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FAMILIES OF REGGE-TRAJECTORIES AT NONVANISHING ENERGY I.

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Abstract

The parent-daughter trajectory phenomenon is studied for spinless particles in the framework of the Bethe-Salpeter /BS/ equation. If the kernel of the equation satisfies some natural conditions, the partial wave amplitude is shown to possess multiple poles in the ℓ -plane, corresponding to a Regge-pole family. Under the same conditions a perturbation formula is given for the derivatives of trajectory functions and residua at zero momentum transfer as a function of $\varkappa$, the order of daughter trajectories.

1. Introduction

The existence of daughter trajectories [1] [2] [3] [4] implies the existence of more particles with identical discrete quantum numbers, if Regge poles are linear for high momentum transfer [5] and the daughters are parallel /or approximately parallel/ to them. However as far as we know SL (2,C) symmetry can state something about daughter trajectories only at t=0, namely here they are integrally spaced below the parent trajectory in the angular momentum plane. In this paper we shall show that group theoretical considerations can be extended to the derivatives of the amplitude at t=0.

A perturbation expansion was proposed in our first paper[1] about this subject, without explicitly evaluating the perturbation formulae. In another paper[4] one of us has given an explicit expression for the behaviour of daughter trajectories near t=0, for small violation of SL(2,C) symmetry, using the Fredholm method of solution of the BS equation:

$$\alpha_{\varkappa}(t) \approx \alpha_0 - \varkappa + [\alpha_1 + \alpha_2(\alpha_0-\varkappa)(\alpha_0-\varkappa+1)]t, \quad /1/$$

where the index $\varkappa$ enumerates the trajectories, α_0 is the place of the Lorentz-pole at t=0. Another proof for such a behaviour of daughter trajectories for a special class of kernels was given as well[6].

In this paper first we shall discuss a fundamental question which was not investigated before. The four dimensional symmetry and the existence of Lorentz-poles implies the presence of daughter poles in the partial wave amplitude, T_ℓ , only under two conditions:

1/ The possibility of continuing T_ℓ to the left hand side of the complex ℓ plane[7] [8]. This is a trivial requirement as for t=0 all the even order daughters of any trajectory are situated on the left hand side of the line $\mathrm{Re}\,\ell = -1$, due to the Froissart bound.

2/ As T_ℓ is represented as a sum over contributions containing only one pole, this sum should be at least convergent, if we want to be sure that T_ℓ itself has these poles. On the other hand the above mentioned series for T_ℓ is a Gegenbauer expansion, the convergence of which is secured only for $\mathrm{Re}\,\ell > -3/2$, even for functions analytic in the $\cos\vartheta$ plane[9].

We shall show the convergence of the above series under fairly general conditions, not only for T_ℓ at t=0, but for derivatives of arbitrary order, $\left(\frac{d}{dt}\right)^{(i)} T_\ell$ at t=0.

After such a preparation we are able to construct a method for calculating the $\varkappa$ dependence of the derivatives of $\alpha_\varkappa(t)$ and of the residue function. The dependence is universal in all cases /it is determined by the group structure[10/ 11/], only some unknown parameters appear, which can be calculated easily for any given kernel.

In Sec. 2. we list our assumptions and after that we prove the existence of daughter poles in the partial wave amplitude. In Sec.3. an expansion for the derivatives of the partial wave amplitude is proved to exist and useful expressions will be given for parameters of the daughter trajectories. Sec.4. contains our main result: after a construction of a perturbation formula for the scattering amplitude, using the results of Sec. 3. we arrive at our perturbation expansion of the parameters of daughter trajectories.

A different type of treatment /working with the wave function instead of the scattering amplitude/ of the above problem, emphasizing the group theoretical features and an extension to fermion trajectories will be given in a forthcoming paper[11/].

2. The Existence of Daughter Trajectory Poles of T_ℓ near t=0

We discuss in this paper the scattering of spinless particles with initial momenta p_1 and p_2 and final momenta p_3 and p_4 . We examin the amplitude off the mass shell. The scattered particles may have different discrete quantum numbers. Introducing the independent four-momenta $p=\frac{1}{2}(p_1-p_2)$, $p'=\frac{1}{2}(p_3-p_4)$ and $E=p_1+p_2=p_3+p_4$ the BS equation for the scattering amplitude can be written as

$$T(p,p',E)=K(p,p',E)+\int d^4p''\,\bar{K}(p,p'',E)\,T(p'',p',E),$$

where $T(p,p',E)$ is the scattering amplitude, $\bar{K}(p,p',E)$ is the kernel of the equation and $K(p,p',E)$ is the free term or inhomogeneity. K and $\bar{K}$ differ in the product of propagators of the scattered particles. Throughout this paper we shall work in Euclidean metrics, assuming the possibility of Wick's rotation of the contour of integration in p_0'' . K and $\bar{K}$ are assumed to be even functions of E , otherwise we apply a transformation of the problem, described in the Appendix, which leads to an integral equation with a kernel, symmetric in E .

