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SOME PROPERTIES OF CYLINDRICAL ELECTROVAC FIELDS

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BUDAPEST

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It is shown that the static cylindrical electric and magnetic fields are in a certain sense linearly connected. Furthermore the known properties of the stationary case are summarized: an extension of Weyl's theorem is proved and the field equations obtained by this theorem are presented in a special coordinate system.

I. Introduction

In this paper we shall present a "classical" treatment of the cylindrical stationary electrovac fields /That is fields consisting of empty space-time and electromagnetism[1]/. This means that we shall not use such modern tools of relativists as the spinor calculus or others. For the static case in Section II. we shall show that the electrical and magnetic fields obey a strong restriction, especially that they are in a certain sense linearly connected.

In Section III. and IV. we summarize the results scattered in the literature about the stationary field: we present the extension of a nice theorem of Weyl and others. This theorem holdsoriginally for static fields and will be extended to stationary fields having the properties described in Section III. The theorem states in both case that the number of

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1/ J.L. Syngle: The General Relativity, North-Holland Publishing Co., 1960, p. 367.

nonvanishing independent metrical field variables can be reduced by one. Finally we present the field equations obtained from the more general metric by use of the proved theorem.

On the static field

A particular exact solution of the Einstein-Maxwell equations[2]

$$R^{\mu}_{\nu} = \varkappa T^{\mu}_{\nu} \; ; \qquad /1/$$

$$T^{\mu}_{\nu} = (1/4\pi)\left[F^{\mu\alpha}F_{\nu\alpha} - (1/4)\,\delta^{\mu}_{\nu}F^{\alpha\rho}F_{\alpha\rho}\right]; \qquad /2/$$

$$(g^{1/2}F^{\mu\alpha})_{,\alpha} = 0 \; ; \qquad /3/$$

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \qquad /4/$$

was found by G.Tauber[3] in the static cylindrical case. This solution allows both nonvanishing electric and magnetic fields. The electromagnetic potential vector has the form $A_{\mu} = (0,0,\phi,\psi)$, where ϕ generates the magnetic and ψ the electrical strain components. The line element is taken

$$ds^2 = -e^{\lambda}(dr^2 + dz^2) - e^{-\rho}r^2 d\varphi^2 + e^{\rho}dt^2. \qquad /5/$$

The equations /3/ permit us the following substitution[3,4]:

$$\phi_1 e^{\rho}/r = \Phi_2 \; ; \qquad \phi_2 e^{\rho}/r = -\Phi_1 . \qquad /6/$$

Then we get from /3/:

$$\left.\begin{aligned} \Delta\Phi &= \rho_1\Phi_1 + \rho_2\Phi_2 \; ; \\ \Delta\psi &= \rho_1\psi_1 + \rho_2\psi_2 . \end{aligned}\right\} \qquad /7/$$

2/ Greek indices run from 1 to 4; $x^1 = r$, $x^2 = z$, $x^3 = \varphi$, $x^4 = t$. In this work always will be assumed that the field variables do not depend on φ and t.

3/ G. Tauber, Canad. Journal of Phys. 35, 477 /1957/.

4/ From now on in this Section the lower indices denote partial derivatives with respect to $x^1 = r$ and $x^2 = z$.

Here Δ is the Laplace operator in cylindrical coordinate system: $\Delta f \equiv f_{11}+f_{22}+(1/r)f_1$. In the following we shall use the equation arising from $R^3_4 = \varkappa T^3_4$:

$$\Phi_1 \Psi_2 = \Phi_2 \Psi_1 . \qquad /8/$$

In the particular case found by Tauber, the new electromagnetical potential components are linearly dependent: $\Phi = a\Psi + b$, where a and b are real constants.

Now we shall prove the following

<u>Theorem</u>: In a continuous field domain there exist only such static cylindrical electromagnetic fields that

$$\Phi = a\Psi + b \qquad /9/$$

/a, b are constants/ be fulfilled. The $\Phi = 0$ case corresponds to a = b = 0, the $\Psi = 0$ case to the limit $a \to \infty$.

