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LITTLE GROUPS, ANALYTICITY AND FAMILIES OF REGGE TRAJECTORIES

(An elementary approach)

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C O N T E N T S

I. Introduction

The purpose of this talk is to give a brief review of some ideas which have played a certain role in recent developments of what we usually call Regge pole theory. These developments lead to a better understanding of the properties of "composite states" (what all known hadrons are believed to be) and at the same time resolve some difficulties that have accumulated in the past few years.

Historically, these developments originate from at least four sources.

1. It was discovered by Gribov and Volkov [1] and Goldberger, Grisaru, MacDowell and Wong [2] that certain constraints have to be imposed on helicity amplitudes if one wants to have the sacred principles of analyticity, crossing symmetry and Lorentz invariance to coexist peacefully. In Regge theory these constraints can be fulfilled only if one invokes several Regge poles and assumes some definite relations between their residues and/or trajectories. ("Conspiracy" of Regge poles).

2. Model calculations showed [3,4] that the structure of the bound state spectrum in relativistic theories is more complicated than one would conjecture by a naive extrapolation from Schrödinger theory.

3. Toller and his group studied the group theoretical structure of scattering amplitudes [5] and discovered that a two-body, elastic scattering amplitude at vanishing momentum transfer admits a partial wave expansion according to the representations of the homogeneous Lorentz group instead of the "ordinary" partial wave expansion. Toller was the first to point out that this generalised partial wave expansion automatically satisfies the kinematic constraints discovered in [1, 2], by correlating an <u>infinite</u> number of "ordinary" partial waves.

4. Freedman and Wang [6] tried to force an unequal mass, spinless scattering amplitude to have Regge asymptotic behaviour and found that this was possible only if several Regge poles were arranged according to the scheme described in [3, 5].

There followed an immense number of papers, at various degrees of mathematical sophistication and by now a unified pattern begins to emerge from these - at first glance uncorrelated - fields of research.

It turned out that the analyticity properties and group structure of scattering amplitudes are closely related to each other and all the difficulties of Regge theory (expressed partly in the form of kinematic constraints, partly by the appearance of unwanted singularities in Regge-pole contributions) can be nicely resolved by essentially algebraic methods.

After a brief review of some aspects of the representation theory of the Poincaré group (P) we shall study the group properties of "composite states" and show how the diseases of Regge pole theory can be traced back to certain properties of their spin algebra (the little group of P.). This will give us immediately the recipe to cure the diseases, and at the same time it leads to some interesting physical consequences, of which a few will be briefly mentioned in the last section.

We shall try to emphasize throughout the underlying physical ideas, often at the expense of mathematical rigour. Also, technical details will be omitted whenever possible, in order to make the presentation more understandable.

We have discussed all the ideas and results described here together with P. Surányi. The only reason why his name does not appear under the title of this report, is that he has not seen the present version of the MS and therefore cannot be held responsible for it.

2. Local representations of the little group

In this section we summarise some results which are well - known - or at least should be.

We require that our theory be invariant under inhomogeneous Lorentz transformations; these transformations are generated by the operators P_μ and $M_{\mu\nu}$ which satisfy the usual commutation relations (CR):

$$[P_\mu, P_\nu] = 0 \qquad \begin{pmatrix} \mu,\nu = 0,1,2,3 \\ i,j,k = 1,2,3 \end{pmatrix} \tag{1a}$$

$$[M_i, M_j] = i\,\varepsilon_{ijk} M_k \tag{1b}$$

$$[M_i, N_j] = i\,\varepsilon_{ijk} N_k \tag{1c}$$

$$[N_i, N_j] = -i\,\varepsilon_{ijk} M_k \tag{1d}$$

$$[M_i, P_j] = i\,\varepsilon_{ijk} P_k \tag{1e}$$

$$[M_i, P_0] = 0 \tag{1f}$$

$$[N_i, P_0] = i P_i \tag{1g}$$

$$[N_i, P_k] = i\,\delta_{ik} P_0 \tag{1h}$$

(Here we put $M_i = \frac{1}{2}\varepsilon_{ijk} M_{jk}$, $N_i = M_{0i}$.)

We want to study <u>irreducible states</u>, i.e. those which transform according to an irreducible representation (IR) of the algebra (1).

Our main concern will be the relation of such irreducible states to analytic properties of scattering amplitudes, in which those states enter as intermediate states or exchanged ones.

2.1. <u>Little groups</u>

What are the quantum numbers characterising a state with definite four - momentum? This question leads to the concept of the little group (in the mathematical literature: the isotropy group of the four-momentum).

The answer can be found by constructing the most general set of operator which commute with P_μ . For such a construction one has to go "outside" the algebra (1) and consider elements of its <u>universal enveloping algebra</u> (UEA). For physicists' purposes, a definition of the universal enveloping algebra is the following: consider not only the elements P_μ , $M_{\mu\nu}$, but their "ordinary" products, like $P_\alpha M_{\mu\nu}$, $P_\alpha P_\beta$ etc.and give a meaning to the commutator (the operation which played the role of the "product" in the algebra (1)) in the usual way, i.e. if A, B, C are elements of the algebra, then AB is an element of its UEA, and

$$[A.B, C] \stackrel{\text{def}}{=} A[B,C] + [A,C]B .$$

Now it is easy to show that

$$S_\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}$$

has the property that

$$[S_\mu, P_\alpha] = 0 .$$

Also, only three components of S_μ are independent since $S_\mu P_\mu = P_\mu S_\mu = 0$, The operators S_μ do not form a subalgebra of the UEA of (1) by themselves, since

$$[S_\mu , S_\nu] = i \, \varepsilon_{\mu\nu\rho\sigma} P_\rho S_\sigma . \qquad (2)$$

However, if I consider only those states, which are eigenstates of P_α , I can regard the P_α -s as c -numbers and then the S_μ do form an algebra.

