KFKI
23 / 1968

3-DIMENSIONAL „RELATIVITY" FOR AXISYMMETRIC STATIONARY SPACE-TIMES

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BUDAPEST

3- DIMENSIONAL "RELATIVITY" FOR AXISYMMETRIC STATIONARY SPACE-TIMES

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The equivalence of the axisymmetric stationary vacuum gravitational field problem to a 3-dimensional "relativity theory" in the presence of a certain scalar matter field is shown. An invariant classification can be achieved with respect to the algebraic structure of the 3-dimensional trace-free Ricci-tensor. The extension of these results to electrovac spaces is also discussed.

1. Introduction

H. Levy [1] found recently a 3-dimensional stress tensor for axisymmetric stationary gravitational fields. In the present work we shall extend his result to a 3-dimensional covariant formulation of the problem. The basic conception will be developed in Section 2. Here we shall show that the axisymmetric stationary gravitational field problem in vacuo is completely equivalent to a 3-dimensional relativity theory in the presence of a certain "matter field" and endowed with axial symmetry. In Section 3 we propose an invariant classification of the related space-times based on the algebraic properties of the 3-dimensional Ricci tensor. Section 4 deals with the electrovac problem, while in the summary we shall discuss the various new possibilities which are offered by our method for the search of the axially symmetrical space-times.

2. Foundation of the 3-dimensional "relativity"

The line element of an arbitrary axisymmetric stationary vacuum gravitational field may be written [2]:

$$dS^2 = f^{-1}\, ds^2 - f\left(dt + \omega d\Phi\right)^2 \qquad /1/$$

where

$$ds^2 = e^{2\gamma}\left(d\rho^2 + dz^2\right) + \rho^2\, d\Phi^2 \qquad /2/$$

and f, ω, γ are functions of $x^1=\rho$ and $x^2=z$ only.
Using the notation of F.J. Ernst [3], we introduce the function φ by

$$\nabla\varphi = -f^2 \rho^{-1}\, \hat{n} \times \nabla\omega . \qquad /3/$$

Here ∇ is the 3-dimensional gradient operator and $\hat{n}$ stands for a unit vector which points in the $d\Phi$ direction. We remark that φ agrees with Papapetrou's scalar function [4] A. The asymptotic conditions for the field of a bounded rotating source with mass m and angular momentum $a.m$ are:

if $r = \left(\rho^2 + z^2\right)^{1/2} \to \infty$,

$$f \to 1 - \frac{2m}{r}\ ; \quad \varphi \to \frac{amz}{r^3}\ ; \quad \gamma \to -\frac{m^2}{2}\,\frac{\rho^2}{r^4} . \qquad /4/$$

The field equations constitute two groups, the first of which is easily put down without referring to any particular coordinate system:

$$\left.\begin{aligned} f\,\Delta f &= \nabla f\,\nabla f - \nabla\varphi\,\nabla\varphi, \\ f\,\Delta\varphi &= 2\nabla f\,\nabla\varphi \quad . \end{aligned}\right\} \qquad /5/$$

Δ denotes the Laplace operator in 3-dimensional Euclidean space. The second group of the field equations determines γ in terms of f and φ :

$$\left.\begin{aligned} \gamma_1/\rho &= \frac{1}{4f^2}\left(f_1^2 - f_2^2 + \varphi_1^2 - \varphi_2^2\right) , \\ \gamma_2/\rho &= \frac{1}{2f^2}\left(f_1\, f_2 + \varphi_1\,\varphi_2\right) \end{aligned}\right\} \qquad /6/$$

/Lower indices denote partial derivatives /. The right hand sides of the system /6/ can be written as the $-T_{11}$ and $-T_{12}$ component, respectively, of the 3-dimensional symmetric tensor

$$T_{ik} = -\frac{1}{2f^2}\left\{ f_{,i}\, f_{,k} + \varphi_{,i}\,\varphi_{,k} - \frac{1}{2} g_{ik}\left[(\nabla f\ \nabla f) + (\nabla\varphi\ \nabla\varphi)\right]\right\} . \qquad /7/$$

This tensor in a slightly different form was found first by H. Lévy [1], who has shown that the divergence of T_{ik} vanishes, and stated that T_{ik} has the properties of a gravitational stress tensor. At this point it is natural to ask, whether a generalization of the definition /7/ to a curved 3-space V_3 exists. Then equations /6/ would become the gravitational equations in V_3 , and eq.s /5/ would appear as the "material field equations".