It is clear that this theorem imposes a strong restriction on the shape of the static cylindrical electromagnetic fields.

<u>Proof</u>. First we solve /7/ for ρ_1, ρ_2 as a linear algebraic equation system:

$$\left.\begin{aligned} \rho_1 &= (\Psi_2 \Delta\Phi - \Phi_2 \Delta\Psi)/(\Phi_1\Psi_2 - \Phi_2\Psi_1); \\ \rho_2 &= (\Phi_1 \Delta\Psi - \Psi_1 \Delta\Phi)/(\Phi_1\Psi_2 - \Phi_2\Psi_1). \end{aligned}\right\} \qquad /10/$$

Glancing at equ. /8/ we see that the denominators vanish. So the continuity of ρ requires:

$$\Psi_2 \Delta\Phi = \Phi_2 \Delta\Psi; \quad \Phi_1 \Delta\Psi = \Psi_1 \Delta\Phi , \qquad /11/$$

from which we get by use of /8/:

$$\Psi_1(\Phi_{11} + \Phi_{22}) = \Phi_1(\Psi_{11} + \Psi_{22}); \qquad /12a/$$

$$\Psi_2(\Phi_{11} + \Phi_{22}) = \Phi_2(\Psi_{11} + \Psi_{22}) . \qquad /12b/$$

Partial differentation of /8/ with respect to r and z and elimination of the mixed derivatives yields the equations

$$\Psi_1\left(\Phi_{22}-\frac{\Phi_1\Psi_1}{\Phi_2\Psi_2}\Phi_{11}\right)=\Phi_1\left(\Psi_{22}-\frac{\Phi_1\Psi_1}{\Phi_2\Psi_2}\Psi_{11}\right) \qquad /13a/$$

and

$$\Psi_2\left(\Phi_{11}-\frac{\Phi_2\Psi_2}{\Phi_1\Psi_1}\Phi_{22}\right)=\Phi_2\left(\Psi_{11}-\frac{\Phi_2\Psi_2}{\Phi_1\Psi_1}\Psi_{22}\right). \qquad /13b/$$

We get from /12/ and /13/ simply by taking the difference of equ.s /a/ and then of equ.s /b/:

$$\Psi_1\Phi_{11}\left(1+\frac{\Phi_1\Psi_1}{\Phi_2\Psi_2}\right)=\Phi_1\Psi_{11}\left(1+\frac{\Phi_1\Psi_1}{\Phi_2\Psi_2}\right); \qquad /14a/$$

$$\Psi_2\Phi_{22}\left(1+\frac{\Phi_2\Psi_2}{\Phi_1\Psi_1}\right)=\Phi_2\Psi_{22}\left(1+\frac{\Phi_2\Psi_2}{\Phi_1\Psi_1}\right). \qquad /14b/$$

Now there are two cases.

a/ If $\frac{\Phi_1\Psi_1}{\Phi_2\Psi_2}\neq-1$ then

$$\Psi_i\Phi_{ii}=\Phi_i\Psi_{ii}\ ;\quad i=1,2 \qquad /15/$$

must hold. Integrating these equations we have:

$$\ln\Phi_i=\ln a\Psi_i\ ;\quad a=\text{const.}, \qquad /16/$$

and further integration yields $\Phi=a\Psi+b$, where b = const.

b/ If $\frac{\Phi_1\Psi_1}{\Phi_2\Psi_2}=-1$ then with /8/ we get

$$\Phi_1^2+\Phi_2^2=\Psi_1^2+\Psi_2^2=0 \qquad /17/$$

that is all the potential components vanish. In this case equ. /9/ is fulfilled with b = 0.

III. The stationary case*

Here we shall make use of the fact that in the cylindrical stationary case it is always fulfilled

$$F^{12}=0; \qquad /18/$$
$$F_{34}=0. \qquad /19/$$

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* The results presented in Section III. and IV. are not new, but are found independently by a number of authors.

