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BUDAPEST

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I. Introduction

In the past two years the Regge theory has regained its reputation in physics. The engrowing number of experimental data could be fitted by it in very good agreement. In the same time great many physicists started again to investigate some bothering problems of the Regge theory, mainly those which brought the theory to a dead-centre some years ago. Some of the problems turned out to be "paper-tiger" /e.g. the shrinkage/ but others led to a better understanding of the problem. It became clear that the ordinary three dimensional partial wave analysis can be generalized as a decomposition of the scattering amplitude in terms of the irreducible representations of the little group, determined by the given kinematical situation. This allowed to group the set of Regge-poles in families at special momentum configurations. It also became clear that the spin is not a trivial problem when reggeizing; the crossing relation forces conspiracies among Regge-poles. The unwanted singularity at s=o in the unequal mass scattering amplitude was also in the lamelight. To make it disappear, required again conspiracies among Regge-poles. It was very nice to see how the two types of conspiracies /due to the spin and due to the unequal masses/ were on the same footing how the types of constraints could be met by the same expression.

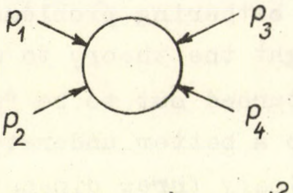
The aim of our paper is to give an introduction into the generalized partial wave analysis. After having summarized the kinematics we recapitulate the Regge theory and later discuss the four dimensional partial wave analysis of the scattering amplitude both in equal and unequal mass case. We examine the analytic properties of a Lorentz-pole contribution in detail. For those who are not familiar with the Lorentz group we have summed up its main properties in the Appendices.

Our primary point of view will be to get free from the unwanted s=o singularity; we shall not investigate the spin-type conspiracy, but/except in II.2./ we shall not neglect the spin of the particles. We think our paper is closed in itself, but for those who are interested in every details we give the references to the original papers.

It was not our aim to publish new results here but we have tried to collect every important formulae for practical calculation. Here we note that our formula for the boost function seems us to be the simplest we know.

II.1. Kinematics

Let us summarize first the kinematics of the quasielastic scattering of particles with different masses. /Everywhere in this paper the components of a vector will be given in the order /o,x,y,z/; $g_{oo}=1$, $g_{ii}=-1$./



Let it be $/p_1+p_2/^2 = s$, $/p_1+p_3/^2=t$, $/p_1+p_4/^2 = u$ and because of the energy-momentum conservation $p_1+p_2=-p_3-p_4$. We work in the CM system: $\underset{\sim}{p}_1 = -\underset{\sim}{p}_2 = /o,o,p/$

$$p = \sqrt{\frac{\left[s-(m_1-m_2)^2\right]\left[s-(m_1+m_2)^2\right]}{4s}} = \frac{1}{2\sqrt{s}}\,\Delta^{1/2}\left(s,m_1{}^2,m_2{}^2\right) \qquad /1/$$

let us denote the angle between $\underset{\sim}{p}_1$ and $-\underset{\sim}{p}_3$ by ϑ, then we can write:

$$p_1 = \left(\frac{s+m_1^2-m_2^2}{2\sqrt{s}}, o,o,p\right) \quad ; \quad -p_3 = \left(\frac{s+m_3^2-m_4^2}{2\sqrt{s}}, o, p'\sin\vartheta, p'\cos\vartheta\right)$$

$$p_2 = \left(\frac{s-m_1^2+m_2^2}{2\sqrt{s}}, o,o,-p\right) \quad ; \quad -p_4 = \left(\frac{s-m_3^2+m_4^2}{2\sqrt{s}}, o, -p'\sin\vartheta, -p'\cos\vartheta\right) \qquad /2/$$

where

$$p' = \frac{1}{2\sqrt{s}}\,\Delta^{1/2}\left(s, m_3^2, m_4^2\right) \qquad /3/$$

and

$$\cos\vartheta = \frac{s\left(t-u\right) + \left(m_3^2 - m_4^2\right)\left(m_1^2 - m_2^2\right)}{\Delta^{1/2}\left(s,m_1^2,m_2^2\right)\Delta^{1/2}\left(s,m_3^2,m_4^2\right)} \qquad /4/$$

It will be usefull to introduce the variables q and q'

$$p_1 = P/2 + q \qquad p_3 = P/2 + q'$$
$$p_2 = P/2 - q \qquad p_4 = P/2 - q' \qquad /5/$$

It is obvious that

$$\underset{\sim}{P} = 0, \quad \cos\vartheta = \underset{\sim}{q}\,\underset{\sim}{q}' \;/\; |\underset{\sim}{q}|\,|\underset{\sim}{q}'|$$

Using the previous formula we can write

$$q_o = m_1^2 - m_2^2 \;/\; 2\sqrt{s} \;;\quad |\underset{\sim}{q}| = p \;;\quad q^2 = 2\left(m_1^2 + m_2^2\right) - s \;/\; 4 \qquad /6/$$

the appropriate components of q' we get by $m_1 \to m_3$, $m_2 \to m_4$,

We shall work in the physical t-channel CMS as well. For equal mass scattering the complex Lorentz transformation what connects the s and t channel CM systems is

$$\mathcal{L} = \begin{pmatrix} 0 & 0 & \sqrt{\frac{u}{t-4}} & \sqrt{\frac{s}{t-4}} \\ 0 & 1 & 0 & 0 \\ -\sqrt{\frac{u}{s-4}} & 0 & -\sqrt{\frac{st}{(s-4)(t-4)}} & \sqrt{\frac{tu}{(s-4)(t-4)}} \\ \sqrt{\frac{t}{s-4}} & 0 & \sqrt{\frac{su}{(s-4)(t-4)}} & \sqrt{\frac{-u}{(s-4)(t-4)}} \end{pmatrix} = R_x\left(\alpha + \frac{\pi}{2}\right) B_z\left(\frac{i\pi}{2}\right) R_x\left(\frac{\pi}{2} - \gamma\right) \qquad /7/$$

where R_x is a complex rotation around the x axis, B_z is a complex boost along the z axis, and

$$\cos\alpha = \sqrt{\frac{-u}{s-4}} \;;\quad \cos\gamma = \sqrt{\frac{-u}{t-4}}$$

That is to say if we apply /7/ onto the momenta of /2/ for $m_i = 1$, and $p_i^c = \mathcal{L}^{-1} p_i$ we obtain:

$$p_1^c = \left(\sqrt{t}/2,\, 0,\, 0,\, p^c\right) \;;\quad p_2^c = \left(-\sqrt{t}/2, 0, -p^c \sin\vartheta_c, -p^c \cos\vartheta_c\right)$$
$$p_3^c = \left(\sqrt{t}/2,\, 0,\, 0, -p^c\right) \;;\quad p_4^c = \left(-\sqrt{t}/2, 0,\; p^c \sin\vartheta_c,\; p^c \cos\vartheta_c\right) \qquad /8/$$

As we see, to different scattering angles in a given s-channel CMS correspond different CMS's in the t-channel. The appropriate value of the crossed parameters can be obtained from the direct ones if we exchange s and t.

Under the crossing the scattering amplitude behaves as follows:

$$\langle p_1 s_1 \lambda_1 ,\ p_2 s_2 \lambda_2 \ |T|\ p_3 s_3 \lambda_3 \ ,\ p_4 s_4 \lambda_4 \rangle = \sum_{\nu_1 \nu_2 \nu_3 \nu_4} d^{s_1}_{\nu_1 \lambda_1}(X_1)\, d^{s_2}_{\nu_2 \lambda_2}(X_2)\ .$$

$$.\ d^{s_3}_{\nu_3 \lambda_3}(X_3)\, d^{s_4}_{\nu_4 \lambda_4}(X_4) \langle p_1 s_1 \lambda_1 ,\ \overline{p_3 s_3 \lambda_3}\ |T|\ p_2 s_2 \lambda_2 \ ,\ p_4 s_4 \lambda_4 \rangle \ ;$$

/9/

$$\sin X_1 = 2m_1 \sqrt{\phi(s,t)} \,/\Delta^{1/2}\left(t, m_1^2, m_2^2\right) \Delta^{1/2}\left(t, m_3^2, m_4^2\right) \quad /\text{cycle}/\ ;$$

$$\phi(s,t) = stu - s\left(m_2^2 - m_4^2\right)\left(m_1^2 - m_3^2\right) - t\left(m_1^2 - m_2^2\right)\left(m_3^2 - m_4^2\right) - \left(m_1^2 m_4^2 - m_2^2 m_3^2\right)\left(m_1^2 + m_4^2 - m_2^2 - m_3^2\right)$$

After having performed the complex Lorentz transformation we are not yet in the physical domain of the crossed channel, to get there we have to continue s and t to their physical values. The whole process can be done without getting into trouble with singularities and cuts. [1, 2].

For convenience we introduce the k_μ and k'_μ vectors of unit length, pointing into the direction

$$q_\mu - \frac{\left(m_1^2 - m_2^2\right)}{P^2} P_\mu \quad ; \quad q'_\mu - \frac{\left(m_3^2 - m_4^2\right)}{P^2} P_\mu \qquad /10/$$

respectively /Wightmann-Gårding coordinates/. As can be seen,

$k_o = k'_o = 0$, $\underline{k} = \underline{q}/|\underline{q}|$, $\underline{k}' = \underline{q}'/|\underline{q}'|$ in the CMS.

