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C. J. Hegedüs

LAURENT EXPANSION OF THE INVERSE OF A FUNCTION MATRIX

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by

C.J.Hegedüs

Central Research Institute for Physics,Budapest
Computing Techniques Department

KIVONAT

Egy általános eljárást javasolunk, amely az $F(z)$ függvénymátrix Taylor együtthatóiból $F^{-1}(z)$ Laurent együtthatóit állitja elő valamely izolált pólushelyen. Megmutatjuk, hogy $F^{-1}(z)$ B_j Laurent együtthatójának kiszámitásához $F(z)$ A_o, A_1, ..., A_{2n+j} Taylor együtthatói szükségesek, ahol n a pólus rendje.

Az 1. Tétel a pólus rendjének eldöntését teszi lehetővé, mig a 2. Tétel azt is megmutatja, az általános esetben hogyan számithatók ki a Laurent együtthatók.

РЕЗЮМЕ

В данной работе подсказывается общий способ для разложения в ряд Лорана $F^{-1}(z)$ около изолированного полюса на основе разложение матричной функции $F(z)$ в ряд Тэйлора. Доказано, что коэффициенты Лорана B_j вычисляются всегда через коэффициенты Тэйлора A_o, A_1, ... A_{2n+j}, где и - порядок особой точки. С помощью Теоремы I определяется порядок полюса тока. Теорема 2 показывает как можно определить коэффициенты Лорана в общем случае.

Abstract This paper suggests a general procedure, based on the Taylor expansion of a function matrix F(z), for calculating the Laurent expansion of $F^{-1}(z)$ around an isolated pole. It is shown that in order to compute the Laurent coefficient matrices B_j of $F^{-1}(z)$, one needs in any case the Taylor coefficients $A_o, A_1, \dots, A_{2n+j}$ of F(z), where n is the order of the pole.

Theorem 1 helps to decide the order of the pole, while Theorem 2 shows how the Laurent coefficients may be computed in the general case, too.

1. Introduction

In recent years the question has arisen in the numerical evaluation of certain integrals [1] as to how to determine the residual value of functions of the type

$$f(x) = (\underline{a}, F^{-1}(x)\underline{a}) \tag{1.1}$$

at a singularity point $x=x_o$, where $\underline{a}$ is a constant vector and F(x) is a given real symmetrical NxN matrix, dependent on x, having no inverse at $x=x_o$.

Since the behaviour of f(x) depends only on the matrix F(x), we may restrict ourselves to the investigation of F(x).

At first for a general treatment we reformulate the problem with a complex function matrix F(z). Let us make some definitions and conventions.

Definition 1: A function matrix $\Phi(z)$ is analytic in z if each element of the matrix is an analytic function of z in a common complex region.

Definition 2: $\Phi(z)$ has a kth order zero in a point ζ, if any of its elements has a zero of at least kth order in ζ.

Definition 3: A function matrix $\Phi(z)$ has a pole of nth order in a point ζ, if any of its elements has a pole of at most nth order in ζ.

The order of the pole may be defined alternatively as the power n for which

$$\lim_{z\to\zeta}(z-\zeta)^n\Phi(z)\neq 0 \qquad (1.2)$$

is finite, since in this case for every $\ell>n$

$$\lim_{z-\zeta}(z-\zeta)^\ell\Phi(z)=0 \qquad (1.3)$$

By the derivative matrix $\Phi'(z)=d\Phi(z)/dz$, we mean the matrix consisting of the derivated elements of $\Phi(z)$.

F(z) is assumed to have the following properties:

a) F(z) is analytic in z_o.

b) F(z) has a kth order zero in z_o, $k\geq 0$.

c) $F^{-1}(z)$ exists in a neighbourhood of z_o, except z_o, hence $F^{-1}(z)$ is analytic there.