By choosing a representative of P_μ for $P^2 \lesseqgtr 0$, one can verify that the algebra (2) is isomorphic to SU(2), E(2), SU(1,1) respectively. Wigner achieved this by choosing the representative vectors as $(0,0,0,W)$, $(0,0,p,p)$, $(0,0,p,0)$.
However, we cannot be satisfied with such a choice. Indeed, our main objective will be to study the behaviour of (2) as the momentum changes continuously from the timelike region through the lightlike one to spacelike values.

A suitable choice is the following: $P_\mu = (0,0,p,\sqrt{p^2+W^2})$.
(There are other choices possible, of course)
(We shall not be concerned with the exceptional case $P_\mu = 0$ because of reasons which will become clear later on.)

Let us write out the explicit form of the S_μ with this choice:

$$\tilde{S}_1 = \frac{1}{W}\left(M_1 \cdot \sqrt{p^2+W^2} - N_2 p\right),$$
$$\tilde{S}_2 = \frac{1}{W}\left(M_2 \sqrt{p^2+W^2} + N_1 p\right),$$
$$\tilde{S}_3 = \frac{1}{W}\left(M_3 \sqrt{p^2+W^2}\right),$$
$$\tilde{S}_0 = \frac{1}{W} M_3 p .$$

(We divided through all the operators by $W=(-P^2)^{-1/2}$ because of dimensional reasons.) Also, let us perform a similarity transformation on all the parameters by multiplying them through with $\left(W/\sqrt{p^2+W^2}\right)^{-1}$, then we are left with the following algebra:

$$\begin{aligned} S_1 &= M_1 - v N_2 , \\ S_2 &= M_2 + v N_1 , \\ S_3 &= M_3 , \\ S_0 &= v M_3 = v S_3 , \end{aligned} \tag{3}$$

where $v = \dfrac{p}{\sqrt{p^2+W^2}}$.

Now, if I keep p fixed, then

$|v| < 1$ corresponds to timelike

$|v| = 1$ corresponds to lightlike

$|v| > 1$ corresponds to spacelike

momenta. (There is a branch point at $W^2 = -p^2$; this is unimportant for us, since we shall be interested in the behaviour near $W=0$).

I have the CR between the independent operators (S_1, S_2, S_3) :

$$[S_1, S_2] = i\,(1-v^2)\,S_3\,,$$
$$[S_3, S_2] = -i\,S_1\,,\qquad (4)$$
$$[S_3, S_1] = i\,S_2\,.$$

The Casimir operator, C can be written as $C = S_\mu S_\mu$; it is obviously Poincaré invariant:

$$[M_{\mu\nu}, C] = [P_\alpha, C] = 0.$$

(In fact, it is one of the Casimirs of (1), the other being $P_\mu P_\mu = -W^2$).

2.2. Standardisation of the algebra of Wigner oparators, S_μ.

A representation of the CR (4) can be easily constructed; thereby we shall obtain a representation of the whole algebra (1) if we start out (as we always did so far) from eigenstates of P_μ.

Introduce $S_\pm = S_1 \pm i\,S_2$, then (4) is rewritten as

$$[S_3, S_\pm] = \pm S_\pm\,, \qquad (5a)$$

$$[S_+, S_-] = 2\,(1-v^2)\,S_3\,. \qquad (5b)$$

On introducing $\varepsilon = \mathrm{sgn}\,(1-v^2)$ ($\varepsilon = 0$ for $v = \pm 1$) and $J_\pm$, J_3 with the definition: $J_\pm \cdot \sqrt{|1-v^2|} = S_\pm$, $J_3 = S_3$, I have

$$[J_3, J_\pm] = \pm J_\pm$$
$$[J_+, J_-] = 2\,\varepsilon\,J_3\,. \qquad (6)$$

(This standardised form of the CR clearly shows the structure of the algebra we just talked about.)

It is instructive to write out the value of the Casimir operator in terms of the J :

$$C = |1-v^2|\,\left(J_1^2 + J_2^2 + \varepsilon\,J_3^2\right).$$

The real question is now: what are the realisations of this algebra (5) we are working with, and how do they behave as v passes through ± 1 . (A parenthetical remark is in order here. In our earlier work [7] we started from an explicit realisation of the algebra (5) for the timelike case and performed an analytic continuation afterwards. Although this procedure is, in principle correct, it gave rise to some misunderstandings. I am grateful to Professors G. Feldman and P.T. Matthews for some critical remarks on this subject.)

2.3. Representations

The standardised form of the generators (6) allows one to construct the representations of the little group by entirely elementary techniques (NB that we are talking about the little group and not about little groups corresponding to the different Wigner classes.)