I.2. Three-dimensional partial wave analysis.

First we reconsider the steps of the Regge-theory to see the role of the formalism what will be used later. For the time being we neglect the spin of the particles.

Let us consider a $|p_1 , p_2\rangle$ two-particle state. Only 3 are independent out of its 8 components of momenta, if we are in an arbitrary CMS. /The 5 relations are $p_1^2 = m_1^2$, $p_2^2 = m_2^2$, $\underline{p}_1 + \underline{p}_2 = 0$ ./ The three independent ones can be chosen as: P_o, ϑ, ϕ where ϑ, ϕ fix the direction of $\underline{q}$./

If we denote by $\overset{o}{q}$ the unit vector parallel to the z axis, and by $R(\vartheta,\varphi)$ the rotation from the z-direction into the (ϑ,φ) direction, a two-particle state has the form $| P_o, R(\theta,\phi)\overset{o}{q} >$. It is obvious, that the group of the R-s is a little group of the P vector.

Making use of the rotation invariance of T, and the little group property of P, we can write:

$$< p_1, p_2 |T| p_3, p_4 > = \delta(P-P') < P_o, R\overset{o}{q} |T| P_o, R'\overset{o}{q} > =$$

$$= \delta(P-P') < P_o, R''\overset{o}{q} |T| P_o, \overset{o}{q} > = F(P^2, R'') = F(P^2; \vartheta'', \phi'') \qquad /11/$$

This way, we can consider the scattering amplitude as a function over the rotation group. If this function meets some convergence criteria it can be decomposed by the irreducible representations of this group. /We note that the decomposition of a function being defined over a group is called Fourier-transformation over the group, see e.g. [3] /

If $f(g) = f(\phi_1, \vartheta, \phi_2)$ is a square-integrable function over the rotation group, its decomposition is

$$f(\phi_1,\vartheta,\phi_2) = \sum_{jmm'} f^j_{mm'} D^J_{mm'}(\phi_1,\vartheta,\phi_2)$$

where $D^j_{mm'}(\phi_1,\vartheta,\phi_2) = e^{i\phi_1 m} d^j_{mm'}(\vartheta) e^{i\phi_2 m'}$ is a unitary representation of the group, $f^J_{mm'}$ are the Fourier coefficients

$$f_{mm'} = \int_G f(g)\, e^{i\phi_1 m}\, e^{i\phi_2 m'}\, d\phi_1\, d\phi_2$$

$$f^j_{mm'} = \int_G f_{mm'}(\vartheta)\, d^j_{mm'}(\vartheta)\, d\cos\vartheta$$

For scalar-scalar scattering obviously $f_{mm'} = f\,\delta_{mo}\,\delta_{m'o}$ so the Fourier decomposition of /7/ is simply

$$F(P^2,R'') = \sum_l f_l(P^2)\, P_l(\cos\vartheta'') = \sum_l f_l(P^2)\, P_l(k_\mu k'_\mu)$$

$$= \sum_l f_l(P^2)\, P_l(k_\mu R''_{\mu\nu} k_\nu) \qquad /12/$$

because in our special frame $k_\mu = (0,0,0,1)$

The f_l Fourier-coefficient is just the l^{th} partial wave amplitude:

$$F(R'') = \sum_{lml'm'} <P_o, R''\overset{o}{q} | P_o, lm>< P_o lm|T|P_o l'm'><P_o l'm' | P_o \overset{o}{q}>$$

$$= \sum_{lml'm'} D^l_{om}(R'')\, D^l_{m',o}(I)\, \delta_{l,l'}\, \delta_{m,m'} \,. \qquad /13/$$

$$< P_o, l ||T|| l, P_o > = \sum_l f_l\, P_l(\cos\vartheta'')$$

We need the rotation invariance at this point /when we find the physical meaning of f_l / again. We call the attention to it, because this is generally not emphasized.

Let us perform now the Watson-Sommerfeld transformation at the physical values of t and s /i.e. $s > s_o$, $t < 0$/:

$$F(p_1p_2p_3p_4) = \frac{i}{2}\int_{-1/2-i\infty}^{-1/2+i\infty} \frac{(2l+1)dl}{\sin\pi l}\, f_l(P^2)\, P_l\left(-k_\mu R_{\mu\nu}k_\nu\right) - \sum_n \pi\frac{\xi(2\alpha_n+1)}{\sin\pi\alpha_n}\beta(P^2,\alpha_n)\, P_{\alpha_n}\left(-k_\mu R_{\mu\nu}\, k_\nu\right) \qquad /14/$$

α is the place, β is the residue of the pole, ξ is the signature factor. After that let us go to the t-channel CMS; i.e. having done the complex Lorentz transormation /7/ we continue analitically in s and t.

$$F\left(p_1^c\, p_3^c\, p_2^c\, p_4^c\right) = \frac{i}{2}\int \frac{(2l+1)dl}{\sin\pi l}\, P_l(P^2)\, P_l\left(-k_\mu^c R_{\mu\nu}^c\, k_\nu^c\right) - \sum_n \pi\frac{\xi(2\alpha_n+1)}{\sin\pi\,\alpha_n}\beta\left(P^{c2},\alpha_n\right) P_{\alpha_n}\left(-k_\mu^c\, K_{\mu\nu}^c\, k_\nu^c\right) \qquad /15/$$

It is obvious that $p^{c2} = p^2 = s$. What is $R^c_{\mu\nu}$? $R_{\mu\nu}$ was an element of the little group of P_μ transforming $\underline{k}_\mu$ into $\underline{k}'_\mu$. So $R^c_{\mu\nu}$ will be an element of the little group of P^c_μ. The little group of P_μ was SO (3), but of P^c_μ is SO(2,1). If all masses are equal, this can be checked at once with the help of eqs. /7,8/

$$P^c = \begin{bmatrix} 0 \\ 0 \\ -p^c\sin\Theta_c \\ -p^c(\cos\Theta_c - 1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\Theta_c/2 & -\sin\Theta_c/2 \\ 0 & 0 & \sin\Theta_c/2 & \cos\Theta_c/2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -2p^c\sin\Theta_c/2 \\ 0 \end{bmatrix} = UP^{c'} \qquad /16/$$

and as $R_{\mu\nu}$ left invariant P_μ, $\mathcal{L}^{-1}R\mathcal{L}$ does the same for P^c_μ $U^{-1}\mathcal{L}^{-1}R\mathcal{L}U$ for $P^{c\prime}$, what is trivially an SO(2,1) transformation.

This way, what we have in eq./15/ is nothing else that the decomposition of the t-channel scattering amplitude in terms of the representations of the SO(2,1) group in the crossed channel /s-channel/.

It is a striking feature that in eq./15/ together with the unitary representations of SO(2,1) appear some non-unitary ones as well. This is the consequence of the fact that in the given region of the s-channel variables /$s<o$, $\cos\Theta > 1$/ the function is not square-integrable over the group SO(2,1) /if it were, unitary representations would be enough/, whereas in the physical domain of the s-channel it is /c.f. the cross section is finite/. However, the question arises how to decompose a "bad" function over a group. It has not been solved yet. The only compass we have are the experiments what we want to fit with the help of Regge-poles /i.e. nonunitary representations/.

At the direct channel partial wave analysis /PWA / we exploited the rotation invariance of T. To obtain the SO(2,1) decomposition we made use even of the analyticity of the scattering amplitude. As the elements of the above mentioned SO(2,1) group belong to different t values, it does not form a subgroup of the real Lorentz transformations.

II.3. The spectrum generating group

An old problem of the ordinary Regge theory was to cope with the unwanted singularity at s=o. If the masses of the scattered particles are unequal from eq./4/ one can see that if $s \to o$, $t \to \infty$ $\cos\Theta$ tends to unity instead of infinity, as it does in the case of equal mass scattering. The $\cos\vartheta =$ $=-1$ point is a singular point of the representation functions of the appropriate groups unless α is integer, so the scattering amplitude has developed a singularity. The first attempt to eliminate it was done by Freedman and Wang [4], but the very root of the problem and its remedy was found by Domokos [5] and Toller [6] and their coworkers. We shall follow the line of Domokos and Tindle. [5]

In the previous section we have shown that the Regge decomposition is nothing else that the decomposition of the scattering amplitude in terms of the little group of P_μ. At s=o the P_μ vector can be either a zero-vector or a timelike vector, Its little group is SL(2,C) or E(2), respectively. So at the critical point the structure of the little group has changed, that is to say it has contracted. It is proven that such a change in the group structure always creates singularities in the representations. Hence the remedy is to find a group what contains all the listed ones but does not contract in the examined domain of s. An appropriate choice is SL(2,C) itself. Accordingly, the only way to avoid the unwanted singularities at s=o if at this point the spectrum /i.e. the set of poles/ shows an extra symmetry. It is not necessary to demand this symmetry for the scattering amplitude because the background integral can be neglected anyway. However, it could turn out that this group is a symmetry group as it will be the case if the masses are pairwise equal. We shall call a group with the above mentioned property a <u>spectrum generating group</u>. What we have found now, can be told with other words as well: at s=o the pole terms are forced to conspire in such a way that they form a representation of SL(2,C). Thus, the poles will appear in families; the pole of the highest angular momentum is called mother, the others: daughters. /At this point we note that this conspiracy is due to the unequal masses. There is an other type of conspiracy among Regge-poles due to the spin of the scattered particles [7]. We shall not treat this problem here. However, our final form for the scattering amplitude will meet the "spin-type" constraints as well [8]./

To have a better insight, first we consider the scattering of equal mass particles; here the spectrum generating group is a symmetry group at the same time.