It follows from Cramer's rule that the elements of $F^{-1}(z)$ are meromorphic functions. The only function analytic throughout a region having a zero of infinite order is the constant 0 [4]. On our assumption the determinant is not identically 0, thus it follows from properties a), b), c) that $F^{-1}(z)$ has a nth order pole at $z=z_o$, where n is finite.

If F(z) and $F^{-1}(z)$ fulfil the conditions a), b), c), they may be expanded as

$$F(z)=\sum_{j=k}^{\infty}A_j(z-z_o)^j\ ,\ k\geq 0\ ,\ A_j=\frac{1}{j!}F^{(j)}(z_o) \qquad (1.4)$$

$$F^{-1}(z)=\sum_{\ell=-n}^{\infty}B_\ell(z-z_o)^\ell\ ,\ \ n>0 \qquad (1.5)$$

where B_ℓ ($\ell=-n,\dots,0,1,\dots$) are the Laurent coefficient matrices of $F^{-1}(z)$. It can be shown (see e.g. [4]) that the Laurent coefficients may be determined as

$$B_\ell = \frac{1}{(n+\ell)!} \lim_{z \to z_0} \frac{d^{n+\ell}}{dz^{n+\ell}} \left[\frac{F(z)}{(z-z_0)^n} \right]^{-1} \tag{1.6}$$

where, analogously to the scalar case, the matrix B_{-1} may be called the residual matrix.

With the aid of (1.4), (1.5) and unit matrix I we may write

$$I - F(z)F^{-1}(z) = I - \sum_{j=k}^{\infty} \sum_{\ell=-n}^{\infty} A_j B_\ell (z-z_0)^{j+\ell} = 0 \tag{1.7}$$

where the equality holds when the matrix product is written in reversed order, too. The coefficient of each power in (1.7) must be zero, giving the system of equations (δ is the Kronecker symbol):

$$\sum_{i=0}^{n+j} A_{k+i} B_{j-i} = \delta_{k+j,0} I \tag{1.8a}$$

$$, \quad j \geq -n$$

$$\sum_{i=0}^{n+j} B_{j-i} A_{k+i} = \delta_{k+j,0} I \tag{1.8b}$$

Note that n and k are not necessarily equal.

Definition 4: The number $d=n-k$ will be called the defect of $F(z)$ in the point of singularity z_0.

In the following we shall be concerned with the solution of the system of equations (1.8). Our aim is to find a numerical procedure that determines the connection between the Taylor coefficients of $F(z)$ and the Laurent coefficients of $F^{-1}(z)$ at a pole z_0, and so the Laurent expansion of $f(x)$ in (1.1) can be constructed, too.

Success of the treatment rests on the general assumption that a numerical procedure for finding the point of singularity is given. A procedure of this kind has been presented by Bach [2]; this finds the complex root of a transcendent function with whose help the zero point of the determinant can be sought. Of course, the real case is simpler and should be handled by one of the usual root-finding procedures.

The determination of the residual matrix, as of the other coefficients of the Laurent expansion, depends mainly on whether the first non-vanishing derivative (henceforth abbreviated as f.n.v.d.) of F(z) has an inverse at $z=z_o$. First we shall be concerned with the simple case when the f.n.v.d. (or the coefficient matrix A_k) is non-singular at this point. Later we shall show that the alternative case (A_k is singular) can be reduced to the former.

Notations. The unit matrix of size N will be denoted by I_N, though sometimes N will be omitted; we write $F(z)\big|_{z=z_o}$ instead of $F(z_o)$ if we want to emphasize the insertion. In addition the following abbreviations will be used: Taylor coefficients = T.c., Laurent coefficients = L.c. The notations of the introduction will be maintained in the subsequent sections.

2. The Simple Case

Before formulating Theorem 1 we begin with a lemma interesting in itself.

Lemma 1.