Since in most of the cases we shall have to do with non-unitary representations (Regge poles), it is more convenient to realise the algebra (6) by means of differential operators rather than by matrices (since the wave functions we shall be dealing with do not satisfy convenient boundary conditions and completeness relations). We start out from the "ordinary" unitary representations for the timelike case, ($\varepsilon = +1$)and choose the functions

$$D^j_{m\lambda}(\phi\,\theta\,-\phi) \qquad (2j = \text{integer})$$

as a basis. Then it is straightforward to show that we obtain a realisation of (6) by the following operators:

$$J_1 = i\left\{\mathrm{ctg}\,\theta\,\cos\phi\,\frac{\partial}{\partial\phi} + \sin\phi\,\frac{\partial}{\partial\theta}\right\} + \lambda\,\frac{\cos\phi}{\sin\theta}\,(1-\cos\theta),$$

$$J_2 = i\left\{\mathrm{ctg}\,\theta\,\sin\phi\,\frac{\partial}{\partial\phi} - \cos\phi\,\frac{\partial}{\partial\theta}\right\} + \lambda\,\frac{\sin\phi}{\sin\theta}\,(1-\cos\theta), \qquad (8)$$

$$J_3 = -i\,\frac{\partial}{\partial\phi} + \lambda \,. \qquad (\varepsilon = +1)$$

In order to obtain representations of the group generated by the $\mathfrak{S}\mathfrak{L}$ it is sufficient to perform a transformation on the parameters.

In order to obtain a proper basis for any value of j (and hence, to be able to produce representations for the spacelike case, $|\nu| > 1$ (or $\varepsilon = -1$) we proceed by following Andrews and Gunson [8].

We define for arbitrary j (not necessarily half-integer), the basis functions as follows:

$$\mathfrak{D}^{j}_{m\lambda}(\phi\,\Theta\,-\phi) = -\frac{1}{\pi}\,\mathrm{tg}(j-\lambda)\pi\; E^{-j-1}_{-m-\lambda}(\phi\,\Theta\,-\phi),$$

where the E-s are functions of the 2nd kind.

For the reader's convenience we quote the corresponding definitions from Andrews and Gunson, op. cit.

$$D^{j}_{m,m'}(\alpha\beta\gamma) = e^{i(m\alpha+m'\gamma)}\, d^{j}_{mm'}(z),$$

$$(z = \cos\beta).$$

For $\quad m-m' \geq 0, \quad m+m' \geq 0 \quad$ we have:

$$d^{j}_{mm'}(z) = \left(\frac{1+z}{2}\right)^{\frac{1}{2}(m+m')}\left(\frac{1-z}{2}\right)^{\frac{1}{2}(m-m')}\left[\frac{\Gamma(j+m+1)\,\Gamma(j-m'+1)}{\Gamma(j+m'+1)\,\Gamma(j-m+1)}\right]^{1/2}\frac{F(-j+m,\,j+m+1;\,1-m-m';\,\frac{1-z}{2})}{\Gamma(m-m'+1)}$$

To continue to other regions, use the following relations:

Region:	Relation:
$m-m' \leq 0$, $m+m' \geq 0$	$d^{j}_{mm'} = (-1)^{m-m'}\, d^{j}_{m',m}$
$m-m' \leq 0$, $m+m' \leq 0$	$d^{j}_{mm'} = (-1)^{m-m'}\, d^{j}_{-m,-m'}$
$m-m' \geq 0$, $m+m' \leq 0$	$d^{j}_{mm'} = d^{j}_{-m'\,-m}$

$$E^{j}_{mm'}(\alpha\beta\gamma) = e^{i(m\alpha + m'\gamma)}\, e^{j}_{mm'}(z),$$

$$e^{j}_{mm'}(z) = \frac{\pi}{2\sin\pi(j-m)}\left\{ e^{-i\pi(\operatorname{sgn}\operatorname{Im} z)(j-m)}\, d^{j}_{mm'}(z) - d^{j}_{m-m'}(-z)\right\},$$

$$e^{j}_{mm'}(z) = \frac{1}{2}\left[\Gamma(j+m+1)\Gamma(j-m+1)\Gamma(j+m'+1)\Gamma(j-m'+1)\right]^{1/2} \times$$

$$\times \left(\frac{1+z}{2}\right)^{\frac{1}{2}(m+m')}\left(\frac{1-z}{2}\right)^{-\frac{1}{2}(m-m')}\left(\frac{z-1}{2}\right)^{-j-m'-1} \frac{F\left(j+m+1, j+m'+1; 2j+2; \frac{2}{1-z}\right)}{\Gamma(2j+2)},$$

$$\lim_{2j=\text{integer}}\left(-\frac{1}{\pi}\operatorname{tg}\pi(j-m)\, e^{-j-1}_{-m,-m'}(z)\right) = d^{j}_{mm'}(z).$$

Products of the $\mathcal{D}^j$ reduce with ordinary Clebsch-Gordan coefficients (CGC) (Recipe: take Racah's symmetrised expression for the CGC and replace the factorials by Γ -functions for arbitrary j -s). Then

$$\mathcal{D}^{j_1}_{m_1m_1'}(\alpha\beta\gamma)\,\mathcal{D}^{j_2}_{m_2m_2'}(\alpha\beta\gamma) = \sum_{mm'}\sum_{j=-\infty}^{j_1+j_2}\mathcal{D}^{j}_{mm'}(\alpha\beta\gamma)(j_1m_1 j_2m_2|jm)(j_1m_1' j_2m_2'|jm').$$

The $\mathcal{D}^j$-s are identical to the functions entering in Mandelstam's Reggeization procedure.

