)|K^-(p_k)> = \frac{1}{(2\pi)^{3/2}} \frac{i\, p_{k\lambda}}{\sqrt{2E_k}}\, \phi\left(m_k^2\right) \;, \qquad /165/$$

and the absolute values of $f\left(m_\pi^2\right)$ and $\phi\left(m_k^2\right)$ could be calculated from the experimental values of the $\pi^- \to \mu\nu$, $K^- \to \mu\nu$ and $\mu \to e\nu\bar{\nu}$ decay rates, because, as we know,

$$f\left(m_\pi^2\right) = \frac{F_\pi}{g} \;, \quad \phi\left(m_k^2\right) = \frac{F_k}{g} \;. \qquad /166/$$

The two expressions /162/ and /164/ for the $<0|J^+_{H\lambda}(0)|\pi^->$ matrix element can be equated to each other if we go in eq. /164/ to the SU(3) symmetry limit. Then, of course

$$\frac{p_{\pi,\lambda}}{\sqrt{2E_\pi}}\, f\left(m_\pi^2\right) \longrightarrow \frac{p_\lambda}{\sqrt{2E}}\, \tilde{f}\left(m^2\right) \;; \qquad /167/$$

similarly

$$\frac{p_{k,\lambda}}{\sqrt{2E_k}}\, \phi\left(m_k^2\right) \longrightarrow \frac{p_\lambda}{\sqrt{2E}}\, \tilde{\phi}\left(m^2\right) \;. \qquad /168/$$

m is the unknown, common SU(3) symmetric mass of the pseudoscalar octet, and we have taken into account that the functional form of f and ϕ may also change. Eq. /162/ - /165/, /167/ and /168/ then give

$$\frac{d_A}{c_A} = \frac{\tilde{\phi}(m^2)}{\tilde{f}(m^2)} \qquad /169/$$

If we suppose that when going to the SU(3) limit $f(m_\pi^2)$ and $\phi(m_k^2)$ or at least their ratio remain unchanged, we find

$$\frac{d_A}{c_A} = \frac{\tilde{\phi}(m^2)}{\tilde{f}(m^2)} = \frac{\phi(m_k^2)}{f(m_\pi^2)} = \frac{F_k}{F_\pi} \ . \qquad /170/$$

From the experimental value of $|F_k : F_\pi|$ we get then

$$\frac{d_A}{c_A} = \pm\left(0{,}27545 \pm 0{,}00038\right) \qquad /171/$$

And with the universality hypothesis

$$\sin_A = \pm\left(0{,}2655 \pm 0{,}0006\right) \qquad /171'/$$

Of course this and the following similar results are valid only in the approximation if the SU(3) breaking effects in the form factors can be neglected. In general we shall always be forced to adopt this hypothesis, because no reliable method for the calculation of the breaking effects is known. As a measure for the expected deviations caused by the SU(3) breaking the relative mass breaking in the baryon octet can be used; then $(10 \sim 30)$ % departures from the symmetry limit are possible.

2/ <u>The $P_1 \to P_2 \ell\nu$ decay</u>

The comparison of the $\pi^- \to \pi^0 e\nu$ and $K^+ \to \pi^0 e\nu$ decay rates gives the value of $|d_V : c_V|$ much in the same way as $\pi \to \ell\nu$ and $K \to \ell\nu$ gave $|d_A : c_A|$ above. In the exact SU(3)limit we have

$$<\pi^0(k_2)|J_{H\lambda}(0)|\pi^+(k_1)> = c_V<p_{\pi^0}(k_2)|\sqrt{2}\,V^8_{\pi^-}(0)|-P_{\pi^+}(k_1)> =$$

$$= -\sqrt{2}\,c_V\left[\begin{pmatrix} 8 & 8 & 8s \\ \pi^+ & \pi^- & \pi^0 \end{pmatrix}\left(8k_2||V^8_\lambda(0)||8k_1\right)_s + \right.$$

$$\left. + \begin{pmatrix} 8 & 8 & 8a \\ \pi^+ & \pi^- & \pi^0 \end{pmatrix}\left(8k_2||V^8_\lambda(0)||8k_1\right)_a\right] =$$

$$= -\sqrt{2}\,c_V\,\frac{1}{\sqrt{3}}\left(8k_2||V^8_\lambda(0)||8k_1\right)_a \qquad /172/$$

/The first clebsh is zero./ Similarly, we find

$$<\pi^0(k_2)|J_{H\lambda}(0)|K^+(k_1)> =$$

$$= -\frac{1}{\sqrt{2}}\,d_V\left[\sqrt{\frac{3}{5}}\left(8k_2||V^8_\lambda(0)||8k_1\right)_s + \frac{1}{\sqrt{3}}\left(8k_2||V^8_\lambda(0)|8k_1\right)_a\right] \qquad /173/$$

The most general form of the reduced matrix elements is (see for comparison eq. /124/):

$$\left(8k_2\|V^8_\lambda(0)\|8k_1\right)_{a,s} = \frac{(2\pi)^{-3}}{\sqrt{4E_1 E_2}}\left[f_+^{a,s}(q^2)(k_2+k_1)_\lambda + f_-^{a,s}(q^2)(k_2-k_1)_\lambda\right] \qquad /174/$$

From $\partial^\lambda V^8_{i,\lambda}(x) = 0$ we find

$$f_-^a(q^2) = f_-^s(q^2) = 0 \qquad \text{for} \quad q^2 < 0 \qquad /175/$$

and from the relation

$$\int d\underline{x} \, \langle\pi^-(k_2)|V_{3,o}(\underline{x},0)|\pi^-(k_1)\rangle = \delta(\underline{k}_2 - \underline{k}_1) =$$

$$= \int d\underline{x} \begin{pmatrix} 8 & 8 & 8a \\ \pi^+ & \pi^o & \pi^+ \end{pmatrix} \left(8k_2\|V^8_o(x,0)\|8k_1\right)_a = \frac{1}{\sqrt{3}} f_+^a(0)\, \delta(\underline{k}_2 - \underline{k}_1) \qquad /176/$$

and from a similar relation for $\int d\underline{x}\langle\pi^+|Y_o(\underline{x},0)|\pi^+\rangle = 0$ we get

$$\frac{1}{\sqrt{3}} f_+^a(0) = 1 \,, \quad f_+^s(0) = 0 \qquad /177/$$

Thus at $q^2 = 0$ we have in the SU(3) limit:

$$\langle\pi^o(k_2)|J_{H\lambda}(0)|\pi^+(k_1)\rangle = -\sqrt{2}\, c_V \frac{(2\pi)^{-3}}{\sqrt{4E_1 E_2}}(k_2 + k_1)_\lambda$$

$$\langle\pi^o(k_2)|J_{H\lambda}(0)|K^+(k_1)\rangle = -\frac{1}{\sqrt{2}} d_V \frac{(2\pi)^{-3}}{\sqrt{4E_1 E_2}}(k_2 + k_1)_\lambda \quad . \qquad /178/$$

On the other hand, for the physical case we have from Lorentz invariance

$$\langle\pi^o(k_{\pi^o})|J_{H\lambda}(0)|\pi^+(k_{\pi^+})\rangle = \frac{(2\pi)^{-3}}{\sqrt{4E_{\pi^o} E_{\pi^+}}}\left[f_+^\pi(q^2)(k_{\pi^o} + k_{\pi^+})_\lambda + f_-^\pi(q^2)(k_{\pi^o} - k_{\pi^+})_\lambda\right], \qquad /179/$$

$$\langle\pi^o(k_{\pi^o})|J_{H\lambda}(0)|K^+(k_k)\rangle = \frac{(2\pi)^{-3}}{\sqrt{4E_{\pi^o} E_k}}\frac{1}{2}\left[f_+^k(q^2)(k_{\pi^o} + k_k)_\lambda + f_-^k(q^2)(k_{\pi^o} - k_k)_\lambda\right] \qquad /180/$$

(The factor 1/2 introduced into eq. /180/ will be convenient.) In the physical case $f_-^\pi(q^2)$, $f_-^k(q^2)$ are not necessarily zero. However they are always multiplied by $(k_2 - k_1)_\lambda = (k_e + k_\nu)_\lambda$, which gives a factor m_e when multiplied by the lepton current. Thus, unless $f_-^\pi(q^2)$, $f_-^k(q^2)$ are very large, (and this is very unlikely since they are zero in the SU(3) limit), they can be surely neglected. Then the physical decay rates are determined by $f_+^\pi(q^2)$ and $f_+^k(q^2)$. The physical region of

q^2 for $f_+^\pi(q^2)$ is so small that the q^2 dependence may be neglected. For $f_+^k(q^2)$ the experimental analysis yields

$$f_+^k(q^2) = f_+^k(0)\left(1 + \lambda_+ \frac{q^2}{m_\pi^2}\right) , \quad \lambda_+ = \left(0{,}020 \pm 0{,}005\right) . \qquad /181/$$

Thus λ_+ is small and in good approximation we can take $f_+^k(q^2) = f_+^k(0)$. Then, if we suppose that when going to the SU(3) limit $f_+^\pi(0)$ and $f_+^k(0)$ (or at least their ratio) remain unchanged, we find from eq. /177/ - /180/ that

$$\frac{d_V}{c_V} = \frac{\tilde{f}_+^k(0)}{\tilde{f}_+(0)} = \frac{f_+^k(0)}{f_+^\pi(0)} . \qquad /182/$$

From the measured $K^+ \to \pi^o e\nu$ decay rate* we find

$$\frac{d_V}{c_V} = \pm\left(0{,}2364 \pm 0{,}0032\right) \qquad \sin\theta_V = \pm\left(0{,}230 \pm 0{,}003\right) \qquad /183/$$

If the q^2 dependence of $f_+^k(q^2)$ is taken into account according to eq. /181/, one finds

$$\frac{d_V}{c_V} = \pm\left(0{,}2516 \pm 0{,}0087\right) \qquad \sin\theta_V = \pm\left(0{,}224 \pm 0{,}008\right) \qquad /183'/$$

3/ The $B_1 \to B_2 \ell\nu$ decays

Let us remind the reader that a value for $c_V = g_V/g$ has already been derived from the experimental value of the coupling constant g_V measured in superallowed nuclear Fermi decays and from the muon life time, which gives g. Supposing that sign g_V = sign g we find from eq. /45/ and /74/ :

$$c_V = 0{,}9778 \pm 0{,}0018 \qquad \sin\theta_V = \pm\left(0{,}2095 \pm 0{,}0086\right) \qquad /184/$$

It is also possible - at least in principle - to derive the values of c_V, c_A, d_V, d_A from the leptonic baryon decays and from the muon decay which gives again g. Indeed, in these decays both $\Delta S = 0$ and $\Delta S = -1$ transitions occur, and in both of them vector and axial parts may be present. However, because of experimental and theoretical uncertainties, this program cannot be carried out completely at present. The point is that even in the

---

*The value of $f_+^\pi(0)$ is known from IVC better, than from direct $\pi^+ \to \pi^o e\nu$ experiments.

exact SU(3) limit we have 12 form factors, and many auxiliary hypotheses have to be introduced to reduce the number of the unknown parameters. Below we shall outline how this analysis of the $B_1 \to B_2\, \ell\nu$ data may be done.