Choosing /2/ as the line element of V_3 , and calculating

$$G_{ik} = R_{ik} - \frac{1}{2} g_{ik} R , \qquad /8/$$

one gets

$$G_{11} = -G_{22} = -\gamma_1/\rho \ ; \ G_{12} = -\gamma_2/\rho \ ; \ G_{33} = -\rho^2 e^{-2\gamma}\left(\gamma_{11}+\gamma_{22}\right), \qquad /9/$$

the remaining components vanish. Eq. /6/ turns out to be the /11/ and /12/ component of the gravitational equations

$$G_{ik} = T_{ik} , \qquad /10/$$

while $G_{33} = T_{33}$ is a consequence of the field eq.s /5/, /6/. The definition of T_{ik} remains formally /7/, using now the line element /2/. The covariant divergence of T_{ik} again vanishes and the form of the field eq.s /5/ may be maintained changing the definition of the Laplace operator to

$$\Delta f \stackrel{\text{def}}{=\!=} g^{ij} f_{;i} f_{;j} .$$

It is easily seen from the line element /1/ that the abstract space V_3, in which the "second relativization" of the problem proceeds, is equivalent to the hypersurfaces $dt + \omega d\Phi = 0$ up to a conform factor f^{-1}.

Following F.J. Ernst, we introduce the complex "material field" in V_3 by

$$\varepsilon = f + i\varphi . \qquad /11/$$

Then eq.s /5/, /6/ may be written:

$$(\mathrm{Re}\ \varepsilon)\varepsilon^{;i}{}_{;i} = \varepsilon^{;i}\varepsilon_{;i} , \qquad /12/$$

$$R_{ik} - \frac{1}{2} g_{ik} R = - \frac{1}{4(\mathrm{Re}\varepsilon)^2}\left(\varepsilon_{;i}\varepsilon^*_{;k} + \varepsilon^*_{;i}\varepsilon_{;k} - g_{ik}\varepsilon_{;r}\varepsilon^{*;r}\right) . \qquad /13/$$

These equations can be derived from the Lagrangian

$$L = R - \frac{1}{4}\frac{\varepsilon_{;i}\varepsilon^{*;i}}{(\mathrm{Re}\ \varepsilon)^2} \qquad /14/$$

by using the variational principle.

Sometimes it is more convenient to introduce the function ξ /Ref. [3], see also Sec. 4./ by

$$\xi = \frac{1+\varepsilon}{1-\varepsilon} . \qquad /15/$$

We remark that in terms of the field variable ξ the stress tensor is expressed as follows:

$$T_{ik} = -\left(\xi\xi^* - 1\right)^{-2}\left(\xi_{;i}\xi^*_{;k} + \xi^*_{;i}\xi_{;k} - g_{ik}\xi_{;r}\xi^{*;r}\right) . \qquad /16/$$

3. Invariant classification

As is well known, in V_3 the Weyl tensor C_{ijkl} vanishes, so that the relationship between the curvature tensor and Ricci tensor reads [5]:

$$R_{ijkl} = -g_{il}R_{jk} + g_{ik}R_{jl} - g_{jk}R_{il} + g_{jl}R_{ik} + \frac{1}{2}R\left(g_{il}g_{jk} - g_{ik}g_{jl}\right) . \qquad /17/$$

The classification with respect to the algebraic structure of the curvature tensor is therefore completely equivalent in V_3 with that of R_{ij} [6]. We shall deal with the tracefree part P^k_i of the Ricci tensor:

$$P^k_i = R^k_i - \frac{1}{3}\delta^k_i R \quad . \qquad /18/$$

It is convenient to use in the calculations the line element /2/. In this coordinate system we find:

$$\left[P^k_i\right] = \frac{e^{-2\gamma}}{3}\begin{bmatrix} \gamma_{11}+\gamma_{22}-3\gamma_1/\rho & -3\gamma_2/\rho & 0 \\ -3\gamma_2/\rho & \gamma_{11}+\gamma_{22}+3\gamma_1/\rho & 0 \\ 0 & 0 & -2\gamma_{11}-2\gamma_{22} \end{bmatrix} \qquad /19/$$