The representations we obtained here are representations of the Lie algebra (5) of the little group. It is important to emphasize again that they are in general representations of the algebra and can be continued to global representations only in particular cases (eg. for Θ real, one must have $2j =$ integer in order to preserve the topological properties of the group).

2.4. Inclusion of space reflection

This is straightforward for the generators (3), using $P\underset{\sim}{M}P^{-1} = \underset{\sim}{M}$, $P\underset{\sim}{N}P^{-1} = -\underset{\sim}{N}$, and elementary geometrical considerations for the Euler angles: under P, φ goes over to $-\varphi$ and ϑ goes over to $\pi-\vartheta$; if - as in the following - λ means a helicity index, then $\lambda \to -\lambda$ under P. Thus the functions

$$\mathcal{D}^{j\pm}_{m\lambda}(\phi\,\theta) = 2^{-\frac{1}{2}}\left[\mathcal{D}^{j}_{m\lambda}(\phi\;\theta - \phi) \pm \eta\,\mathcal{D}^{j}_{m-\lambda}(-\phi, \pi-\theta, \phi)\right] \tag{8}$$

form a basis with $S_\mu S_\mu$, S_3 and P diagonal. (η is a suitable phase factor.)

3. Partial wave analysis, single "particle" terms, pseudostates

This subject is by now well - known [9] and it means to decompose a scattering amplitude or vertex with respect to IR-s of the Poincaré group. Clearly, every matrix element of an invariant scattering operator, unless taken between one-particle states, transforms according to a reducible representation of the Poincaré group. Indeed, if we consider a two-particle state, it transforms according to $(S_1 m_1) \otimes (S_2 m_2)$ and similarly for many--particle states. The Clebsch - Gordan decomposition is the most conveniently written in a mixed basis, where the product state is characterised by the one-particle helicities, $\lambda_1, \lambda_2, \ldots$ and the "intermediate" state by the projection of the spin onto a fixed axis, m . Symbolically then we can write e.g. for a two-particle amplitude:

$$\langle p_3 s_3 \lambda_3\; p_4 s_4 \lambda_4 | T | p_1 s_1 \lambda_1\; p_2 s_2 \lambda_2\rangle = \sum \int \langle p_3 s_3 \lambda_3\; p_4 s_4 \lambda_4 | p' j' m' \alpha'\rangle \times$$

$$\times \langle \alpha' p' j' m' | T | \alpha p j m\rangle \langle \alpha p j m | p_1 s_1 \lambda_1\; p_2 s_2 \lambda_2\rangle \tag{1}$$

The first and last factors are CGC of $\mathcal{P}$, α, α' are degeneracy labels, the summation and integration goes over every dummy label. Further, by Wigner-Eckart

$$\langle \alpha' p' j' m' | T | \alpha p j m\rangle = \langle \alpha' \| T(p^2, j) \| \alpha\rangle\, (p j m, 000 | p' j' m') \tag{2}$$

(since T is an invariant operator)

and the reduced matrix element is what we commonly call the partial wave amplitude. Explicit forms of the CGC have been first calculated by Ioos [9]. The existence of a partial wave expansion for a scattering amplitude is a fundamental property and follows essentially from its Lorentz (i.e. Poincaré) invariance and unitarity.

Now such a partial wave expansion is fine until I begin to combine it with some other properties of the scattering amplitude.

The properties which turn out to be "dangerous" are equally fundamental: they are: crossing and analyticity properties.

Instead of going into a full treatment of the problem, let us describe the situation qualitatively.

I can perform a partial wave decomposition of a two-body scattering amplitude in three channels. In one channel the total four-momentum is timelike (this is the expansion we have just written down) while in both the other ones it is spacelike.

Now, since by assumption, a scattering matrix element (1) is an analytic function of the Mandelstam invariants with known analytic properties, there is a path in the complex s,t space along which an expansion given by (1) and (2) can be continued to a spacelike expansion. (The helicity amplitudes we are working with have more complicated analytic properties than the spinor amplitudes - the so-called invariant amplitudes - do; a full analysis of these kinematic singularities has been given recently in the beautiful papers by Cohen-Tanondji et al [10]. Here we are concerned with a slightly different problem; it is the following.

In the process of continuing the scattering amplitude from a "direct" to a "crossed" channel, the structure of the little group changes from SU(2) to SU(1,) as shown by eqs. (2.5-6). Correspondingly, one must continue the expansion (1), (2) by means of the Watson-Sommerfeld method. Now, unless the scattering amplitude is physically uninteresting, in the process of the Watson-Sommerfeld continuation one will pick up singularities of the partial wave amplitude $\langle\alpha'\| T(p^2;j)\|\alpha\rangle$, and we shall be interested in the relation of the singularity structure of the partial wave amplitude and its analyticity properties. In order to simplify the treatment, we shall assume that the singularities correspond to "bound states" in the two-particle amplitude.