II.4. <u>Four dimensional partial wave analysis</u>

Let us consider the $|p_1 s_1 \lambda_1, p_2 s_2 \lambda_2 >$ two-particle state. Instead of the 12 commuting operators $/P^{(1)}_\mu, W^{(1)}_\mu W^{(1)}_\mu, P^{(1)}_\mu W^{(1)}_\mu, P^{(2)}_\mu, W^{(2)}_\mu W^{(2)}_\mu, P^{(2)}_\mu W^{(2)}_\mu/$ we choose a base being labelled by the following 12 commuting operators: $P_\mu = P^{(1)}_\mu + P^{(2)}_\mu$, $Q_\mu = P^{(1)}_\mu - P^{(2)}_\mu$, $W_\mu W_\mu$, $P_\mu W_\mu$, $P_\mu W^{(1)}_\mu$, $P_\mu W^{(2)}_\mu$; $\left(W_\mu = W^{(1)}_\mu + W^{(2)}_\mu \right)$.

The connection between the two bases is simple at rest:

$$|\mathring{p}, \mathring{q}, s\, \lambda\, \lambda_1 \lambda_2 > = \sum < s_1 \lambda_1, s_2 - \lambda_2 | s\, \lambda > | p_1 s_1 \lambda_1, p_2 s_2 \lambda_2 > \qquad /17/$$

In the following we supress $\lambda_1\ \lambda_2$ indices.

In the CMS $P_\mu = (\sqrt{s}, 0,0,0)$, if $s \to 0$, the little group of P_μ is SL(2,C); we want to decompose the two-particle state in terms of this group. First we seek an R transformation, $R \in SL(2,C)$, for what $q_\mu = R_{\mu\nu} \mathring{q}_\nu$. As at s=o $q^2 = m^2 > o$ /c.f. eq./6//, and as R does not make the lenght changed $\mathring{q}$ must have the form $(\sqrt{q^2}, 0,0,0)$. However, as one can see from eq. /2/ $q = (0,0,0, im)$, and there is no real SL(2,C) transformation connecting q and $\mathring{q}$ but there is an O /4/ transformation /up to unitary equivalence/ of the required property:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ im \end{pmatrix} = \begin{pmatrix} \sqrt{i} & & & \\ & 1 & & \\ & & 1 & \\ & & & \sqrt{i} \end{pmatrix} \begin{pmatrix} \cos\pi & 0 & 0 & -\sin\pi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\pi & 0 & 0 & \cos\pi \end{pmatrix} \begin{pmatrix} \sqrt{i} & & & \\ & 1 & & \\ & & 1 & \\ & & & \sqrt{i} \end{pmatrix}^{+} \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad /18/$$

We remind the reader the two little group, SO(2,1) and SO (3) /we have met when examining the ordinary Regge decomposition/, could be embedded only into the complex Lorentz group, L(C). The O (4) group what we have found now is a compact subgroup of L(C). As we are not in the physical domain of s, we have encountered with it instead of L(R). A detailed analysis of O(4) in this context can be found in a work of Freedman and Wang [9], here we give only the main points. An $R(\alpha,\Theta,\phi) \in O(4)$ transformation can transform a $\overset{o}{p} = (\ p^2\ ,0,0,0)$ vector so, that $(R\overset{o}{p})$ will point to Θ, ϕ direction, the magnitude of $(R\overset{o}{p})_o$ depends on α /c.f. I.6./. We shall use the notation R(p) for this, if $p = R\overset{o}{p}$. The O(4) decomposition of a two particle state reads:

$$|P_\mu = 0,\ q_\mu,\ s\lambda\rangle = R(q)|\ P_\mu = 0,\ \overset{o}{q},\ s\lambda\rangle = \frac{1}{2\pi\sqrt{2s+1}} \cdot$$

$$\cdot \sum_{j_o\sigma jm} \left[\sigma^2 - j_o^2 \right]^{\frac{1}{2}} D^{j_o}_{jms\lambda}(q)|\ P_\mu\ ;\ j_o\ \sigma jm;\ s\ \rangle \qquad /19/$$

In eq. /19/ $D^{jo}_{jms\lambda}(q)$ is a finite dimensional unitary representation of O(4), representing R(q), $|j_o\,\sigma\, j\, m\rangle$ is a basis vector of O(4).

Using the addition theorem of Appendix A eq. /21/ and eq. /19/, the scattering amplitude will have the form:

$$T_{\lambda_1\lambda_2\lambda_3\lambda_4} = \langle p_1 s_1\lambda_1,\ p_2 s_2\lambda_2\ |T|\ p_3 s_3\lambda_3,\ p_u s_u\lambda_u\rangle\big|_{s=o} =$$

$$= \frac{1}{(2\pi)^3} \sum_{s=|s_1-s_2|}^{|s_1+s_2|} \sum_{s'=|s_3-s_4|}^{|s_1+s_4|} \sum_{j_o=\min(s,s')}^{\min(s,s')} \sum_{\sigma=j_o+1}^{\infty} \frac{\sigma^2 - j_o^2}{\sqrt{(2s+1)(2s'+1)}} \qquad /20/$$

$$\cdot \langle s_1\lambda_1,\ s_2-\lambda_2|s\lambda\rangle\langle s'\lambda'|\ s_3\lambda_3,\ s_4-\lambda_4\rangle\, D^{j_o\sigma}_{s\lambda s'\lambda'}\left(R^{-1}(q)R(q')\right) T^{j_o\sigma}_{s,s'}(p^2)$$

When obtaining eq./20/, we have employed the result of II.5. that T is diagonal in the $O(4)$ quantum numbers at $s=o$; the reduced matrix element of T is denoted by $T^{j_o\sigma}_{s,s'}$.

Now we perform a Watson-Sommerfeld transformation on the σ -plane. If we discard the contribution of the semicircle at infinity and write separately the possible pole terms, the result will be quite similar to eq./15/. The poles on the σ -plane are called Lorentz-poles.

Similar as in the ordinary Regge theory, in the physical domain of the t- channel the W-S-transformed form of eq./20/ can be interpreted as a decomposition in terms of $O(3,1)$ group; the background integral runs over the infinite dimensional unitary representations of $O(3,1)$, the pole terms appear to be non-unitary, infinite dimensional representations /reducible if $\sigma - j_o$ is integer/. The contribution of a Lorentz pole to the scattering amplitude has the form:

$$\frac{\sigma^2 - j_o^2}{\sqrt{(2s+1)(2s'+1)}} < s_1\lambda_1, s_2-\lambda_2 | s\lambda><s'\lambda' | s_3\lambda_3, s_4-\lambda_4 > .$$

$$. D^{j_o\sigma}_{s\lambda s'\lambda'}\left(R_q^{-1} R_{q'}\right) \Gamma^{j_o\sigma}_{s,s'}$$

$\Gamma^{j_o\sigma}_{s,s'}$ is the residue of the pole. In sec. II.6. we discuss how the argumenta of $D^{j_o\sigma}_{s\lambda s'\lambda'}$ depend on the quantum numbers of the scattered particles and in sec. III. we discuss its analitical properties.

A family of physical particles belongs to definite parity, so we ought to have diagonalized $T_{\lambda_1\lambda_2\lambda_3\lambda_4}$ in parity as well. The method is again analogous to what is used in ordinary Regge theory [10], for Lorentz poles it is described e.g. in [11]. An other question we shall not treat here in detail either is the introduction of Lorentz signature, what can be found in [12].

Salam et al [13] following the line of Toller [6], try to avoid the complications due to the analytic continuation, and they introduce the scattering amplitude as a function over the group $O(3,1)$ in the crossed channel. It can be done, because the $< p_1^c ; p_2^c |T| p_3^c ; p_4^c >$ amplitude in the crossed channel, but at $s = \left(p_1^c + p_2^c\right)^2 \geq s_o > 0$, $t = \left(p_1^2 - p_2^2\right)^2 \leq 0$ can be roughly written as $< -p_4 ; p_2 |T| p_3 ; -p_1 >$ where the $< -p_4 ; p_2 |$ state is not physical, i.e. $-p_4$ negativ timelike vector. If $p_2 - p_4 = 0$ /t=o/, then by eq. /5/ $q=q'$ what presupposes that $\cos\Theta = 1$, $m_1 = m_3$, $m_2 = m_4$, as can be seen from eqs. /1/, /2/, /3/. This way

$$p_1+p_3 = P+2q = \left(s+m_1^2-m_2^2/\sqrt{s},0,0,\ \Delta^{1/2}\left(s,m_1^2,m_2^2\right)/\sqrt{s}\right);\ \left(p_1+p_3\right)^2 = 4m_1^2$$

Now $p_{1o}+p_{3o}/2m_1 > 1$ so this vector can be obtained from a timelike unit vector with the help of a real Lorentz-transformation. Hence the scattering amplitude is a function over 0/3,1/, and if $t=o$, using only the Lorentz invariance of T, it can be decomposed in terms of the irreducible representation of the Lorentz group. Having crossed eq. /20/ we get back their results in the appropriate domain of s and t .