The hadronic matrix elements of the $B_1 \to B_2\, \ell\nu$ decays contain in most cases the adjoint of the operator $J_{H\lambda}$. In the exact SU(3) limit we find

$$<B_{(\nu_2)}(p_2)|J^+_{H\lambda}(0)|B_{(\nu_1)}(p_1)>$$

$$= -\sqrt{2}\left\langle B_{(\nu_2)}(p_2)\left|c_V V^8_{\pi^+,\lambda}(0) + d_V V^8_{K^+,\lambda}(0) + c_A A^8_{\pi^+,\lambda}(0) + d_A A^8_{K^+,\lambda}(0)\right|B_{(\nu_1)}(p_1)\right\rangle =$$

$$= -\sqrt{2}\sum_{\gamma=a,s}\left\{\left[c_V\begin{pmatrix}8 & 8 & 8\gamma\\ \nu_1 & \pi^+ & \nu_2\end{pmatrix} + d_V\begin{pmatrix}8 & 8 & 8\gamma\\ \nu_1 & K^+ & \nu_2\end{pmatrix}\right]\Big(B(p_2)||V^8_\lambda(0)||B(p_1)\Big)_\gamma + \right.$$

$$\left. + \left[c_A\begin{pmatrix}8 & 8 & 8\gamma\\ \nu_1 & \pi^+ & \nu_2\end{pmatrix} + d_A\begin{pmatrix}8 & 8 & 8\gamma\\ \nu_1 & K^+ & \nu_2\end{pmatrix}\right]\Big(B(p_2)||A^8_\lambda(0)||B(p_1)\Big)_\gamma\right\} \qquad /185/$$

A similar expression holds for the matrix element $<B_{(\nu_2)}(p_2)|J_{H\lambda}(0)|B_{(\nu_1)}(p_1)>$ with $-\sqrt{2} \to \sqrt{2}$, $\pi^+ \to \pi^-$ and $K^+ \to K^-$. (The constants c_V, c_A, d_V, d_A are real if T violation is neglected.)

The reduced matrix elements in eq. /185/ may be explicited in the usual way:

$$\Big(B(p_2)||V^8_\lambda(0)||B(p_1)\Big)_{a,s} = \frac{\bar{B}(p_2)}{(2\pi)^{3/2}}\left[\gamma_\lambda F_1^{f,d}(q^2) - i\sigma_{\lambda\nu}\frac{(p_2-p_1)^\nu}{2M}F_2^{f,d}(q^2) + \frac{(p_2-p_1)_\lambda}{2M}F_3^{f,d}(q^2)\right]\frac{B(p_1)}{(2\pi)^{3/2}} \qquad /186/$$

$$\Big(B(p_2)||A^8_\lambda(0)||B(p_1)\Big)_{a,s} = -\frac{\bar{B}(p_2)}{(2\pi)^{3/2}}\left[\gamma_\lambda H_1^{f,d}(q^2) - i\sigma_{\lambda\nu}\frac{(p_2-p_1)^\nu}{2M}H_2^{f,d}(q^2) + \frac{(p_2-p_1)_\lambda}{2M}H_3^{f,d}(q^2)\right]i\gamma_5\frac{B(p_1)}{(2\pi)^{3/2}}$$

In these equations the „f" form factors correspond to the $\gamma = a$ case, the „d" form factors to the $\gamma = s$ case. These labels refer to the antisymmetric $f_{ik\ell}$ and to the symmetric $d_{ik\ell}$ couplings. M stands for the unknown mass of the baryon octet in the exact SU(3) limit. Notice that in that limit the equality

$$<B_{(\nu_2)}(p_2)|J^+_{H\lambda}(0)|B_{(\nu_1)}(p_1)> = <B_{(\nu_1)}(p_1)|J_{H\lambda}(0)|B_{(\nu_2)}(p_2)>^* \qquad /188/$$

applied e.g. to the $|B_{(\nu_1)}(p_1)> = -|p(p_1)>$, $|B_{(\nu_2)}(p_2)> = -|n(p_2)>$ case leads to the reality conditions

$$\Big(B(p_2)||V^8_\lambda(0)||B(p_1)\Big)_{a,s} = \Big(B(p_2)||A^8_\lambda(0)||B(p_1)\Big)^*_{a,s} \qquad /189/$$

$$\left(B(p_2)\,||A^8_\lambda(0)||\,B(p_1)\right)_{a,s} = \left(B(p_2)\,||A^8_\lambda(0)||\,B(p_1)\right)^*_{a,s} \qquad /190/$$

or, expressed in terms of the form factors, to

$$F^{f,d}_{1,2}(q^2) = F^{f,d}_{1,2}(q^2)^*, \qquad F^{f,d}_3(q^2) = -F^{f,d}_3(q^2)^*, \qquad /191/$$

$$H^{f,d}_{1,3}(q^2) = H^{f,d}_{1,3}(q^2)^*, \qquad H^{f,d}_2(q^2) = -H^{f,d}_2(q^2)^*. \qquad /192/$$

All the hadronic matrix elements of the possible $B_1 \to B_2\,\ell\nu$ decays can now be easily written down in the exact SU(3) limit. For the neutron decay we find (see table 7 and eq. /185/)

$$\langle p(p_2)|J^+_{H\lambda}(0)|n(p_1)\rangle = \langle -B_p(p_2)|J^+_{H\lambda}(0)|-B_n(p_1)\rangle =$$

$$= -\sqrt{2}\sum_{\gamma=a,s}\left[c_V\begin{pmatrix}8 & 8 & 8\gamma\\ n & \pi^+ & p\end{pmatrix}\left(B(p_2)||V^8_\lambda(0)||B(p_1)\right)_\gamma + c_V\begin{pmatrix}8 & 8 & 8\\ n & \pi^+ & p\end{pmatrix}\left(B(p_2)||A^8_\lambda(0)||B(p_1)\right)_\gamma\right] \qquad /193/$$

From the Particle Data Tables

$$\begin{pmatrix}8 & 8 & 8\\ n & \pi^+ & p\end{pmatrix} = \begin{pmatrix}8 & 8 & 8\gamma\\ 1\,\frac{1}{2}\,-\frac{1}{2} & 0\,1\,1 & 1\,\frac{1}{2}\,\frac{1}{2}\end{pmatrix} = \left(\frac{1}{2}\,-\frac{1}{2};\;1\;1\middle|\frac{1}{2}\,\frac{1}{2}\right)\left(K\Sigma|N\right)_\gamma =$$

$$= \begin{cases} -\frac{\sqrt{3}}{2}\,\frac{1}{2} = -\frac{1}{\sqrt{6}} & \text{for} \quad \gamma = a \\ -\sqrt{\frac{2}{3}}\,\frac{3\sqrt{5}}{10} = -\sqrt{\frac{3}{10}} & \text{for} \quad \gamma = s \end{cases} \qquad /194/$$

Introducing the shorthand notations

$$\left(B(p_2)||V^8_\lambda(0)||B(p_1)\right)_{a,s} \equiv V^{a,s}, \qquad \left(B(p_2)||A^8_\lambda(0)||B(p_1)\right) \equiv A^{a,s}, \qquad /195/$$

$$\langle p(p_2)|J^+_{H\lambda}(0)|n(p_1)\rangle \equiv \langle p|J^+|n\rangle, \qquad /196/$$

we obtain for the $n \to p$ matrix element, and by a similar calculation for the other matrix elements, the following expressions:

$$<p|J^+|n> = c_V\left(\frac{1}{\sqrt{3}}V^a + \sqrt{\frac{3}{5}}V^s\right) + c_A\left(\frac{1}{\sqrt{3}}A^a + \sqrt{\frac{3}{5}}A^s\right)$$

$$<\Lambda|J^+|\Sigma^-> = c_V\left(0 + \sqrt{\frac{2}{5}}V^s\right) + \;.\;.\;.\;.\;.\;.\;.\;V \to A\;.\;.\;.$$

$$<\Lambda|J|\Sigma^+> = c_V\left(0 - \sqrt{\frac{2}{5}}V^s\right) + \;.\;.\;.\;.\;.\;.\;.\;.\;V \to A\;.\;.\;.$$

$$<\Sigma^o|J^+|\Sigma^-> = c_V\left(\sqrt{\frac{2}{3}}V^a + 0\right) + \;.\;.\;.\;.\;.\;.\;.\;V \to A\;.\;.\;.$$

$$<\Xi^o|J^+|\Xi^-> = c_V\left(\frac{1}{\sqrt{3}}V^a + \frac{3}{5}V^s\right) + \;.\;.\;.\;.\;V \to A\;.\;.\;.$$

$$<p|J^+|\Lambda> = d_V\left(-\frac{1}{\sqrt{2}}V^a - \frac{1}{\sqrt{10}}V^s\right) + \;.\;.\;.\;.\;V \to A\;.\;.\;.$$

$$<n|J^+|\Sigma^-> = d_V\left(-\frac{1}{\sqrt{3}}V^a + \sqrt{\frac{3}{5}}V^s\right) + \;.\;.\;.\;.\;V \to A\;.\;.\;.$$

$$<\Lambda|J^+|\Xi^-> = d_V\left(\frac{1}{\sqrt{2}}V^a - \frac{1}{\sqrt{10}}V^s\right) + \;.\;.\;.\;.\;.\;.\;V \to A\;.\;.\;.$$

$$<\Sigma^-|J^+|\Xi^-> = d_V\left(\frac{1}{\sqrt{6}}V^a + \sqrt{\frac{3}{10}}V^s\right) + \;.\;.\;.\;.\;V \to A\;.\;.$$

$$<\Sigma^+|J^+|\Xi^o> = d_V\left(-\frac{1}{\sqrt{3}}V^a - \sqrt{\frac{3}{5}}V^s\right) + \;.\;.\;.\;.\;V \to A\;.\;.\;.$$

/197/

We also find, of course, that the selection rules $\Delta I \leq 1$, $\Delta S = \Delta Q$, $|\Delta S| \leq 1$ are respected

$$<n|J|\Sigma^+> = <\Sigma^-|J|\Xi^o> = <n|J^+|\Xi^-> = <p|J^+|\Xi^o> = 0 \quad . \qquad /198/$$

We shall now shortly describe how the formulae /197/ can be compared with the experimental results. First of all, we try to go back from the SU(3) symmetric expressions /197/ to the physical ones. To this end in the reduced matrix elements /186/ and /187/ we put $M \to M_1$ in $B(p_1)$, $M \to M_2$ in $\bar{B}(p_2)$ and $2M \to M_2 + M_1$ in the $(p_2 - p_1)\,/\,2M$ factors. Also we use the appropriate physical masses in the relations $p^o = \sqrt{p^2 + M^2}$. The result of these symmetry breaking "corrections" is the same as if in the physical matrix elements of the $B_1 \to B_2 \ell\nu$ decays we were to insert the appropriate SU(3) form factors in the place of the physical form factors. To give an example, it is easy to see that the substitutions

$$F_i(q^2) \to \frac{1}{\sqrt{3}} F_i^f(q^2) + \sqrt{\frac{3}{5}}\, F_i^d(q^2)$$

$$H_i(q^2) \to \frac{1}{\sqrt{3}} H_i^f(q^2) + \sqrt{\frac{3}{5}}\, H_i^d(q^2)$$

lead from eq. /51/ to the first "corrected" eq. /197/ . In the functional form of the form factors $F_i^{f,d}$, $H_i^{f,d}$ we do not know how to break the SU(3) symmetry, and we are forced to neglect this breaking. We shall also neglect the q^2 dependence of these form factors and we shall work with their values at $q^2 = 0$. Then we have only 12 numbers to determine instead of 12 functions. However even these are too many for the available experimental data. Thus we shall use all the possible external information on the form factors.
The vector form factors can be easily connected with the isovector and isoscalar form factors. Indeed, if we calculate the matrix elements $\langle p(p_2)|V_{3,\lambda}(0)|p(p_1)\rangle$ and $\langle p(p_2)|\frac{1}{2} Y_\lambda(0)|p(p_1)\rangle$ in the SU(3) formalism and then compare the result with eq. /99/ and /100/ taken in the SU(3) limit (i.e. with $M_p \to M$) we find

$$\frac{1}{\sqrt{3}} F_i^f(q^2) = \frac{1}{2}\left[F_i^v(q^2) + 3F_i^s(q^2)\right] \quad ,$$

$$\sqrt{\frac{3}{5}}\, F_i^d(q^2) = \frac{3}{2}\left[F_i^v(q^2) - F_i^s(q^2)\right] \; . \qquad /199/$$

From the known properties of the $F_i^{v,s}$ form factors we immediately find

$$\frac{1}{\sqrt{3}} F_1^f(0) = 1 \qquad\qquad F_1^d(0) = 0$$

$$\frac{1}{\sqrt{3}} F_2^f(0) = \mu_p + \frac{1}{2}\mu_n \qquad\qquad \sqrt{\frac{3}{5}}\, F_2^d(0) = -\frac{3}{2}\mu_n$$

$$F_3^f(q^2) = 0 \qquad\qquad F_3^d(q^2) = 0$$

(As far as SU(3) breaking effects are neglected, the values $\mu_p = 1{,}793$ and $\mu_n = -1{,}913$ are unchanged.)