What is a "bound state"?
Unitarity always allows us to write $\langle\alpha'\| T\|\alpha\rangle$ in the form:

$$\langle\alpha'\|T\|\alpha\rangle = \sum_{\alpha''}\langle\alpha'\|N\|\alpha''\rangle\langle\alpha''\|D^{-1}\|\alpha\rangle$$

$$= \frac{1}{\Delta}\sum_{\alpha''}\langle\alpha'\|N\|\alpha''\rangle\langle\alpha''\|[D]\|\alpha\rangle,$$

where $\Delta = \mathrm{Det}\, D$, and $\langle\alpha'\|[D]\|\alpha\rangle$ is the minor corresponding to the element $\langle\alpha'\|D\|\alpha\rangle$ of the matrix D.

We shall say that there is a bound state in the two-particle system we are investigating, if, (on writing $p^2 = -s$) $\Delta(s,j)$ has a simple zero in such a way that,

a) $\Delta(s,j)$ is regular in some (sufficiently large) domain in the complex s,j space

b) The equation $\Delta(s,j) = 0$ can be solved either in the form $s = s(j)$ or $j = \alpha(s)$

c) The residue matrix of the corresponding pole in T has rank one (factorised residues).

The bound states defined in this way share many (although not all) properties of an ordinary particle.

In particular, one finds that the residue-factors in the T -matrix play very much the same role as form factors (or wave functions) of a particle; since the same bound state is coupled - through unitarity - to any physical particle system, (unless the coupling is zero because of some selection rule) which, by assumption form a complete set of states, one can even define a "state vector" to every point of the surface $\Delta(s,j) = 0$, in such a way that the residue factors can be considered as scalar products (i.e. "wave functions") in the usual way.

Correspondingly, an infinitesimal Lorentz transformation on the physical states induces an infinitesimal transformation on the state vector of the bound state. Looking at the CG decomposition of T (eqs 1. and 2) and at the explicite form of the CGC one can verify in particular that the infinitesimal transformations of the little group are realised by the differential operators, S_i, constructed in the previous Section.

As long as we consider infinitesimal transformations, there is no real difference between wave function which transform according to unitary or non-unitary representations of the little group. Differences arise only when we start considering <u>finite</u> transformations, since the wave functions [13] corresponding to non-unitary representations do not satisfy proper boundary conditions, therefore in this case we shall speak about pseudostates.

In the following section we shall investigate the following problem: given the contribution of a pseudostate (or Regge pole, in more conventional terms) to the scattering amplitude, how does it behave when the mass of the pseudostate passes through zero. Before going into the investigation, we add a few remarks about the nature and importance of this problem. It is well known that when a pseudostate is exchanged in a scattering amplitude, (i.e. the contribution of the pseudostate is written in the form of a Regge-pole) the mass of that pseudostate is tied together with an external variable, namely the invariant momentum transfer. Therefore, depending on the external mass configuration, the point where the mass of the pseudostate vanishes is either on the boundary or in the inside of the physical region but (provided there are no massless physical particles) certainly in the inside of the regularity domain of T (or rather, of the amplitude freed of kinematical singularities).

We have seen in the previous Section that the sturcture of the (strictly speaking, complexified) little group changes as the velocity v passes through one. Now, if one looks in particular at the form of the little group opearators expressed through the $\mathcal{F}_i$ -s (eq. 2.6), it is evident that the structure of the group (or algebra) changes discontinuously at $|v|=1$. (The CR involve the step function $\varepsilon = \mathrm{sgn}\,(1-v^2)$). The question naturally arising is whether such a discontinuous change in the group structure as we pass through the light cone of the four-momentum of the exchanged pseudostate does not give rise to unwanted singularities in the scattering amplitude?

If the answer is in the affirmative (as we shall see, it is) then we either should abandon the concept of one-"particle" exchange at all- which is hard to do, since after all it is based on physically very clear concepts, or else, we should modify the group structure of a "bound state" so as to make analyticity, unitarity and Lorentz invariance compatible.

4. The trouble on the light cone and how to avoid it

4.1. The trouble: contraction of the little group

Let us investigate the change in the structure of the little group as realised by the operators (2.6, 2.8). The corresponding operators S_i can be defined as an analytic function of the parameter $\xi = (1-v^2)^{1/2}$ namely

$$S_{1,2}(\xi) = \xi \, \mathcal{J}_{1,2}$$

$$S_3(\xi) = \mathcal{J}_3$$

Therefore, as $\xi \to 0$, naively we would put $S_{1,2}(\xi) \to 0$; of course, whether limiting procedure is justified, depends on the way as $\mathcal{J}_{1,2}$ acts on the basis vectors. There are several possibilities,

a/ If $\mathcal{J}_{1,2}$ carries every basis vector into another one with a finite coefficient, then the Lie algebra of the little group contracts (Wigner, Inönü [11]) into the algebra generated by the single element S_3.

b/ If the coefficients in $\xi \mathcal{J}_{1,2} |jm\rangle$ are such that they tend to a finite limit as $\xi \to 0$, then the algebra contracts into E(2), as shown in Sec. 2.

