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A METHOD FOR CONSTRUCTING CERTAIN
AXIALLY SYMMETRICAL EINSTEIN-MAXWELL FIELDS

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It is shown that the existing cylindrical static electric and magnetic fields are in a certain linear connection, simplifying the field equations. Furthermore a method is given to translate results of the cylindrical static Einstein-Maxwell theory to the cylindrical stationary free gravitational case and inversely. As an example of the use of this method, the gravitational field of a magnetic dipole is obtained from the Kerr metric.

The Einstein-Maxwell equations of interacting gravitational and electromagnetic fields

$$R^{\mu}_{\nu} = -\kappa T^{\mu}_{\nu} \qquad (\kappa > 0) ; \qquad |1|$$

$$T^{\mu}_{\nu} = -F^{\mu\alpha} F_{\nu\alpha} + (1/4)\,\delta^{\mu}_{\nu} F^{\alpha\beta} F_{\alpha\beta} ; \qquad |2|$$

$$\left(g^{1/2} F^{\mu\alpha}\right)_{,\alpha} = 0 ; \qquad F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \qquad |3|$$

permit to reduce the static cylindrical line element to the form

$$ds^2 = -e^{\lambda(\rho,z)}\left[d\rho^2 + dz^2\right] - e^{-\nu(\rho,z)}\rho^2 d\varphi^2 + e^{\nu} dt^2 . \qquad |4|$$

By adding a gradient field to the electromagnetic potential vector, we may put: $A_{\mu} = (0,0,\phi,\psi)$. The field variables depend on $x^1 = \rho$ and $x^2 = z$. The Maxwell equations /3/ allow us the following substitution: [1]

$$\phi_1 e^{\nu}/\rho = \Phi_2 ; \qquad \phi_2 e^{\nu}/\rho = -\Phi_1$$

/Here and in what follows, lower indices denote partial derivatives /. In terms of this new potential /3/ may be written:

$$\left.\begin{aligned} \Delta\Phi &= \nu_i \Phi_i , \\ \Delta\psi &= \nu_i \psi_i . \end{aligned}\right\} \quad i = 1,2. \qquad |5|$$

Here Δ denotes the Laplace operator in cylindrical coordinate system: $\Delta f = f_{ii} + (1/\rho) f_1$, and the summation convention holds for i. We shall use the equation obtained from $R^{34} = -\kappa T^{34}$:

$$\Phi_1 \psi_2 = \Phi_2 \psi_1 . \qquad |6|$$

G. Tauber [1] found some exact solutions of the cylindrical static field equations with nonvanishing Φ and ψ. These solutions have the property $\Phi = A\psi + B$ /A,B being real constants./ Now we shall prove the following

Theorem: There exist only such static cylindrical electromagnetic fields for which

$$\Phi = A\psi + B \qquad |7|$$

holds. The $\Phi \equiv 0$ case corresponds to A=B=0, the $\psi \equiv 0$ case to the limit $A \to \infty$.

It is clear that this theorem imposes a strong restriction on the shape of the static cylindrical electromagnetic fields. A further consequence of the theorem is that the relevant field equations can always be reduced to the $\psi \equiv 0$ special form.

Proof: The meaning of equ. /6/ is that /5/, regarded as an inhomogeneous linear algebraic system for ν_i, is singular. So the relations $\psi_i \Delta\Phi = \Phi_i \Delta\psi$ hold, from which, by using partial derivatives of equ. /6/ we get:

$$\left.\begin{aligned} (\psi_{11}\Phi_1 - \Phi_{11}\psi_1)\,\Phi_i^2 &= 0 ; \\ (\psi_{22}\Phi_2 - \Phi_{22}\psi_2)\,\Phi_i^2 &= 0 . \end{aligned}\right\} \qquad |8|$$

The $\Phi_i^2 \equiv 0$ case corresponds to the absence of the electromagnetic field /See equ. /6//, and if we take $\Phi_i^2 \not\equiv 0$, we arrive at the theorem by simple integration.

Applying this result to the field equations, we have:

$$\left.\begin{aligned} \Delta\nu &= \kappa' (e^{\nu}/\rho^2)\,\Phi_i^2 , \\ \left[(e^{\nu}/\rho)\Phi_i\right]_i &= 0 , \end{aligned}\right\} \qquad |9|$$

where $\kappa' = \kappa(1+A^2)$. Now we see that the generalization from the $\psi \equiv 0$ case /when only the magnetic field is present/ to $\psi \not\equiv 0$ causes only the change

$\kappa \to \kappa'$ in these equations. The remaining field equations yield λ by means of simple line integrals. A more interesting relation will be established between the axially symmetrical static and stationary fields in our second

Theorem: A change in the sign of the gravitational constant κ causes that the cylindrical static Einstein-Maxwell field problem goes over to a source-free cylindrical stationary gravitational field one; the inverse statement holds too with nonphysical sign of the gravitational constant.