Thus in this approximation the vector current form factors are known. Let us turn to the axial form factors. The contribution of H_2 and H_3 may be neglected. H_2 is supposed to be small because it is pure imaginary in the SU(3) limit (see eq. /192/) and real if time invariance holds. Notice that the same argument is applicable to F_3, but F_3 is zero, also due to current conservation. F_3 and H_2 are also known as "second class current" form factors, in contradistinction to F_1, F_2, H_1 and H_3 which belong to "first class currents" [1], p. 408. Strong interactions cannot induce second class current if originally only first class currents, e.g. $\bar{\psi}_2\gamma_\lambda\psi_1$ and $-\bar{\psi}_2 i\gamma_\lambda\gamma_5\psi_1$ were present in the weak hadron current. H_2 is often neglected on this basis. Concerning H_3, we know that it is multiplied by the factor $i\gamma_5(p_2-p_1)_\lambda/(M_1+M_2)$. When contracted with the lepton current, $(p_2-p_1)_\lambda$ gives m_ℓ ($\ell = \mu$ or e), while $i\gamma_5$ gives a factor $0 \leq |p_2|/2M_2 < M_2-M_1/2M_2$. From PCAC $H_3(0) \approx 200 H_1(0)$ (see in the next chapter), and the factor which multiplies it is smaller then 5.10^{-3} for muonic decays and smaller then 5.10^{-5} for electron decays. Thus H_3 may also be neglected. The only free parameters coming from the 12 form factors are then $H_1^f(0)$ and $H_1^d(0)$. Together with the four Cabibbo constants c_V, c_A, d_V and d_A we have in this approximation six real parameters. For convenience we shall write out the relevant formulae for those $B_1 \to B_2\ \ell\nu$ decays which are measured. Introducing the notations

$$\frac{1}{\sqrt{3}} H_1^f(0) \equiv F \qquad \sqrt{\frac{3}{5}}\, H_1^d(0) \equiv D \quad , \qquad /201/$$

we find from eq. /186/, /187/, /195/ and /197/ the coefficients O_1, O_2, O_3, given in Table 8. They correspond to the contributions of the vector, weak magnetism, and axial vector form factors, respectively.

Table 8.

$B_1 \to B_2$ matrix elements in the theory of Cabibbo at $q^2 = 0$.

Decay	O_1 (vector)	O_2 (weak magnetism)	O_3 (axial)
$n \to p$	c_V	$c_V(\mu_p - \mu_n)$	$c_A(F+D)$
$\Sigma^\pm \to \Lambda$	0	$-\sqrt{\frac{3}{2}}\, c_V\, \mu_n$	$\sqrt{\frac{2}{3}}\, c_A\, D$
$\Lambda \to p$	$-\sqrt{\frac{3}{2}}\, d_V$	$-\sqrt{\frac{3}{2}}\, d_V\, \mu_p$	$-\sqrt{\frac{3}{2}}\, d_A\left(F+\frac{1}{3}D\right)$
$\Sigma^- \to n$	$-d_V$	$-d_V(\mu_p+2\mu_n)$	$-d_A(F-D)$
$\Xi^- \to \Lambda$	$\sqrt{\frac{3}{2}}\, d_V$	$\sqrt{\frac{3}{2}}\, d_V(\mu_p+\mu_n)$	$\sqrt{\frac{3}{2}}\, d_A\left(F-\frac{1}{3}D\right)$
$\Xi^- \to \Sigma^0$	$\frac{1}{\sqrt{2}}\, d_V$	$\frac{1}{\sqrt{2}}\, d_V(\mu_p-\mu_n)$	$\sqrt{\frac{1}{2}}\, d_A(F-D)$

The expression for a $B_1 \rightarrow B_2$ matrix element is now easily written down. For instance, for the $n \rightarrow p$ decay we have

$$\langle p(p_2)|J^{+}_{H\lambda}(0)|n(p_1)\rangle =$$

$$= \frac{\bar{p}(p_2)}{(2\pi)^{3/2}}\left[\gamma_\lambda\, O_1 - i\sigma_{\lambda\nu}\,\frac{(p_2-p_1)_\nu}{M_p + M_n}\, O_2 - i\gamma_\lambda\,\gamma_5\, O_3\right]\frac{n(p_1)}{(2\pi)^{3/2}} \qquad /202/$$

with $O_1 = C_V$, $O_2 = C_V\ (\mu_p - \mu_n)$, and $O_3 = C_A(F+D)$ and so on.

"A New Fit of the Parameters for Cabibbo's Theory" has been recently published by F. Eisele et al. in Z. Physik, 225, 383 1969. The universality hypothesis has been used both in its precise ($C_V = C_A = \cos\Theta$, $d_V = d_A = \sin\Theta$; "one-angle fit") and weakened ($c_V = \cos\Theta_V$, $c_A = \cos\Theta_A$, $d_V = \sin\Theta_V$, $d_A = \sin\Theta_A$; two-angle fit) forms. The authors go beyond the approximation /202/ in two respects. 1/ They do not neglect the H_3 form factor and calculate it from H_1 with the help of PCAC. 2/ They take into account the q^2 dependence of the form factors in the linear approximation, and calculate the slopes in the following way: for $F_1(q^2)$ and $F_2(q^2)$ the phenomenological formulae /108, v'/ are used. For $H_1(q^2)$ the slope is taken to be equal to the half of the slope of $F_1(q^2)$,because the q^2 dependence is supposed to be dominated by the vector meson octet for F_1 and by the axial meson octet for H_1, and the ratio of the masses squared is $M_V^2 : M_A^2 \approx 1 : 2$. The slope of H_3 is then calculated from that of H_1 using PCAC. Thus no new free parameters are introduced into the theory, all the slopes are "known". We notice that the authors apply the SU(3) formalism only at the point $q^2 = 0$, the slopes are not splitted into "f" and "d" parts.

The experimental input data of the "New Fit" are given in Table 9. The fitted values of the free parameters in the case of the one-angle fit (confidence level = 51 %) are

$$\Theta = 0{,}239 \pm 0{,}006 \qquad \sin\Theta = 0{,}2367 \pm 0{,}0058$$

$$F = 0{,}488 \pm 0{,}020 \qquad D = 0{,}743 \pm 0{,}020 \qquad /203/$$

while for the two-angle fit (confidence level = 45 %) they are

$$\Theta_V = 0{,}232 \pm 0{,}013 \qquad \sin\Theta_V = 0{,}2299 \pm 0{,}0126$$

$$\Theta_A = 0{,}250 \pm 0{,}018 \qquad \sin\Theta_A = 0{,}2474 \pm 0{,}0174$$

$$F = 0{,}478 \pm 0{,}023 \qquad D = 0{,}757 \pm 0{,}028 \qquad /204/$$

The predicted values of the input data for the one-angle fit are also given in Table 9. The agreement with the experimental values is quite good.

Table 9.

<u>Experimental data and prediction of the Cabibbo theory for leptonic baryon decays</u>

Decay	Branching ratios		$O_3 : O_1$	
	Experiment	Theory	Experiment	Theory
$n \rightarrow pe\bar{\nu}$	100 %	100 %	$1,23 \pm 0,01$	1,23
$\Sigma^- \rightarrow \Lambda e\bar{\nu}$	$6,04 \pm 0,60\ .10^{-5}$	$6,4.10^{-5}$	$O_1 : O_3 = -0,29 \pm 0,20$	$O_1 : O_3 = 0$
$\Sigma^+ \rightarrow \Lambda e\nu$	$2,11 \pm 0,45\ .10^{-5}$	$1,9.10^{-5}$	-	$O_1 : O_3 = 0$
$\Lambda \rightarrow pe\bar{\nu}$	$8,50 \pm 0,81\ .10^{-4}$	$8,8.10^{-4}$	$0,97\ {}^{+\,0,22}_{-\,0,14}$	0,73
$\Lambda \rightarrow p\mu\bar{\nu}$	$1,35 \pm 0,60\ .10^{-4}$	$1,45.10^{-4}$	-	0,73
$\Sigma^- \rightarrow ne\bar{\nu}$	$1,100 \pm 0,048\ .10^{-4}$	$1,06.10^{-3}$	$\lvert O_3 : O_1 \rvert = 0,28\ {}^{+\,0,14}_{-\,0,13}$	-0,26
$\Sigma^- \rightarrow n\mu\bar{\nu}$	$4,51 \pm 0,45\ .10^{-4}$	$5,0.10^{-4}$	-	-0,26
$\Xi^- \rightarrow \Lambda e\bar{\nu}$	$9,0\ {}^{+\,7,1}_{-\,4,3}\ .10^{-4}$	$5,5.10^{-4}$	-	0,24
$\Xi^- \rightarrow \Sigma^o e\bar{\nu}$	$6,2\ {}^{+\,2,0}_{-\,3,0}\ .10^{-4}$	$0,8.10^{-4}$	-	1,23

Fitted parameters: $\theta = \Theta_A = \Theta_V = 0,239 \pm 0,006$

$F = 0,488 \pm 0,020$

$D = 0,743 \pm 0,020$

Only the $\Xi^- \to \Sigma^0 \ell \nu$ branching ratio has a discrepancy exeeding 3 standard deviations. The different measured energy and angular distributions compare also very well with the theoretical distributions, calculated with the fitted parameters.

Nevertheless, one should keep an open mind concerning the good results of this "new fit". The ambiguity of the present situation is well illustrated by the values of the Cabibbo angle, calculated from various sources /Table 10/.

Table 10.

The values of the Cabibbo angle calculated from the decays of mesons (M), nuclei (N) and baryons (B)

Transition involved	Fitted quantities	Cabibbo angle
$K \to \mu\nu$ $\pi \to \mu\nu$	d_A / c_A	$\sin\Theta^M_A = 0{,}2655 \pm 0{,}0006$
${}^{*}K \to \pi\ell\nu$ $\pi \to \pi e\nu$	d_V / c_V	$\sin\Theta^M_V = 0{,}224 \pm 0{,}008$
Superallowed nuclear Fermi transitions	c_V	$\sin\Theta^N_V = 0{,}2095 \pm 0{,}0086$
Leptonic decays of the baryon octet	F, D, Θ_V, Θ_A	$\sin\Theta^B_V = 0{,}2299 \pm 0{,}0126$ $\sin\Theta^B_A = 0{,}2474 \pm 0{,}0174$
	$F, D, \Theta = \Theta_V = \Theta_A$	$\sin\Theta^B = 0{,}2367 \pm 0{,}0058$

* $\Gamma\, \pi \to \pi e\nu$ calculated from IVC. The value of $\sin\Theta^M_V$ corresponds to $\lambda_+ = 0{,}02$ (see the text).

These values are compatible with each other if we take into account the imperfection of our theoretical methods, namely the lack of a consistent theory of the breaking of the SU(3) symmetry. Since the SU(3) breaking effects may give corrections of the order of 20%, it is impossible to see whether the SU(3) current theory of Cabibbo and Gell-Mann is itself only approximately valid, or whether it is basically correct and the 20% discrepancies betwen the values of the Cabibbo angle are due only to the uncalculable SU(3) breaking. If the mechanism of this breaking will be found, then the Cabibbo--Gell-Mann theory may be more precisely tested and if needed modified. We can go then from the 20% level of precision to the 2% level, i.e. to the level of the SU(2) breaking effects and of the radiative corrections, which are also uncalculable at present. Thus our conclusion is that, although the basic ideas of the SU(3) current theory and of the universality of the weak current may be correct, it is illusory to test them with a precision which exceeds the level of the precision of the available symmetry breaking mechanism. Further progress in the SU(3) theory of the weak interaction hinges on the progress in the theory of the symmetry breaking.