Remark: case b/ is realised in the Wigner-Inönü procedure of contraction by taking a sequence $\xi_n \to 0$ and letting j become a function of ξ_n , in such a way that $2j(\xi_n)$ = integer (n=1,2,....) and $\lim_{n\to\infty} \xi_n j(\xi_n)$ = finite.

c/ Finally, it is possible that $\xi \mathcal{J}_{1,2} |jm\rangle$ has no limit as $\xi \to 0$.

Clearly, when we are considering our pseudostates, only cases a/ or c/ can be realised, since the Froissart bound on the scattering amplitude does not allow a Regge pole to go to infinity at zero mass (barring pathological behaviours of the residues).

Now let us see how this situation is reflected in the properties of scattering amplitudes involving pseudostates.

The contribution of a Regge pole to a two-particle scattering amplitude is proportional to the quantity:

$$R = \sum_{m\, \nu_1 \dots \nu_4} \langle p_3 s_3 \lambda_3 \;\; p_4 s_4 \lambda_4 | p, \alpha(s), m\, \nu_3 \nu_4 \rangle \, \beta^*_{\nu_3 \nu_4}(s,\alpha) \, \beta_{\nu_1 \nu_2}(s,\alpha) \times$$

$$\times \langle p, \alpha(s), m\, \nu_1 \nu_2 | p_1 s_1 \lambda_1 \;\; p_2 s_2 \lambda_2 \rangle \, , \qquad (1)$$

where the first and lest factors are CGC of the Poincaré group, β is a residue factor. If (as we want to assume) the pseudostate corresponding to the Regge pole is an eigenstate of parity, then the corresponding linear combinations of the CGC are just proportional to the functions $\mathcal{D}^{\alpha(s)\pm}$ defined in Sec. 2.

The $\mathcal{D}^{\alpha\pm}$ express the effect of a finite transformation generated by the algebra (2.5) (for the timelike case, a pure rotation in the rest frame), which brings the relative momentum over to a vector parallel to p. If we denote the Euler angles of this rotation in the rest frame by $(\phi, \Theta, -\phi)$, and the polar angles of the relative momentum in the frames we have chosen (i.e. $p_3 \neq 0$, fixed) by Θ' and ϕ', then by elementary kinematics we find $\phi' = \phi$ and $\mathrm{tg}\frac{\Theta}{2} = [(1-v)/(1+v)]^{1/2}\, \mathrm{tg}\frac{\Theta'}{2}$ (actually the $\mathcal{D}^{\alpha\pm}$ are linear combinations of two "ordinary" rotations, one by $(\phi, \Theta, -\phi)$, the other by $(\phi, \pi - \Theta, -\phi)$).

If one works out the details, one finds that in the Regge-pole contribution there appears a singularity (a branch point) at vanishing mass of the pseudostate, as a consequence of the contraction of the little group, unless the spin of the state at vanishing mass is physical ($2\alpha(0)$ = integer). [7].

In other words, one finds that for the known Regge-pole (with the possible exception of the Pomeranchuk, provided it passes through exactly one) case c/ above is realised.

This means that the contracted little group cannot be represented on the pseudostates at zero mass - and as a consequence, there appear unwanted singularities in the scattering amplitude. (By "unwanted" we mean that the singularities in question cannot be removed by factoring out the kinematic singularities of the helicity amplitudes, or in other words that they are present in the spinor amplitudes which are free of kinematic singularities.)

These singularities are indeed unwanted: they appear at a point (vanishing invariant momentum transfer) which is an internal point of the regularity domain of the amplitude!

4.2. .... and how to avoid it: enlarging the little group

Our problem is now to eliminate the unwanted singularities from the scattering amplitudes. Clearly, the singularities arise because the Regge-poles (and not some particular scattering amplitudes) have unpleasant properties; therefore a consistent elimination of all the unwanted singularities is possible only if one changes the group structure of Regge-poles.

In ref [7] this was achieved on the basis of a no-contraction theorem [12]; we searched namely for the algebra which does not contract at $|v| = 1$ and showed - in fact by inspection - that the spin group generated by the non-contracting algebra gave rise to well-behaved Regge terms in the scattering amplitude. Here we follow a slightly different approach [13], which has the advantage that it is both more rigorous and clearer intuitively. The final result is, of course, equivalent to the previously obtained one.

We saw that all our trouble came from the fact that the matrix elements of the operators of the little group were singular at $|v| = 1$. In order to cure this disease, let us remember the definition (2.3) of the generators of the little group, and define the following operators:

$$E_1 = M_1 - N_2 \,, \quad F_1 = M_1 + N_2 \,, \qquad (2)$$
$$E_2 = M_2 + N_1 \,, \quad F_2 = M_2 - N_1 \,.$$

The three indepent operators S_i can be expressed with these operators as follows:

$$S_1 = \tfrac{1}{2}(1+v)E_1 + \tfrac{1}{2}(1-v)F_1 \,,$$
$$S_2 = \tfrac{1}{2}(1+v)E_1 + \tfrac{1}{2}(1-v)F_2 \,,$$
$$S_3 = M_3 \quad \text{(as before).}$$

In order to investigate the behaviour at $v = +1$, we introduce a parameter which is slightly more convenient than v. We write namely

$$S_3(\lambda) = M_3 \,, \quad S_1(\lambda) = (E_1 + \lambda F_1)\tfrac{1}{2} \,, \quad S_2(\lambda) = (E_2 + \lambda F_2)\tfrac{1}{2}$$

with $\lambda = \dfrac{1-v}{1+v}$ (and we didn't give a new name to the operators $S_i/(1+v)$)

Now, the crucial observation is the following, which we formulate in the form of a theroem.