II.5. If masses are unequal ...

Last /but not least/ we discuss the general mass case. As we have learnt from the previous sections, now SL/2,C/ is not a symmetry group, what it was in the equal mass case. The only way to avoid the unwanted singularities at $s=0$ is to classify the spectrum according to this group. The method is elaborated in (5), we follow the steps of this work.

The contribution of the Regge poles to the scattering amplitude is

$$T_{\lambda_1\lambda_2\lambda_3\lambda_4\,\text{pole}} \sim \sum_i \beta_{\lambda_1\lambda_2\lambda_3\lambda_4}\left(s,\alpha_i(s)\right) D^{\alpha_i(s)}_{\lambda_3-\lambda_4,\lambda_1-\lambda_2}\left(k\ R\ k\right)\xi_i \qquad /15'/$$

where

$$\xi_i = 1 \pm \exp\ -i\Pi\alpha\ /\sin\Pi\alpha_i$$

We can write $\lambda_1-\lambda_2 = \lambda$, $\lambda_3-\lambda_4 = \lambda'$

$$T_{\lambda_1\lambda_2\lambda_3\lambda_4\,\text{pole}} \sim \sum_{i,m} \beta^i_{\lambda_1\lambda_2}\,\beta^i_{\lambda_3\lambda_4}\,\xi_i\ .$$

$$.\ D^{\alpha_i}_{\lambda'm}\left(k\ R\ k\right).\ D^{\alpha_i}_{m\lambda}\left(k\ I\ k\right) =$$

$$= \sum_{im} \langle P,q,s\lambda|\ P\alpha_i\ m\lambda\rangle.\ \beta^i_{\lambda,\lambda_2}\ \beta^i_{\lambda_3\lambda_4} \langle P\alpha_i\ m\lambda'|\ Pq,s'\lambda'\rangle\ .$$

$$\langle s_1\lambda_1,\ s_2-\lambda_2|s\lambda\rangle.\ \langle s'\lambda'|\ s_3\lambda_3\ ,\ s_4-\lambda_4\rangle\ . \qquad /23/$$

The statement that the spectrum is classified according to SL/2,C/, or what is the same, the poles "conspire" to give the basis of SL/2,C/ representations, means that at $s=0$ the set of poles can be grouped into subsets, and for each subset

$$\alpha_{i_2} = \alpha_{i_o} - \kappa = \sigma_i - \kappa \qquad /24/$$

and

$$\beta_{\lambda_1\lambda_2}\left(s=0,\ \alpha_{i_\kappa}\right)\beta_{\lambda_3\lambda_4}\left(s=0,\ \alpha_{i_\kappa}\right) =$$

$$= \Gamma_s^{j_o\sigma_i}\ \Gamma_{s'}^{j_o\sigma_i} \sum_{jm} \langle\alpha_{i_\kappa} m| j_o\sigma_j\ jm\rangle \langle j_o\sigma_i\ jm|\alpha_{i_\kappa}\ m\rangle$$

where $\langle j_o\sigma jm|\ \alpha_i, m\rangle$ is a generalized Clebsch-Gordan coefficient, decomposing an SL/2,C/ basis vector to SU/2/ or SU/1,1/, respectively. If we write back eq. /24/ into eq. /23/:

$$T_{\lambda_1\lambda_2\lambda_3\lambda_4\,\mathrm{pole}} \sim \sum_{ijm} \langle Pqs\lambda\ |P,\sigma_i j_o j_m, s\rangle \cdot \Gamma_s^{j_o\sigma_i}\ \Gamma_{s'}^{j_o\sigma_i}\ \xi_i\ .$$

$$.\ \langle P,\sigma_i j_o jm, s'\ |\ P\, q' s' \lambda'\rangle\ .$$

$$.\ \langle s_1\lambda_1,\ s_2-\lambda_2\ |\ s\lambda\rangle\langle s'\lambda'\ |\ s_3\lambda_3,\ s_4-\lambda_4\rangle \qquad /24/$$

Remembering to eq. /19/: $\langle P,q,s\lambda\ |\ P, j_o\sigma j_m, s\rangle \sim D^{j_o\sigma}_{s\lambda jm}(q)$ and applying the addition formula of eq. /A.21/, we get:

$$T_{\lambda_1\lambda_2\lambda_3\lambda_4\,\mathrm{pole}} \sim \sum \langle s_1\lambda_1,\ s_2-\lambda_2\ |\ s\lambda\rangle\langle s'\lambda'\ |\ s_3\lambda_3,\ s_4-\lambda_4\rangle\ .$$

$$.\ D^{j_o\sigma}_{s\lambda s'\lambda'}\left(R_q^{-1}\ R_{q'}\right)\Gamma_s^{j_o\sigma}\ \Gamma_{s'}^{j_o\sigma}\ .\ \xi \qquad /25/$$

As we have seen now, the leading pole terms of the scattering amplitude have the same structure, both in equal and unequal mass scattering, however the values of their parameters are different. As we shall see in the next section, eq./25/ is regular around $s=o$.

We ought to tell, there are some tricky steps in this derivation of eq. /25/. E.g. eq. /19/ is valid at $s=o$. At this point q is singular in the general mass case. To avoid the singularity we got to $s=o+\varepsilon$. But there q_o is not independent of $\underline{q}$ and P_μ, if we are on the mass shell; so we have to go off-shell. As far as these delicate steps are concerned we suggest the reader to turn to the original papers [5].

Up to now we have seen how the SL/2,C/ non-unitary representations are built from SU/1,1/ ones, I.e. how a family is built from single poles - from daughters. This procedure is reversible, we can decompose an SL/2,C/ representation into SU/1,1/ ones, to get back the contribution of a single pole to the scattering amplitude, or to calculate the coupling ratio of the daughters. For the details see [14] and [15].

II.6. The D-function parameters

In what follows we give the parameters of the $D^{j_0 \sigma}_{jmjm'}$ functions. We choose the z axis to be parallel to the incoming particles, hence

$$q = U(\alpha)\,\overset{o}{q} \quad , \qquad q' = U(\beta, \vartheta, \phi)\,\overset{o}{q}$$

The form of the U matrices are given in the Appendix, together with their formula of addition. As we have told there, the formulae are good both for O/3,1/ and O/4/.

Now we apply them for

$$\frac{q_o}{q} = \mathrm{ch}\,\alpha \quad , \quad \frac{q'_o}{q'} = \mathrm{ch}\,\beta \quad , \quad \frac{\underline{q}\cdot\underline{q}'}{|\underline{q}|\,|\underline{q}'|} = \cos\vartheta \; .$$

Let us assume that $m_1 > m_2$, $m_3 > m_4$ then

$$\mathrm{ch}\alpha = \frac{m_1^2 - m_2^2}{\sqrt{s}\left[2\, m_1^2 + m_2^2 - s\right]}$$

$$\mathrm{sh}\alpha = \frac{\sqrt{\left[s-(m_1 - m_2)^2\right]\left[s-(m_1 + m_2)^2\right]}}{\sqrt{s}\,\left[2\left(m_1^2 + m_2^2\right) - s\right]} \qquad /26/$$

$$\begin{matrix}\mathrm{ch}\beta \\ \mathrm{sh}\beta\end{matrix} = \begin{matrix}\mathrm{ch}\alpha \\ \mathrm{sh}\alpha\end{matrix}\begin{pmatrix} m_1 \to m_3 \\ m_1 \to m_4 \end{pmatrix}$$

where $s > 0$, and

$$\cos\Theta_s = \frac{s\left(t-u\right) + \left(m_1^2 - m_2^2\right)\left(m_3^2 - m_4^2\right)}{\sqrt{\left[s-(m_1-m_2)^2\right]\left[s-(m_1+m_2)^2\right]\left[s-(m_3-m_4)^2\right]\left[s-(m_3+m_4)^2\right]}} \qquad /27/$$

m_i /i=1,.....4/ in the formulae /26/ and /27/ will denote the physical masses of the particles scattered. Thus when realizing the expansion /25/ of the scattering amplitude at least what concerns the functions $D\left(U^{-1}_{L_q} \cdot U_{L_q}\right)$ we exchange two limits. Our procedure is firstly a technical simplicity: we do not want to introduce two complex variables instead of $m_1^2 - m_2^2$ and $m_3^2 - m_4^2$. This way we are led to the correct $t^{\sigma-1}$ behaviour and moreover we can get formulae depending on s . These may be advantageous in a theory with broken SL/2,C/.

We continue from the physical region through values $\mathrm{Jm}\,s > 0$ to the vicinity of $s = 0$, remaining in the region $t < 0$, in accordance with our picture where the "partial wave" expansion is made in the s-channel and here the Regge poles mean bound states.