In obtaining the various values of the Cabibbo angles given in table 10 we neglected the SU(3) breaking in the form factors. It is also possible to introduce a unique Cabibbo angle θ and to attribute the source of the discrepancies in table 10 to the form factors. Then θ is usually called the bare Cabibbo angle, while the angles in table 10 are called the renormalized Cabibbo angles. See e.g. [8], chapter 5.

§5. The universality hypothesis

As we mentioned already in chapter III, §1, the universality hypothesis in the framework of the IVC theory consisted in the requirement that the $V_\lambda^{s=o} = V_{1,\lambda} - i\, V_{2,\lambda}$ current take part in the weak interaction with the same strength as the lepton current, in full analogy with the electromagnetic interaction, where the $V_{3,\lambda}$ current has also the same weight as the electromagnetic lepton current. In our notation this universality hypothesis simply means that

$$c_V = 1 \qquad /205/$$

As we know, the experimental results show that c_V is smaller than 1 by a few percents. This deviation is, however, of the same order of magnitude as the radiative corrections and the SU(2) symmetry breaking effects; therefore from this point of view the universality hypothesis /205/ can be considered as consistent with the experimental data. A serious difficulty arises only when strangeness changing decays are investigated. Indeed, the only natural extension of /205/ would be to suppose that the strangeness-changing currents are also coupled with the same strength, i.e. that

$$d_V = 1 \qquad /206/$$

Of course 10 % deviations from this value would be tolerated, due to renormalization effects. However as we know the experimental results indicate that d_V is much smaller than 1.

A reformulation of the universality hypothesis became possible and was offered by Gell-Mann when the octet current and the existance of two kinds of neutrinos was discovered. The basic observation was that the weak lepton current with two neutrinos may be written in the following way: $(\bar{e} \equiv \bar{\psi}_e(x)$,etc.)

$$j_\lambda(x) = \bar{e}\gamma_\lambda\left(1 - i\gamma_5\right)\nu_e + \bar{\mu}\gamma_\lambda\left(1 - i\gamma_5\right)\nu_\mu =$$

$$= 2\left[\left(\bar{\nu}_e, \bar{e}\right)\gamma_\lambda \frac{1-i\gamma_5}{2}\frac{1}{2}\left(\tau_1 - i\tau_2\right)\begin{pmatrix}\nu_e\\ e\end{pmatrix} + \left(\bar{\nu}_\mu, \bar{\mu}\right)\gamma_\lambda \frac{1-i\gamma_5}{2}\frac{1}{2}\left(\tau_1 - i\tau_2\right)\begin{pmatrix}\nu_\mu\\ \mu\end{pmatrix}\right] \equiv$$

$$\equiv 2\left[c^\ell_{1,\lambda}(x) - i\, c^\ell_{2,\lambda}(x)\right] . \qquad /207/$$

It is easy to verify that the leptonic charges

$$c^\ell_1(t) \equiv \int d\underline{x}\; c^\ell_1(\underline{x},t) , \qquad c^\ell_2(t) \equiv \int d\underline{x}\; c^\ell_2(\underline{x},t) ,$$

$$c^\ell_3(t) \equiv -i\left[c^\ell_1(t), c^\ell_2(t)\right] \qquad /208/$$

define an SU(2) group, and that the currents

$$c^\ell_{1,\lambda}(x), \quad c^\ell_{2,\lambda}(x) \quad \text{and} \quad c^\ell_{3,\lambda}(\underline{x},t) \equiv -i\left[c^\ell_1(t), c^\ell_{2,\lambda}(\underline{x},t)\right]$$

are the isocurrent densities for these charges.

The new universality requirement is that the weak hadron current be also of the form

$$2\left[c^H_{1,\lambda}(x) - i\, c^H_{2,\lambda}(x)\right] \qquad /209/$$

where $c^H_{1,\lambda}(x)$, $c^H_{2,\lambda}(x)$ should generate an SU(2) group of hadron charges $c^H_i(t)$ /i = 1,2,3/ in the same way as the lepton charges were generated by $c^\ell_{1,\lambda}$ and $c^\ell_{2,\lambda}$. It is easy to show that our octet current

$$J_{H,\lambda} = c_V\left(V_{1,\lambda} - iV_{2,\lambda}\right) + c_A\left(A_{1,\lambda} - iA_{2,\lambda}\right) + d_V\left(V_{4,\lambda} - iV_{5,\lambda}\right) + d_A\left(A_{4,\lambda} - iA_{5,\lambda}\right) \qquad /210/$$

will meet these conditions if and only if

$$c_V = c_A , \; d_V = d_A , \; c_V^2 = d_V^2 = 1 = c_A^2 + d_A^2 \qquad /211/$$

This universality hypothesis is of course compatible with the experimental result

$$c_V = 0{,}9778 \pm 0{,}0018 \qquad /212/$$

which is now considered not as a troublesome deviation from an exact theoretical value $C_V = 1$; on the contrary, it is welcomed that C_V is smaller than but near to 1, because this gives then for

$$d_V = \sqrt{1 - c_V^2} \qquad /213/$$

the plausible value $0 < d_V \ll 1$, and both the existence and the damping of the $\Delta S \neq 0$ decays is thereby accounted for.

It is important to point out that if only one kind of neutrino existed, the new universality hypothesis would indeed not work; then we would have

$$j_\lambda(x) = \bar{e}\gamma_\lambda(1 - i\gamma_5)\nu + \bar{\mu}\gamma_\lambda(1 - i\gamma_5)\nu =$$

$$= 2\sqrt{2}\left(\bar{\nu}, \frac{\bar{e} + \bar{\mu}}{\sqrt{2}}\right)\gamma_\lambda \frac{1 - i\gamma_5}{2}\,\frac{1}{2}\left(\tau_1 - i\tau_2\right)\begin{pmatrix}\nu \\ \frac{e + \mu}{\sqrt{2}}\end{pmatrix} \equiv$$

$$\equiv 2\sqrt{2}\left(\tilde{c}^{\ell}_{1,\lambda}(x) - i\,\tilde{c}^{\ell}_{2,\lambda}(x)\right) \qquad /214/$$

where now the $\tilde{c}^{\ell}_{i,\lambda}$ would play the role of the $c^{\ell}_{i,\lambda}$. Then the universality hypothesis would require that the octet current /210/ enter the weak interaction with the overall weight factor $\sqrt{2}$, because only in this case would the strength of the lepton and the hadron current be the same and only then would the full weak current have definite SU(2) properties. Due to this extra $\sqrt{2}$ the experimental data would require $c_V \approx \frac{1}{\sqrt{2}}$ instead of $c_V \approx 1$, and then $d_V = \sqrt{1-c_V^2} \approx \frac{1}{\sqrt{2}}$. This would mean that the strangeness-changing decays ahve the same strength as the strangeness-conserving ones, in contradiction with the experimental data.

Thus we see that while the old form of the universality hypothesis /205/ must be abandoned, the new form /211/ is in agreement both with the fact that $\nu_\ell \neq \nu_\mu$, and with the experimental decay rates in the leptonic hadron decays.

V. OPEN PROBLEMS

§1. Basic problems

One of the most serious problems of principle in the weak interaction is the lack of reliable methods for the calculation of higher-order weak corrections to the lowest order matrix elements (see chapter I). Several attempts have been made to invent such a method, but no satisfactory solution has been reached as yet.

Another important problem is the lack of satisfactory methods for the calculation of the internal symmetry breaking effects. As we have seen the modern current-current theory of the weak interaction is based on an $SU(3) \otimes SU(3)$ algebra which is surely broken, because the physical hadron states do not belong to exact multiplets of this algebra. The departures from the symmetry limit cannot be calculated, and only ad hoc and arbitrary procedures exist which "take into account" the symmetry breaking (see e.g. chapter IV, §4). The solution of this problem would be of the greatest value not only for the theory of the weak interaction, but also for the theory of the strong and electromagnetic interactions, where broken $SU(3)$ and $SU(2)$ symmetries play an important role.

A interesting problem of the octet current-current theory of the weak interaction is the origin of the Cabibbo angle. We have seen that the universality hypothesis of Gell-Mann naturally leads to the introduction of this angle, but the value of the angle is not predicted by this hypothesis. There are several interesting attempts to calculate the Cabibbo angle on the basis of various theoretical considerations. The main difficulty on the way to the solution again comes from the fact that ultraviolet divergencies and internal symmetry breaking effects cannot be systematically managed. It is probable that a satisfactory explanation of the origin and value of the Cabibbo angle hinges upon the solution of these basic problems.

The discussion of these questions lies outside the scope of our notes. The interested reader should consult the original papers in the recent literature. The most important contributions are listed in [5] and [6].

§2. The non leptonic weak decays. The PCAC hypothesis

Let us now come to the problem of the non-leptonic weak decays of the hadrons. An excellent review of the status in this field has been given in [2] in 1967, and the situation has not changed too much since then. Nevertheless, we shall present here a brief discussion of the subject.

In §2 of chapter IV we have seen that the current-current theory of Cabibbo allows both $\Delta I = 1/2$ and $\Delta I = 3/2$ transitions for the non leptonic decays. On the other hand, the experimental results in hyperon and kaon decays indicate that the $\Delta I = 3/2$ channel is strongly damped, the $\Delta I = 3/2$ amplitude being $\lesssim 5\%$ of the $\Delta I = 1/2$ amplitude. In Table 11 we give the simplest predictions for pure $\Delta I = 1/2$ transitions and the corresponding experimental results. The calculation is elementary and involves only the W-E theorem for the SU(2) group. A further triangle relation may be obtained between the three $\Sigma^{\pm} \to N\pi$ decay amplitudes and, with SU(3), an other triangle relation between the $\Xi \to \Lambda\pi$, $\Sigma \to N\pi$ and $\Lambda \to N\pi$ amplitudes. These triangle rules are also supported by the experimental results.

In table 11 the symbols Λ^o_o , Λ^o_- , Ξ^o_o and Ξ^-_- refer to the $\Lambda \to n\pi^o$ $\Lambda \to p\pi^-$, $\Xi^o \to n\pi^o$ and $\Xi^- \to \Lambda\pi^-$ decays, respectively. The asymmetry parameters

$$\alpha = \frac{2\mathrm{Re}\, s^* p}{|s|^2 + |p|^2} \qquad \qquad = \frac{|s|^2 - |p|^2}{|s|^2 + |p|^2} \qquad /213/$$

contain the s and p wave amplitudes, which enter the decay amplitude F for the $B_1 \to B_2 \pi$ decay at rest in the following way:

$$F = \chi_2^+ \left[s + p \frac{\sigma\, q_2}{|q_2|} \right] \chi_1 \; . \qquad /214/$$

In /214/ χ_1 and χ_2 stand for the Pauli spinors of the baryons B_1 and B_2 , while q_2 is the momentum of B_2 in the rest frame of B_1 . The asymmetry parameters α and γ can be measured in experiments with polarized baryons.

Since the current-current theory of Cabibbo does not forbid the $\Delta I = 3/2$ transitions by a selection rule, only a detailed calculation of the matrix elements of the non leptonic decays could answer the question whether the current-current theory is applicable or not to these decays. However, the explicit form of the weak hadron current is unknown, and the modifications caused by the strong interaction, i.e. the q^2 dependence of the weak form factor, is also poorly known. In spite of these difficulties remarkable results are obtained for the non-leptonic decays with the help of the current algebra relations given in chapter IV §8 and of the partially conserved

Table 11.