We introduce as independent invariants p^2, p'^2, $E^2 = t$. $(p-p')^2$, pE and $p'E$. Going into the CMS of scattered particles T will depend on the following quantities:

$$p^2,\ p'^2,\ t,\ s=(p-p')^2,\ p_0^2 t,\ p_0'^2 t,\ p_0 p_0' t.$$

Our assumptions about the functions, K, $\bar{K}$, and T are the following:

1./ T, K and $\bar{K}$ are analytic functions of t, pE, p'E if p and p' are in the interval of integration and $|t| < \varepsilon$.

2./ For $|t|<\varepsilon'$ T, K and $\bar{K}$ are analytic functions of $s=(p-p')^2$ in the s plane with two cuts, they can be represented as

$$T(s,\dots) = \frac{1}{\pi}\int_{s_0}^{\infty} \frac{t_s(s',\dots)}{s'-s}\,ds' - \frac{1}{\pi}\int_{u_0}^{\infty}\frac{t_u(u',\dots)}{u'+s+t+2p^2+2p'^2}\,du', \quad /3/$$

with possible subtractions, $s_0 > 0$, $u_0 > 0$. For t=0 t_s and t_u are polynomially bounded if $s \to \infty$: $|t_s(s,\dots)| < c\cdot s^{\alpha}$, $|t_u(u,\dots)| < c' u^{\alpha}$ for some c, c' and α, if p and p' are bounded. Similar representations are assumed to hold for the partial derivatives of T in the invariants pE and p'E at t=0 and for K and $\bar{K}$ and their derivatives as well.

3./ We assume the boundedness of the kernel for $|t|<\varepsilon''$.

4./ The partial wave amplitude, which can be obtained from the Froissart-Gribov form from eq.3. can be continued for arbitrary complex values Re $\ell > \alpha_1$, where α_1 is the power in the bound of the spectral functions of $\bar{K}$. We assume the possibility of continuing T_ℓ beyond this bound using a trick similar to that of Mandelstam's one [7/ 8/]. T_ℓ is assumed to be analytic near t=0 even for such values as well /except for $\ell=\alpha_0-\varkappa$, when T_ℓ has a pole at t=0/.

5./ For t=0, according to assumption 2. we can expand the amplitude in Gegenbauer functions C_n^1 (cos γ), where γ is the four dimensional relative angle between p and p'. Using representation 3. we can define the positive and negative O(4) signature amplitudes by

$$T_0^{n,\pm}(p^2,p'^2) = \frac{1}{2pp'}\Big[\int ds'\, D_n^1\Big(\frac{p^2+p'^2+s'}{2pp'}\Big)$$

$$\times\, t_s(p^2,p'^2,s') \pm \int du'\, D_n^1\Big(\frac{p^2+p'^2+u'}{2pp'}\Big)\, t_u(p^2,p'^2,u')\Big], \quad /4/$$

where D^1_n is a Gegenbauer function of second kind. Representation 4. can be used for an analytic continuation of $T_o^{n,\pm}$ in the half plane $\mathrm{Re}\, n > \alpha$ Due to the larger analyticity domain of $\bar{K}_o^n$ we can use the BS equation for a further continuation. Of course in the region $\alpha > \mathrm{Re}\, n > \alpha_1$ T_o^n may have poles. We assume the presence of a pole in this region at $n = \alpha_o$, and the possibility of continuing T_o^n by some method to arbitrary negative values of n. Similarly we assume the possibility of continuing the projections of K, $\bar{K}$ and of the partial derivatives of T, K and $\bar{K}$ in the invariants pE and p'E to negative n values. All these functions are assumed to be analytic at $n = \alpha_o + k$, where k is an arbitrary integer, $k \neq 0$.

In connection with the above five assumptions we merely remark that assumption 1. can be proved if we restrict ourselves to normal singularities in the mass variables. Assumption 3. is not absolutely necessary, we could substitute it by weaker ones. We included it only for sake of simplicity.

After such a preparation we can prove the following theorem:

If conditions 1., to 5., are satisfied, then the partial wave amplitude has infinitely many daughter poles in the angular momentum plane, provided the Lorentz pole is not at integer or half integer values of n.