Theorem

The contribution of a pseudostate is regular at $v = 1$ (i.e. $\lambda = 0$), if and only if the "wave function" of pseudostate furnishes a representation of the operators

$$S_i \equiv S_i(0) \quad \text{and} \quad T_i^n \equiv \left[\frac{d^n S_i(\lambda)}{d\lambda^n}\right]_{\lambda=0}$$

as well as the algebra formed by taking all the possible commutators between them.

The proof of this theorem is straightforward and we skip it here, adding only three comments.

a/ Evidently, the last part of the statement is necessary since we have to generate finite transformation by "exponentiation".

b/ If the conditions of the theorem are satisfied then we can expand in a power series in λ every matrix element or equivalently, expand in powers of $W = \sqrt{-p^2}$, this is just what we need.

c/ The case $\nu \to -1$ can be treated analogously by changing the rôles of the E_i and F_i . The problem is now to find the algebra in question and its representations.

Evidently, $S_1(0) = E_1/2, S_2(0) = E_2/2$, $S_3(0) = M_3$ generate the little group of a single pseudostate at $\lambda = 0$; further we have

$$T_1^1 = F_1/2, \quad T_2^1 = F_2/2, \quad T_i^k = 0 \ , \quad (k = 2, 3, \ldots) .$$

The commutators can be worked out by using the definition (2) and the standard CR of the Lorentz group. As a result, we need one more element besides the operators we had so far. We have namely:

$$\begin{aligned}
&[E_1, F_1] = [E_2, F_2] = 2i N_3 , \\
&[E_1, F_2] = [F_1, E_2] = 2i M_3 , \\
&[E_1, E_2] = [F_1, F_2] = [M_3, N_3] = 0 , \qquad\qquad (3) \\
&[M_3, E_2] = -i E_1 \ , \quad [M_3, F_2] = -i F_1 , \\
&[M_3, E_1] = i E_2 \ , \quad [M_3, F_1] = i F_2 ,
\end{aligned}$$

$$[N_3, E_1] = -iE_1 , \quad [N_3, E_2] = -iE_2 ,$$

$$[N_3, F_1] = +iF_1 , \quad [N_3, F_2] = +iF_2 .$$

We recognize the CR of the algebra SL(2,C) in a root vector basis.

In order to show that a single "ordinary" pseudostate in general does not span a representation of (3) it is sufficient to observe that e.g. the operator F_1 does not commute with the Casimir operator C at $\lambda = 0$, indeed, $C(\lambda=0) \sim E_1^2 + E_2^2$, and $[E_1^2+E_2^2, F_1] = 2i(\{E_1, N_3\} - \{E_2, M_3\}) \neq 0$.

Representations can be constructed, however, by taking "ladders" of irreducible representations of the little group, and this leads in a well-known way to the "families" of Regge <u>intercepts</u> at zero mass.

In order to represent <u>space inversion</u> $\mathcal{P}$, a doubling of the number of states is necessary. Indeed, using again the relations

$$[P, M_i] = \{P, N_i\} = 0,$$

one finds that if we build up a representation of (3) based on a ladder of IR-s of the algebra E_1, E_2, M_3, then space reflection brings it over to a different ladder based on the IR-s of the (isomorphic) algebra F_1, F_2, M_3. It follows in particular that if one labels the IR-s of SL(2,C) by the two quantum numbers (σ, j_0), then a space-reflected representation is characterised by $(\sigma, -j_0)$ and parity eigenstates can be constructed by forming suitable linear combinations of the basis vectors. This parity doubling was again and again discovered and re-discovered (and mystified) by various authors working on the so-called conspiracy-theory of Regge poles. (Sometimes one wonders how useful this "neo -primitivism" in the physical literature really is: there exist some thirty-or-so pages long papers which effectively could be substituted by a single reference to a good graduate text on group theory.....)

5. Selected topics from the theory of "conspiracy"

It is impossible to review in a paper of this size all the results which emerged from the idea of Regge-families, conspiracies, etc. We can but briefly mention a few - and perhaps not the most commonly known - aspects of the theory.

5.1. What is a "particle"? Lorentz poles

We have shown in the previous section that in order to secure the correct analyticity properties of scattering amplitudes, the bound state contributions must span a representation of SL(2,C). We did not prove, however that they must span an irreducible representation. Nevertheless, dynamical models suggest to consider a set of states transforming according to an IR of SL(2,C) as representing a single - "particle" "state" , in other words to consider the various IR-s of E(2) contained in the IR of SL(2,C) as various states of the same "Lorentz-pole" (as opposed to a Regge-pole), or "particle". (The reader should recall the analogous situation in - say - the nonrelativistic domain, we say that "the electron" of spin 1/2 has two states, with "spin up" and "spin down" instead of speaking about an "up-electron" and a "down-electron" although the "up-electron and "down-electron" transform according to different IR-s of U(1), both contained in the single IR of SU(2). In the presence of a magnetic field, SU(2) is not even a symmetry group of the problem! However, there again come analyticity arguments, starting like: "If I turn off the magnetic field, ... "etc.).