The $\Delta I = 1/2$ rule in $\Lambda \to N\pi$ and $\Xi \to \Lambda\pi$ decays

Quantity measured	Experiment	Theory
$\Gamma(\Lambda^o_-) : [\Gamma(\Lambda^o_-) + \Gamma(\Lambda^o_o)]$	0,640 ± 0,014	2/3
$\alpha(\Lambda^o_o) : \alpha(\Lambda^o_-)$	1,10 ± 0,27	1
$\gamma(\Lambda^o_o) : \gamma(\Lambda^o_-)$	1,04 $^{+0,33}_{-0,21}$	1
$\Gamma(\Xi^o_o) : \Gamma(\Xi^-_-)$	0,548 ± 0,036	1/2
$\alpha(\Xi^o_o) : \alpha(\Xi^-_-)$	0,82 ± 0,19	1
$\gamma(\Xi^o_o) : \gamma(\Xi^-_-)$	0,97 ± 0,17	1

axial current (PCAC) hypothesis. This hypothesis asserts that the divergence of the axial current $\partial_\lambda A^\lambda_i(x)$ is proportional to the operator $\varphi_i(x)$ of the pseudoscalar octet. In these notes we shall work only with the SU(2) part of this hypothesis, i.e. we shall restrict it to 1, 2 and 3. Then the PCAC hypothesis reads

$$\partial_\lambda A^\lambda_i(x) = c_\pi \varphi^\pi_i(x) \qquad i = 1,2,3 \qquad /215/$$

where $\varphi^\pi_i(x)$ stands for the hermitean components of the pion field operator.

The PCAC relation may hold because the singularity structure (the location of the poles and of the residues) of the matrix element of the operator $\partial_\lambda A^\lambda_i(x)$ between any two states is the same as the singularity structure of the matrix element of the operator $\varphi_i(x)$ between these states. This follows from the fact that both operators are pseudoscalars and have identical internal quantum numbers. However the strength of these singularities (the residues of the poles and the spectral functions of the cuts) could be different. The PCAC hypothesis asserts that even the residues and the spectral functions are identical for $\partial_\lambda A^\lambda_i(x)$ and $\varphi_i(x)$ up to a common multiplicative constant. The very strong restriction imposed by this condition, which is the simplest possible relation between $\partial_\lambda A^\lambda_i(x)$ and $\varphi_i(x)$, is obvious, and it is also clear that it may have far reaching consequences. A clear-cut answer to the question whether the PCAC relation is exact or not is not available at present, because of the poor knowledge of the spectral functions to be compared and of other factors entering the formulae to be tested. Thus we shall assume that the PCAC relation is an exact one up to electromagnetic corrections, and we shall explore some of the consequences of this assumption.

The value of the constant c_π can be easily expressed through other constants of the theory of the weak interaction. To see this we write

$$\partial_\lambda \langle 0|J_H^{\lambda+}(x)|\pi^-(p_\pi)\rangle = c_A\, \partial_\lambda \langle 0|A_1^\lambda(x) + i\, A_2^\lambda(x)|\pi^-(p_\pi)\rangle =$$

$$= c_A\, c_\pi \langle 0|\varphi_1^\pi(x) + i\,\varphi_2^\pi(x)|\pi^-(p_\pi)\rangle =$$

$$= c_A\, c_\pi\, \sqrt{2}\, \langle 0|\varphi^\pi(x)^+|\pi^-(p_\pi)\rangle = c_A\, c_\pi\, \sqrt{2}\, \frac{e^{-ip_\pi x}}{\sqrt{2p_\pi^o}\,(2\pi)^{3/2}} \qquad /216/$$

On the other hand from eq. /20/ we find

$$\partial_\lambda \langle 0|J_H^{\lambda+}(x)|\pi^-(p_\pi)\rangle = m_\pi^2\, f(m_\pi^2)\, \frac{e^{-ip_\pi x}}{\sqrt{2p_\pi^o}\,(2\pi)^{3/2}} \qquad /217/$$

and finally

$$c_\pi = m_\pi^2\, \frac{f(m_\pi^2)}{\sqrt{2}\, c_A} \equiv m_\pi^2\, f_\pi \; . \qquad /218/$$

Thus the PCAP equation /215/ takes the form

$$\partial_\lambda\, A_i^\lambda(x) = m_\pi^2\, f_\pi\, \varphi_i^\pi(x) \; .$$

Let us point out that the PCAC hypothesis breaks the $SU(3) \otimes SU(3)$ symmetry . Indeed, in the exact symmetry limit both the vector and axial currents must be conserved. We see that the symmetry is broken by the pion mass. If this mass were zero, the axial charge would be exactly conserved. Since the pion mass (more generally the mass of the pseudoscalar octet) is small as compared to the other hadron masses the current conservation is almost respected. This is why one speaks of a partially conserved axial current.

The PCAC equation /219/ has many important consequences. A group of these is obtained if we apply the $(\Box - m_\pi^2)\,\partial_\lambda$ operator to the equation

$$\langle p(p_2)|A_1^\lambda(x) + i\, A_2^\lambda(x)|n(p_1)\rangle =$$

$$= -e^{i(p_2-p_1)x}\, \frac{\bar{p}(p_2)}{(2\pi)^{3/2}} \left[\gamma^\lambda\, H_1(q^2) - i\sigma^{\lambda\nu}\, \frac{(p_2-p_1)_\nu}{M_p+M_n}\, H_2(q^2) + \frac{(p_2-p_1)^\lambda}{M_p+M_n}\, H_3(q^2)\right] i\gamma_5\, \frac{n(p_1)}{(2\pi)^{3/2}} \qquad /220/$$

taking into account that

$$(\Box - m_\pi^2)\, \varphi_i^\pi(x) = j_i^\pi(x) \qquad /221/$$

we find

$$m_\pi^2 f_\pi \langle p(p_2)|j_1^\pi(0) + i\, j_2^\pi(0)|n(p_1)\rangle =$$

$$= \left(q^2 - m_\pi^2\right)\left[\left(M_p + M_n\right) H_1(q^2) + \frac{q^2}{M_p+M_n} H_3(q^2)\right] \frac{\bar{p}(p_2)}{(2\pi)^{3/2}} \gamma_5 \frac{n(p_1)}{(2\pi)^{3/2}} \qquad /222/$$

The most general structure of the matrix element $\langle p(p_2)|j_1^\pi(0) + i\, j_2^\pi(0)|n(p_1)\rangle$, allowed by Lorentz invariance, is

$$\langle p(p_2)|j_1^\pi(0) + i\, j_2^\pi(0)|n(p_1)\rangle =$$

$$= 2g_{\pi NN}\, K(q^2) \frac{\bar{p}(p_2)}{(2\pi)^{3/2}} \gamma_5 \frac{n(p_1)}{(2\pi)^{3/2}} . \qquad /223/$$

In this equation $g_{\pi NN}$ is the conventional πNN coupling constant and the form factor $K(q^2)$ is normalized in the usual way:

$$\frac{g_{\pi NN}^2}{4\pi} = 14{,}37 \pm 0{,}3$$

$$|g_{\pi NN}| = 13{,}59 \pm 0{,}14 \qquad K(m_\pi^2) = 1 \qquad /224/$$

With eq. /222/ and /223/ we arrive at the PCAC relation

$$\left(M_p + M_n\right) H_1\left(q^2\right) + \frac{q^2}{M_p + M_n} H_3\left(q^2\right) = 2g_{\pi NN}\, f_\pi\, m_\pi^2 \frac{K\left(q^2\right)}{m_\pi^2 - q^2} . \qquad /225/$$

From the analytic S matrix theory the form factors $H_1(q^2)$, $H_3(q^2)$ and $K\left(q^2\right)$ are known to have a cut in the complex q^2 plane going from $q^2 = 9\,m_\pi^2$ to $q^2 = +\infty$; the form factor H_3 has also a pole at $q^2 = m_\pi^2$. These are the only singularities of these functions in the finite q^2 plane. Thus $H_3\left(q^2\right)$ may be written in the form

$$H_3\left(q^2\right) = \frac{R}{m_\pi^2 - q^2} + \tilde{H}_3(q^2) , \qquad /226/$$

where the constant R is the residue at the pole, and $\tilde{H}_3\left(q^2\right)$ has only the cut. Similarly, we may rewrite the left hand side of eq. /225/ to separate the pole:

$$\frac{K\left(q^2\right)}{m_\pi^2 - q^2} = \frac{1}{m_\pi^2 - q^2} + \tilde{K}\left(q^2\right) \qquad /227/$$

where

$$\tilde{K}(q^2) \equiv \frac{K\left(q^2\right)-1}{m_\pi^2 - q^2} \qquad /228/$$

is a regular function at $q^2 = m_\pi^2$, due to the normalization $K(m_\pi^2) = 1$. From eq. /225/, /228/ it is easy to see that

$$R = 2g_{\pi NN}\, f_\pi \left(M_p + M_n\right) . \qquad /229/$$

Let us now take the eq. /225/ at the point $q^2 = 0$, where all the form factors are regular. We find:

$$\left(M_p + M_n\right) H_1(0) = 2g_{\pi NN}\, f_\pi\, K(0) \; ; \qquad /230/$$

this is the famous Goldberger-Treiman relation. All the quantities but $K(o)$ are known in eq. /230/, and we find for the latter

$$K(0) = \pm \left(0{,}90 \pm 0{,}13\right) \qquad /231/$$

If we choose the + sign, we see that our result supports the general belief that far from their singularities the form factors are slowly varying "smooth" functions of the corresponding variable. Morevoer, we see also from eq. /227/, /228/ and /230/ that at the point $q^2 = 0$ the contribution of the pole in /225/ is of 90%. Thus PCAC* tells us that the function

$$G(q^2) \equiv \left(M_p + M_n\right) H_1(q^2) + \frac{q^2}{M_p + M_n} H_3(q^2) \;, \qquad /232/$$

which is essentially a matrix element of the divergence of the axial current, is dominated by the pole term in a neighbourhood of the pole which in particular contains the point $q^2 = 0$. This property of the function G is called the pole dominance of the divergence of the axial current (PDDAC) . We derived this property from PCAC* and a smoothness condition (because we supposed that the function $K(q^2)$ is smooth; this property was only indicated, but not proved by the PCAC result $|K(o)| = 0{,}9$). Some times PDDAC is postulated independently as a basic hypothesis and used instead of PCAC. In that case one generally supposes that a dispersion relation without substraction can be written for the function $G(q^2)$ (see [2]).

With the help of eq. /226/, /229/ and /230/ we can find an expression for $H_3(0)$. Indeed,

$$H_3(0) = \frac{R}{m_\pi^2} + \tilde{H}_3(0) = 2g_{\pi NN}\, f_\pi\, \frac{M_p + M_n}{m_\pi^2} + \tilde{H}_3(0) =$$

$$= \frac{\left(M_p + M_n\right)^2}{m_\pi^2} \frac{1}{K(0)} H_1(0) + \tilde{H}_3(0) \; . \qquad /233/$$

---

*supplemented by the experimental value of K(o) coming from / 230/

If we now suppose that the form factor $H_3(q^2)$ is also dominated by the pion pole term at $q^2 = 0$, i.e. that $|R/m_\pi^2| >> |\tilde{H}_3(0)|$, then we find the approximate relation used in chapter IV §4:

$$H_3(0) \approx \frac{(M_p + M_n)^2}{m_\pi^2} \frac{1}{K(0)} H_1(0) = 210\ H_1(0) . \qquad /234/$$

This result is often exploited when one neglects the contribution of $H_3(q^2)$. Indeed, as a rule $H_3(q^2)$ is multiplied by kinematical factors much smaller than 1/200. We would like to point out that this result depends on a PDDAC hypothesis for $H_3(q^2)$ and that this is a separate requirement, which does not follow from PDDAC for $G(q^2)$. Indeed, from eq. /232/ we see that $\tilde{H}_3(q^2)$ is multiplied by q^2 , therefore near $q^2 = 0$ its behaviour cannot be inferred from the behaviour of $G(q^2)$, even if $H_1(q^2)$ were known.