At E=0 the BS equation has the following form

$$T_o(p,p') = K_o(p,p') + \int \bar{K}_o(p,p'')\, T(p'',p')\, d^4p''. \qquad /5/$$

As $T_o(p,p')$, $\bar{K}_o(p,p')$ and $K_o(p,p')$ depend on p^2, p'^2 and $(p-p')^2$ only, and the dependence on $(p-p')^2$ is analytic, they can be expanded in Gegenbauer polynomials[9]

$$T_o(p^2, p'^2, (p-p')^2) = \frac{2}{\pi}\Big[\sum_{n=\mathrm{even}} (n+1)\, C_n^1(\cos\gamma)\, T_o^{n,+}(p^2,p'^2)$$

$$+ \sum_{n=\mathrm{odd}} (n+1)\, C_n^1(\cos\gamma)\, T_o^{n,-}(p^2,p'^2)\Big], \qquad /6/$$

$$\vdots$$

Expansions 6. converge uniformly in the integration interval. Using the addition theorem of Gegenbauer polynomials[9] and exchanging the order of summation and integration in eq. 6 /this is allowed due to the uniform convergence and analycity of the series/ we obtain an equation for the amplitude $T_o^{n,\pm}$

$$T_0^{n,\pm}(p^2,p'^2) = K_0^{n,\pm}(p^2,p'^2)$$
$$+ \int_0^\infty dp''\, p''^3\, \bar{K}_0^{n,\pm}(p^2,p''^2)\, T_0^{n,\pm}(p''^2,p'^2). \qquad /7/$$

Using assumption 3., expansion 6. and the addition theorem we can prove easily that the kernel $p''^3 \bar{K}_0^{n,\pm}(p^2,p''^2)$ is bounded as well. The solution of eq.7. can be written in the following form

$$T_0^{n,\pm}(p^2,p'^2) = \int_0^\infty R^{n,\pm}(p^2,p''^2) K_0^{n,\pm}(p''^2,p'^2)\, p''^3 dp'', \qquad /8/$$

where $R^{n,\pm}$ is the resolvent of eq.7. In what follows we shall write formally $T_0^{n,\pm} = R^{n,\pm} K_0^{n,\pm}$. The aforementioned pole of the scattering amplitude $T_0^{n,\pm}$ is a pole of the resolvent operator.

The partial wave amplitude can be continued to the left half plane according to assumption 4. On the other hand, from representation 6., we can project out the partial waves by exchanging the order of summation and integration /again this is allowed under the conditions of our theorem/ and obtain

$$T_{0,\ell}(p^2,p'^2,p_0,p_0') = \sum_{\varkappa} P^{\ell+\varkappa,\ell}\left(\frac{p_0}{p}\right) T_0^{\ell+\varkappa,\pm}(p^2,p'^2)\, P^{\ell+\varkappa,\ell}\left(\frac{p_0'}{p'}\right), \qquad /9/$$

where the functions $P^{\ell+\varkappa,\ell}(\cos\gamma)$ are orthonormal in the interval $0 \le \gamma < \pi$ with weight function $\sin^2\gamma$

$$P^{\ell+\varkappa,\ell}(\cos\gamma) = C_\varkappa^{\ell+1}(\cos\gamma)\,(\sin\gamma)^\ell$$
$$\times\, 2^{\ell+1/2}\left(\frac{(\ell+1+\varkappa)\,\varkappa!}{\pi\,\Gamma(\varkappa+2\ell+2)}\right)^{1/2}\Gamma(\ell+1). \qquad /10/$$

As $T_{0,\ell}$ is an analytic function of p_0 and p_0' if $|p_0| < p$ and $|p_0'| < p'$, it can be expanded in a Gegenbauer series $C_\varkappa^{\ell+1}\left(\frac{p_0}{p}\right)$ for complex ℓ values as well if $\mathrm{Re}\,\ell > -3/2$ [13]. Such an expansion coincides with expansion 9. for integer values of ℓ and it serves as an analytic continuation of eq. 9 as well. As we remarked in the preceeding section, the daughter trajectories lie at negative values of ℓ at t=0, so the poles of $T_0^{\ell+\varkappa}$ at $\ell+\varkappa = \alpha_0$ will be poles of $T_{0,\ell}$ as well if series 9. are convergent for such ℓ values.

In fact, expansion 9. remains absolutely and uniformly convergent for arbitrary ℓ values. In order to prove this statement we consider the part of the sum in eq.9. for which $\varkappa > \alpha_1 - \mathrm{Re}\,\ell$ and $\varkappa$ is even. Similarly one can consider the part of expansion 9. in which $\varkappa$ is an odd number. Using the spectral representation 4. it is easy to get an upper bound for T_0^n if $p_1 < N$ and $p_2 < N$ are satisfied[9]

$$\left| T_0^{n,\pm}(p^2, p'^2) \right| < \frac{c}{n+1}\left[1 + \frac{\min(s_0, u_0)}{N^2}\right]^{-n+\alpha_1},$$

for some constant c . On the other hand for non integer ℓ and even $\varkappa$ we have the following bound for $C_{\varkappa}^{\ell+1}(x)$ [9]:

$$\left| C_{2m}^{\ell+1} \right| \leq \frac{\Gamma(\ell+1+m)}{m!\,\Gamma(1+\ell)} .$$

