KFKI
31/1968

ON A SYMMETRY OF THE DIRAC EQUATION

Á. Sebestyén

HUNGARIAN ACADEMY OF SCIENCES
CENTRAL RESEARCH INSTITUTE FOR PHYSICS

BUDAPEST

ON A SYMMETRY OF THE DIRAC EQUATION

Á. Sebestyén
Central Research Institute for Physics, Budapest, Hungary.

INTRODUCTION

The requirement of invariance of a theory under a certain group may be satisfied in various ways depending on the form of the axioms or principles which serve as a basis for the theory in question. If e.g. covariance is demanded and one has a Lagrangian then it has to be written down in a covariant form, while in an S matrix theory the relevant generators must commute with the scattering operator.

It is interesting however that in the case of a Lagrangian formalism if only covariance is required some further rather implicit symmetry group of the problem may be present. This symmetry group is to be distinguished from the Poincaré group though its existence is a consequence of invariance of the theory under the inhomogeneous Lorentz group, thus constraints of covariance yield definitive conditions on its mathematical structure.

The existence of such an implicit symmetry group is shown in this paper. A few comments are made in part I concerning a special type of representations of groups. Part II contains some general properties of the supposed symmetry group. In part III a symmetry group is actually constructed and some features of its mathematical structure are clarified. Part IV gives the relationship with the homogeneous Lorentz transformations.

In the paper we use Lagrangians of the classical form well known from textbooks although it is readily admitted, that their role in the description of elementary particle processes seems dubious in the light of recent research.

Throughout the paper the notations and form of matrices of ref [1] are used. The sign of summation is dropped everywhere, it is shown by repeated indices. Latin indices always take the values 0,1,2,3 and they occur in co- or contravariant position except if they denote spinor components when they take the values 1,2,3 and 4. Greek indices run through 1,2,3 and their position is irrelevant. The metric is $g^{oo} = 1$; $g^{11} = g^{22} = g^{33} = -1$.

I.

Consider now a field $\psi_i(x)$ having n components /i = 1,.....,n/ completely characterized by giving the value of i and coordinates of the point x in space-time. The field characterized thus by n functions may be regarded in an other way too, namely it is an infinite set of numbers each being in a one to one correspondence to the five numbers $/i,x^o,x^1,x^2,x^3/$. In other words $\psi_i(x)$ may be regarded as a component of an infinite dimensional vector ψ of an infinite, more exactly continuously infinite dimensional vector space. In this picture i and the numbers $/x^o, x^1, x^2, x^3/$ play the role of one compound index, i having the range 1,...,n and $/x^o, x^1, x^2, x^3/$ all having the range $(+\infty, -\infty)$. A matrix D of this space has two indices $\{i,x\}$ and $\{k,x'\}$ i.e. it is of the form $D_{ik}(x,x')$ and if it is applied to a "vector" ψ one gets

$$\psi_i'(x) = \int D_{ir}(x,x')\ \psi_r(x')\ d^4x'$$

where besides the sum over r there is a "sum" over the continuous part of the index x' which is actually an integral.

Suppose now that a Lie group G is given and there is a one to one correspondence between elements $g\epsilon G$ and the matrices D :

$$g \to D_{ik}^{(g)}(x,x')$$

The product of two elements $g, h\epsilon G$ is mapped in the following manner:

$$gh \to \int D_{ir}^{(g)}(x,x'')\, D_{rk}^{(h)}(x'',x')\, d^4x'' .$$

For the unit element $e\epsilon G$ one obviously has

$$e \to \delta_{ik}\ \delta(x - x') .$$

The element in the neighbourhood of e is mapped into a matrix of the form

$$e \to \delta_{ik}\ \delta(x - x') + \delta\eta^s\ G_{ik}^s\ (x,x')$$

if the parameters $\delta\eta^s$ /s=1,..., k/ are small enough. The matrices G_{ik}^s correspond to the generators of the associated Lie algebra. The commutation relations of the group may be written down also without any difficulty:

$$\int \left[G^a_{ir}(x,x'')\, G^b_{rk}(x'',x') - G^b_{ir}(x,x'')\, G^a_{rk}(x'',x') \right] d^4x'' =$$

$$= C_{abs}\, G^s_{ik}(x,x')$$

with C_{abc} the structure constants of the algebra.