Let the sign of the square root be positive in /26/ and /27/, and

$$0 < s < \min \left\{ \frac{\left(m_1^2 - m_2^2\right)^2}{2\left(m_1^2 + m_2^2\right)} , \frac{\left(m_3^2 - m_4^2\right)^2}{2\left(m_3^2 + m_4^2\right)} \right\} , \qquad /28/$$

and let us choose $t < 0$ so, that $|\cos\Theta_s| < 1$ be fulfilled. Applying the formulae of addition /A21/, the arguments of the function

$$D^{j_o\sigma}_{s\lambda,s'\lambda'}\left(U^{-1}_{Lq'}\ U_{Lq}\right) = \sum_\mu d^s_{\lambda\mu}(\chi)\, d^{j_o\sigma}_{s\mu s'}(\xi)\, d^{s'}_{\mu\lambda'}(\psi)$$

are the following:

$$\mathrm{ch}\xi = \frac{u - t}{\sqrt{\left[2\left(m_1^2+m_2^2\right)-s\right]\left[2\left(m_3^2+m_4^2\right)-s\right]}} \qquad /29/$$

$$\mathrm{sh}\xi = \frac{\sqrt{\left(m_1^2+m_2^2 - m_3^2 - m_4^2\right)^2 - 4ut}}{\sqrt{\left[2\left(m_1^2+m_2^2\right)-s\right]\left[2\left(m_3^2+m_4^2\right)-s\right]}}$$

$$\cos\chi = \frac{\left(m_1^2-m_2^2\right)\left[2\left(m_3^2+m_4^2\right)-s\right] - \left(m_3^2-m_4^2\right)(u-t)}{\sqrt{\left[s-(m_3-m_4)^2\right]\left[s-(m_3+m_4)^2\right]\left[\left(m_1^2+m_2^2 - m_3^2-m_4^2\right)^2 - 4ut\right]}}$$

$$\sin\chi = 2\sqrt{\Phi(s,t)}\sqrt{\frac{2\left(m_3^2 + m_4^2\right) - s}{\left[s-(m_3-m_4)^2\right]\left[s-(m_3+m_4)^2\right]\left[\left(m_1^2+m_2^2-m_3^2-m_4^2\right)^2 - 4ut\right]}} \qquad /30/$$

$$\cos\psi = -\cos\chi\begin{pmatrix} m_1 \leftrightarrow m_3 \\ m_2 \leftrightarrow m_4 \end{pmatrix}$$

$$\sin\psi = \sin\chi\begin{pmatrix} m_1 \leftrightarrow m_3 \\ m_2 \leftrightarrow m_4 \end{pmatrix} , \qquad /31/$$

where

$$\Phi(s,t) = stu - s(m_1^2-m_3^2)(m_2^2-m_4^2) - t(m_1^2-m_2^2)(m_3^2-m_4^2) + (m_1^2-m_2^2-m_3^2+m_4^2)(m_2^2\, m_3^2-m_1^2\, m_4^2).$$

All the square roots in /29/-/31/ are positive in the domain chosen for s and t . For other values of s , t the arguments of $D(U_{Lq}^{-1}, U_{Lq})$ are defined by the analitic continuations of formulae /29/-/31/.

For the case of equal masses our condition /28/ makes no sence, and only the prescription $|\cos\theta_s| < 1$ has to be considered and the analogons of the expressions /29/-/31/ are derived by extending the addition formulae /A21/ to 0/4/. The same expressions are got by extending /29/-/31/ to the case $m_1=m_2$, $m_3=m_4$, in accordance with the results of Domokos and Tindle [5] who have shown that at least concerning the Regge-poles there is no difference between the cases of equal and unequal masses.

We note that by the sole inspection of /28/ it may seem that the choice $t < 0$ has an essential role. Actually this is not the case as a choice of $u < 0$ together with $|\cos\theta_s| < 1$ would lead to an approach of $s = 0$ through $s < 0$ retaining $\operatorname{ch}\xi \geq 1$.

III. Some analytic properties and asymptotic behaviour

Let us consider now the contribution of the Regge-poles of the scattering amplitude in the crossed channel.

The Regge-poles in the direct channel for $s = 0$ are the following:

$$< p_3\, s_3\, \lambda_3,\, p_4\, s_4\, \lambda_4\, |T|\, p_1\, s_1\, \lambda_1,\, p_2\, s_2\, \lambda_2 >_{\text{pole}} =$$

$$= \lim_{\substack{s\to 0\\ \operatorname{Jm}s\to +0\\ \operatorname{Jm}t\to -0}} \sum_i \beta\big(s, \sigma_i(s)\big) \frac{1 \pm \exp\big(-i\Pi\, \sigma_i(s)\big)}{\sin\Pi\, \sigma_i(s)} \times$$

$$\times \sum_{jj'} < s_3\lambda_3, s_4 - \lambda_4 | j\lambda > D^{j_0\, \sigma_i(s)}_{j\lambda, j'\lambda'}\big(\chi, \xi, \psi\big) < j'\lambda' |\, s_1\lambda_1,\, s_2 - \lambda_2 > . \quad /32/$$

Our aim is to examine $D^{j_0\sigma_i(s)}_{j\lambda, j'\lambda'}\big(\chi, \xi, \psi\big)$ for the case $s\to 0$ /$\operatorname{Jm}s\to -0$/, $t\to\infty$/$\operatorname{Jm}t\to +0$/. For this end we quote some well known facts concerning the crossing of helicity amplitudes [1].

A helicity amplitude may be built up of invariant amplitudes combined with kinematical factors. For example in the case of s-t crossing we have to continue analitically from the region of the s-channel remembering that the physical amplitude is the limit $s+i\varepsilon$, $t-i\varepsilon$ in the s-channel and is the limit $s-i\varepsilon$, $t+i\varepsilon$ in the t channel, respectively. This prescription means that when crossing one intersects the real axis. We may do the continuation so that $\mathrm{Jm}\, s = -\mathrm{Jm}\, t$ be always fulfilled, thus the path of continuation intersects the real $s-t$ plane. If this crossing point is chosen to be inside the triangle of the Mandelstam plane defined by $s < s_o$, $t < t_o$ $u < u_o$ then the crossing of any cut of the amplitude is avoided. /Here s_o, t_o and u_o are the branching points of the lowest unitarity-cuts of the s,t,u channels./ One may be sure, that such a path of crossing does not lead onto an unphysical Riemann sheet.

When performing the crossing in the expansion /32/ the aforementioned points should be taken into account. As there are some restrictions for the path of continuation attention must be payed for the kinematical singularities emerging in the expansion. As the singularities of the Lorentz residuum $\beta\left(s,\sigma_i\,(s)\right)$ are unknown, $\beta\left(s{=}o,\ \sigma_i\,(o)\right)$ is a parameter fitted to experimental data, and its values for the s and t channels are generally not the same.

Note from the formulae /30/, /31/ that for the variables χ and ψ $|\cos\chi|$, $|\cos\psi| \to 1$ or, in the equal mass case, $|\sin\chi|$, $|\sin\psi| \to 1$ are fulfilled independentily on the path of continuation, thus they have no significance for the high-energy behaviour in the t channel. Examine now the variable

$$X \equiv e^{-2\xi} = \frac{\left[u-t - \sqrt{\left(m_1^2 + m_2^2 - m_3^2 - m_4^2\right)^2 - 4ut}\right]^2}{\left[2\left(m_1^2 + m_2^2\right) - s\right]\left[2\left(m_3^2 + m_4^4\right) - s\right]} \qquad /33/$$

for the case $s \simeq 0$ and $|t| \to \infty$. The sign of the square root in the $t(u)$ - channel is defined by the path of continuation. Expanding the square root in terms of $1/t$ about $s=0$ we get:

$$X \sim \frac{\left[t \mp |t| \pm \mathrm{sign}(t)\, \frac{\left(m_1^2+m_2^2\right)\left(m_3^2+m_4^2\right)}{2t}\right]^2}{\left(m_1^2+m_2^2\right)\left(m_3^2+m_4^2\right)} \qquad /34/$$

where the upper sign is to be taken if the cut of the square root is crossed and the lower if not. As there is no constraint on the choice of the crossing point within the triangle in the Mendelstam plane the asymptotic expansion e.g. in the t channel may emerge either at $X \sim t^2$ or at $X \sim 1/t^2$. In the appendix it is shown that

$$e^{j_o}_{j\mu j'}\,(x)_{x\to o} \sim x^{1/2\left(|\mu - j_o| - \sigma + 1\right)} \qquad \text{if } |\mu| \geq j_o$$

and

$$d^{j_o \sigma}_{j\mu j'}\,(x) \sim d^{j_o -\sigma}_{j'\mu j}\left(\frac{1}{x}\right) .$$

This shows as

$$d^{j_o\sigma}_{j\mu j'}(x) = e^{j_o\sigma}_{j\mu j'}(x) + (-1)^{j'-j}\, e^{-j_o-\sigma}_{j'\mu j}(x)$$

that for both cases we get the $t^{\sigma-1}$ behaviour for the contribution of one Lorentz-pole. If we introduce mirror poles similarly to the expansion given by Toller [6] in the equal mass case, writing thus the functions of second kind instead of $D^{j_o}_{j\mu,j'\mu'}$ the cases $x \to o$ and $x \to \infty$ get distinguished i.e. in one case the "poles" and in the other the "mirror poles" give the usual Regge-behaviour.