Using the above estimation and the definition of $P^{\ell+\varkappa,\ell}$ in eq. 10 we see that the m-th term in expansion 9. is majorized by

$$\left| P^{\ell+2m,\ell}\left(\frac{p_0}{p}\right) T_0^{\ell+2m} P^{\ell+2m,\ell}\left(\frac{p_0'}{p'}\right) \right| < \frac{c}{\pi}\, 2^{2\ell+1}$$

$$\times \frac{\Gamma(2m+1)}{(\Gamma(m+1))^2} \; \frac{(\Gamma(\ell+m+1))^2}{\Gamma(2\ell+2m+2)} \left[1 + \frac{\min(s_0, u_0)}{N^2}\right]^{-\ell-2m-1} \qquad /11/$$

for arbitrary $\left|\frac{p_0'}{p'}\right|, \left|\frac{p_0}{p}\right| < 1$. As the r.h.s. of eq. 11 is a term of a convergent series in m, we have proved our statement about the absolute and uniform convergence of expansion 9. Thus we conclude that a pole of $T_0^{n,\pm}$ in the n plane at $n = \alpha_0$ induces poles of $T_{0,\ell}$ in the ℓ plane at $\ell = \alpha_0 - \varkappa$, $\varkappa = 0,1,\ldots$. It is easy to see that the first terms of expansion 9. which were neglected in our proof of convergence may have singularities /poles and branch points/ at integer and half integer values of n so in general such values of α_0 need further considerations.

3. <u>A representation for the partial derivatives of T_ℓ and of the Regge pole parameters</u>

For $\ell > -3/2$ we have representations of the partial derivatives of the amplitude with respect to the invariants t, $p_0^2 t$, $p_0'^2 t$, $p_0 p_0' t$, which are similar to representation 9. On the other hand we can express the derivative $\frac{dT_\ell}{dt}$ at $t=0$ as

$$\left.\frac{dT_\ell}{dt}\right|_{t=0} = \left.\frac{\partial T_\ell}{\partial t}\right|_{t=0} + p_0^2 \left.\frac{\partial T_\ell}{\partial p_0^2 t}\right|_{t=0} + \ldots \qquad /12/$$

In eq. 12. we distinguished between the total derivative and the partial derivative in t. Substituting the expansion similar to 9. into eq. 12. we obtain a representation for $dT_l/dt\,|_{t=0}$ for negative values of l as well. The combinations $p_0\, P^{l+\varkappa,l}(p_0/p)$ and $p_0^2\, P^{l+\varkappa,l}(p_0/p)$, which appear in such a way, can be rewritten by making use of the recurrence relations of functions $P^{l+\varkappa,l}$ [1], which are valid even for negative values of l.

$$zP^{l+\varkappa,l}(z) = \frac{1}{2}P^{l+\varkappa+1,l}(z)\sqrt{\frac{(\varkappa+1)(\varkappa+2l+2)}{(\varkappa+l+1)(\varkappa+l+2)}}$$

$$+\ \frac{1}{2}P^{l+\varkappa-1,l}(z)\sqrt{\frac{\varkappa(\varkappa+2l+1)}{(\varkappa+l)(\varkappa+l+1)}}\,. \qquad /13/$$

At the end we arrive at an expansion of $dT_l/dt\,|_{t=0} \equiv T_{1,l}$

$$T_{1,l}(p^2,p'^2,p_0,p_0') = \sum_{\varkappa,\varkappa'}^{\infty} P^{l+\varkappa,l}\left(\frac{p_0}{p}\right) T_{1,l}^{l+\varkappa,l+\varkappa'}(p^2,p'^2)\, P^{l+\varkappa',l}\left(\frac{p_0'}{p'}\right), \qquad /14/$$

where $T_{1,l}^{l+\varkappa,l+\varkappa'}$ can be expressed by the expansion coefficients of the invariant functions appearing on the r.h.s. of eq. 12. By construction and from eq. 13, $T_{1,l}^{l+\varkappa,l+\varkappa'}$ differs from zero only for $\varkappa=\varkappa'$ or $\varkappa=\varkappa'\pm 2$ and expansion 14. is uniformly and absolutely convergent. In a similar way we may represent a derivative of arbitrary order of T_l at $t=0$ by a uniformly and absolutely convergent series:

$$T_{i,l} = \sum_{\varkappa,\varkappa'}^{\infty} P^{l+\varkappa,l}\left(\frac{p_0}{p}\right) T_{i,l}^{l+\varkappa,l+\varkappa'}\, P^{l+\varkappa',l}\left(\frac{p_0'}{p'}\right), \qquad /15/$$

where $T_{i,l}$ is defined as $\left(\frac{d}{dt}\right)^{(i)} T_l\big|_{t=0}$. $T_{i,l}^{l+\varkappa,l+\varkappa'}$ is defined by eq. 15., it differs form zero if $\varkappa-\varkappa'$ = even and $|\varkappa-\varkappa'|\leq 2i$ are satisfied.