This kind of representations obviously bears a formally striking resemblance with that of the ordinary types.

II

Suppose now that a group G is represented in the manner introduced in the preceeding section in the space of the field quantities. If the field is an operator at the same time in the space of physical states then it satisfies the well known relation

$$U(\Lambda,a)\, \psi_i(x)\, U^+(\Lambda,a) = S_{ir}(\Lambda^{-1})\, \psi_r(\Lambda x + a) \qquad /1/$$

where $U(\Lambda,a)$ is a unitary representation of the element (Λ,a) of the Poincaré group; Λ being a homogeneous Lorentz transformation and a a translation:

$$x' = \Lambda x + a$$

The matrix $S_{ir}(\Lambda)$ represents Λ in the component space of ψ. For e.g. a Lorentz transformation along the positive z axis in the case of Dirac spinor it reads:

$$S_{ik}(\Lambda) = \mathrm{ch}\,\frac{\phi}{2}\,\delta_{ik} - (\gamma^o\gamma^3)_{ik}\cdot \mathrm{sh}\,\frac{\phi}{2} \quad .$$

Now if for an element g of our group the corresponding transformation is

$$\psi'_i(x) = \int D^{(g)}_{ir}(x,x')\, \psi_r(x')\, d^4x' \qquad /2/$$

then the relation /1/ will give restrictions on the structure of $D^{(g)}_{ir}(x,x')$. We consider first only translation; homogeneous Lorentz transformations being more involved will be considered presently /see part IV/. If we put $\Lambda = I$ then /1/ reads:

$$U(I,a)\, \psi_i(x)\, U^+(I,a) = \psi_i(x + a) \quad . \qquad /3/$$

Applying this relation to /2/ we get:

$$U(I,a)\ \psi'_i(x)\ U^+(I,a) = \psi'_i\ (x + a) =$$

$$= \int D^{(g)}_{ir}(x,x')\ U\ (I,a)\ \psi_r(x')\ U^+(I,a)\ d^4x' =$$

$$= \int D^{(g)}_{ir}(x,x')\ \psi_r\ (x' + a)\ d^4x'$$

which by substituting $x'+a \to x'$ yields

$$\psi'_i(x + a) = \int D^{(g)}_{ir}(x,x'-a)\ \psi_r(x')\ d^4x' \qquad /4/$$

while putting $x+a$ instead of x in /2/ we get

$$\psi'_i(x + a) = D^{g}_{ir}(x+a,x')\ \psi_r(x')\ d^4x'$$

which together with /4/ gives the translation invariance of the function $D^{(g)}_{ik}$ i.e. it is the function of only $x - x'$ and thus has the form

$$D^{(g)}_{ik}(x - x')\ .$$

Obviously the generators are also translation invariant; so for small $\delta\eta^s$ parameters we get

$$D^{(g)}_{ik}(x-x') \sim \delta_{ik}\ \delta(x-x') + \delta\eta^s\ G^s_{ik}\ (x-x')\ .$$

Now we pick up the problem of explicit construction of the generators for the simple case of a spin one-half free particle.

III

Let $\psi(x)$ be the field of a free Dirac particle and let the Fourier transforms of the generators fulfill the relation:

$$\left[F^s(k)\ ,\ \hat{k}\right] = 0 \qquad /5/$$

where $G^s(x) = \int F^s(k)\ e^{ikx}\ d^4k$ $\hat{k} = \gamma^o k^o - \bar{\gamma}\ \bar{k}$
The function

$$\psi'(x) = \psi(x) + d\eta^s \int G^s(x-x')\,\psi\ (x')\ d^4x' \qquad /6/$$

will be then a solution of the Direac equation provided

$$i\gamma^n \frac{\partial\psi(x)}{\partial x^n} - m\psi(x) = 0 \qquad /7/$$

is satisfied as for such $G^s(x)$ the equation

$$\int \left[\gamma^n \frac{\partial G^s(x-x')}{\partial x^n} \psi(x') - G^s(x-x')\gamma^n \frac{\partial \psi(x')}{\partial x'^n} \right] d^4x' = 0 \qquad /8/$$

holds.