Let the function matrix $\Phi^{-1}(z)$ exist in a neighbourhood of z_o, except z_o. Then the limit matrix

$$\lim_{z\to z_o} (z-z_o)^k \Phi^{-1}(z) = \Phi_o \qquad (2.1)$$

is finite and non-singular if and only if $\Phi(z)$ has the form:

$$\Phi(z) = (z-z_o)^k \Gamma(z) \qquad (2.2)$$

where the limit of the matrix $\Phi(z)$ has an inverse in z_o too.

Proof (Necessity) According to the limiting process (2.1) we can write

$$(z-z_o)^k \Phi^{-1}(z) = \Phi_o + E(z), \quad E(z)\Big|_{z=z_o} = 0 \tag{2.3}$$

which is equivalent to

$$(z-z_o)^{-k}\left[I + \Phi_o^{-1}E(z)\right]\Phi(z) = \Phi_o^{-1}. \tag{2.4}$$

Taking the limit of both sides, we get

$$\lim_{z\to z_o}(z-z_o)^{-k}\Phi(z) = \Phi_o^{-1} \tag{2.5}$$

Thus if $\Gamma(z) = (z-z_o)^{-k}\Phi(z)$, then the limit of $\Gamma(z)$ has an inverse and it is equal to Φ_o.

The proof of sufficiency is trivial. It may be remarked, though, that the analyticity of $\Phi(z)$ was not supposed here.

Theorem 1.

Let F(z) have the properties a), b), c) of sect. 1. Then $F^{-1}(z)$ has a pole of kth order at $z=z_o$ if and only if

$$\frac{d^k}{dz^k}F(z)\Big|_{z=z_o}$$

can be inverted.

Proof Condition b) ensures that F(z) may be produced in the form

$$F(z) = (z-z_o)^k\, G(z) \tag{2.6}$$

from which it follows by l'Hospital's rule that

$$\lim_{z\to z_o} G(z) = \lim_{z\to z_o}\frac{F(z)}{(z-z_o)^k} = \frac{1}{k!}\frac{d^k}{dz^k}F(z)\Big|_{z=z_o} \tag{2.7}$$

If this can be inverted, an application of Lemma 1 completes the proof.

Calculation of the Laurent Coefficients in the Simple Case

This time the determination of the order of the pole by Theorem 1 is rather simple because it only involves finding the f.n.v.d. of $F(z)$ in z_o. Let this be the kth derivative. If it has an inverse, then by Theorem 1 $F^{-1}(z)$ has a pole of kth order at $z=z_o$.

There are two possibilities for computing the Laurent coefficients.

On one hand we may use (1.6) and the identity

$$\frac{d}{dz} T^{-1}(z) = -T^{-1}(z)\frac{dT(z)}{dz} T^{-1}(z) \tag{2.8}$$

In fact (1.6) contains $G(z)$ of (2.6):

$$B_\ell = \frac{1}{(k+\ell)!} \lim_{z\to z_o} \frac{d^{k+\ell}}{dz^{k+\ell}} G^{-1}(z) \tag{2.9}$$

Thus

$$B_{-k} = \lim_{z\to z_o} G^{-1}(z) \tag{2.1o}$$

This is the inverse of (2.7), which on our assumption exists. For the other coefficients we must apply (2.8) in (2.9) as many times as is necessary. Here the derivatives of $G(z)$ will appear also, e.g.

$$B_{-k+1} = \lim_{z\to z_o} -G^{-1}(z)G'(z)G^{-1}(z) \tag{2.11}$$

where each term has finite limit in z_o.