5.2. Where are Regge's daughters? The trajectory formula

The analogy with the nonrelativistic problem just mentioned, lead us to speculate [4,14] about the fate of "daughter"-trajectories as one goes off the point W=0. It is clear from model calculations that the family of trajectories can no more be described by an irreducible representation of SL(2,C) (if it is at W=0), but by a reducible one. Fortunately, the operator which perturbs the W=0 representation has simple transformation properties: it is just a four-vector, and therefore quite definite statements can be made about trajectories and residues. In particular, the known nucleon and delta trajectories (and the others, so far not known) form one family in the sense that they shrink to a single IR of SL(2,C) at W=0.

If I expand the trajectory function in powers of the mass (á la Chew-Frantschi), a four parameter formula describes an <u>infinite</u> family of nucleon trajectoires. Fitting in four known states we were able to predict the masses of <u>all</u> the known $I=1/2$ πN reasonances within some 15 percents and the intercept of N_α at $t=0$ to about the same accuracy (provided one is willing to accept the fits to backward πN scattering.....). The situation is similar in the $I=3/2$ channel [15], and somewhat less spectacular, but still encouraging with the strange baryon resonances. So, it seems that those daughters have been living with us all the time - we only did not recognise them.

5.3. Partial wave expansion with respect to a non-invariance group; SL(2,C) as a Trajectory Generating Algebra

The group we just found and which classifies the Regge trajectories (in the sense that the individual pseudostates should be put together to form the rungs of a ladder) is - generally speaking - <u>not a little group</u> of the four-momentum transfer. Indeed, unless the masses of external particles joining a Regge vertex are equal, the momentum transfer vector is lightlike at $t=0$ but not a null-vector. Therefore the term "trajectory generating algebra" was proposed for SL(2,C) in this role: its operators when applied to a pseudostate, produce another one, not identical to the original.

Nonetheless, the contribution of the Lorentz-pole can be cast into a form which is strictly analogous to an ordinary Regge-pole contribution. The only problem is to couple a two-particle state (transforming according to a product representation of $\mathcal{P}$) to a pseudostate $|\sigma j_0\, jm\rangle$ being related to the eigenvalue of the Casimir operator $E_1^2+E_2^2$ and M_3 in the usual way.

Now the kinematic part of the bracket $\langle\sigma j_0\, jm|\, p_1 s_1\lambda_1\; p_2 s_2\lambda_2\rangle$ can be determined according to the same Frobenius-Jacob-Wick- method as one has learnt it in constructing the CGC of $\mathcal{P}$.

1. One recouples the product state into states of the type $|qs\lambda, Pjm\rangle$ where q is the relative momentum, S a total spin (defined by ordinary angular momentum coupling from $(s_1\lambda_1)\otimes(s_2\lambda_2)$ in the frame where $\underline{q}=0$,Pjm the total momentum, angular momentum and its projection, respectively.

2. One generates $|qs\lambda, Pjm\rangle$ by a boost $L(q)$ from a standard state: $U(L(q))\,|\,\underline{q}=0,\, s\lambda;\, Pjm\rangle = |qs\lambda,\; Pjm\rangle$.

3. One uses the fact that $\langle \sigma j_0\, jm|$ is a basis vector of an IR of SL(2,C), i.e. finally

$$\langle \sigma j_0\, jm | q s \lambda\; P\, jm \rangle = \mathcal{D}^{(\sigma j_0)}_{jm\, s\lambda}\left(L(q) \right) \cdot \gamma \,,$$

where the factor γ contains normalisation factors and the "radial part" of the wave function, i.e. the reduced matrix element. Hence the contribution of a Lorentz-pole - after performing standard manipulations - to the scattering amplitude will appear in the form [7, 16].

$$L \sim \bar{\gamma}(s'\lambda')\, \mathcal{D}^{(\sigma j_0)}_{s'\lambda'\, s\lambda}\left(L^{-1}(q')\, L(q) \right) \gamma(s\lambda)$$

i.e. precisely as if the amplitude were invariant with respect to SL(2,C). Indeed, what we find is that from the point of view of bound state contributions the behaviour of the amplitude is uniform on the whole light cone (the apex of the light cone, where the little group is SL(2,C), not being a distinguished point).

This has as a particularly pleasant consequence - among other things - that a Lorentz-pole contribution is stable with respect to small variations of the external masses, whereas a single Regge-pole contribution is not. (A single Regge-amplitude would lead to the absurd result that while a neutral ρ - contribution to NN scattering behaves as $s^{\alpha_\rho(0)}$ at $t=0$, the corresponding charged ρ -contribution to charge exchange scattering does not, since the mass of the neutron differs from that of the proton by a few MeV-s!)

6. Some philosophy

"I have come to agree with his (M.von Lane's) answer that the recognition that almost all rules of spectroscopy follow from the symmetry of the problem is the most remarkable result". (E.P. Wigner, Preface to the English Translation of his "Group Theory"). Perhaps further comments are unnecessary.

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