As an illustration let us take the NN forward scattering: $m_i = m$, $i=1,2,3,4$. In the s and u-channel the process is an $N\bar{N} \to N\bar{N}$ scattering, in the t-channel it is $NN \to NN$. The crossing point must be chosen inside the following triangle in the Mandelstam plane / μ is the mass of the pion:/

$$\begin{aligned} s &\le 4\mu^2 \\ t &\le 4m^2 \\ u &\le 4\mu^2 \end{aligned} \qquad /35/$$

For the points of the triangle $s_{min} = -4\mu^2$ $t_{min} = 4m^2 - 8\mu^2$, $u_{min} = 4\mu^2$. Taking s and t as independent variables let us examine the branching points of the square root in eq. /33/. There is a branching point on the complex s plane the position of which depends on the value of t: $s = 4m^2 - t$. For the region of the triangle the position of the branching point varies between $\bar{s}_{max} = 8\mu^2$ and $\bar{s}_{min} = 0$. It may be seen taking the actual values of the masses of the pion and the nucleon that the coordinates / s, t / of the crossing point can be chosen so that the cut of eq./33/ be or be not crossed. Eq. /34/ gives that in the first case $x \sim m^4/t^2$ and in the second $x \sim t^2/m^4$ if $s = 0$, $t \to \infty$.

As the behaviour of the functions $d^{j_o\sigma}_{j\mu j'}$ is similar for $x \to o$ and $x \to \infty$ it also follows that the singularities of the residue-functions $\beta(s)$ must not depend on the choice of the crossing point.

IV. Summary and outlook

To summarize we may say that there are two approaches to the question of Lorentz-poles. Toller's one is from the side of generalized partial wave analysis. He applies a very large amount of mathematics. It directs the attention to the very difficult problems of Fourier-analysis on non-compact groups. New mathematical theorems would be needed on this subject. Another difficulty is that this approach needs more than the usual premise about the invariance of scattering amplitude under Poincaré-group. People /at least we/ are not very familiar with the meaning of invariance under complex Lorentz group.

The approach of Domokos and Tindle and its reformulation by Domokos [22] is a rather elegant approximation to the subject, to see the main point in it is less tedious than in the former case. They need only the fundamental principle of analiticity and the natural requirement on getting a minimal non-contracting classification group. Nevertheless, we think that for the "practical life" it is very useful to know the steps of Toller's approach, at least so much as we have presented here.

In the future, in our opinion, we shall have a more unified picture of Regge-poles. Of course the $s=o$ point has a distinguished role,because the SL/2,C/ symmetry is badly violated at other points, but it would be nice to have a more uniform technique for working at least near $s=o$. The first step was taken by Domokos and Surányi [20] and by Salam and Strathdee [23]. Domokos and Surányi have worked out methods for breaking the symmetry and have got nice fits for Regge trajectories of πN resonances [21]. The last two authors have achieved the same results for the symmetry breaking, working in the field theoretical Born-approximation. They applied it only to extract general outgrowths for the scattering amplitude similar to Domokos and Surányi who did the same with Bethe-Salpeter equation of arbitrary kernel, satisfying only general conditions. The most interesting thing in Salam and Strathdee's work is that they describe the objects in a field theoretic language.

A further possibility to enlarge the symmetry group with the SU/3/ group. The first step was already done [24], [25] the previous results are promising.

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Appendix A.

We have seen that the representations of the elements of the proper homogeneous orthocronous Lorentz group play an important role in the preceeding argumentations. We shall simply call this group the Lorentz group as no other Lorentz group as the inhomogeneous or the one containing reflections will be spoken about.

In the following we encounter some properties of and facts about the Lorentz group and its representations. For further details and proofs see e.g. refs. [17], [13], [19].

The commutation rules of the generators $J_{\mu\nu} = -J_{\mu\nu}$ can be written in the following form

$$\left[J_{\mu\nu}, J_{\lambda\rho}\right] = i\left[g_{\nu\lambda} J_{\mu\rho} - g_{\mu\lambda} J_{\nu\rho} + g_{\mu\rho} J_{\nu\lambda} - g_{\nu\rho} J_{\mu\lambda}\right]. \qquad /A1/$$

Introducing $M_i = i\, \varepsilon_{ikl} \overline{M}_l \quad ; \quad /i,k,l = 1,2,3\,/$

and $N_i = J_{oi}$

we get the usual SU/2/ Lie algebra

$$\left[M_i\, ,\, M_k\right] = i\, \varepsilon_{ikl}\, J_{kl} \qquad /A2/$$

and

$$\left[M_i\, ,\, N_k\right] = i\, \varepsilon_{ikl}\, N_l \qquad /A3/$$

which shows that the generators N_i apart from sign and factor constitute an irreducible vector-operator triplet with respect to the algebra /A2/, and the relation

$$\left[N_i, N_k\right] = -i\, \varepsilon_{ikl}\, M_l \qquad /A4/$$

expressing the non-compactness of the Lorentz group.

The Casimirians of the group are

$$\frac{1}{2} J_{\mu\nu} J^{\mu\nu} = \underline{M}^2 - \underline{N}^2 = j_o^2 + \sigma^2 - 1 \quad ; \quad \frac{1}{4} \varepsilon_{\mu\nu\lambda\rho} J^{\mu\nu} J^{\lambda\rho} = \underline{M} \cdot \underline{N} = i j_o \sigma. \qquad /A5/$$

The unitary irreducible representations are infinite dimensional and may be grouped in two classes

a./ principal series: $j_o = 0, \frac{1}{2}, 1, \ldots;$ Reσ =0, $-\infty <$ Jmσ + ∞

b./ supplementary series: $j_o=0;$ o < Reσ < 1; Jmσ = 0 .

The finite dimensional /except the trivial one are non-unitary/ representations are characterised by $j_o=0, \frac{1}{2}, 1, \ldots$; $\sigma=j_o+n$, n>1 . For all the other values of σ we have infinite dimensional non-unitary representations. Strictly speaking the unitary representations mentioned above are the one valued representations of SL/2,C/ which is the so called universal covering group of the Lorentz group.

An element of a representation, corresponding to a pair of values j_o, σ may be characterised by the SU/2/ quantum numbers, corresponding to the generators M^2, M_3 . We shall denote such an element by $|j_o\sigma; jm>$. Solving the algebra /A2-A4/ of the generators we get their matrix elements:

$$M^{\pm} |j_o\sigma; jm> = \sqrt{(j\mp m)(j\pm m+1)}\, |j_o\sigma; jm\pm 1> \qquad /A6/$$

$$M_3 |j_o\sigma; jm> = m|j_o\sigma; jm>$$

$$N^{\pm} |j_o\sigma jm> = \mp \sum_{j'=j-1}^{j+1} < jm, 1\pm 1 | j' m\pm 1> \rho_j^{j'} | j_o\sigma; j' m\pm 1>$$

$$\sqrt{2}\, N_3 |j_o\sigma; jm> = \sum_{j'=j-1}^{j+1} < jm, 1\, 0 |j' m> \rho_j^{j'} |j_o\sigma; j'm> \qquad /A7/$$

where

$$\rho_j^{j+1} = \frac{2}{\sqrt{(2j+2)(2j+3)}} \left[(j+1)^2 - \sigma^2\right]\left[(j+1)^2 - j_o^2\right]^{1/2}$$

$$\rho_j^j = i j_o \sigma \frac{2}{j(j+1)}$$

$$\rho_j^{j-1} = -\frac{2}{\sqrt{2j(2j-1)}} \left[\left(j^2-\sigma^2\right)\left(j^2-j_o^2\right)\right]^{1/2} .$$

In the relations /A7/ the remark, folloving eq./A3/ is manifested.

As to a finite element of the Lorentz group has six Euler-parameters it can be represented in the following form:

$$g = g_z(\theta_1)\, g_y(\phi_1)\, g_z(\theta_1')\, \bar{g}_z(\xi)\, g_y(\phi_2)\, g_z(\theta_2)$$

where the parameters may take the values

$$0 \le \theta_1,\ \theta_1',\ \theta_2 < 2\Pi\ ;\quad 0 \le \phi_1,\ \phi_2 < \Pi;\quad 0 \le \xi$$

and g_z, g_y mean rotations about the axes z and y, respectively; whereas $\bar{g}_z$ means a Lorentz transformation along the z axis.