Calculate now the general form of $F^s(k)$. Being a four by four matrix it can be written in the form:

$$F^s(k) = A^s + B^s_r \gamma^r + C^s_{rt}\gamma^r\gamma^t + D^5_r \gamma^5\gamma^r + E^s\gamma^s \qquad /9/$$

To retain covariance we claim the quantities A, B_r, C_{rt}, D_r and E to be scalar, vector, antisymmatric tensor, vector and scalar functions of k respectively, this will play an essential role in part IV. After substituting /9/ into /5/ we follow the procedure of multiplying by various elements of the γ algebra and taking traces. Thus we get the following conditions:

$$C^s_{rt}\, k^t = 0 \ ; \ D^s_r\, k^r = 0 \qquad /10/$$

whereas A^s is arbitray, B^s_r is proportional to k_r and $E^s = 0$, these last terms all being equivalent in the transformation /2/ to a multiplication by a number will be left out from the consideration.

If the field $\psi(x)$ is an operator field then the natural question of invariance of the commutation relations arises. Commuting the transformed functions $\psi'(x)$ and $\psi'(y)$ the transformation being infinitesimal /see (6) we get:

$$[\psi'(x), \bar{\psi}'(y)] = \frac{1}{i} S(x-y) +$$

$$+ \frac{\delta\eta^2}{i} \int d^4x' \left[G^s(x-x')\, S\,(x'-y) + S(x-x')\, \bar{G}^s(y-x') \right]$$

where the function $\frac{1}{i} S(x)$ is the anticommutator function of $\psi(x)$:

$$[\psi(x) \; , \; \bar{\psi}(y)] = \frac{1}{i} S(x-y)$$

Now the transformation leaves the anticommutator relation invariant if

$$\int d^4x' \left[G^s(x-x') \, S\,(x'-y) + S(x-x') \, \bar{G}^s(y-x')\right] = 0$$

The necessary and sufficient condition for this equation to be satisfied is that the functions C^s_{rt} and D^s_r be real. This can be seen by taking Fourier transforms in the last equation and using /5/ and the expression [1]:

$$S(x) = \frac{i}{(2\pi)^3} \int e^{ikx} \, \delta\left(k^2-m^2\right) \varepsilon\left(k^0\right) \left(\hat{k}-m\right) d^4k$$

Thus from now on we shall use real quantities for C^s_{rt} and D^s_r and put $A=B_r=E=0$ /see the remark following equation /10/ /.

Now we show that the matrices $D^s_r \gamma^5 \gamma^r$ and $C^s_{rt} \gamma^r \gamma^t$ constitute a Lie algebra. The number of linearly independent vectors - necessarily space-like - satisfying $D^s_r k^r = 0$ is three, denote them by $D^\alpha (\alpha=1,2,3)$. Also the number of independent antisymmetric tensors C^s_{rt} fulfilling $C^s_{rt} k^t = 0$ is three, denote them by C^α. The number of parameters of our group is consequently six. If we take

$$D^\alpha_r \, D^{\beta,r} = -\frac{1}{4} \, \delta^{\alpha\beta} \tag{11}$$

then it is easy to show that the matrices

$$N^\alpha = i \, D^\alpha_r \gamma^5 \gamma^r \quad ; \quad M^\alpha = i \, \varepsilon^{\alpha\rho\sigma} \, D^\rho_r \, D^\sigma_t \, \gamma^r \gamma^t$$

$$\left(D^\alpha_r \, k^r = 0\right) \tag{12}$$

where $\varepsilon^{\alpha\beta\gamma}$ is the three dimensional Levy Civita symbol, will follow the commutation rules

$$\left[M^\alpha, \, M^\beta\right] = i \, \varepsilon^{\alpha\beta\rho} \, M^\rho \; ; \; \left[N^\alpha, \, N^\beta\right] = i \, \varepsilon^{\alpha\beta\rho} \, M^\rho$$

$$\left[M^\alpha, \, N^\beta\right] = i \, \varepsilon^{\alpha\beta\rho} \, N^\rho \tag{13}$$

and the vectors D_r^α and tensors $C_{rt}^\alpha = \varepsilon^{\alpha\rho\sigma} D_r^\rho D_t^\sigma$ will satisfy /10/ and consequently /5/. The algebra is obviously isomorphic to an SO/4/ algebra.