The limits of the derivatives of $G(z)$ can be given with the aid of the Taylor expansion (1.4). In the case of a kth order zero one gets

$$\lim_{z\to z_o} \frac{d^m}{dz^m} G(z) = \lim_{z\to z_o} \frac{d^m}{dz^m} \frac{F(z)}{(z-z_o)^k} = \frac{m!}{(k+m)!} F^{(k+m)}(z_o) \tag{2.12}$$

The formulae of the residual matrices for the cases $k=1,2$ are

$$B_{-1} = \left[F'(z_o)\right]^{-1} \text{, if } k=1, \tag{2.13}$$

$$B_{-1} = -\frac{2}{3} \left[F''(z_o)\right]^{-1} F^{(3)}(z_o)\left[F''(z_o)\right]^{-1}\text{, if } k=2 \tag{2.14}$$

The alternative way yields a more convenient numerical algorithm for the computation of the Laurent coefficients. Since in this case k=n,(1.8) has a simple solution:

$$B_{-k} = A_k^{-1}, \quad B_j = -B_{-k} \sum_{i=1}^{k+j} A_{k+i} B_{j-i}, \quad j > -k \qquad (2.15)$$

That is, every equation of (1.8) can be solved directly using the solution of the preceding equations with smaller j,as in the case of scalar functions.

It is easy to check that for the computation of B_j one needs the Taylor coefficients $A_0, A_1, \ldots, A_{2k+j}$.

3.The Case of Decrease in Rank

Here we are concerned with the case when the f.n.v.d. of F(z) ,or equivalently A_k in (1.4),is a singular matrix. We must prepare for Theorem 2 with three further lemmas.

Lemma 2.

Let F(z) have the properties a),b),c) of sect.1.If the f.n.v.d. of F(z) is a singular matrix in z_o,then $F^{-1}(z)$ has a higher order pole in z_o than the order of the zero of F(z) ,

$$k < n , \qquad (3.1)$$

and vice versa.

Proof Let us consider

$$\lim_{z \to z_o} \left[(z-z_o)^{-k} F(z) \right] \left[(z-z_o)^n F^{-1}(z) \right] = \lim_{z \to z_o} (z-z_o)^{n-k} I \qquad (3.2)$$

Here the brackets on the left side have finite limits (see (1.4),(1.5));consequently the same must hold for the right side,too.The case k=n,in accordance with Theorem 1, does not result in a decrease of the rank of the f.n.v.d. This gives the assertion.

In the following we shall need an inversion procedure called the inversion by partitioning.This procedure,which is also called the Frobenius-Schur relation,may be found in any textbook of linear algebra (e.g.[3]);we apply it here in a product form,which is more adequate for our purposes.

We assume further on that F(z) has the properties a),b),c) and that its f.n.v.d. has a rank ϱ in z_o. Then,by interchanging rows and columns,F(z) may be partitioned into four submatrices:

$$F(z) = P \left[\begin{array}{c|c} F_1(z) & F_2(z) \\ \hline F_3(z) & F_4(z) \end{array}\right] Q. \tag{3.3}$$

This is done in such a way that $F_1^{(k)}(z)$ be a $\varrho \times \varrho$ matrix with a definite inverse in z_o.The permutation matrices P and Q represent the operations of interchanging rows and columns.Introducing the notations

$$X(z) = Q^T \left[\begin{array}{c|c} F_1^{-1}(z) & 0 \\ \hline 0 & 0 \end{array}\right] P^T, \tag{3.4}^*$$

$$U(z) = Q^T \left[\begin{array}{c} F_1^{-1}(z)F_2(z) \\ \hline -I_{N-\varrho} \end{array}\right], \tag{3.5}$$

$$V(z) = \left[\begin{array}{c|c} F_3(z)F_1^{-1}(z) & -I_{N-\varrho} \end{array}\right] P^T, \tag{3.6}$$

$$D(z) = F_4(z)-F_3(z)F_1^{-1}(z)F_2(z), \tag{3.7}$$

it can be checked directly that the inverse of (3.3) can be decomposed in a neighbourhood of z_o as

$$F^{-1}(z) = X(z) + U(z)D^{-1}(z)V(z). \tag{3.8}$$

The following lemmas contain assertions relating to D(z) of (3.7).

---

[*] Q^T denotes the transpose of $Q, Q^T=Q^{-1}$.

Lemma 3.