After this point we are allowed to work only with the expansion coefficients $T_{i,l}^{l+\varkappa,l+\varkappa'}$ as $T_{i,l}$ can be constructed from these functions.

As we estabilished in the preceeding section, T_l has poles corresponding to a Regge-family at $t=0$. Then using assumption 4. about the analicity of T_l at $t=0$, T_l must have poles for $|t|<\delta$ as well, where δ is some positive number. The trajectory functions and residua must be analytic functions in this region /the residua of daughter trajectories do not have the singularities discussed by Freedman and Wang[3] due to the different definition of t and the off-shell approach/.

We choose r and δ' so small that inside the circle C defined by $\alpha_0-\varkappa+re^{i\varphi}$, $0\leq\varphi<2\pi$ the $\varkappa$-th daughter pole is the only singularity of T_ℓ and $|\alpha_\varkappa(t)-\alpha_0+\varkappa|<r$ if $|t|<\delta'$.

Then we can evaluate the following contour integral

$$\oint_C (\ell-\alpha_0+\varkappa)^k \, T_\ell^{\ell+\lambda,\,\ell+\lambda'}(p^2,p'^2,t)\,d\ell$$
$$= \gamma_\varkappa^{\ell+\lambda,\,\ell+\lambda'}(p^2,p'^2,t)\,\left(\alpha_\varkappa(t)-\alpha_\varkappa(0)\right)^k , \qquad /16/$$

where $\gamma_\varkappa^{\ell+\lambda,\,\ell+\lambda'}$ is the analytically continued projection of the $\varkappa$ -th residue function by the orthogonal functions $P^{\ell+\lambda,\ell}(\frac{p_0}{p}),\ P^{\ell+\lambda',\ell}(\frac{p_0'}{p'})$. From eq. 16. we obtain easily the required representations of the derivatives of daughter trajectory parameters if we make use of the uniform convergence of the integral on the l. h. s. of eq. 16.

$$\oint_C T_{i,\ell}^{\ell+\lambda,\,\ell+\lambda'}(p^2,p'^2)\,d\ell = \gamma_{i,\varkappa}^{\ell+\lambda,\,\ell+\lambda'}(p^2,p'^2) , \qquad /17/$$

$$\left(\frac{d}{dt}\right)^{(i)} \frac{\oint_C (\ell-\alpha_\varkappa(0))\,T_\ell^{\ell+\varkappa,\,\ell+\varkappa}(p^2,p'^2,t)\,d\ell}{\oint_C T_\ell^{\ell+\varkappa,\,\ell+\varkappa}(p^2,p'^2,t)\,d\ell} = \alpha_\varkappa^{(i)}(0) , \qquad /18/$$

where we introduced the notation

$$\gamma_{i,\varkappa}^{\ell+\lambda,\,\ell+\lambda'}(p^2,p'^2) = \left(\frac{d}{dt}\right)^{(i)} \gamma_\varkappa^{\ell+\lambda,\,\ell+\lambda'}(p^2,p'^2,t)\Big|_{t=0} .$$

4. Perturbation formulae for daughter trajectory parameteres

Eqs. 17. and 18. can be used for the determination of $\gamma_{i,\varkappa}^{\ell+\lambda,\ell+\lambda'}$ and $\alpha_\varkappa^{(i)}(0)$ if we can calculate $T_{i,\ell}^{\ell+\lambda,\,\ell+\lambda'}$. This can be done using the BS equation. As T, K and $\bar{K}$ are analytic functions of t they can be expanded into power series of t

$$T = T_0 + T_1 t + T_2 t^2 + \dots ,$$
$$K = K_0 + K_1 t + K_2 t^2 + \dots , \qquad /19/$$
$$\bar{K} = \bar{K}_0 + \bar{K}_1 t + \bar{K}_2 t^2 + \dots .$$

Substituting expansions 19. into eq. 2. we obtain equations for T_i . We may write down the solutions of these equations in terms of K_i, $\bar{K}_i$ and the resolvent of the equation at $t=0$, R.