Using the relation

$$\gamma^{\mu+} = \gamma^o \gamma^\mu \gamma^o$$

it can easily be checked that

$$\bar{M}^\alpha = \gamma^o M^{\alpha+} \gamma^o = M^\alpha$$

$$\bar{N}^\alpha = \gamma^o N^{\alpha+} \gamma^o = N^\alpha$$

The Casimirians are $M^\rho M^\rho + N^\rho N^\rho$ and $M^\rho N^\rho$.
The first may readily be calculated:

$$M^\rho M^\rho + N^\rho N^\rho = \frac{3}{2} .$$

As to the second we note the identities:

$$M^\rho N^\rho = \varepsilon^{\rho\sigma\tau} D_r^\rho D_s^\sigma D_t^\tau \gamma^5 \gamma^r \gamma^s \gamma^t =$$

$$= \frac{1}{6} \varepsilon^{\rho\sigma\tau} \varepsilon_{qrst} \varepsilon^{quvz} D_u^\rho D_v^\sigma D_z^\tau \gamma^5 \gamma^r \gamma^s \gamma^t$$

and

$$\frac{1}{6} \varepsilon_{qrst} \gamma^5 \gamma^r \gamma^s \gamma^t = \gamma_q .$$

Thus

$$M^\rho N^\rho = \varepsilon^{\rho\sigma\tau} \varepsilon^{quvz} D_u^\rho D_v^\sigma D_z^\tau \gamma_q .$$

Now the vector $\varepsilon^{\rho\sigma\tau} \varepsilon^{quvz} D_u^\rho D_v^\sigma D_z^\tau$ is orthogonal to any of the vectors D and is not a zero vector so it must be parallel to k i.e.

$$M^\rho N^\rho = c\hat{k}$$

where c is a function of m only.

IV

Now we proceed to give a more explicit form of the vectors D^{α}, and strive to satisfy the relation /1/ for homogeneous Lorentz transformations. To construct explicitly the orthogonal system D^1, D^2, D^3 one must start with the linearly independent set e^1, e^2, e^3 and k, where for the vectors e^{α} one can take spacelike unit vectors coinciding with the coordinate axes in a specific Lorentz frame:

$$e^{\alpha,r} = \delta^{\alpha r} \quad ; \quad e^{\alpha}_{r} = -\delta^{\alpha r}$$

$$e^{\alpha}_{r}\, e^{\beta,r} = -\delta^{\alpha\beta} \qquad /14/$$

$$/\alpha,\beta = 1,\ 2,\ 3\ /$$

Now it is well known that the helicity operator has the form $c_{rs}\,\gamma^r\gamma^s$ and it commutes with $\hat{k}$, so it is possible to take the helicity for M^3. In doing so a matrix must be constructed having a covariant structure and being proportional to the helicity in the frame /14/:

$$M^3 \sim \varepsilon^{\rho\sigma\tau}(e^{\tau}k)\, e^{\rho}_{r}\, e^{\sigma}_{t}\, \gamma^r\gamma^t \qquad /15/$$

In the frame /14/, /15/ gives

$$\varepsilon^{\rho\sigma\tau}\,(e^{\tau}k)\, e^{\rho}{}_{r}\, e^{\sigma}_{t}\, \gamma^r\gamma^t = -\left(k^1\gamma^2\gamma^3 + k^2\gamma^3\gamma^1 + k^3\gamma^1\gamma^2\right)$$

being just the helicity. The next step is to construct $N^3 = i\, D^3_r\, \gamma^5\gamma^r$. D^3 has the form:

$$D^3 = \alpha k + \beta_1 e^1 + \beta_2 e^2 + \beta_3 e^3$$

(k, e^1, e^2, e^3) being a linearly independent system. The following relations serve to fix the coefficients α, β_1, β_2, β_3

$$D^3_r\, D^{3,r} = -\frac{1}{4} \quad ; \quad \left[M^3, N^3\right] = 0 \quad ; \quad D^3_r\, k^r = 0$$

yielding:

$$N^3 = \pm \frac{i}{2m} \sqrt{\frac{(e^\tau k)(e^\tau k)}{m^2+(e^\rho k)(e^\rho k)}} \left(k_r - \frac{m^2}{(e^\sigma k)(e^\sigma k)} (e^\chi k)\, e_r^\chi \right) \gamma^r \qquad /16/$$

here $(e^\rho k) = e_r^\rho\, k^r$.