/19/ immediately follows from /4/. In order to obtain /18/ we integrate the first two Maxwell equations

$$(g^{1/2}F^{21})_{,1}=0\ ;\qquad (g^{1/2}F^{12})_{,2}=0 \qquad /20/$$

and so we obtain $g^{1/2}F^{12}$ = const. We shall assume that at the infinity $F^{12} = 0$. From this we have const = 0 and so /if the metric is regular/ everywhere in the space is statisfied $F^{12} = 0$. Then we may choose the vector potential to have the following form:

$$A_\mu = (0,0,\phi,\psi)\,. \qquad /21/$$

Furthermore from /18/ and /19/ we may conclude that the cylindrical stationary electromagnetic field has the property

$$T^1_1+T^2_2=T^3_3+T^4_4=0, \qquad /22/$$

the advantage of which fact will be taken in the proof of the following theorem.

In the presence of an origin-centered nonrelativistic electric monopole and magnetic dipole there is a flow of impulse in the $d\varphi$ direction. Now we shall consider the analogue relativistical fields. Then we have $T_{14} = T^{14} = T_{24} = T^{24} = 0$, from which it can be shown that $g_{14} = g^{14} = g_{24} = g^{24} = 0$.

If we use canonical co-ordinates, the metric has the form

$$[g_{\mu\nu}] = \begin{bmatrix} \alpha^2 & & & \\ & \alpha^2 & & \\ & & \beta^2 & \varepsilon \\ & & \varepsilon & -\gamma^2 \end{bmatrix}. \qquad /23/$$

We stress that it isn't necessary the use of /23/, exact solutions of the electrovac equations are known which have a very simple form in other co-ordinate systems[5]. Nevertheless, it can be shown[6] that any cylindrical stationary metric can be put into the form /23/. /23/ can always be diagonalized to have the signature (+ + + -).

5/ E.T.Newman, E.Couch, K.Chinnapared, A.Exton, A.Prakash and R.Torrence, Journal of Math.Phys. 6, 918 /1965/.

6/ A.Papapetrou, Ann.Inst.Henri Poincaré, IV. 83 /1966/.

The computation of Christoffel symbols yields the following nonvanishing components[7/]:

$$\Gamma_{11}^{1} = \Gamma_{12}^{2} = -\Gamma_{22}^{1} = \frac{\alpha_1}{\alpha}; \qquad \Gamma_{22}^{2} = \Gamma_{12}^{1} = -\Gamma_{11}^{2} = \frac{\alpha_2}{\alpha};$$

$$\begin{aligned}
\Gamma_{i3}^{3} &= (\gamma^2\beta\beta_i + \varepsilon_i\varepsilon/2)/(\beta^2\gamma^2+\varepsilon^2); & \Gamma_{33}^{i} &= -\beta\beta_i/\alpha^2;\\
\Gamma_{i3}^{4} &= (\varepsilon\beta\beta_i - \varepsilon_i\beta^2/2)/(\beta^2\gamma^2+\varepsilon^2); & \Gamma_{34}^{i} &= -\varepsilon_i/(2\alpha^2);\\
\Gamma_{i4}^{3} &= (\varepsilon_i\gamma^2/2 - \varepsilon\gamma\gamma_i)/(\beta^2\gamma^2+\varepsilon^2); & \Gamma_{44}^{i} &= \gamma\gamma_i/\alpha^2;\\
\Gamma_{i4}^{4} &= (\varepsilon_i\varepsilon/2 + \beta^2\gamma\gamma_i)/(\beta^2\gamma^2+\varepsilon^2); & &
\end{aligned} \qquad /24/$$

$$\Gamma_{i\mu}^{\mu} = 2\alpha_i/\alpha + (\gamma^2\beta\beta_i + \beta^2\gamma\gamma_i + \varepsilon\varepsilon_i)/(\beta^2\gamma^2+\varepsilon^2).$$