Let us now come to the application of PCAC to the non leptonic weak decays. Here PCAC, together with the current algebra relations of chapter IV §3, gives an expression of the amplitudes of the type $H \to H' + \pi$ through the amplitudes $\tilde{H} \to H'$ and $H \to \tilde{H}'$, where $\tilde{H}$ and $\tilde{H}'$ contain the same hadrons as H and H', respectively, but possibly in other charge configuration. E.g. if $H' = \pi^+ \pi^0$, then $\tilde{H}'$ may be $\pi^0 \pi^0$ or $\pi^+ \pi^-$ and so on. Unfortunately the relations between the amplitudes are obtained at the non physical point where the four momentum k_λ of the pion in the $H \to H' + \pi(k)$ amplitude vanishes: $k_\lambda \to 0$; hence $k^2 \to 0$, the pion is not on its mass shell $k^2 = m_\pi^2$. No unambiguous methods of analytic continuation back to the mass shall exist at present. In general one adopts the working hypothesis that the continuation would not change the results drastically, i.e. the results for the non physical point are supposed to be approximately valid at the physical point too.

The derivation of the basic formula is quite simple and goes as follows:

$$(2\pi)^{3/2} \sqrt{2k^0} <\pi_i(k)H'|L_H(0)|H> =$$

$$= -i\left(k^2 - m_\pi^2\right) \int dx\ e^{ikx} <H'|T(\varphi_i(x)\ L_H(0))|H> = \qquad /235/$$

$$= -i \frac{k^2 - m_\pi^2}{f_\pi m_\pi^2} \int dx\ e^{ikx} <H'|T(\partial_\lambda A_i^\lambda(x)\ L_H(0))|H> = \qquad /236/$$

$$= -k_\lambda \frac{k^2 - m_\pi^2}{f_\pi m_\pi^2} \int dx\, e^{ikx} \langle H'|T\left(A_i^\lambda(x)\, L_H(0)\right)|H\rangle +$$

$$+ i \frac{k^2 - m_\pi^2}{f_\pi m_\pi^2} \int dx\, e^{ikx}\, \delta(x_0) \langle H'|\left[A_i^0(x),\, L_H(0)\right]|H\rangle \quad . \qquad /237/$$

Eq. /235/ is the well known LSZ reduction formula for the pion state $\langle \pi_i(k)|$; eq. /236/ comes from the PCAC eq. /219/, while eq. /237/ is obtained by partial integration where as usual the surface term is assumed to be zero and is not written out. In these equations L_H stands for the Lagrangean of the hadronic weak interactions, which may be the L_{HH} part of our current-current Lagrangean (see eq. /7/), but may be also an other one. Its only property used in eq. /235/ - /237/ is that it is a local operator $L_H(x)$.

Let us now take eq. /237/ in the limit $k_\lambda \to 0$. Then the second term reduces to

$$- \frac{i}{f_\pi} \langle H'|\left[I_i^A(0),\, L_H(0)\right]|H\rangle \quad , \qquad /238/$$

where $I_i^A(0)$ are the axial charges at $t = 0$ defined in eq. /143/.

Let us suppose that we work with the current-current theory and that the universality hypothesis of Gell-Mann holds in its precise form. Then $L_H = L_{HH}$ contains only $V_{i,\lambda}^{(+)} = \frac{1}{2}\left(V_{i,\lambda} + A_{i,\lambda}\right)$ currents, and then

$$\left[I_i^{(-)}(0),\, L_H(0)\right] \equiv \frac{1}{2}\left[I_i(0) - I_i^A(0),\, L_H(0)\right] = 0 \quad , \qquad /239/$$

i.e.

$$- \frac{i}{f_\pi} \langle H'|\left[I_i^A(0),\, L_H(0)\right]|H\rangle = - \frac{i}{f_\pi} \langle H'|\left[I_i,\, L_H(0)\right]|H\rangle \quad . \qquad /240/$$

In eq. /240/ we suppressed the time argument in I_i because we neglect the small SU(2) breaking effects.

The isospin operators I_i are known to act only on the third component of the isospin of a hadron state. Thus we have

$$\langle H'|\left[I_i,\, L_H(0)\right]|H\rangle = \langle \tilde{H}'|L_H(0)|H\rangle - \langle H'|L_H(0)|\tilde{H}\rangle \quad , \qquad /241/$$

where

$$\tilde{H}'\rangle = I_i^+ |H'\rangle \quad , \quad \tilde{H}\rangle = I_i |H\rangle \qquad /242/$$

Concerning the first term of eq. /237/, it is proportional to k_λ and thus vanishes in the $k_\lambda \to 0$ limit unless the integral has a pole at $k_\lambda = 0$. From the known analytic properties one finds that such a pole turns up in the hyperon decays $B_1 \to B_2 \pi$ but not in the kaon decays $K \to 3\pi$, $K \to 2\pi$. Anyhow, our final result is:

$$(2\pi)^{3/2} \lim_{k_\lambda \to o} \sqrt{2k^o} \langle \pi_i(k)\, H'|L_H(0)|H\rangle =$$

$$= \frac{1}{f_\pi} \lim_{k_\lambda \to o} k_\lambda \int dx\, e^{ikx} \langle H'|T\left(A_i^\lambda(x),\, L_H(0)\right)|H\rangle +$$

$$+ \frac{1}{if_\pi} \left[\langle \tilde{H}'|L_H(0)|H\rangle - \langle H'|L_H(0)|\tilde{H}\rangle\right] . \qquad /243/$$

If we apply this formula to $K \to 3\pi$ decay, then, as we said above, the first term is missing and we have an expression of the $K \to 3\pi$ decay amplitude through $K \to 2\pi$ amplitudes. Applying eq. /243/ once more, we arrive at the $K \to \pi$ matrix elements, and applying it again we go down to the $\langle 0|L_H(0)|K\rangle$ matrix element. Thus we have a $K \to 3\pi \Rightarrow K \to 2\pi \Rightarrow K \to \pi \Rightarrow K \to$ vacuum chain, and at each step interesting predictions can be made on the corresponding amplitudes. These predictions are supported by the experimental results. For details the reader is referred to [2] and [3]. Here we notice only that the last loop of the chain is obviously a pure $\Delta I = 1/2$ transition, even if $L_H(o)$ itself contains $I = 3/2$ parts. Coming back along the chain up to the $K \to 2\pi$ and $K \to 3\pi$ amplitudes, we see that a possible explanation of the $\Delta I = 1/2$ rule emerges in the kaon decays. Of course our chain works at a non-physical point, but the $\Delta I = 1/2$ rule itself is known to hold only approximately, as shown by the very existence of the $K^+ \to \pi^+ \pi^o$ decay, which is a pure $\Delta I = 3/2$ transition.

The application of eq. /243/ to the hyperon decays is more difficult because of the pole terms at $k_\lambda \to 0$. The origin of the $\Delta I = 1/2$ rule is also more obscure in this case, and in many calculation it is imposed by hand, i.e. one simply requires that the Lagrangean $L_H(o)$ transform as an $I = 1/2$ operator. Practically this means that the current-current theory of Cabibbo is abandoned. We point out that in the derivation of our basic formula /243/ the current-current picture was not really exploited, only the locality and the universality property /239/ of the Lagrangean were needed. Thus it is possible to abandon the current-current structure and to require only these properties and the $\Delta I = 1/2$ rule to hold. In many calculations of the non-leptonic weak decays this procedure is adopted. This does not prove, of course, that the current-current theory is inapplicable to the non-leptonic decays;

but, unfortunately, the exploitation of the current-current structure involves either the introduction of field theoretical models where non renormalizable divergencies occur, or the introduction of a complete system of states, the bulk of which must be neglected because their contributions are uncalculable. In spite of these difficulties, the current-current theory is also used to calculate non-leptonic hyperon decays, and with a "reasonable" cut off of the divergent integrals or of the intermediate state spectrum, results of the same quality as without the current-current theory are reached; in particular, in hyperon decays, the $\Delta I = 1/2$ rule is imitated due to "accidental" cancellations between the unwanted $\Delta I = 3/2$ contributions. Unfortunately, these encouraging results are based on calculations which contain too many uncontrollable approximations and therefore they cannot be considered as a proof of the applicability of the current-current theory to the non-leptonic decays.

To conclude these notes we would like to express the opinion that the serious problems in the theory of the weak interaction exposed in this chapter must not overshadow the brilliant successes of this theory, presented in the other chapters. Hopefully further progress will before long allow us to rewrite this last chapter in the spirit of the preceding ones.

APPENDIX

§.1. Definitions and notations

In these notes the following basic notations and definitions have been used.

Non-zero components of the metric tensor:

$$g_{oo} = 1 = -g_{11} = -g_{22} = -g_{33} \qquad \text{A-1}$$

Scalar product of two four vectors a and b :

$$(ab) \equiv a^{\lambda} b_{\lambda} \equiv a^{o} b_{o} + a^{1} b_{1} + a^{2} b_{2} + a^{3} b_{3} \; ; \qquad \text{A-2}$$

with

$$a^{\lambda} = g^{\lambda\mu} a_{\mu} \qquad \text{A-3}$$

$$(ab) = g^{\lambda\mu} a_{\mu} b_{\lambda} = a_{o}b_{o} - a_{1}b_{1} - a_{2}b_{2} - a_{3}b_{3} = a_{o}b_{o} - \underline{\underline{ab}} \qquad \text{A-4}$$

Differential operators:

$$\partial \equiv \frac{\partial}{\partial x_{\lambda}} \qquad \square \equiv -\partial^{\lambda} \partial_{\lambda} \qquad \text{A-5}$$

Matrices of Dirac:

$$[\gamma_{\mu}, \gamma_{\nu}]_{+} = 2g_{\mu\nu} \qquad \gamma^{\mu} \equiv g^{\mu\nu} \gamma_{\nu} \; ; \qquad /\mu,\nu = 0,1,2,3/ \qquad \text{A-6}$$

A frequently used representation of the 4 x 4 Dirac matrices, adopted also in our notes is:

$$\gamma_{o} = \begin{pmatrix} I & O \\ O & -I \end{pmatrix} \qquad \gamma_{i} = \begin{pmatrix} O & \sigma_{i} \\ -\sigma_{i} & O \end{pmatrix} . \qquad /i = 1,2,3/ \qquad \text{A-7}$$

In eq. /7/

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} ; \quad \sigma_{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \sigma_{2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \sigma_{3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{A-8}$$

We also define the matrix γ_5 :

$$\gamma_5 \equiv \gamma_o \gamma_1 \gamma_2 \gamma_3 = -i \begin{pmatrix} O & I \\ I & O \end{pmatrix} \quad . \qquad \text{A-9}$$

In this representation

$$\gamma_o \gamma_\lambda^+ \gamma_o = \gamma_\lambda \qquad /\lambda = 0,1,2,3,5/ \quad . \qquad \text{A-10}$$

Notation: for any four vector a

$$\hat{a} \equiv \gamma_\lambda a^\lambda = \gamma_o a_o - \gamma_1 a_1 - \gamma_2 a_2 - \gamma_3 a_3 = \gamma_o a_o - \underline{\gamma a} \qquad \text{A-11}$$

especially,

$$\hat{\partial} \equiv \gamma_\lambda \partial^\lambda . \qquad \text{A-12}$$

§2. Zero spin field

Below the operators $\varphi(x)$, $a(k)$, $b(k)$ stand for "in" fields, but the label "in" has been suppressed. The same relations holds also between the out operators, but of course $\varphi_{in}(x) \neq \varphi_{out}(x)$, except for non-interacting fields.