If we now want to construct D^1 and D^2 it can easily be checked that if

$$\beta_\rho^1 (e^\rho k) = 0 \;; \quad \beta_\rho^1\, \beta_\rho^1 = \frac{1}{4} \;; \quad \beta_\alpha^2 = \frac{1}{\sqrt{(e^\rho k)(e^\rho k)}}\, \varepsilon^{\alpha\chi\sigma} (e^\sigma k)\, \beta_\chi^1$$

then putting $D^1 = \beta_\rho^1\, e^\rho$, $D^2 = \beta_\rho^2\, e^\rho$ and using the formulae /12/ for the construction N^1, N^2, M^1, M^2 and M^3 these quantities together with /16/ will satisfy the commutation rules /13/.

For the sake of completeness we write down the generators in the frame /14/

$$M^1 = i\, \frac{|\bar{k}|}{m}\, \gamma^o\, \bar{D}^2\, \bar{\gamma} + \frac{k^o}{2m}\, \bar{D}^1\, (\bar{\gamma} \times \bar{\gamma})$$

$$M^2 = -i\, \frac{|\bar{k}|}{m}\, \gamma^o\, \bar{D}^1\, \bar{\gamma} + \frac{k^o}{2m}\, \bar{D}^2\, (\bar{\gamma} \times \bar{\gamma})$$

$$M^3 = \frac{i}{4}\, \frac{\bar{k}}{|\bar{k}|}\, (\bar{\gamma} \times \bar{\gamma})$$

$$N^1 = i\, \gamma^5\, \bar{D}^1\, \bar{\gamma}$$

$$N^2 = i\, \gamma^5\, \bar{D}^2\, \bar{\gamma}$$

$$N^3 = -i\, \frac{|\bar{k}|}{2m}\, \gamma^5 \gamma^o + \frac{i k^o}{2m|\bar{k}|}\, \gamma^5\, \bar{k}\, \bar{\gamma} \quad .$$

As the vectors D^1, D^2 have no time components we can say that $\bar{D}^1$, $\bar{D}^2$ and $\bar{k}$ are mutually orthogonal vectors in the ordinary three dimensional space.

Consider now the homogeneous Lorentz transformations. /1/ reads now:

$$U(\Lambda,o)\ \psi_i(x)\ U^+(\Lambda,o) = S_{ir}\ (\Lambda^{-1})\psi_r(\Lambda x)$$

giving for Fourier transforms

$$U(\Lambda,o)\ \psi_i(k)\ U^+\ (\Lambda,o) = S_{ir}\ (\Lambda^{-1})\psi_r(\Lambda k)$$

If the generators are defined by means of the triplet e^1, e^2, e^3 /see the previous procedure/ then for an infinitesimal transformation we should write precisely:

$$\psi_i'(x) = \psi_i(x) + \delta\eta^s \int G^s_{ir}\left(e^1, e^2, e^3, (x-x')\right) \psi_r(x')\, d^4x' \qquad /17/$$

Showing explicitly that the form of the generators depends on the triplet. Now applying $U(\Lambda,o)$ /17/ gives:

$$U(\Lambda,o)\ \psi_i'(x)\ U^+(\Lambda,o) = U(\Lambda,o)\ \psi_i(x)\ U^+(\Lambda,o) +$$

$$+ \delta\eta^s \int G^s_{ir}\left(e^1, e^2, e^3, (x-x')\right)\ U\ (\Lambda,o)\ \psi_r(x')\ U^+\ (\Lambda,o)\ d^4x' =$$

$$= S_{ir}(\Lambda^{-1})\ \psi_r(\Lambda x) + \delta\eta^s \int G^s_{ir}\ \left(e^1, e^2, \epsilon^3, (x-x')\right) S_{rt}\left(\Lambda^{-s}\right)\ \psi_t(\Lambda x')\ d^4x'$$

or

$$\psi_i'(\Lambda x) = \psi_i(\Lambda x) + \delta\eta^s \int S_{ir}(\Lambda)\ G^s_{rt}\left(e^1, e^2, e^3, (x-x')\right) S_{tz}\left(\Lambda^{-1}\right)\ \psi_z\ (\Lambda x')\ d^4x' =$$

$$= \psi_i(\Lambda x) + \delta\eta^s \int S_{ir}(\Lambda)\ G^s_{rt}\left(e^1, e^2, e^3, (x-\Lambda^{-1}\ x')\right) S_{tz}\ \left(\Lambda^{-1}\right)\ \psi_z(x')\ d^4x'\ . \qquad /18/$$