$$T_0 = R K_0 ,$$
$$T_1 = R K_1 + R\bar{K}_1 R K_0 ,$$
$$T_2 = R K_2 + R\bar{K}_2 R K_0 + R\bar{K}_1 R K_1 + R\bar{K}_1 R\bar{K}_1 R K_0 , \qquad /20/$$
$$\vdots$$

where the product of operators is defined as the product of the two corresponding function and integration over the four dimensional p-space. The partial wave projection of eq. 20 may be expanded in the series of orthonormal functions $P^{\ell+\lambda,\ell}(p_0/p)$ and $P^{\ell+\lambda',\ell}(p_0'/p')$

$$T_0^{\ell+\lambda} = R^{\ell+\lambda} K_0^{\ell+\lambda} ,$$
$$T_{1,\ell}^{\ell+\lambda,\ell+\lambda'} = R^{\ell+\lambda} K_{1,\ell}^{\ell+\lambda,\ell+\lambda'} + R^{\ell+\lambda} \bar{K}_{1,\ell}^{\ell+\lambda,\ell+\lambda'} R^{\ell+\lambda'} K_0^{\ell+\lambda'} ,$$
$$T_{2,\ell}^{\ell+\lambda,\ell+\lambda'} = R^{\ell+\lambda} K_{2,\ell}^{\ell+\lambda,\ell+\lambda'} + R^{\ell+\lambda} \bar{K}_{2,\ell}^{\ell+\lambda,\ell+\lambda'} R^{\ell+\lambda'} K_0^{\ell+\lambda'} , \qquad /21/$$
$$+ \sum_{\lambda''} R^{\ell+\lambda} \bar{K}_{1,\ell}^{\ell+\lambda,\ell+\lambda''} R^{\ell+\lambda''} K_{1,\ell}^{\ell+\lambda'',\ell+\lambda'}$$
$$+ \sum_{\lambda''} R^{\ell+\lambda} \bar{K}_{1,\ell}^{\ell+\lambda,\ell+\lambda''} R^{\ell+\lambda''} \bar{K}_{1,\ell}^{\ell+\lambda'',\ell+\lambda'} R^{\ell+\lambda'} K_0^{\ell+\lambda'} ,$$
$$\vdots$$

The product of operators in eq. 21. is defined as multiplication of the product of corresponding functions by p^3 and integration over p in the interval $0 < p < \infty$. $K_{i,\ell}^{\ell+\lambda,\ell+\lambda'}$ and $\bar{K}_{i,\ell}^{\ell+\lambda,\ell+\lambda'}$ are defined similarly to $T_{i,\ell}^{\ell+\lambda,\ell+\lambda'}$. They can be expressed by the expansion coefficients of the derivatives of K and $\bar{K}$ with respect to invariants t. pE and p'E. As $K_{i,\ell}^{\ell+\lambda,\ell+\lambda'}$ connects only λ and λ' values, for which $\lambda-\lambda'$ is an even number and $|\lambda-\lambda'| \leq 2i$, the sums in eq. 21. are finite.

Eq. 21. can be continued to arbitrary negative values of ℓ . If we substitute it into eqs. 17. and 18. we arrive at our final result. On the l.h.s. of eqs. 17. and 18. we have to extract the residua of simple and double poles of $T_{i,l}^{l+\lambda,l+\lambda'}$ at $l=\alpha_{\varkappa}(0)$. This can be done using eq. 21. if we take into account that the poles of $T_{i,l}^{l+\lambda,l+\lambda'}$ come from the poles of $R^{l+\lambda}$ at $l+\lambda=\alpha_0$. Therefore we expand $R^{l+\lambda}$ in MacLorain's series and $K_{i,l}^{l+\lambda,l+\lambda'}$ and $\bar{K}_{i,l}^{l+\lambda,l+\lambda'}$ in Taylor's series around $l=\alpha_{\varkappa}(0)$ /according to assumption 5. R^n and the expansion coefficients of the partial derivatives of K and $\bar{K}$ with respect to the invariants are regular functions of n at $n=\alpha_0$ except the simple pole of R^n. at $n=\alpha_0$ /. The coefficients of recurrence relations are regular as well, if α_0 is not equal to an integer of half-integer number.

Using Dirac's notations we may write the series for R^n as follows

$$R^n(p^2,p'^2)=\frac{|\varphi(p^2)\rangle\langle\varphi(p'^2)|P_0}{n-\alpha_0}+R_1(p^2,p'^2)+\dots\,, \qquad /22/$$

where $|\varphi\rangle$ is the solution of the homogeneous equation

$$|\varphi\rangle=\bar{K}_0^{\alpha_0}|\varphi\rangle\quad,\quad\langle\varphi|=\langle\varphi|P_0K_0^{\alpha_0},$$

and P_0 is the product of propagators of scattered particles at E=0. Comparing eqs. 17. 18. 21. and 22. we obtain finally for the derivatives of $\alpha_{\varkappa}(t)$ at t=0

$$\alpha_{\varkappa}'(0)=\langle\varphi|P_0\bar{K}_{1,\alpha_0-\varkappa}^{\alpha_0,\alpha_0}|\varphi\rangle,$$