The Klein-Gordon equation reads:

$$\left(\Box - M^2\right) \varphi(x) = 0 \quad . \qquad \text{A-13}$$

The plane wave decomposition of $\varphi(x)$ can be written as follows:

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\underline{k}}{\sqrt{2k^o}} \left(e^{ikx} \; b^+(\underline{k}) + e^{-ikx} \; a(\underline{k})\right) \qquad \text{A-14}$$

In eq. /14/ the condition

$$k_o = \left|\sqrt{\underline{k}^2 + M^2}\right| \qquad \text{A-15}$$

holds.

If the field $\varphi(x)$ is non-hermitean, i.e. if $\varphi^+(x) \neq \varphi(x)$, then the operators $a(\underline{k})$ and $b(\underline{k})$ are quantized as follows:

$$\left[a(\underline{k}), a^+(\underline{k}')\right]_- = \left[b(\underline{k}), b^+(\underline{k}')\right]_- = \delta\left(\underline{k} - \underline{k}'\right) \qquad \text{A-16}$$

all the other commutators of a, a^+, b, b^+ vanish identically. From eq. /14/ - /16/ the well-known canonical commutation relation

$$\left[\varphi(x),\ \partial_0\, \varphi^+(y)\right]_-^{x_0=y_0} = -i\delta(\underline{x} - \underline{y}) \qquad \text{A-17}$$

is easily deduced.

The physical interpretation of the operators a and b is:

$a(\underline{k})$ absorbs a particle with momentum $\underline{k}$
$a^+(k)$ creates a particle with momentum $\underline{k}$
$b(k)$ absorbs an antiparticle with momentum $\underline{k}$
$b^+(k)$ creates an antiparticle with momentum $\underline{k}$

If $\varphi^+(x) = \varphi(x)$, then $b(\underline{k}) = a(\underline{k})$. The only non-zero commutator then is

$$\left[a(\underline{k}),\ a^+(\underline{k}')\right]_- = \delta(\underline{k} - \underline{k}') \qquad \text{A-18}$$

and the antiparticle is identical with the particle. The canonical commutation relation takes the form

$$\left[\varphi(x),\ \partial_0\, \varphi(y)\right]_-^{x_0=y_0} = -i\,(\,\underline{x} - \underline{y}\,)\,; \qquad \text{A-19}$$

notice that $\left[\varphi(x),\ \partial_0\, \varphi(y)\right]_-^{x_0=y_0}$ is zero if $\varphi^+(x) \neq \varphi(x)$,

§3. 1/2 spin field

The Dirac equation for the field operator (with the label "in" suppressed) reads:

$$(i\hat{\partial} - m)\,\psi(x) = 0 \qquad \text{A-20}$$

Eq. /20/ is a shorthand notation for

$$\left(i\partial^\lambda\, \gamma_\lambda - mE\right)_{\alpha\beta}\, \psi_\beta(x) = 0 \qquad \text{A-21}$$

where E is the 4×4 unit matrix and α,β are spinor indices running through 1, 2, 3, 4. In /21/ a summation over the index β is understood.

Plane wave decomposition:

$$\psi_\alpha(x) = \frac{1}{(2\pi)^{3/2}} \int dp \left(e^{ipx} \sum_s v_\alpha^s(\underline{p})\, d_s^+(\underline{p}) + e^{-ipx} \sum_{s=1,2} u_\alpha^s(\underline{p})\, c_s(\underline{p})\right) \qquad \text{A-22}$$

In eq. /22/ we have

$$p_o = |\sqrt{\underline{p}^2 + m^2}| \quad . \qquad \text{A-23}$$

The Dirac equation imposes for the spinor amplitudes u and v the equations

$$(\hat{p} - m)\, u(\underline{p}) = 0 \qquad (\hat{p} + m)\, v(\underline{p}) = 0 \ . \qquad \text{A-24}$$

We normalize the spinors as follows:

$$v^{s+}(\underline{p})\, v^{r}(\underline{p}) = u^{s+}(\underline{p})\, u^{r}(\underline{p}) = \delta_{rs} \qquad \text{A-25}$$

Notice that in general

$$v^{s+}(\underline{p})\, u^{r}(\underline{p}) \neq \delta_{rs} \ , \qquad \text{A-26}$$

since v and u are solutions of different equations.

For spinor fields all the known antiparticles are different from their particles. Thus we give the quantization only for the $\psi^{+}_{\alpha}(x) \neq \psi_{\alpha}(x)$ case:

$$\left[c_s(\underline{p}),\, c^{+}_{r}(\underline{p})\right]_{+} = \left[d_s(\underline{p}),\, d^{+}_{r}(\underline{p})\right]_{+} = \delta_{sr}\ \delta(\underline{p} - \underline{p}') \ . \qquad \text{A-27}$$

All the other anticommutators are zero. From eq. /22/ and /27/ the usual canonical quantization

$$\left[\psi_{\alpha}(x),\, \psi^{+}_{\beta}(y)\right]_{+}^{x_o = y_o} = \delta_{\alpha\beta}\ \delta(\underline{x} - \underline{y}) \qquad \text{A-28}$$

is easily obtained. The physical interpretation of the operators c and d is the following:

$c_s(\underline{p})$ absorbs a particle with momentum $\underline{p}$ and polarization s

$c^{+}_{s}(\underline{p})$ creates a particle with momentum $\underline{p}$ and polarization s

$d_s(\underline{p})$ absorbs an antiparticle with momentum $\underline{p}$ and polarization s

$d^{+}_{s}(\underline{p})$ creates an antiparticle with momentum $\underline{p}$ and polarization s

Let us now look at the polarization states in more detail. The equation $(\hat{p} - m)\, u(\underline{p}) = 0$ has two linearly independent solutions for a given $\underline{p}$ and with $p_o = |\sqrt{\underline{p}^2 + m^2}|$. This is why in eq. /22/ we have two polarization states $s = 1,2$ for $u(\underline{p})$. The same is true for the equation

$(\hat{p} + m)v(\underline{p}) = 0$. The remaining solutions of these equations belong to the $p_o = -|\sqrt{\underline{p}^2 + m^2}|$ case, and do not enter our eq. /22/ due to eq. /23/.

For particles in motion the polarization is conveniently parametrized in terms of helicity eigenstates (spin parallel or antiparallel to the direction of the motion), while for particles at rest the projection of the spin on a fixed direction can be used (spin parallel or antiparallel with respect to a fixed axe of quantization).

The components of the spin operator S_i^p for a particle (i.e. on the space of the spinors $u(p)$) reads:

$$S_i^p = \frac{1}{2}\Sigma_i \equiv \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \qquad i = 1,2,3 \qquad \text{A-29}$$

while for an antiparticle (i.e. on the $v(\underline{p})$-s)

$$S_i^{\bar{p}} = -\frac{1}{2}\Sigma_i = -\frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \qquad i = 1,2,3 \;; \qquad \text{A-30}$$

the helicity operator for a particle of momentum $\underline{p}$ is

$$\eta^p(\underline{n}) = \Sigma\,\underline{n} \quad , \qquad \underline{n} = \frac{\underline{p}}{|\underline{p}|} \qquad \text{A-31}$$

while for an antiparticle with momentum $\underline{p}$

$$\eta^{\bar{p}}(\underline{n}) = \Sigma\,\underline{n} \qquad \underline{n} = \frac{\underline{p}}{|\underline{p}|} \qquad \text{A-32}$$

In the representation /7/ of the γ matrices the two eigenstates of S_3^p , satisfying the eq. $(\hat{p} - m)u(\underline{p})=0$ with $p = 0$ may be choosen to be

$$u^1(0) \equiv u^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u^2(0) \equiv u^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{A-33}$$

while for an antiparticle we may choose

$$v^1(0) \equiv v^1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad v^2(0) \equiv v^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \text{A-34}$$

Here $u^1(v^1)$ describes a particle (antiparticle) with spin parallel to the third axe $\left(\text{i.e. } S_3^p\, u^1 = \frac{1}{2}u^1 ,\quad S_3^{\bar{p}}\, v^1 = \frac{1}{2}v^1\right)$, while $u^2(v^2)$ describes a particle (antiparticle) with spin antiparallel to the third axe $\left(\text{i.e. } S_3^p\, u^2 = -\frac{1}{2}u^2 \quad S_3^{\bar{p}}\, v^2 = -\frac{1}{2}v^2\right)$.

For the helicity eigenstates in our representation we find $\left(\underline{n} = \underline{p}/|\underline{p}|\right)$:

$$u^{\uparrow}(\underline{p}) = \sqrt{\frac{m + p}{2p_o}} \begin{bmatrix} I \\ \dfrac{\underline{\sigma}\,\underline{p}}{p_o + m} \end{bmatrix} \frac{1}{\sqrt{2(1+n_3)}} \begin{pmatrix} 1 + n_3 \\ n_1 + in_2 \end{pmatrix} \qquad \text{A-35}$$

$$u^{\downarrow}(\underline{p}) = \sqrt{\frac{m+p_o}{2p_o}} \begin{pmatrix} I \\ \frac{\sigma \underline{p}}{p_o + m} \end{pmatrix} \frac{1}{\sqrt{2(1+n_3)}} \begin{pmatrix} -n_1 + in_2 \\ 1 + n_3 \end{pmatrix} \qquad \text{A-36}$$

$$u^{\uparrow}(p) = \sqrt{\frac{m+p_o}{2p_o}} \begin{pmatrix} \frac{\sigma \underline{p}}{p_o + m} \\ I \end{pmatrix} \frac{1}{\sqrt{2(1+n_3)}} \begin{pmatrix} -n_1 + in_2 \\ 1 + n_3 \end{pmatrix} \qquad \text{A-37}$$

$$v^{\downarrow}(\underline{p}) = \sqrt{\frac{m+p_o}{2p_o}} \begin{pmatrix} \frac{\sigma \underline{p}}{p_o + m} \\ I \end{pmatrix} \frac{1}{\sqrt{2(1+n_3)}} \begin{pmatrix} 1 + n_3 \\ n_1 + in_2 \end{pmatrix} \qquad \text{A-38}$$

$u^{\uparrow}(\underline{p})$ $\left(v^{\uparrow}(\underline{p})\right)$ describes a particle (antiparticle) with positive helicity $\left(\text{i.e. } \chi^{p}(\underline{n})\, u^{\uparrow}(\underline{p}) = u^{\uparrow}(\underline{p}),\ \chi^{\bar{p}}(\underline{n})\, v^{\uparrow}(\underline{p}) = v^{\uparrow}(\underline{p})\right)$, while $u^{\downarrow}(\underline{p})$ $\left(v^{\downarrow}(\underline{p})\right)$ describes a particle (antiparticle) with negative helicity $\left(\text{i.e. } \chi^{p}(\underline{n})\, u^{\downarrow}(\underline{p}) = -u^{\downarrow}(\underline{p}),\ \chi^{\bar{p}}(\underline{n})\, v^{\downarrow}(\underline{p}) = -v^{\downarrow}(\underline{p})\right)$. In eq. /35/ - /38/ we have of course spinors of one column and four rows, e.g.

$$u^{\uparrow}(\underline{p}) = \sqrt{\frac{m+p_o}{2p_o}} \frac{1}{\sqrt{2(1+n_3)}} \begin{pmatrix} 1+n_3 \\ n_1+in_2 \\ \frac{|\underline{p}|(1+n_3)}{p_o + m} \\ \frac{|\underline{p}|(n_1+in_2)}{p_o + m} \end{pmatrix} \qquad \text{A-39}$$

and so on.