Finally by very tedious calculation we get the components of the Ricci tensor. For later use we put down the proper linear combinations of the components:

$$R_{11} = \sum_i\left[(\ln\alpha)_{ii} + (-1)^i\frac{1}{2}\,\frac{\alpha_i N_i}{\alpha N}\right] + \frac{(\sqrt{N})_{11}}{\sqrt{N}} - \frac{1}{2N}\left[(\beta^2)_1(\gamma^2)_1 + (\varepsilon_1)^2\right]; \qquad /25/$$

$$R_{22} = \sum_i\left[(\ln\alpha)_{ii} - (-1)^i\frac{1}{2}\,\frac{\alpha_i N_i}{\alpha N}\right] + \frac{(\sqrt{N})_{22}}{\sqrt{N}} - \frac{1}{2N}\left[(\beta^2)_2(\gamma^2)_2 + (\varepsilon_2)^2\right]; \qquad /26/$$

$$R_3^3 + R_4^4 = \frac{1}{\alpha^2}\sum_i \frac{(\sqrt{N})_{ii}}{\sqrt{N}}; \qquad /27/$$

$$R_3^3 - R_4^4 = \frac{1}{\alpha^2N^2}\sum_i\left[\beta^4\gamma^4\left(\frac{\beta_{ii}}{\beta} - \frac{\gamma_{ii}}{\gamma}\right) + \beta^2\gamma^2\varepsilon^2\left(\frac{\beta_{ii}}{\beta} - \frac{\gamma_{ii}}{\gamma} + \frac{\beta_i^2}{\beta^2} - \frac{\gamma_i^2}{\gamma^2} - \frac{\beta_i\varepsilon_i}{\beta\varepsilon} - \frac{\gamma_i\varepsilon_i}{\gamma\varepsilon}\right)\right]; \qquad /28/$$

$$R_{12} = -\frac{1}{2}\,\frac{\alpha_1 N_2}{\alpha N} - \frac{1}{2}\,\frac{\alpha_2 N_1}{\alpha N} + \frac{(\sqrt{N})_{12}}{\sqrt{N}} - \frac{1}{2N}\left[\frac{(\beta^2)_1(\gamma^2)_2 + (\beta^2)_2(\gamma^2)_1}{2} + \varepsilon_1\varepsilon_2\right]; \qquad /29/$$

7/ Here and in what follows is assumed to have $i = 1,2$ and the summation convention holds for i . Moreover see footnote 4.

$$R_{34} = \frac{\varepsilon}{2\alpha^2 N} \sum_i \left[\beta^2\gamma^2 \left(\frac{\varepsilon_{ii}}{\varepsilon} - \frac{\beta_i \varepsilon_i}{\beta \varepsilon} - \frac{\gamma_i \varepsilon_i}{\gamma \varepsilon} + 4 \frac{\beta_i \gamma_i}{\beta \gamma} \right) + \varepsilon^2 \frac{\varepsilon_{ii}}{\varepsilon} \right]; \qquad /30/$$

where $N \equiv \beta^2\gamma^2 + \varepsilon^2.$

IV. Extension of Weyl's theorem

Theorem. In a stationary cylindrical universe having the metric form /23/ the number of independent nonvanishing components of the metric tensor can be reduced from four to three. Either of the unknown functions in /23/ except α can be eliminated.

Proof. We shall use the consequence of /22/:

$$R^3_3 + R^4_4 = 0. \qquad /31/$$

/24/ can be written by means of this equation in the remarkably simple form:

$$\sum_i \left[(\beta^2\gamma^2 + \varepsilon^2)^{1/2} \right]_{ii} = 0 \qquad /32/$$

From now on we may follow the order of ideas found by Weyl and others[8]. Equ. /32/ means that $r' = (\beta^2\gamma^2 + \varepsilon^2)^{1/2}$ is a harmonic function of r and z , furthermore there exist a conjugate harmonic function $z'(r,z)$ such that $r' + iz' = f(r + iz)$. Making use of the conformal transformation $(r,z) \to (r',z')$ one can eliminate one of the functions β, γ or ε .