Let F(z) have the properties a),b),c) of sect.1 and let $F^{(k)}(z_o)$ be a singular matrix.If D(z) of (3.7) is obtained from a partitioning made according to the rank of the f.n.v.d. of F(z)(see (3.3)),then $D^{-1}(z)$ has the same order of pole in z_o as $F^{-1}(z)$.

Proof In (3.8) the matrix X(z) of (3.4) has by Theorem 1 a kth order pole in z_o.In U(z) of (3.5) $F_2(z)$ has a zero of at least kth order,thus U(z) has a finite and non-zero limit in z_o.The same reasoning can be repeated for the V(z) of (3.6).Since by Lemma 2 $F^{-1}(z)$ has a pole of higher order than k in z_o,$D^{-1}(z)$ remains to have the same order of pole as $F^{-1}(z)$.

The lemma can be extended to any partition where $F_1^{-1}(z)$ has a kth order pole.

Lemma 4.

Under the same conditions as in Lemma 3,D(z) has a defect less that F(z).

Proof Let us introduce $G_i(z)=(z-z_o)^{-k}F_i(z)$,i=1,2,3,4 similarly to (2.6),$G_i(z_o)$ being the limit of $G_i(z)$.

According to (2.7) and the partitioning chosen,the column vectors of $G_2(z_o)$ and $G_4(z_o)$ may be expressed as a linear combination of the column vectors of $G_1(z_o)$ and $G_3(z_o)$:

$$\left[\begin{array}{c} G_2(z_o) \\ \hdashline G_4(z_o) \end{array}\right] = \left[\begin{array}{c} G_1(z_o) \\ \hdashline G_3(z_o) \end{array}\right]\left[\begin{array}{c} L \end{array}\right] \tag{3.9}$$

or

$$\begin{aligned} G_2(z_o) &= G_1(z_o)\, L \\ G_4(z_o) &= G_3(z_o)\, L \end{aligned} \tag{3.1o}$$

which gives

$$G_4(z_o) = G_3(z_o)\, G_1^{-1}(z_o)\, G_2(z_o) \tag{3.11}$$

On comparison with (3.7) this yields

$$\lim_{z\to z_o} (z-z_o)^{-k} D(z) = 0 \ , \qquad (3.12)$$

which means that D(z) has a zero of higher order than k. By Lemma 3 $F^{-1}(z)$ and $D^{-1}(z)$ have poles of the same order, hence the assertion is proved (see Def.4).

These investigations reveal that the defect may be conceived as the structural part of the singularity, since it arises from the structure of the matrix.

Theorem 2.

The inverse of an analytic matrix F(z) which exists in a neighbourhood of z_o, except z_o, can be expressed as a finite sum

$$F^{-1}(z) = \sum_{i=0}^{q} \frac{H_i(z)}{(z-z_o)^{k_i}} \ , \ 0 \leq k_o \leq \ldots \leq k_q = n, \ q \leq n \qquad (3.13)$$

where the $H_i(z)$ arising from a series of decompositions have finite limits in z_o and a rank which decreases with increasing k_i.

Proof In the simple case the assertion is trivial. Supposing F(z) has non-zero defect in z_o, let $k=k_o$, and let the matrices X,U,V,D of (3.8) be indexed by O. If $D(z) = D_o(z)$ has defect $d_o > 0$, the decomposition (3.8) can be repeated for $D_o^{-1}(z)$:

$$D_o^{-1}(z) = X_1(z) + U_1(z) D_1^{-1}(z) V_1(z) \qquad (3.14)$$

and this procedure can be continued for any $D_i^{-1}(z)$ having defect $d_i > 0$, (see Lemma 2).

From Lemma 4 it follows that for any $i < j, d_i > d_j$ holds, but by Lemma 3 any $D_i^{-1}(z)$ has a pole of nth order. Hence the reiteration must break down at a $D_q(z)$ with $d_q=0, q \leq n$.