In the calculation of transition probabilities, expressions of the form

$$|\bar{w}^{s}(\underline{p})\, O(p,p')\, w^{s'}(\underline{p}')|^2 \qquad \text{A-40}$$

must be evaluated. In eq. /40/ $O(p,p')$ stands for a 4×4 matrix, w stands for u or v, and

$$\bar{w}^{s}(\underline{p}) \equiv w^{s+}(\underline{p})\, \gamma^{o} \quad . \qquad \text{A-41}$$

The calculation of eq. /40/ is straightforward and leads to the well known result:

$$|\bar{w}^{s}(\underline{p})\, O\, w^{s'}(\underline{p})|^2 =$$

$$= \mathrm{Tr}\left\{ \chi_{w^{s}(\underline{p})} \frac{\hat{p} \pm m}{2p_o}\, O\, \chi_{w^{s'}(\underline{p}')} \frac{\hat{p}' \pm m}{2p_o'}\, \gamma^{o} O^{+} \gamma^{o} \right\} \qquad \text{A-42}$$

In eq. /42/ $\chi_{w^s}(p)$ stands for the helicity or spin projection operator on the state $w^s(p)$. Thus

$$\chi_{u^{\uparrow}(p)} = \frac{1 + \Sigma n}{2} \qquad \chi_{u^{\downarrow}(p)} = \frac{1 - \Sigma n}{2} \qquad \text{A-43}$$

$$\chi_{v^{\uparrow}(p)} = \frac{1 - \Sigma n}{2} \qquad \chi_{v^{\downarrow}(p)} = \frac{1 + \Sigma n}{2} \qquad \text{A-44}$$

$$\chi_{u^1} = \frac{1 + \Sigma_3}{2} \qquad \chi_{u^2} = \frac{1 + \Sigma_3}{2} \qquad \text{A-45}$$

$$\chi_{v^1} = \frac{1 - \Sigma_3}{2} \qquad \chi_{v^2} = \frac{1 + \Sigma_3}{2} \qquad \text{A-46}$$

In the factor $\hat{p} \pm m$ the sign + (-) must be choosen if p refers to the momentum of a particle (antiparticle). The same prescription holds for $\hat{p}' \pm m$.

The relevant formulae for unpolarized cases are easily obtained from eq. /42/ - /46/ . Thus

$$\sum_{s=1,2} |\bar{w}^s(p)\, O w^{s'}(p')|^2 = \mathrm{Tr}\left\{ \frac{\hat{p} \pm m}{2p_o}\, O \chi_{w^{s'}(p')}\, \frac{\hat{p} \pm m'}{2p_o}\, \gamma^o O^+ \gamma^o \right\} , \qquad \text{A-47}$$

$$\sum_{s=1,2} |\bar{w}^s(p)\, O w^{s'}(p')|^2 = \mathrm{Tr}\left\{ \chi_{w^s(p)}\, \frac{\hat{p} \pm m}{2p_o}\, O\, \frac{\hat{p}' \pm m}{2p_o'}\, \gamma^o O^+ \gamma^o \right\} , \qquad \text{A-48}$$

$$\sum_{\substack{s=1,2\\ s'=1,2}} |\bar{w}^s(p)\, O w^{s'}(p)|^2 = \mathrm{Tr}\left\{ \frac{\hat{p} \pm m}{2p_o}\, O\, \frac{\hat{p}' \pm m'}{2p_o}\, \gamma^o O^+ \gamma^o \right\} . \qquad \text{A-49}$$

As well known, for the polarization index which refers to the initial state still a factor 1/2 must be introduced because we have to take the average of the two polarization states. For the final state the summation is correct as it stands in eq. /47/ - /49/, because both polarization states contribute to the unpolarized transition probability.

§4. Helicity and handedness

In our representation /7/ - /9/ of the γ matrices, the matrix $\underline{\Sigma}$ can be written in the following way:

$$\underline{\Sigma} \equiv \begin{pmatrix} \underline{\sigma} & 0 \\ 0 & \underline{\sigma} \end{pmatrix} = i\,\gamma_5\,\gamma_o\,\underline{\gamma} \qquad \text{A-50}$$

Now it is easy to verify that for a zero mass particle the following relations hold:

$$\gamma_o\,\underline{\gamma}\,\underline{n}\,u^{\uparrow}(\underline{p}) = u^{\uparrow}(\underline{p})$$

$$\gamma_o\,\underline{\gamma}\,\underline{n}\,u^{\downarrow}(\underline{p}) = u^{\downarrow}(\underline{p})$$

$$\gamma_o\,\underline{\gamma}\,\underline{n}\,v^{\uparrow}(\underline{p}) = -v^{\uparrow}(\underline{p})$$

$$\gamma_o\,\underline{\gamma}\,\underline{n}\,v^{\downarrow}(\underline{p}) = -v^{\downarrow}(\underline{p}) \qquad \text{A-51}$$

Thus if $m = 0$ the helicity operator is $i\gamma_5$ for the particle and $-i\gamma_5$ for the antiparticle.
The projection operators for the helicity then become:

$$\chi_{u^{\uparrow}(\underline{p})} = \frac{1 + i\gamma_5}{2} \qquad \chi_{u^{\downarrow}(\underline{p})} = \frac{1 - i\gamma_5}{2}$$

$$\chi_{v^{\uparrow}(\underline{p})} = \frac{1 - i\gamma_5}{2} \qquad \chi_{v^{\downarrow}(\underline{p})} = \frac{1 + i\gamma_5}{2} \qquad \text{A-52}$$

The operator $i\gamma_5$ on the space of the $u(\underline{p})$-s and the operator $-i\gamma_5$ on the space of the $v(\underline{p})$-s are called handedness operators*, irrespective to the mass value of the spinor field. A particle which is in an eigenstate of $i\gamma_5$ with eigenvalue +1 (-1) is called a right-handed (left-handed) particle. For the zero mass case a right (left) handed particle is also a positive (negative) helicity particle. An antiparticle which is in an eigenstate of $-i\gamma_5$ with eigenvalue +1 (-1) is called a right-handed (left-handed) antiparticle. For the zero mass case a right (left) handed antiparticle is also a positive (negative) helicity particle.

As we have seen in these notes, in the V-A theory of the weak interaction all the lepton fields in the leptonic current are multiplied by the $(1 - i\gamma_5)$ operator. Indeed, the lepton current may be written as follows:

---

* or chirality operators

$$j_\lambda(x) = \bar{\psi}_e \gamma_\lambda (1 - i\gamma_5) \psi_{\nu e} + e \longrightarrow \mu =$$

$$= \frac{1}{2} \psi_e^+ \gamma_o \gamma_\lambda (1 - i\gamma_5)^2 \psi_{\nu e} + e \longrightarrow \mu =$$

$$= \frac{1}{2} \left[(1 - i\gamma_5) \psi_e\right]^+ \gamma_o \gamma_\lambda (1 - i\gamma_5) \psi_{\nu e} + e \longrightarrow \mu . \qquad \text{A-53}$$

Thus in the weak interactions only left handed leptons and right handed antileptons take part. For the neutrinos this means also that only negative helicity neutrinos and positive helicity antineutrinos may interact. For the electron and the muon, a handedness eigenstate contains both positive and negative helicity states.

§5. Decay rates

Let a particle A with four-momentum p_A and polarization s_A decay into r particles with four-momenta $p_1, p_2, \ldots, p_r$ and polarizations $s_1, s_2, \ldots, s_r$. We define the transition amplitude F for the decay $A \to r$ through the expression

$$\langle r,\text{out}|A,\text{in}\rangle = F \frac{\delta\left(p_A - \sum_{K=1}^{r} p_K\right)}{\sqrt{2p_A^o\, 2p_1^o \ldots 2p_r^o}} \qquad \text{A-54}$$

The amplitude F is of course a function of all the variables p_A, s_A, p_K, s_K $(K = 1, \ldots, r)$. The states $|A,\text{in}\rangle$ and $|r,\text{out}\rangle$ in eq. /54/ stand for

$$|A,\text{in}\rangle = a^+_{s_A}(p_A)_{\text{in}} |0\rangle \ , \qquad \text{A-55}$$

$$|r,\text{out}\rangle = a^+_{s_1}(p_1)_{\text{out}}\, a^+_{s_2}(p_2)_{\text{out}} \ldots a^+_{s_r}(p_r)_{\text{out}} |0\rangle \ . \qquad \text{A-56}$$

The operators $a(p)$ obey the quantization rules /18/ or /27/ . Of course they hold only between "in" or between "out" operators. The commutator between an "out" and an "in" operator depends on the interaction and its calculation involves the solution of the equations of the interacting fields. In fact this was our main task for the weak interactions in these notes when we calculated the different decay amplitudes F.

The differential decay rate $d\Gamma(A \to r)$ for the decay $A \to r$ is expressed through the decay amplitude F in the following way:

$$d\Gamma(A \to r) = \frac{1}{2\pi} \frac{1}{2p^o_A} |F|^2 \, \delta\left(p_A - \sum_{K=1}^{r} p_k\right) \frac{dp_1 \ldots\ldots dp_r}{2p^o_1 \ldots\ldots 2p^o_r} . \qquad \text{A-57}$$

If among the decaying particles k are identical and they decay to states with all quantum numbers identical except the momenta, then the right hand side of eq. /57/ must be divided by $k!$.

Supposing a pure exponential decay low for the particle A, the probability $dW(A \to r)$ that the decay $A \to r$ from the state specified by p_A, s_A to the configuration p_1, s_1; p_2, s_2; ...; p_i; s_r takes place in the time interval t, $t + dt$ is given by

$$dW(A \to r) = d\Gamma(A \to r)\, e^{-t\Gamma A} dt \qquad \text{A-58}$$

In eq. /58/ $\Gamma(A)$ stands for the full decay rate of the particle A. If the channel $A \to r$ is the only one, then $\Gamma(A)$ is simply obtained by integration over all the momenta $p_1 \ldots, p_r$ and summation over all the polarizations s_A, s_1, ... s_r in eq. /57/. (For s_A the average must be taken, not the sum!). Thus in this case $\Gamma(A) = \Gamma(A \to r)$. If there are N decay channels, then

$$\Gamma(A) = \sum_{n=1}^{N} \Gamma(A \to r_n) . \qquad \text{A-59}$$

It is easy to see that eq. /58/ is correctly normalized, because

$$\int_0^{+\infty} \Gamma(A)\, e^{-t\Gamma(A)} dt = 1 \qquad \text{A-60}$$

i.e. the full probability that the particle A will decay in the time interval $(0, +\infty)$ is equal to 1 . Also it is easy to see that $\Gamma(A)$ is the inverse of the mean lifetime of the particle:

$$\tau(A) = \int_0^{+\infty} t\Gamma(A)\, e^{-t\Gamma(A)} dt = \frac{1}{\Gamma(A)} . \qquad \text{A-61}$$

REFERENCES

[1] T.D. Lee and C.S. Wu: Weak Interactions. Ann. Rev. Nucl. Sci., 15, 381, /1965/.

[2] M. Jacob: Algebre des Courants. Rapport CEA-R3346, 1967.

[3] Topical Conference on Weak Interactions. CERN, Geneva, 14-17 January 1969. CERN 69-7.

[4] J.J. de Swart: Symmetries of Strong Interactions. Proceedings of the 1966 CERN School of Physics, Noordwijk aan Zee, June 8-18, 1966. CERN 66-29, Vol. II p.1.

[5] T.D. Lee: A Possible Way to Remove Divergencies in Physics. p.427 in ref. [3].

[6] R. Gatto: Cabibbo Angle and SU_2XSU_2 Breaking. Springer Tracts in Modern Physics, Vol. 53. Springer-Verlag, Berlin, New-York.

[7] Particle Data Group: Review of Particle Properties. Rev. Mod. Phys., 42, No 1, January 1970. UCRL 8030-Pt 1, (Rev.).

[8] R.E. Marshak, Riazuddin, C.P. Ryan: Theory of Weak Interactions in Particle Physics. Monographs and Texts in Physics and Astronomy, Vol. 24, Wiley-Interscience, 1969.

Printed in the Central Research Institute
for Physics, Budapest, Hungary

Kiadja a Könyvtár Kiadói Osztály
O.v. dr. Farkas Istvánné
Szakmai lektor: Marx György
Nyelvi lektor: Hraskó Péter
Példányszám: 305   Munkaszám: 5123
Készült a KFKI házi sokszorositójában
F.v.: Gyenes Imre