While, taking directly all the vectors in the transformed frame /17/ gives

$$\psi_i'(\Lambda x) = \psi_i(\Lambda x) + \delta\eta^s \int G^s_{ir}\ \left(e^{1'}, e^{2'}, e^{3'}, (\Lambda x - x')\right) \psi_r(x')\ d^4x' \qquad /19/$$

where $e^{\alpha'} = \Lambda e^{\alpha}$

/18/ and /19/ give

$$G^{s}_{ik}\left(e^{1'},e^{2'},e^{3'},(\Lambda x-x')\right) = S_{ir}(\Lambda)\ G^{s}_{rt}\left(e^{1},e^{2},e^{3},(x-\Lambda^{-1}x')\right)\ S_{tk}(\Lambda^{-1}) \qquad /20/$$

as the necessary and sufficient condition for the constraint of covariance to be met.

If we take Fourier transforms in /20/ then

$$\int F^{s}\left(e^{1'},e^{2'},e^{3'},k\right) e^{ik(\Lambda x-x')}\, d^4k =$$

$$= \int S(\Lambda)F^{s}\left(e^{1},e^{2},e^{3},k\right) S(\Lambda^{-1})\, e^{ik\left(x-\Lambda^{-1}x'\right)}\, d^4k$$

where the spinor indices are supressed. Substituting $\Lambda k \to k$ and taking into account $k\Lambda^{-1}x=\Lambda kx$ we get the condition for the Fourier transforms:

$$S(\Lambda)\ F^{s}\left(\Lambda^{-1}e^{1'},\Lambda^{-1}e^{2'},\Lambda^{-1}e^{3'},\Lambda^{-1}k\right) S\left(\Lambda^{-1}\right) = F^{s}\left(e^{1'},e^{2'},e^{3'},k\right)$$

which in turn is obvious as $S(\Lambda)\gamma\, S\left(\Lambda^{-1}\right)=\Lambda^{-1}\gamma$ and the matrices F^{s} where built up covariantly from the vectors k, e^{1}, e^{2}, e^{3}, and the matrices γ^{i}.

Finally we remark that the vectors e^{α} in the construction might be taken to be any triplet for which the set k, e^{α} is independent.

Conclusions

We have seen that in a Poincaré covariant theory of a free Dirac particle an implicit symmetry group G emerges. Its representation is of the type

$$g(\epsilon G) \to D^{(g)}_{ik}(x - x')$$

i.e. the effect of a transformation associated with the group element g on the spinor ψ is the following:

$$\psi_i'(x) = \int D_{ir}^{(g)}(x - x')\, \psi_r(x')\, d^4x'$$

By requiring the Lagrangian to be invariant under these transformations we have constructed the generators explicitly and have seen that they constitute a six parameter Lie algebra and the neighbourhood of the unity is mapped into "matrices" of the kind

$$\delta_{ik}\, \delta(x-x') + \delta\eta^s\, G_{ik}^s(x-x')\ .$$

The algebra is locally isomorphic to the SO/4/ algebra.

By construction the requirements of covariance are fulfilled.

Further relationship with the algebra of currents and theory of interactions will be published presently.

Acknowledgements

The author wishes to express sincere thanks to Mr.J. Kóta and Dr. A.Frenkel for valuable discussions.

R e f e r e n c e

1. Bogol'ubov and Shirkov
Introduction into the Theory of Quantized Fields
Interscience Publishers 1959.

Printed in the Central Research Institute for Physics, Budapest
Kiadja a Könyvtár- és Kiadói Osztály. O.v.: dr.Farkas Istvánné
Szakmai lektor: Frenkel Andor. Nyelvi lektor: Perjési Zoltán
Példányszám: 210 Munkaszám: KFKI 4136 Budapest, 1968.december 18.
Készült a KFKI házi sokszorositójában. F.v.: Gyenes Imre