Finally we shall present the field equations in a case when they take a very simple form. The components of the metric are chosen in the following manner:

$$\left. \begin{aligned} \alpha^2 &= f^2 \\ \beta^2 &= h^2 r^2 \\ \varepsilon &= -h^2 r^2 \omega \\ &\text{and so} \\ \gamma^2 &= 1/h^2 - h^2 r^2 \omega^2 \end{aligned} \right\} \qquad /33/$$

Then the line element has the form:

$$ds^2 = f^2(dr^2 + dz^2) + r^2h^2(d\varphi - \omega\, dt)^2 - (1/h^2)\, dt^2. \qquad /34/$$

8/ Synge: The General Relativity, p. 311.

The Maxwell equations remaining to be solved are:

$$\left\{ r \left[\left(\frac{1}{h^2 r^2} - h^2 \omega^2 \right) \Phi_i - h^2 \omega \psi_i \right] \right\}_i = 0 ; \qquad /35/$$

$$\left\{ r \left[h^2 \omega \Phi_i + h^2 \psi_i \right] \right\}_i = 0 . \qquad /36/$$

/36/ is equivalent with the statement

$$\left. \begin{aligned} h^2 r \omega \Phi_1 + h^2 r \psi_1 &= \Psi_2 ; \\ h^2 r \omega \Phi_2 + h^2 r \psi_2 &= - \Psi_1 . \end{aligned} \right\} \qquad /37/$$

Then the Maxwell equations may be written

$$\left(\frac{\Phi_i}{h^2 r} \right)_i = \omega_1 \Psi_2 - \omega_2 \Psi_1 ; \qquad /38/$$

$$\left(\frac{\Psi_i}{h^2 r} \right)_i = - \omega_1 \Phi_2 + \omega_2 \Phi_1 . \qquad /39/$$

/39/ is the consequence of the identity $\psi_{12} = \psi_{21}$.

The Einstein equations are in this case:

$$\Delta \ln f - \frac{2}{r} \frac{f_1}{f} + \frac{2}{r} \frac{h_1}{h} + 2 \frac{h_1^2}{h^2} - \frac{1}{2} h^4 r^2 \omega_1^2 = \frac{K}{h^2 r^2} \left(\Phi_1^2 - \Phi_2^2 + \Psi_1^2 - \Psi_2^2 \right) ; \qquad /40/$$

$$\Delta \ln f + 2 \frac{h_2^2}{h^2} - \frac{1}{2} h^4 r^2 \omega_2^2 = - \frac{K}{h^2 r^2} \left(\Phi_1^2 - \Phi_2^2 + \Psi_1^2 - \Psi_2^2 \right) ; \qquad /41/$$

$$- \frac{1}{r} \frac{f_2}{f} + \frac{1}{r} \frac{h_2}{h} + 2 \frac{h_1 h_2}{h^2} - \frac{1}{2} h^4 r^2 \omega_1 \omega_2 = \frac{2K}{h^2 r^2} \left(\Phi_1 \Phi_2 + \Psi_1 \Psi_2 \right) ; \qquad /42/$$

$$\Delta \ln h + \frac{1}{2} h^4 r^2 \omega_i^2 = \frac{K}{h^2 r^2} \left(\Phi_i^2 + \Psi_i^2 \right) ; \qquad /43/$$

$$\left(\omega_i h^4 r^3 \right)_i = - 4K \left(\Phi_1 \Psi_2 - \Phi_2 \Psi_1 \right) . \qquad /44/$$

where K denotes $\varkappa/8\pi$. These equations demonstrate the possibility of the significant simplification of the stationary field equations in consequence of the proved theorem. The only known solution of these equations with nonvanishing Φ, ψ and ω is obtained by the methods of the gravitational radiation theory and the transformation of it to our coordinate system yields very complicated expressions.

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