The eigenvalue-problem

$$P^k_i\, v^i = \lambda\, v^k \qquad /20/$$

leads to the following characteristic equation:

$$\left[2(\gamma_{11}+\gamma_{22}) + \Lambda\right]\left[(\gamma_{11}+\gamma_{22} - \Lambda)^2 - 9(\gamma_1^2 + \gamma_2^2)/\rho^2\right] = 0 \qquad /21/$$

where $\Lambda = 3e^{2\gamma}\lambda$. The solutions of eq. /21/ are:

$$\Lambda_o = -2(\gamma_{11} + \gamma_{22}) \quad ,$$
$$\Lambda_\pm = \gamma_{11} + \gamma_{22} \pm 3\rho^{-1}\left(\gamma_1^2 + \gamma_2^2\right)^{1/2} \qquad /22/$$

Using the field eq.s /12/, /13/, the eigenvalues can be written in an invariant form:

$$\left.\begin{aligned} \lambda_o &= \frac{1}{6}\left|\frac{\nabla\varepsilon}{\mathrm{Re}\ \varepsilon}\right|^2 \quad , \\ \lambda_\pm &= -\frac{1}{12}\,\frac{|\nabla\varepsilon|^2 \mp 3|\nabla\varepsilon\ \nabla\bar{\varepsilon}|}{(\mathrm{Re}\ \varepsilon)^2} \quad . \end{aligned}\right\} \qquad /23/$$

The relation $\lambda_o+\lambda_++\lambda_- = 0$ is satisfied because $\left[P^k_i\right]$ is tracefree.

If we restrict ourselves to physically realistic spaces, for which the asymptotic conditions /4/ hold, we have asymptotically:

$$\lambda_o = \lambda_+ = -2\lambda_- = \frac{2}{3}\frac{m^2}{r^4} \quad . \qquad /24/$$

The type of the Ricci tensor is asymptotically degenerate /D/. The alternative possibility is that the degeneracy /24/ does not occur, in that case the type is called general /G/. All other possibilities, as e.g. $\lambda=0$ are excluded by the asymptotic conditions /4/ for physically interesting spaces.

Now we have shown that the axisymmetric stationary graviational fields may be classified in an invariant manner, with respect to the algebraic structure of their Ricci tensor in the corresponding space V_3 . The possible types are D and G.*

From eq. /23/ is learned the necessary and sufficient condition of the degeneracy:

$$f_1 \varphi_2 - f_2 \varphi_1 = 0 \quad . \qquad /25/$$

So the class of the type D vacuum solutions consists exhaustively of the static /Weyl/ spaces [5] and the Papapetrou's solutions [2] for that one has

$$\Lambda_o = \Lambda_+ = \frac{1}{2}\left(\chi_{12}^2 + \chi_{22}^2\right) \qquad /26/$$

where χ is an arbitrary harmonic function. However, we recall the well known fact that Papapetrou's solutions are physically unacceptable because they do not satisfy the asymptotic conditions /4/ /the mass monopole term is lacking/.

All other relevant spaces, among others the Kerr metric [7], being now the only known solution of the field eq.s /12/, /13/ which can really represent the gravitational field of a bounded rotating source, are of type G. For the Kerr solution one has

$$\left.\begin{aligned} \lambda_o &= \frac{2}{3}\,\frac{m^2}{\left(r^2-2mr+a^2\cos^2\theta\right)^2}\left(1+\frac{2a^2\sin^2\theta}{r^2-2mr+a^2\cos^2\theta}\right) \\ \lambda_{\pm} &= -\frac{1}{3}\,\frac{m^2}{\left(r^2-2mr+a^2\cos^2\theta\right)^2}\left(1 \mp 3 + \frac{2a^2\sin^2\theta}{r^2-2mr+a^2\cos^2\theta}\right) \end{aligned}\right\} \qquad /27/$$

where the functions $r(\rho,z)$ and $\theta(\rho,z)$ are defined by the relations

$$\left.\begin{aligned} \rho &= \left(r^2-2mr+a^2\right)^{1/2}\sin\theta \;, \\ z &= (r-m)\cos\theta \;. \end{aligned}\right\} \qquad /28/$$

It is seen from /27/, /28/ also that the space becomes asymptotically of type D.