$$\begin{aligned}\alpha_{\varkappa}''(0)&=\langle\varphi|P_0\bar{K}_{2,\alpha_0-\varkappa}^{\alpha_0,\alpha_0}|\varphi\rangle\\&+2\langle\varphi|P_0\bar{K}_{1,\alpha_0-\varkappa}^{\alpha_0,\alpha_0}R_1\bar{K}_{1,\alpha_0-\varkappa}^{\alpha_0,\alpha_0}|\varphi\rangle\\&+2\langle\varphi|P_0\bar{K}_{1,\alpha_0-\varkappa}^{\alpha_0,\alpha_0-2}R^{\alpha_0-2}\bar{K}_{1,\alpha_0-\varkappa}^{\alpha_0-2,\alpha_0}|\varphi\rangle\\&+2\langle\varphi|P_0\bar{K}_{1,\alpha_0-\varkappa}^{\alpha_0,\alpha_0+2}R^{\alpha_0+2}\bar{K}_{1,\alpha_0-\varkappa}^{\alpha_0+2,\alpha_0}|\varphi\rangle\\&+2\langle\varphi|P_0\frac{d}{dl}\bar{K}_{1,l}^{l+\varkappa,l+\varkappa}\Big|_{l=\alpha_0-\varkappa}|\varphi\rangle\,\alpha_{\varkappa}'(0),\\&\;\vdots\end{aligned}\qquad /23/$$

Similarly obtain the derivatives of the residua of daughter poles. As we know /and can be checked directly to any order in t/ the residua factorize into the product of two vertex functions. For the sake of simplicity we shall write the perturbation formulae for these vertex functions.

Writing the solution of the inhomogeneous equation as

$$T_\ell = \frac{|\Psi(t)\rangle\langle\Psi(t)|P}{\ell-\alpha_\varkappa(t)} \quad + \text{ terms regular at } \quad \ell=\alpha_\varkappa(t)$$

where $|\Psi(t)\rangle$ is the solution of the homogeneous equation

$$|\Psi(t)\rangle = \overline{K}_{\alpha_\varkappa(t)}|\Psi(t)\rangle$$

We expand $|\Psi(t)\rangle$ in powers of t

$$|\Psi(t)\rangle = |\Psi_0\rangle + |\Psi_1\rangle t + |\Psi_2\rangle t^2 + \ldots$$

and $|\Psi_i\rangle$ can be represented as a series of functions $P^{\alpha_0+2s,\alpha_0-\varkappa}(\frac{p_0}{p})$. It is easy to see that $|\Psi_i\rangle$ has only a finite number of expansion coefficients different from zero

$$\begin{aligned} |\Psi_0\rangle &= P^{\alpha_0,\alpha_0-\varkappa}\left(\tfrac{p_0}{p}\right)|\varphi\rangle, \\ |\Psi_1\rangle &= \sum_{s=-1}^{s=1} P^{\alpha_0+2s,\alpha_0-\varkappa}\left(\tfrac{p_0}{p}\right)|\Psi_1^s\rangle, \\ |\Psi_2\rangle &= \sum_{s=-2}^{s=2} P^{\alpha_0+2s,\alpha_0-\varkappa}\left(\tfrac{p_0}{p}\right)|\Psi_2^s\rangle, \\ &\vdots \end{aligned} \qquad /24/$$

The expansion coefficients are expressed by R and K_i, $\bar{K}_i$ as follows

$$\begin{aligned} |\Psi_1^0\rangle &= R_1\overline{K}_{1,\alpha_0-\varkappa}^{\alpha_0,\alpha_0}|\varphi\rangle + \tfrac{1}{2}|\varphi\rangle\langle\varphi|P_0\tfrac{\partial}{\partial\ell}\overline{K}_{1,\ell}^{\ell+\varkappa,\ell+\varkappa}\Big|_{\ell=\alpha_0-\varkappa}|\varphi\rangle, \\ |\Psi_1^1\rangle &= R^{\alpha_0+2}\overline{K}_{1,\alpha_0-\varkappa}^{\alpha_0+2,\alpha_0}|\varphi\rangle, \\ |\Psi_1^{-1}\rangle &= R^{\alpha_0-2}\overline{K}_{1,\alpha_0-\varkappa}^{\alpha_0-2,\alpha_0}|\varphi\rangle, \\ &\ldots \end{aligned} \qquad /25/$$

As we remarked earlier, we can calculate $\alpha_\varkappa^{(i)}(0)$ and $|\Psi_i^s\rangle$ only in the case of knowledge of R and K_i. Nevertheless the general $\varkappa$ dependence of them is very simple, it comes entirely from the $\varkappa$ dependence of coefficients in the recurrence relation 13.