Since the $X_i(z)$ have non-zero blocks with defect 0 and the U_i, V_i-s may be made analytic in z_o, each X_i defines a $H_i(z)$:

$$
\begin{aligned}
&H_o(z)=(z-z_o)^{k_o}X_o(z)\\
&H_1(z)=(z-z_o)^{k_1}U_o(z)X_1(z)V_o(z)\\
&\quad\vdots\\
&H_i(z)=(z-z_o)^{k_i}U_oU_1\dots U_{i-1}X_iV_{i-1}V_{i-2}\dots V_o
\end{aligned}
\tag{3.15}
$$

where finally $X_q(z)=D_q^{-1}(z)$ may be taken.

On account of the decreasing order of the $D_i(z)$ matrices, the $H_i(z)$ have decreasing rank, thus the Theorem is proved.

The Determination of the Laurent Coefficients

The proof of Theorem 2 also shows how to compute the coefficients of the Laurent expansion.

Explicit formulae may be obtained from the $H_i(z)$ where (2.8), (2.12) may be applied.

In the simplest case, when $k_o=0, q=n=1$, one gets a simple solution for the residual matrix:

$$B_{-1}=U_o(z)\left[D_o'(z)\right]^{-1}V_o(z)\Big|_{z=z_o} \tag{3.16}$$

Here the first derivative of $D_o(z)$ can be composed as

$$D_o'(z)=V_o(z)F'(z)U_o(z) \tag{3.17}$$

Numerically one proceeds in the following manner:

1) Compute the Taylor coefficients (T.c.) of $F(z)$.
2) From the $F_1(z)$ part (see (3.3)) of the T.c. compute the Laurent coefficients (L.c.) of $X_1(z)$ according to the direct solution of (2.15).
3) Compute the T.c. of $U_o(z)$ and $V_o(z)$ from the L.c. of $F_1^{-1}(z)$ and the T.c. of $F_2(z)$ and $F_3(z)$ (see (3.5-6)).
4) Determine the T.c. of $D_o(z)$ (see (3.7)) and the L.c. of its inverse from (2.15), if this is possible.
   Then the L.c. of $F^{-1}(z)$ can be constructed on the basis of (3.8).

5) If the L.c. of $D_o(z)$ can not be calculated directly, apply steps 1),2),3),4) for $D_o(z)$ and continue the procedure for every further $D_i(z)$ until we reach a D_q with $d_q=0, q \leq n$.
6) From the L.c. of $D_q(z)$ construct the L.c. of $D_{q-1}(z)$; similarly from the L.c. of $D_{q-1}(z)$ construct the L.c. of $D_{q-2}(z)$ and so on, until the procedure ends with the L.c. of $F^{-1}(z)$.

It follows from (3.7) that the first ℓ Taylor coefficients of D(z) can be calculated from the first ℓ Taylor coefficients of F(z). This assertion can be maintained from any D_i to D_{i+1}, so that in the multiply decomposed case, too, a knowledge of $A_o, A_1, \dots, A_n$ is necessary to decide the existence of a nth order pole and to determine B_{-n}. Finally the coefficients $A_o, A_1, \dots, A_{2n+j}$ are needed here also for the computation of the Laurent coefficients $B_{-n}, B_{-n+1}, \dots, B_j$.

Example

The matrix

$$F(z) = \left[\begin{array}{cc|c} \cos z & e^z & e^z \\ 0 & e^z & \cos z \\ \hline e^z & \cos z & \cos z \end{array}\right] \qquad (3.18)$$

has the determinant $-(e^z-\cos z)^2(\cos z+e^z)$, which displays a double zero at z=0. The matrix may be partitioned as shown by the dashed lines. Let the submatrices be designated as in (3.3). Although the residual matrix may be determined directly, here our aim is to illustrate the general procedure proposed in this section.