It may be seen that as the matrix elements of g_y and g_z are known from the theory of the rotation group it is only the matrix elements of $\bar{g}_z$ that must be calculated. For this purpose we write down the relations

$$e^{i\xi N_3} M_\pm e^{-i\xi N_3} = M_\pm \operatorname{ch}\xi \pm iN_\pm \operatorname{sh}\xi$$

$$-i\, e^{-i\xi N_3} N_3 = \frac{d}{d\xi} e^{i\xi N_3} \qquad /A8/$$

the first of which may be verified by taking derivatives with respect to ξ of any order at $\xi=0$; the second is obvious. Introducing the notation:

$$d^{j_0\sigma}_{jmj'}(\xi) = \langle j_0\sigma;\ jm\,|\, e^{i\xi N_3} \,|\, j_0\sigma\ ;\ j'm \rangle$$

and taking matrix elements for the eqs. /A8/ after we have multiplied the first by $e^{-i\xi N_3}$ on left we get the following system of equations

$$\sqrt{(j'-m)(j'+m+1)}\, d^{j_0\sigma}_{jm+1j'} - i\operatorname{sh}\xi \langle j-1\ m,\ 1\ 1|j\ m+1\rangle \varrho^{j}_{j-1}\, d^{j_0\sigma}_{j-1\ mj'} -$$

$$i\operatorname{sh}\xi\langle j+1\ m,\ 11|jm+1\rangle\, d^{j_0\sigma}_{j+1\ mj'} = \left[\sqrt{(j-m)(j+m+1)}\operatorname{ch}\xi + i\operatorname{sh}\xi\langle jm, 11|jm+1\rangle \varrho^{j}_{j}\right] d^{j_0\sigma}_{jmj'}$$

$$\sqrt{(j'+m)(j'-m+1)}\, d^{j_0\sigma}_{jm-1j'} - i\operatorname{sh}\xi\ \langle j-1\ m, 1-1|\ j\ m-1\rangle \varrho^{j-1}_{j}\, d^{j_0\sigma}_{j-1\ mj'} -$$

$$- i\operatorname{sh}\xi\langle j+1\ m, 1-1|j\ m-1\rangle \varrho^{j+1}_{j}\, d^{j_0\sigma}_{j+1\ mj'} = \left[\sqrt{(j+m)(j-m+1)}\operatorname{ch}\xi + i\operatorname{sh}\xi\langle jm, 1-1|jm-1\rangle \varrho^{j}_{j}\right] d^{j_0\sigma}_{jmj'}, \qquad /A9/$$

$$\langle j-1\ m,\ 10|jm\rangle \varrho^{j-1}_{j}\, d^{j_0\sigma}_{j-1\ mj'} + \langle j+1\ m, 10|jm\rangle \varrho^{j+1}_{j}\, d^{j_0\sigma}_{j+1\ mj'} =$$

$$= i\sqrt{2}\,\frac{d}{d\xi}\, d^{j_0}_{jmj'} - \langle jm,\ 1\ 0|jm\rangle\, \varrho^{j}_{j}\, d^{j_0\sigma}_{jmj'}\ .$$

Here we took account of the eqs. /A6-7/.

We see that the system is a complicated system of differential and recursion equations. It seems more useful to get second order differential equations which can be done in a lot of ways: differentiating equations of eq. /A9/ and making tremendous amount of substitutions with adequate values of j and m at the end we are led to the equations

$$\left[\frac{d^2}{d\xi^2} + 2\mathrm{cth}\,\xi \frac{d}{d\xi} - \left(j_o^2+\sigma^2-1-m^2\right) - \frac{j(j+1)-2m^2+j'(j'+1)}{\mathrm{sh}^2\xi}\right] d^{j_o\sigma}_{jmj'}(\xi) =$$

$$= -\frac{\mathrm{cth}\xi}{\mathrm{sh}\xi}\left[\sqrt{(j+m)(j-m+1)(j'+m)(j'-m+1)}\, d^{j_o\sigma}_{jm-1j'} + \sqrt{(j-m)(j+m+1)(j'-m)(j'+m+1)}\, d^{j_o\sigma}_{jm+1j'}\right]$$

$$\left[m\left(\frac{d}{d\xi} + \mathrm{cth}\xi\right) - j_o\sigma\right] d^{j_o\sigma}_{jmj'}(\xi) = \qquad /A10/$$

$$= \frac{1}{2\mathrm{sh}\xi}\left[\sqrt{(j+m)(j-m+1)(j'+m)(j'-m+1)}\, d^{j_o\sigma}_{jm-1j'} - \sqrt{(j-m)(j+m+1)(j'-m)(j'+m+1)}\, d^{j_o\sigma}_{jm+1j'}\right].$$

A more elegant way to get these equations is to take matrix elements of the expressions

$$e^{-\xi N_3}\ \underline{M}^2 - \underline{N}^2 \ ; \quad e^{-i\xi N_3}\ \underline{M}\cdot\underline{N}$$

and remembering that $\underline{M}^2-\underline{N}^2$ and $\underline{M}.\underline{N}$ are Casimirians of the group /A5/ on one side, and applying repeatedly eqs. /A8/ on the other. We stress that the system /A10/ is not equivalent to the system /A9/ as in deriving /A10/ information is lost.

Both the systems /A9/ and /A10/ are very involved in the general case, and though the solution of the system was already given by a number of authors, a really compact and easily treatable form is still lacking. Toller was able to get an expression for the functions $d^{j_o\sigma}_{jmj'}$ by means of another reasoning.

We shall strive to solve the system /A10/ for the case $m = j$ $j'>j$ /it is obvious that in the general case $|m| < j,j'$ / . If $m \neq j$ we may use the second equation of /A10/ to get the actual function $d^{j_o\sigma}_{jmj'}$ from the known form of $d^{j_o}_{jjj'}$.

Let us suppose that $\mathrm{Re}\sigma = 0$ i.e. we are interested in the functions of the principal series. Our result will be apt to continue to other values of σ /e.g. integer/ as well. Introducing the variables $x=e^{-2\xi}$ we get the following equation:

$$\left\{ 4x^2 \frac{d^2}{dx^2} + 4x\left[1 - (j+1)\frac{1+x}{1-x}\right]\frac{d}{dx} + \left[j(j+1) - j'(j'+1)\right]\left(\frac{1+x}{1-x}\right)^2 - \right.$$

$$\left. -2j_o\sigma\frac{1+x}{1-x} + j'(j'+1) + j - \left(j_o^2 + \sigma^2 + 1\right)\right\} d^{j_o\sigma}_{jjj'}(x) = 0 \qquad /A11/$$

We want to construct a solution which is regular in the point $x = 1$. It is easy to see that the solutions have the form

$$x^{\frac{1}{2}(j+j_o+\sigma+1)} (1-x)^{j'-j} f(1-x) \qquad /A12/$$

where $f(z)$ is a function which fulfills a hypergeometric equation in the variable $z = 1-x$:

$$z(1-z)f'' + \left[c - (a+b+1)z\right] f' - abf = 0. \qquad /A13/$$

The values of the three parameters are

$$a = j'+j_o+1, \qquad b = j'+\sigma+1, \qquad c = 2j'+2$$

showing that the equation is of the so called degenerate type; it has two linearly independent solutions denoted by u_3 and u_6 of the well known Kummer series [18]:

$$u_3 = (-z)^{1-c}(1-z)^{c-a-1} F\left(a+1-c,\ 1-b;\ a+1-b;\ \frac{1}{1-z}\right) \qquad /A14/$$

$$u_6 = z^{1-c}(1-z)^{c-a-b} F(1-a,\ 1-b;\ c+1-a-b;\ 1-z) \qquad /A15/$$

A linear combination regular in the point $z=o$ is of the form:

$$f(z) = e^{i\pi(c-a)} \frac{\Gamma(c)\Gamma(1-b)}{\Gamma(a)\Gamma(c+1-a-b)} u_6 - e^{i\pi(1-a)} \frac{\Gamma(c)\Gamma(1-b)}{\Gamma(c-a)\Gamma(a+1-b)} u_3 \qquad /A16/$$

As $a+1-c = j_o-j'<o$ and $1-a = -j'-j_o<o$ the hypergeometric functions /A14/ and /A15/ are expressible using the Jacobian polinomials [18]

$$P_n^{(\alpha,\beta)}(y) = \binom{n+\alpha}{n}\left(\frac{1}{2}+\frac{1}{2}y\right)^n F\left(-n,\ -n-\beta;\ +1;\ \frac{y-1}{y+1}\right) \qquad /A17/$$

$$P_n^{(\alpha,\beta)}(y) = \binom{n+\beta}{n}\left(\frac{1}{2}y-\frac{1}{2}\right)^n F\left(-n,\ -n-\alpha;\ \beta+1;\ \frac{y+1}{y-1}\right).$$

To get normalization constants we have two relations:

$$d^{j_o\sigma}_{jjj'}\left(x=1\right)=\delta_{jj'}$$

$$\sum_m \int_0^1 \frac{(1-x)^2}{16x^2}\, d^{j'_o\sigma'}_{jjj'}(x)\, d^{j_o\sigma}_{jmj'}(x)\, dx = \frac{\Pi}{4}\, \frac{(2j+1)(2j'+1)}{j_o^2-\sigma^2}\, \delta_{j_o j'_o}\, \delta(i\sigma - i\sigma').$$

By means of these we get the form, corresponding to that used in the literature.

$$d^{j_o\sigma}_{jjj'}(x) = \frac{\sqrt{(2j+1)(2j'+1)}}{j_o+\sigma}\, \frac{\Gamma(1+j_o+\sigma)\Gamma(1-j_o-\sigma)}{\Gamma(1+j+\sigma)\Gamma(1+j'-\sigma)} \times \qquad /A19/$$

$$\times \left[\frac{\Gamma(2j+1)\Gamma(j'+j+1)\Gamma(j'-j_o+1)\Gamma(j'+j_o+1)}{\Gamma(j'-j+1)\Gamma(j-j_o+1)\Gamma(j+j_o+1)}\right]^{1/2} \times$$

$$\times \Bigg[x^{\frac{1}{2}(j-j_o-\sigma+1)} (1-x)^{j_o-j-1} P^{(-j_o-\sigma;\,j_o+\sigma)}_{j'+j_o}\left(\frac{1+x}{1-x}\right)$$

$$- x^{\frac{1}{2}(j+j_o+\sigma+1)} (1-x)^{-j_o-j-1} P^{(j_o+\sigma;\,j_o-\sigma)}_{j'-j_o}\left(\frac{1+x}{1-x}\right)\Bigg]$$

We must stress that the normalization relations do not make the normalization constant unambiguous. An additional phase factor

$$e^{i\frac{\Pi}{2}(j'-j)} \left[\frac{\Gamma(1+j-\sigma)\Gamma(1+j'+\sigma)}{\Gamma(1+j+\sigma)\Gamma(1+j'-\sigma)}\right]^{1/2}$$

would be needed to be in agreement with eqs. /A7/ and /A9/.