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* The present classification differs from that of Lévy [1].

4. The electrovac problem

As we shall see, the extension of the 3-dimensional "relativity" to electrovac spaces yields rather complicated expressions, although the results are very similar to those in the absence of electromagnetism. The only exception is that in the electrovac case the type D metrics are not all known.

Our notation is in agreement with Ref. [8]: A_μ stands for the electromagnetic 4-potential, and the field variables A_3', Φ, φ, ε used here are defined by the relations

$$\hat{n} \times \nabla A_3' = \rho^{-1} f \left(\nabla A_3 - \omega \nabla A_4 \right) , \qquad /29/$$

$$\Phi = A_4 + i A_3' , \qquad /30/$$

$$\hat{n} \times \nabla \varphi = \rho^{-1} f^2 \nabla \omega - 2 \hat{n} \times \mathrm{Im} \left(\Phi^* \nabla \Phi \right) , \qquad /31/$$

$$\varepsilon = f - |\Phi|^2 + i \varphi . \qquad /32/$$

The unit bector $\hat{n}$ is seen to be the Killing vector of the space V_3 with the line element /2/. In properly chosen units the first group of the field equations which govern the axisymmetric stationary electrovac spaces is as follows:

$$\left. \begin{aligned} \left(\mathrm{Re}\varepsilon + |\Phi|^2 \right) \Delta \varepsilon &= \left(\nabla \varepsilon + 2\Phi^* \nabla \Phi \right) \nabla \varepsilon , \\ \left(\mathrm{Re}\varepsilon + |\Phi|^2 \right) \Delta \Phi &= \left(\nabla \varepsilon + 2\Phi^* \nabla \Phi \right) \nabla \Phi . \end{aligned} \right\} \qquad /33/$$

To any solution of the field eq.s /12/, /13/ it is possible to find its "electromagnetic pair" for which $\varepsilon = \frac{\xi' - 1}{\xi + 1}$ is an analytic function of Φ. Then one has [8]

$$\varepsilon = 1 - 2\Phi/q , \qquad /34/$$

where q is a complex number. The field eq.s of Sec. 2 are formally get back by denoting $\xi = \left(1 - qq^*\right)^{-1/2} \xi'$.

For the sake of brevity we introduce the complex 3-vectors

$$\underline{G} = \frac{1}{2}\,\frac{\nabla\varepsilon + 2\Phi^{*}\,\nabla\Phi}{\mathrm{Re}\,\varepsilon + |\Phi|^{2}} \qquad /35/$$

and

$$\underline{H} = \left(\mathrm{Re}\,\varepsilon + |\Phi|^{2}\right)^{-1/2}\,\nabla\Phi\,. \qquad /36/$$

The second group of the electrovac field equations is now equivalent to the "gravitational equations" in V_3 for which the line element is given by eq. /2/ and the stress tensor has the form

$$T_{ik} = -\left\{\left(G_i G_k^{*} + G_i^{*} G_k\right) - \left(H_i H_k^{*} + H_i^{*} H_k\right) - g_{ik}\left(G^r G_r^{*} - H^r H_r^{*}\right)\right\}. \qquad /37/$$

One easily finds now that the corresponding Lagrangian is

$$L = R - G^r G_r^{*} + H^r H_r^{*}\,. \qquad /38/$$

We put down here the eigenvalues of the Ricci tensor also:

$$\lambda_o = \frac{2}{3}\left(|\underline{G}|^2 - |\underline{H}|^2\right),$$
$$\lambda_{\pm} = -\frac{1}{3}\left(|\underline{G}|^2 - |\underline{H}|^2\right) \pm \left\{|\underline{G}^2|^2 + |\underline{H}^2|^2 - 2|\underline{GH}|^2 + 2|\underline{G}^2\underline{H}^2 - (\underline{GH})^2|\right\}^{1/2} \qquad /39/$$