1) The first step is to determine the T.c. of F(z). Here we shall calculate with the first four coefficients and believe it unnecessary to write down them explicitly.

2) The L.c. of $F_1^{-1}(z)$ reduce now to T.c.The coefficients of $F_1^{-1}(z)$ (the labels denote the indices of the coefficients) are:

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}_0, \quad \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}_1, \quad \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}_2, \quad \begin{bmatrix} 0 & 0 \\ 0 & -1/6 \end{bmatrix}_3 \tag{3.19}$$

3) The T.c. of $F_1^{-1}(z)F_2(z)$ are

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}_0, \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}_1, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}_2, \quad \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}_3 \tag{3.20}$$

and the T.c. of $F_3(z)F_1^{-1}(z)$ are

$$[1 \;\; 0]_0, \quad [1 \;\; -2]_1, \quad [1 \;\; -1]_2, \quad [2/3 \;\; -1/3]_3 \tag{3.21}$$

4) The T.c. of $D(z)$ are

$$[0]_0, \quad [0]_1, \quad [-2]_2, \quad [-3]_3 \tag{3.22}$$

This also shows that $F^{-1}(z)$ has a pole of second order at $z=0$. The first two L.c. of $D^{-1}(z)$ are

$$[-1/2]_{-2}, \quad [3/4]_{-1} \tag{3.23}$$

and the first two L.c. of $F^{-1}(z)$ are

$$B_{-2} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}_0 \left[-\frac{1}{2}\right]_{-2} [1 \;\; 0 \;\; -1]_0 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \tag{3.24}$$

$$\begin{aligned} B_{-1} = {} & \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}_1 \left[-\frac{1}{2}\right]_{-2} [1 \;\; 0 \;\; -1]_0 + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}_0 \left[-\frac{1}{2}\right]_{-2} [1 \;\; -2 \;\; 0]_1 + \\ & \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}_0 \left[\frac{3}{4}\right]_{-1} [1 \;\; 0 \;\; -1]_0 = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 3 & 4 & -5 \\ -1 & -4 & 3 \end{bmatrix} \end{aligned} \tag{3.25}$$

4. Appendix

This section examines some additional properties of the function matrices. First we shall derive an upper bound for the order of the pole, after which we present a generalization of the logarithmic derivative.

The following lemma is needed in the estimation:

Lemma 5.

Let the submatrices be indexed at any partitioning as in (3.3). Then, if $D(z)$ is defined by (3.7),

$$\mathrm{Det}\left[P\,F(z)\,Q\right] = \mathrm{Det}\,F_1(z)\;\mathrm{Det}\,D(z) \tag{4.1}$$

Proof Let F_1 have size ϱ. Then

$$\mathrm{Det}\left[P\,F(z)\,Q\right] = \mathrm{Det}\left[\begin{array}{c|c} F_1 & F_2 \\ \hline F_3 & F_4 \end{array}\right] \mathrm{Det}\left[\begin{array}{c|c} F_1^{-1} & 0 \\ \hline 0 & I_{N-\varrho} \end{array}\right] \mathrm{Det}\,F_1 =$$

$$\mathrm{Det}\left[\begin{array}{c|c} I_\varrho & F_2 \\ \hline F_3F_1^{-1} & F_4 \end{array}\right] \mathrm{Det}\,F_1 = \mathrm{Det}\left[\begin{array}{c|c} I_\varrho & 0 \\ \hline F_3F_1^{-1} & F_4-F_3F_1^{-1}F_2 \end{array}\right] \mathrm{Det}\,F_1$$

QED.

A Bound for the Order of the Pole

We assume for $F(z)$ the properties a), b), c) of sect. 1 and introduce $G(z)$ as in (2.6), $G(z_o)$ being the limit of (2.7).

From the inverse matrix formula

$$G^{-1}(z) = \frac{\mathrm{Adj}\,G(z)}{\mathrm{Det}\,G(z)} \tag{4.2}$$

one concludes that the order of the pole is determined by the adjoint matrix element having the smallest order of zero in z_o.