Let us define the functions singular in the point $x=1$:

$$e^{j_o\sigma}_{jjj'}(x) = \frac{\sqrt{(2j+1)(2j'+1)}}{j_o+\sigma}\, \frac{\Gamma(1+j_o+\sigma)\Gamma(1-j_o-\sigma)}{\Gamma(1+j+\sigma)\Gamma(1+j'-\sigma)} \times$$

$$\times \left[\frac{\Gamma(2j+1)\Gamma(j'+j+1)\Gamma(j'-j_o+1)\Gamma(j'+j_o+1)}{\Gamma(j'-j+1)\Gamma(j-j_o+1)\Gamma(j+j_o+1)}\right]^{1/2} \times$$

$$\times\, x^{\frac{1}{2}(j-j_o-\sigma+1)} (1-x)^{(j_o-j-1)} P^{(-j_o-\sigma;\,-j_o+\sigma)}_{j'+j_o}\left(\frac{1+x}{1-x}\right).$$

These have the same relationship to the functions $d^{j_o}_{jj'}$ as those of Toller's second kind boost functions to the first kind ones [6]:

$$d^{j_o\sigma}_{jj'}(x) = e^{j_o\sigma}_{jj'}(x) + \frac{\Gamma(1+j-\sigma)\Gamma(1+j'+\sigma)}{\Gamma(1+j+\sigma)\Gamma(1+j'-\sigma)} \; e^{-j_o-\sigma}_{jj'}(x) \quad .$$

It may be seen from eq. /A19/ that

$$d^{j_o\sigma}_{jj'}\left(\frac{1}{x}\right) \sim d^{j_o\sigma}_{jj'}(x) \; .$$

It also may be seen that for $x \to 0$

$$d^{j_o\sigma}_{jmj'}(x) \sim x^{\frac{1}{2}(m-j_o-\sigma+1)} \quad \text{if} \quad m > j_o > 0 \; .$$

Having got the explicit form of $d^{j_o\sigma}_{jmj'}(\xi)$ we turn our attention towards the calculation of the Euler parameters in $D\left(U^{-1}_{Lq'}, U_{Lq}\right)$. This might be done by means of the composition or multiplication law of the Lorentz group, by means of which one is able to write down the Euler parameters of two subsequent Lorentz transformations as a function of the two transformations. The similar law of the rotation group is well known [17]. It may be seen easily from the parametrized form of a Lorentz transformation that the main problem is to calculate the parameters ϕ, δ, χ as a function of α, β, γ i.e. to solve the equation

$$e^{-i\phi M_2} \; e^{-i\delta N_3} \; e^{-i\chi M_2} = e^{-i\alpha N_3} \; e^{-i\beta M_2} \; e^{-i\gamma N_3}$$

Using the defining representation for each of the matrices M_2, N_3 we get a matrix equation. The composition law may be written down at once:

$$\mathrm{ch}\delta = \mathrm{ch}\alpha\,\mathrm{ch}\beta + \cos\beta\,\mathrm{sh}\alpha\,\mathrm{sh}\gamma \quad ; \quad \mathrm{sh}\delta > 0$$

$$\sin\phi = \frac{\sin\beta\,\mathrm{sh}\gamma}{\mathrm{sh}\delta} \quad ; \quad \cos\phi = \frac{\mathrm{sh}\alpha\,\mathrm{ch}\gamma + \cos\beta\,\mathrm{ch}\alpha\,\mathrm{sh}\gamma}{\mathrm{sh}\delta}$$

$$\sin\chi = \frac{\mathrm{sh}\alpha\,\sin\beta}{\mathrm{sh}\delta} \quad ; \quad \cos\chi = \frac{\mathrm{ch}\alpha\,\mathrm{sh}\gamma + \cos\beta\,\mathrm{sh}\alpha\,\mathrm{ch}\gamma}{\mathrm{sh}\delta} \qquad /A20/$$

It is interesting to note that just as in the case of 0/3/ these formulae also may be got by trigonometrical consideration, only in this case the trigonometry of a hyperboloid of two surfaces must be applied.

All the remaining operations which are needed to the full composition law of the Lorentz group are the operations well known from the theory of 0/3/.

As to the transformation involved in formula /25/ some minor alterations are needed when formula /A20/ is applied to it, i.e.

$$L_{q'}^{-1} L_q = e^{i\beta N_3} e^{i\Theta_s M_2} e^{-i\alpha N_3} = e^{-i\chi M_2} e^{-i\xi N_3} e^{-i\psi M_2}$$

$$ch\xi = ch\alpha\, ch\beta - \cos\Theta_s\, sh\alpha\, sh\beta \quad ; \quad sh\xi > 0$$

$$\sin\chi = \frac{\sin\Theta_s\, sh\alpha}{sh\xi} \quad ; \quad \cos\chi = -\frac{ch\alpha\, sh\beta - \cos\Theta_s\, sh\alpha\, ch\beta}{sh\xi}$$

$$\sin\psi = \frac{\sin\Theta_s\, sh\beta}{sh\xi} \quad ; \quad \cos\psi = \frac{ch\beta\, sh\alpha - \cos\Theta_s\, sh\beta\, ch\alpha}{sh\xi} \qquad /A21/$$

Appendix B

For the sake of easier understanding we quote some interconnections between the groups $SU(2)$ and $SU(1,1)$; O/4/ and the Lorentz group.

Let us consider the following matrix

$$g(\alpha,\beta,\gamma) = \begin{pmatrix} \cos\frac{1}{2}\beta\, e^{i\frac{1}{2}(\alpha+\gamma)} & \sin\frac{1}{2}\beta\, e^{i\frac{1}{2}(\alpha-\gamma)} \\ -\sin\frac{1}{2}\beta\, e^{-i\frac{1}{2}(\alpha-\gamma)} & \cos\frac{1}{2}\beta\, e^{-i\frac{1}{2}(\alpha+\gamma)} \end{pmatrix} \qquad /B1/$$

where $-2\pi \le \alpha+\gamma,\ \alpha-\gamma \le 2\pi$. If $-1 \le z \equiv \cos\beta \le 1$, the matrix $g(\alpha,\beta,\gamma)$ is unitary and unimodular, i.e. it is the element of $SU(2)$ in its defining representation. If $z > 1$ then we get the matrix:

$$\begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}$$

which is unimodular, and is an element of $SU(1,1)$. The group $SU(1,1)$ is non-compact /In /B 1 / the region of parameter β is unbounded/ with unitary representations which are infinite dimensional apart from the trivial one. Let j label an irreducible representation and m characterise an element of it. The classification of the representations of $SU(1,1)$ are the following:

I. Unitary representations:

a./ principal series: j is complex ; $\mathrm{Re}j = -\frac{1}{2}$

$m = 0, \pm 1, \pm 2, \ldots$

or $m = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$

b./ supplementary series: $\mathrm{Jm}j = 0, \quad -\frac{1}{2} < \mathrm{Re}j < 0$

$m = 0, \pm 1, \pm 2, \ldots$

c./ discrete series: $j = -\frac{1}{2}, -1, -\frac{3}{2}, \ldots$

with two possible types:

$j^{+} : m = -j, -j+1, \ldots$

$j^{-} : m = j, j-1, \ldots$

d./ scalar representation: $j = m = 0$.

II. If j is differing from those of I. then the representation is non-unitary.

The matrix elements of $SU(1,1)$ are of the form:

$$D^{j}_{mm'}(\mu, \xi, \nu) = e^{-im\mu-im'\nu} \; d^{j}_{mm'}(\xi) ,$$

here

$0 \leq \mu < 4\pi$

$0 \leq \nu < 2\pi$

$0 \leq \xi < \infty$

and the functions $d^{j}_{mm'}(\xi)$ may be calculated from the corresponding functions $d^{j}_{mm}(\cos\delta)$ of $SU(2)$ by setting $\cos\delta > 1$ and by continuation to the actual value of j.

The connection between 0/4/ and the Lorentz group is similar; taking e.g. a Lorentz transformation along the axis z:

$$\Lambda^{\mu}_{\ \nu} = \begin{pmatrix} \mathrm{ch}\alpha & 0 & 0 & \mathrm{sh}\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \mathrm{sh}\alpha & 0 & 0 & \mathrm{ch}\alpha \end{pmatrix} .$$

If we make α pure imaginary we get a matrix which is equivalent to the usual O/4/ rotation matrix

$$\begin{pmatrix} \cos\beta & 0 & 0 & \sin\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\beta & 0 & 0 & \cos\beta \end{pmatrix},$$

the matrix of equivalence being

$$U = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -i \end{pmatrix}.$$

The connection between the representations of O/4/ and the Lorentz group is quite analogous to that between $SU(2)$ and $SU(1,1)$.

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