Hence the condition of the degeneracy $\lambda_o = \lambda_+$ is seen to be

$$\left(|\underline{G}|^2 - |\underline{H}|^2\right)^2 = |\underline{G}^2|^2 + |\underline{H}^2|^2 - 2|\underline{GH}|^2 + 2|\underline{G}^2\underline{H}^2 - (\underline{GH})^2|\,. \qquad /40/$$

If /40/ holds, the space is of type D, else G. The asymptotically flat static spaces are once again of type D. Known examples of the static electrovac spaces are the solutions of Weyl [9], for that $\varepsilon = \varepsilon(\Phi)$ is assumed, or the space with the line element

$$ds^2 = -N^2\left(r^2-a^2\cos^2\theta\right)^2\left[(r-m)^2-(a^2+m^2)\cos^2\theta\right]^{-3}\left[dr^2\left((r-m)^2-(a^2+m^2)\right)^{-1} + d\theta^2\right] - N^{-2}\left[(r-m)^2-(a^2+m^2)\right]\left[r^2-a^2\cos^2\theta\right]^2\sin^2\theta\, d\Phi^2 + N^2\left(r^2-a^2\cos^2\theta\right)^{-2} dt^2 \qquad /41/$$

and magnetic potential

$$A_3 = 2mar\sin^2\theta/N \qquad /42/$$

/ a, m are real constants, $N = (r-m)^2 - m^2 - a^2\cos^2\theta$ /. This latter solution was obtained from the Kerr metric by using an accidental symmetry of the axisymmetric spaces [10].

The metric found by Newman et al. [8, 11] is the "electromagnetic pair" of the Kerr solution and as such, is naturally of type G. One can construct the "electromagnetic pairs" of Papapetrou's stationary solutions also, which are then of type D. The existence of other type D electrovac spaces that can in addition be interpreted as the fields of some realistic sources, is still an open question.

5. Summary

After the "second relativization" of the stationary axisymmetric gravitational field problem, many of the present methods of general relativity, as those based on the algebraic features of the curvature tensor /Sec. 2/ or on optical properties, can be applied to this particular problem. These may prove useful in finding new axisymmetric stationary solutions. At this point one can conjecture that certain cylindrical sources are privilegized with respect to the rotation around their axis of symmetry, in that these do not lose energy then by gravitational radiation. This would mean an exclusion principle for the axisymmetric gravitational fields with bounded singularities: some of them could not be stationary.

On the other hand, our procedure may shed some light on the question, whether general relativity can fully be "geometrized", because it gives an example of the situation when the material field becomes part of the metric in a higher dimensional empty space theory.

Further investigations are needed in order to find the possible extensions of our results to more general gravitational fields.

Acknowledgments

The author would like to thank dr. F. Károlyházy for valuable discussions and the entire theoretical group of Central Research Institute for Physics for helpful comments.

Literature

[1] H. Lévy, Nuovo Cimento, 56B, 253 /1968/

[2] A. Papapetrou, Ann.Physik 12, 309 /1953/

[3] F.J. Ernst, Phys.Rev. 167, 1175 /1968/

[4] A. Papapetrou, Ann.Inst. Poincaré iv 83 /1966/

[5] A.Z. Petrov: New Methods in the Theory of General Relativity /russian/, "Nauka",Moscow /1966/

[6] For the classification of the Ricci tensor in four dimensions, see Ref [5], p. 378

[7] R. Kerr, Phys.Rev.Letts. 11, 237 /1963/

[8] F.J. Ernst, Phys.Rev. 168, 1415 /1968/

[9] H. Weyl, Ann. der Phys., 54, 117 /1917/

[10] W. Bonnor, Zeitschrift für Phys. 190, 444 /1966/, see also Z. Perjés, Nuovo Cimento 55B, 600 /1968/

[11] E.T. Newman et al., Journal of Math.Phys. 6, 918 /1968/

Printed in the Central Research Institute for Physics, Budapest
Kiadja a KFKI Könyvtár és Kiadói Osztály. O.v.: dr. Farkas Istvánné
Szakmai lektor: Dr.Károlyházi Frigyes. Nyelvi lektor: Sebestyén Ákos
Példányszám 155 Munkaszám: 3898 Budapest, 1968. szeptember 3.
Készült a KFKI házi sokszorositójában. F.v.: Gyenes Imre