$$\alpha'_{\varkappa}(0) = a + b(\alpha_0 - \varkappa)(\alpha_0 - \varkappa + 1),$$

$$\alpha''_{\varkappa}(0) = c + d(\alpha_0 - \varkappa)(\alpha_0 - \varkappa + 1)$$

$$+ e(\alpha_0 - \varkappa)^2(\alpha_0 - \varkappa + 1)^2 + 2f(\alpha_0 - \varkappa + \tfrac{1}{2})\,\alpha'_{\varkappa}(0),$$

$$\vdots$$

$$|\Psi_1^0\rangle = |\alpha\rangle + (\alpha_0 - \varkappa)(\alpha_0 - \varkappa + 1)|\beta\rangle$$

$$+ \frac{1}{2} f(\alpha_0 - \varkappa + \tfrac{1}{2})|\varphi\rangle, \qquad /26/$$

$$|\Psi_1^{-1}\rangle = \sqrt{\left(1 - \frac{(\alpha_0 - \varkappa)(\alpha_0 - \varkappa + 1)}{\alpha_0(\alpha_0 + 1)}\right)\left(1 - \frac{(\alpha_0 - \varkappa - 1)(\alpha_0 - \varkappa)}{\alpha_0(\alpha_0 - 1)}\right)}\,|\gamma\rangle,$$

$$|\Psi_1^{+1}\rangle = \sqrt{\left(1 - \frac{(\alpha_0 - \varkappa + 1)(\alpha_0 - \varkappa + 2)}{(\alpha_0 + 1)(\alpha_0 + 2)}\right)\left(1 - \frac{(\alpha_0 - \varkappa + 2)(\alpha_0 - \varkappa + 3)}{(\alpha_0 + 2)(\alpha_0 + 3)}\right)}\,|\delta\rangle,$$

$$\vdots$$

where a, b, c, d, e, f and $|\alpha\rangle, |\beta\rangle, |\gamma\rangle, |\delta\rangle$ are some independent constants and constant vectors.

5. Conclusion

To summarise, we have established that at nonzero energies the families of Regge trajectories can be classified according to a broken SL(2,C) algebra. The symmetry breaking term has a definite transformation character under the spectrum generating algebra of Regge poles at zero energy and therefore - at least in the framework of the perturbation formalism - we can make quite definite statements about the behaviour of Regge poles and their residues near the "symmetry point". In particular, we rederived the "mass formula" of ref.[4] and established that the Regge vertices can be represented as the superposition of a few irreducible representations of SL(2,C) .

No attempt was made in the present work to push the theory to the point where comparison with experimental results is possible; our aim was rather to explore the analytic structure of the relevant Green's function necessary to apply the analytic theory of group representations[14] ; this could be done without taking into account the complications arising from the spins of <u>external</u> particles. In our subsequent paper we shall take for granted the analyticity properties derived here and develop the group theoretical formalism to the point where the interpretation of the results and comparison with experimental data is possible.

Appendix

Assume, that K and $\bar{K}$ depend on the sign of pE and $p'E$. Then we define $K^{\pm}$, $\bar{K}^{\pm}$ and $T^{\pm}$ by symmetrizing and antisymmetrizing in E,

$$K^{\pm}(p,p',E) = K(p,p',E) \pm K(p,p',-E)\,, \text{ e.t.c.}$$

The amplitudes $T^{\pm}$ satisfy the following equations

$$T^{\pm} = K^{\pm} + \bar{K}^{+} T^{\pm} + \bar{K}^{-} T^{\mp}. \qquad /1/$$

Assuming the existence of the resolvent

$$R^{+} = (1 - \bar{K}^{+})^{-1}$$

and making use of eq.1. we can express T^{-} as

$$T^{-} = R^{+} K^{-} + R^{+} \bar{K}^{-} T^{+}.$$

We obtain the following eq. for T^{+}

$$T^{+} = K^{+} + \bar{K}^{+} T^{+} + \bar{K}^{-} R^{+} \bar{K}^{-} T^{+} + \bar{K}^{-} R^{+} K^{-}. \qquad /2/$$

The solution of eq. 2. can be written as

$$T^{+} = R K^{+} + R \bar{K}^{-} R^{+} K^{-}, \qquad /3/$$

where

$$R = (1 - \bar{K}^{+} - \bar{K}^{-} R^{+} \bar{K}^{-})^{-1}.$$

We can see that R and so eq.3. is symmetric in E.

Similarly we obtain for T^{-}

$$T^{-} = R K^{-} + R \bar{K}^{-} R^{+} K^{+}. \qquad /4/$$

As the resolvents appearing in eqs.3. and 4. are identical, the same poles give contributions to both amplitudes. The Regge-poles depend only on $E^2 = t$, the residua of poles in T^{-}, however, has a linear dependence on E. The problem of giving a perturbation formula for this vertex function will not be discussed in this paper.

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