Let $G(z_0)$ have size N and rank ϱ. If Det $G(z)$ has a pth order zero in z_0, then

$$p \geq N-\varrho , \qquad 0 < \varrho \leq N \tag{4.3}$$

This can be seen applying (4.1) to $G(z)$ with the partitioning described in (3.3), taking into account that the new $D(z) = G_4(z) - G_3(z)G_1^{-1}(z)G_2(z)$ of size $N-\varrho$ is, by (3.11), zero at $z=z_0$.

The possible greatest rank of a $(N-1)x(N-1)$ submatrix is $\varrho - \delta_{\varrho,N}$, thus the adjoint matrix element, defined as the determinant of this submatrix, may have the smallest order of zero in z_0. Concretely, the smallest order of zero may be greater than or equal to $N-1-(\varrho-\delta_{\varrho,N})$, similarly to (4.3).

Finally, one gets for the order of the pole of $F^{-1}(z)$

$$n \leq k+p-N+\varrho +1-\delta_{\varrho,N} \tag{4.4}$$

where k is defined in (1.4). Of course, if $p=N-\varrho$, equality holds. This can be shown by Theorem 1 and Lemma 2.

Generalization of the Logarithmic Derivative

Some further formulae can be derived which, although they contain the order of pole explicitly, may be of lesser importance in determining the order of the pole

First, by substituting $F(z)$ for (z) and n for k in (2.3) on gets

$$\lim_{z \to z_0} (z-z_0)\frac{dE(z)}{dz} = 0 \tag{4.5}$$

and the substitution of $E(z)$ with the aid of (2.8) yields

$$\lim_{z \to z_0} (z-z_0)^{n+1} F^{-1}(z)F'(z)F^{-1}(z) = nB_{-n} \tag{4.6}$$

If B_{-n} has an inverse, it is easy to conclude that B_{-n} can be extracted from both sides of (4.6). There are two possibilities of doing this on the left side, either from the left, or from the right, giving

$$\lim_{z\to z_0}(z-z_0)F^{-1}(z)F'(z) = \lim_{z\to z_0}(z-z_0)F'(z)F^{-1}(z)=nI \quad (4.7)$$

This is a generalization of the well-known property of the logarithmic derivate of scalar functions.

If B_{-n} can not be inverted, it can not be asserted that

$$\lim_{z\to z_0}(z-z_0)F'(z)F^{-1}(z)$$

is finite; in fact, $F'(z)F^{-1}(z)$ may have a higher order pole. The general answer is given by making use of the series (1.4) and (1.5):

$$F'F^{-1} = \sum_{j=k}^{\infty}\sum_{\ell=-n}^{\infty}(z-z_0)^{\ell+j-1}\,jA_jB_\ell = \sum_{j=k-n-1}^{\infty}(z-z_0)^jE_j \quad (4.8)$$

and, with the help of (1.8), the coefficients of the two lowest powers are

$$E_{k-n-1} = kA_k\,B_{-n} = k\,\delta_{n,k}\,I \quad (4.9)$$

$$E_{k-n} = kA_kB_{-n+1}+(k+1)A_{k+1}B_{-n} = k\,\delta_{n,k+1}I + A_{k+1}\,B_{-n} \quad (4.1o)$$

Thus in the case $k \neq n$, $F'(z)F^{-1}(z)$ has a pole of $(n-k)$th order if $A_{k+1}B_{-n} \neq 0$, though the case $n-k=1$ is independent of this condition.

Finally, we note that our results can be extended to negative k and n as well.

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Felelős kiadó: Varga László, a KFKI Számitástechnikai Tudományos Tanácsának elnöke
Szakmai lektor: Zimányi Józsefné
Nyelvi lektor: T. Wilkinson
Példányszám: 180 Törzsszám: 73-8055
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