In order to prove this theorem, we remark that the most general source-free stationary cylindrical metric may be written [2] , [3] :

$$ds^2 = -e^{\mu(\rho,z)}\left[d\rho^2 + dz^2\right] - \rho^2 v(\rho,z)d\varphi^2 + (1/v)\left[dt - w(\rho,z)d\varphi\right]^2. \qquad |10|$$

The gravitational equations read now:

$$\left.\begin{aligned} & v\Delta v - v_i^2 - (1/\rho^2)w_i^2 = 0, \\ & v\left[\Delta w - (2/\rho)w_1\right] - 2v_i w_i = 0, \\ & \mu_1 = (1/2\rho v^2)\left[\rho^2(v_1^2 - v_2^2) - (w_1^2 - w_2^2)\right] + v_1/v, \\ & \mu_2 = (1/\rho v^2)\left[\rho^2 v_1 v_2 - w_1 w_2\right] + v_2/v . \end{aligned}\right\} \qquad |11|$$

For $R = (\rho^2+z^2)^{1/2} \to \infty$ the conditions of asymptotical flatness are: $v\to 1$, $w\to 0$, $\mu\to 0$. The substitution

$$\nu = -2\ln v\ ; \qquad \phi = -\sqrt{-2/\kappa}\, w\ ; \qquad \lambda = 4\mu - 2\ln v \qquad |12|$$

with $\kappa < 0$ brings /11/ to the form /9/. The asymptotic conditions become: $\lambda, \nu, \phi, \to 0$ for $R \to \infty$. The space is asymptotically flat with vanishing electromagnetic field at the infinity.

The physical background of the theorem proved now is obscure; nevertheless, by means of it we can translate results in the static, electromagnetic aspect to the free stationary case and vica versa. We mention that this procedure sometimes fails. This happens whenever the change in Sign(κ) excludes the nontrivial solutions. E.g., as it is easily seen, the solutions of Weyl, having the property [4]

$$e^{\nu} = (\kappa/2)\psi^2 + A\psi + 1 \qquad |13|$$

and the vacuum stationary metrics of Papapetrou [5] go into each other when applying to them our procedure; but no corresponding solution belongs to the special form of the Weyl metrics, for which /13/ is assumed to have the form [6] : $e^{\nu} = (\sqrt{\kappa/2}\,\psi - 1)^2$.

As a first application of the theorem, we shall construct the field which corresponds to the Kerr metric. We remark that the only known cylindrical stationary vacuum solutions are the Kerr metric [7] and the solutions of Papapetrou [5] .

Our starting point is the line element found by R. Kerr:

$$ds^2 = -(r^2 + a^2\cos^2\theta)(d\theta^2 + \sin^2\theta d\varphi^2) - 2(du + a\sin^2\theta d\varphi)(dr + a\sin^2\theta d\varphi) + \left(1 - \frac{2mr}{r^2+a^2\cos^2\theta}\right)(du + a\sin^2\theta d\varphi)^2, \qquad |14|$$

which may be brought to the desired canonical form by means of the transformation

$$\left.\begin{aligned} \rho^2 &= \left[(r-m)^2 + a^2 - m^2\right]\sin^2\theta \,, \\ z &= (r-m)\cos\theta \,, \\ \varphi' &= \varphi + \left(a/\sqrt{a^2 - m^2}\right)\mathrm{arctg}\left[(r-m)/\sqrt{a^2-m^2}\right] \,, \\ t &= u - r - (2m^2/a)(\varphi' - \varphi) - 2m\ln(\rho/\sin\theta) \,. \end{aligned}\right\} \qquad |15|$$

The field equations /11/ are satisfied by complex a also. In order to make ϕ real, we put $a \to ia$. Using /12/, we get the following metric:

$$ds^2 = -N^2(r^2 - a^2\cos^2\theta)^2\left[(r-m)^2 - (a^2+m^2)\right]^{-3}\left[dr^2\left((r-m)^2 - (a^2+m^2)\right) + d\theta^2\right] - N^{-2}\left[(r-m)^2 - (a^2+m^2)\right]\left[r^2 - a^2\cos^2\theta\right]^2\sin^2\theta d\varphi^2 + N^2(r^2 - a^2\cos^2\theta)^2 dt^2 \qquad |16|$$

and magnetic potential

$$\phi = \sqrt{2/\kappa}\; 2\, m\, a\, r\, \sin^2\theta\, /N \qquad /17/$$

with $N = (r-m)^2 - m^2 - a^2\cos^2\theta$. In the far-field approximation we have the field of a magnetic dipole with the momentum $2\sqrt{2\kappa}\,am$ and with a mass proportional to m . If $m = 0$, the space is flat, and we have cylindrical coordinates in ρ and z . For $m \neq 0$ /17/ has singularities at $\rho = 0$, $z = \pm(a^2+m^2)^{1/2}$, which can be interpreted as the location of the magnetic poles.

This solution may be generalized to have nonvanishing ϕ, ψ by using our first theorem. A more detailed analysis of it as well as the results of the current work for obtaining further solutions will be published elsewhere.

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